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Finite subtransitive sets

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Let $\varphi : G \to \mathrm{O}(d)$ be an orthogonal representation of a group G. A finite subset X of a Euclidean space $\mathbf{R}^d$ is said to be *subtransitive under* (G, φ) when it is a subset of an orbit under the action of $\varphi(G)$; furthermore, we say that X is *finitely subtransitive* when it is subtransitive under (G, φ) for some finite group G. If X is subtransitive under a representation $\varphi : G \to \mathrm{O}(d)$ then it is clear that X is spherical; that is, contained in a sphere in the representation space $\mathbf{R}^d$. In connection with Euclidean Ramsey theory, I.B. Leader [**6**] has posed the converse question:

Is is true that every finite spherical set is finitely subtransitive?

In this paper we establish two general results in connection with Leader's question. In a positive direction, we give a characterisation of finite spherical sets in terms of subtransitivity under general finitely generated discrete groups; thus let F_n denote the (non abelian) group on $n \geqslant 2$ generators.

Theorem A. *Let X be a finite spherical set; then*

(i) *X is subtransitive under a representation $\varphi : F_2 \to \mathrm{O}(d)$; furthermore*

(ii) *X is finitely subtransitive if and only if it is subtransitive under a representation $\varphi : F_2 \to \mathrm{O}(d)$ in which* $\mathrm{Ker}(\varphi)$ *has finite index in F_2.*

In a different direction, we first consider the totality of *finite metric spaces*; more particularly, we consider the set $\mathcal{I}(m)$ of *isometry classes* of metric spaces with a given finite cardinal m. We begin by showing that $\mathcal{I}(m)$ has a tractable topological structure; it is the quotient of a semi-algebraic set by a finite group, and so admits a stratification in the sense of Thom–Whitney [**9**, **10**]. It follows that $\mathcal{I}(n)$ admits a parametrization by a locally finite simplicial complex of dimension $m(m-1)/2$ and, in particular, admits a positive measure.

The generic situation in which a metric space of cardinal m occurs as a spherical set is when its points are affinely independent, that is span a Euclidean $(m-1)$-simplex. In this case, a result of Schoenberg [**8**] shows that the set $\mathcal{I}(m)$ of *all* isometry classes of metrics is *a priori* larger than the set of isometry classes arising from imbeddings in orthogonal Γ-orbits. However, as we show in Section 4, the set $\mathcal{E}(m)$ of isometry classes of Euclidean $m-1$ simplices also has a positive measure, since by Schoenberg's Theorem it imbeds in $\mathcal{I}(m)$ as an open subset. Denoting by $\mathcal{E}(m, d)$ the subset of $\mathcal{E}(m)$ represented by isometry classes of sets which imbed subtransitively in an orthogonal Γ-representation $\rho : \Gamma \to \mathrm{O}(d)$, for some finite group Γ, we show:

Theorem B. *If $d < m(m-1)/2$ then $\mathcal{E}(m, d)$ has measure zero in $\mathcal{E}(m)$.*

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Leader has observed [**6**] that a well known argument in Euclidean Ramsey theory (c.f. [**1**]) may be modified to show that any Euclidean 2-simplex (i.e. triangle) imbeds subtransitively in a 3-dimensional orthogonal representation of a suitable finite group. In Section 5 we give a reformulation of Leader's argument which is more straightforwardly group theoretic than the Fränkl–Rödl construction. As a by-product, it follows that, for triangles, the finite group may always be taken to be a product $D_{2n} \times C_2$ where D_{2n} the dihedral group of order $2n$ and C_2 is the cyclic group of order 2.

1. *A general criterion for subtransitivity*

We denote by F_n the free nonabelian group on n generators.

PROPOSITION 1·1. *Every finite spherical set is subtransitive under an orthogonal representatation of* F_2.

Proof. Let $S^{m-1}(r)$ be the sphere of radius r in $\mathbf{R}^m$ and let $X \subset S^{m-1}(r)$ be a finite subset. Without loss, we may suppose that X has $n \geqslant 3$ elements, indexed thus: $\{x_0, x_1, \ldots, x_{n-1}\}$. Since $\mathrm{O}(m)$ acts transitively on $S^{m-1}(r)$, for each index i we choose $T_i \in \mathrm{O}(m)$ such that $T_i(\mathbf{x}_0) = \mathbf{x}_i$. Let F_{n-1} be the free group on generators $t_1, \ldots, t_{n-1}$ and let $\varphi : F_{n-1} \to \mathrm{O}(m)$ be the representation given by $\varphi(t_i) = T_i$. By construction X is subtransitive under the representation (F_{n-1}, φ).

Observe that for $n \geqslant 3$, there is an imbedding $\iota : F_{n-1} \to F_2$ of F_{n-1} onto a subgroup of index $n-2$ in F_2. In particular, if F_2 is the free group on two generators ξ, η then the elements η^{n-1} and $\eta^r \xi \eta^{-r}$ for $0 \leqslant r \leqslant n-3$ freely generate a subgroup isomorphic to F_{n-1}. Taking the induced representation $\Phi : F_2 \to \mathrm{O}(d)$, where $d = m(n-2)$, we see easily that X is subtransitive under (F_2, Φ) via the diagonal imbedding

$$\mathbf{R}^m \longrightarrow \underbrace{\mathbf{R}^m \times \cdots \times \mathbf{R}^m}_{n-2}.$$

THEOREM A. *Let X be a finite spherical set; then the following are equivalent:*

(i) *X is finitely subtransitive;*

(ii) *X is subtransitive under a representation (F_2, φ), $\varphi : F_2 \to \mathrm{O}(d)$ in which $\mathrm{Ker}(\varphi)$ has finite index in F_2.*

Proof. If X is subtransitive under a representation $\varphi : F_2 \to \mathrm{O}(d)$ then it is also subtransitive under the quotient representation $\varphi_* : F_2/\mathrm{Ker}(\varphi) \to \mathrm{O}(d)$. The implication (ii) $\Rightarrow$ (i) is now straightforward.

To prove (i) $\Rightarrow$ (ii) we repeat the proof of Proposition 1·1 but with extra conditions. Suppose that X is subtransitive under (G, φ) where G is finite with $|G| = N$ and $\varphi : G \to \mathrm{O}(m)$. By Cayley's Theorem, imbed G into the group of permutations $G \subset \Sigma_N$ and take the induced representation $\Phi : \Sigma_N \to \mathrm{O}(d)$, where $d = mN$. Then X is still subtransitive under (Σ_N, Φ), again via the diagonal imbedding. However Σ_N is a two generator group, generated by a cycle of length N and a transposition.

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Let $\mu : F_2 \to \Sigma_N$ be a surjective homomorphism, and put $\Psi = \Phi \circ \mu$. Then X is subtransitive under (F_2, Ψ) and $\mathrm{Ker}(\Psi)$ has finite index in F_2.

2. *The space of finite metric spaces*

Fix a finite set X; we denote by $X(2)$ the set of pairs $\{x, y\}$, where $x, y \in X$ and $x \neq y$. For $\alpha, \beta \in X(2)$, we denote by $\alpha \bullet \beta$ the symmetric square

$$\alpha \bullet \beta = \alpha \bigcup \beta - \alpha \bigcap \beta.$$

Let $\mathcal{F}(X(2), \mathbf{R})$ be the set of functions $f : X(2) \to \mathbf{R}$. Then $\mathcal{M}(X)$, the set of metrics on X can be described as follows; if $\mu \in \mathcal{M}(X)$, then the mapping $\{x, y\} \mapsto \mu(x, y)$ defines an element $\tilde{\mu} \in \mathcal{F}(X(2), \mathbf{R})$, and the correspondence $\mu \mapsto \tilde{\mu}$ gives an imbedding $\mathcal{M}(X) \to \mathcal{F}(X(2), \mathbf{R})$. Describing a point $\xi \in \mathcal{F}(X(2), \mathbf{R})$ by $\xi = (\xi_\alpha)_{\alpha \in X(2)}$, then ξ takes the form $\xi = \tilde{\mu}$ for some metric μ on X if and only the following conditions hold:

(i) $0 < \xi_\alpha$ for all $\alpha \in X(2)$; and

(ii) $\xi_{\alpha \bullet \beta} \leqslant \xi_\alpha + \xi_\beta$ whenever $\alpha \bullet \beta \in X(2)$.

If condition (i) is relaxed to requiring only that $0 \leqslant \xi_\alpha$ for all $\alpha \in X(2)$ then ξ is said to be a *semi-metric*. We shall see that semi-metrics arise in the context of group actions by regarding the change from a generic to a singular orbit as a 'metric degeneration' in which certain distances are allowed to become zero. We denote by $\overline{\mathcal{M}(X)}$ the set of semi-metrics on X. It is clear that:

Proposition 2·1. *For any finite set X, both $\mathcal{M}(X)$ and $\overline{\mathcal{M}(X)}$ are semi-algebraic submanifolds of codimension 0 in $\mathcal{F}(X(2), \mathbf{R})$. In particular, $\mathcal{M}(X)$ has dimension $m(m-1)/2$ where $m = |X|$.*

If μ_1, μ_2 are metrics on X, we say that (X, μ_1) is *strictly similar* to (X, μ_2) when there exists a positive real number λ such that, for all $x, y \in X$,

$$\mu_2(x, y) = \lambda \mu_1(x, y).$$

Let $\mathcal{F}_0(X(2), \mathbf{R})$ be the unit sphere in $\mathcal{F}(X(2), \mathbf{R})$ with respect to the Euclidean norm $\|\xi\|^2 = \sum_{\alpha \in X(2)} \xi_\alpha^2$. Then $\mathcal{M}_0(X) = \mathcal{M}(X) \bigcap \mathcal{F}_0(X(2), \mathbf{R})$ represents a complete set of strict similarity classes of metrics on X. Furthermore, the correspondence

$$\xi \longmapsto \left(\frac{\xi}{\|\xi\|}, \|\xi\| \right)$$

gives rise to a direct product structure $\mathcal{M}(X) \leftrightarrow \mathcal{M}_0(X) \times (0, +\infty)$. More generally, if μ_1, μ_2 are metrics on X, we say that (X, μ_1) is *similar* to (X, μ_2) when there exists a positive real number λ and a permutation σ of X such that, for all $x, y \in X$,

$$\mu_2(x, y) = \lambda \mu_1(\sigma(x), \sigma(y)).$$

In the case where $\lambda = 1$ we say that (X, μ_1) and (X, μ_2) are *isometric*. To parametrize isometry classes rather than metrics, we must allow for the action of the permutation group, Σ_X, of X, which acts on the right of $\mathcal{F}(X(2), \mathbf{R})$ by

$$f^\sigma(\{x, y\}) = f(\{\sigma(x), \sigma(y)\}).$$

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$\mathcal{M}(X)$ is invariant under this action, and we may identify the set

$$\mathcal{I}(X) = \mathcal{M}(X)/\Sigma_X,$$

of isometry classes of metrics on X with a subset of $\mathcal{F}(X(2), \mathbf{R})/\Sigma_X$. Moreover, the action of Σ_X is compatible with the product structure on $\mathcal{M}(X)$. Denoting by $\mathcal{I}_0(X)$ the set of similarity classes of metrics on X, we obtain a corresponding product structure $\mathcal{I}(X) \leftrightarrow \mathcal{I}_0(X) \times (0, +\infty)$. The spaces $\mathcal{I}(X)$, $\mathcal{I}_0(X)$ have 'tame topological structures' in the sense of Grothendieck [**3**]. In particular, they admit Whitney–Thom stratifications [**4**, **5**, **9**, **10**], and so are triangulable by locally finite simplicial complexes of finite dimension [**4**]. To see this, first note that if Γ is a finite group acting smoothly on a closed manifold M, then the quotient M/Γ is a Whitney–Thom stratification. Now $\mathcal{M}_0(X)$ is a semi-algebraic subset of codimension zero in a Euclidean sphere and since $\mathcal{M}_0(X)$ is invariant under the action of Σ_X, similar considerations apply to show that the quotient $\mathcal{I}_0(X) = \mathcal{M}_0(X)/\Sigma_X$ admits a Whitney–Thom stratification. It now follows that:

THEOREM 2·3. *Let X be a finite set with $|X| = m$; then $\mathcal{I}(X)$, the space of isometry classes, and $\mathcal{I}_0(X)$, the space of similarity classes of metrics on X admit Whitney–Thom stratifications with respect to which there is a natural product structure*

$$\mathcal{I}(X) = \mathcal{I}_0(X) \times (0, +\infty).$$

Clearly $\mathcal{I}(X)$, being a finite quotient of $\mathcal{M}(X)$, has dimension $m(m-1)/2$.

3. *Euclidean simplices and Schoenberg's Theorem*

Put $X = \{0, 1, \ldots, m-1\}$, and write $\mathcal{M}(m) = \mathcal{M}(X)$, $\mathcal{I}(m) = \mathcal{I}(X)$. By an *Euclidean $(m-1)$-simplex* we mean a metric induced on $\{0, 1, \ldots, m-1\}$ from an imbedding as a set of affinely independent points in a Euclidean space of arbitrary dimension $\geqslant m-1$. Such a simplex is *spherical* when its vertices lie on a sphere with centre at the origin. The classical theorem on escribed circles shows that every Euclidean 2-simplex (i.e. triangle) is isometric to one which is spherical. More generally, one can show (see Fränkl–Rödl [**2**]) that, up to isometry, every Euclidean $(m-1)$-simplex is spherical.

We shall denote by $\mathcal{E}(m)$ the set of isometry classes of Euclidean $(m-1)$-simplices. To describe $\mathcal{E}(m)$ as a subset of $\mathcal{I}(m)$ we appeal to a theorem of Schoenberg [**8**]. To any metric μ on $\{0, 1, \ldots, m-1\}$ we associate the quadratic form in $(m-1)$-variables

$$\mathcal{Q}^\mu(\mathbf{x}) = \sum_{1 \leqslant i,j \leqslant m-1} \left(\mu(0,j)^2 + \mu(0,k)^2 - \mu(j,k)^2\right) \mathbf{x}_i \mathbf{x}_j.$$

THEOREM 3·1 (Schoenberg [**8**]). *$(\{0, 1, \ldots, m-1\}, \mu)$ is isometric with a subset of a Euclidean space if and only if the quadratic form $\mathcal{Q}^\mu$ is positive semi-definite. Furthermore, $(\{0, 1, \ldots, m-1\}, \mu)$ is isometric with a Euclidean $(m-1)$-simplex if and only if $\mathcal{Q}^\mu$ is positive definite.*

The space of positive definite quadratic forms of a given rank k is an open subset in the space of all quadratic forms of rank k. Thus it follows from Schoenberg's Theorem that $\mathcal{E}(m)$ is an open subset of $\mathcal{I}(m)$.

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4. *The mapping induced by a representation*

Let Γ be a finite group and let $\rho : \Gamma \to \mathrm{O}(d)$ be a representation. We obtain a mapping $\nu(\Gamma, \rho) : \mathbf{R}^d \to \overline{\mathcal{M}(\Gamma)}$ by defining

$$\nu(\Gamma, \rho)(\mathbf{x}) = \left(\|g.\mathbf{x} - h.\mathbf{x}\|\right)_{\{g,h\} \in \Gamma(2)} \quad \text{for } \mathbf{x} \in \mathbf{R}^d.$$

Proposition 4·1. *$\mathrm{Im}(\nu(\Gamma, \rho))$ is a subset of dimension $\leqslant d$ in $\overline{\mathcal{M}(\Gamma)}$.*

Proof. Regarding $\mathbf{R}^d$, $\mathcal{F}(\Gamma(2), \mathbf{R})$ as complete real affine varieties, there is a real polynomial mapping $\hat{\nu}(\Gamma, \rho) : \mathbf{R}^d \to \mathcal{F}(\Gamma(2), \mathbf{R})$ given by

$$\hat{\nu}(\Gamma, \rho)(\mathbf{x}) = \left(\|g.\mathbf{x} - h.\mathbf{x}\|^2\right)_{\{g,h\} \in \Gamma(2)}.$$

Then $\mathrm{Im}(\hat{\nu}(\Gamma, \rho))$ is a real algebraic set of dimension $\leqslant d$; in fact, by the theorem of Lojasiewicz [7] there exists a triangulation of $\mathcal{F}(\Gamma(2); \mathbf{R})$ in which $\mathrm{Im}(\hat{\nu}(\Gamma, \rho)$ is triangulated as subcomplex of dimension $\leqslant d$.

Using natural coordinates, identify $\mathcal{F}(\Gamma(2); \mathbf{R})$ with $\mathbf{R}^N$, and let $\mathcal{F}_+$ denote the subset of $\mathcal{F}(\Gamma(2); \mathbf{R})$ corresponding to the generalized quadrant $\mathbf{R}_+^N$, where $\mathbf{R}_+ = \{x \in \mathbf{R} : x \geqslant 0\}$. Then $\mathrm{Im}(\hat{\nu}(\Gamma, \rho)$ and $\overline{\mathcal{M}(\Gamma)}$ are both contained in $\mathcal{F}_+$. However, the mapping $Sq : \mathcal{F}_+ \to \mathcal{F}_+$ given in natural coordinates by

$$Sq((x_1, \ldots, x_N)) = \left(x_1^2, \ldots, x_N^2\right)$$

is a homeomorphism, and it is clear that $Sq \circ \nu(\Gamma, \rho) = \hat{\nu}(\Gamma, \rho)$. Hence $\mathrm{Im}(\nu(\Gamma, \rho))$ is a subset of dimension $\leqslant d$ in $\overline{\mathcal{M}(\Gamma)}$ as claimed.

If X is a set on which Γ acts on the left then a semi-metric μ on X is said to be Γ-equivariant when $\mu(\gamma \cdot \mathbf{x}, \gamma \cdot \mathbf{y}) = \mu(\mathbf{x}, \mathbf{y})$ for all $\gamma \in \Gamma$ and all $\mathbf{x}, \mathbf{y} \in X$. We denote by $\overline{\mathcal{M}_\Gamma(X)}$ (resp. $\mathcal{M}_\Gamma(X)$) the set of all Γ equivariant semi-metrics (resp. Γ equivariant metrics) on X. It is straightforward to check that $\overline{\mathcal{M}_\Gamma(\Gamma)}$ is a closed semi-algebraic subset of $\overline{\mathcal{M}(\Gamma)}$.

If $\rho : \Gamma \to \mathrm{O}(d)$ is a representation and $\mathbf{x} \in \mathbf{R}^d$ it is straightforward to see that $\nu(\Gamma, \rho)(\mathbf{x})$ is Γ-equivariant, so that:

Proposition 4·2. *$\mathrm{Im}(\nu(\Gamma, \rho))$ is contained in $\overline{\mathcal{M}_\Gamma(\Gamma)}$.*

For $\mu \in \overline{\mathcal{M}_\Gamma(\Gamma)}$ put $Z(\mu) = \{\gamma \in \Gamma : \mu(1, \gamma) = 0\}$; then $Z(\mu)$ is a subgroup of Γ. When Δ is a subgroup of Γ, put

$$S(\Gamma, \Delta) = \{\mu \in \overline{\mathcal{M}_\Gamma(\Gamma)} : Z(\mu) = \Delta\}.$$

Then regarding Γ/Δ as Γ-set under left translation, we obtain a bijection $\iota : \mathcal{M}_\Gamma(\Gamma/\Delta) \to S(\Gamma, \Delta)$ by defining $\iota(\mu)(\gamma_1, \gamma_2) = \mu(\gamma_1\Delta, \gamma_2\Delta)$, so identifying $S(\Gamma, \Delta)$ with the set $\mathcal{M}_\Gamma(\Gamma/\Delta)$ of Γ-equivariant metrics on Γ/Δ. Evidently

$$\overline{\mathcal{M}_\Gamma(\Gamma)} = \bigcup_\Delta S(\Gamma, \Delta)$$

where Δ runs through all subgroups of Γ. In this representation, $\mathcal{M}_\Gamma(\Gamma)$ is identified naturally with $S(\Gamma, \{1\})$ so that the union $\bigcup_{\Delta \neq \{1\}} S(\Gamma, \Delta)$ decomposes the frontier $\overline{\mathcal{M}_\Gamma(\Gamma)} - \mathcal{M}_\Gamma(\Gamma)$ into strata.

Let Γ be a finite group; if Δ is a subgroup of Γ and $\mathcal{X}$ is a subset of Γ/Δ there is a 'restriction of metrics' map $r : \mathcal{M}_\Gamma(\Gamma/\Delta) \to \mathcal{M}(\mathcal{X})$ induced from the canonical linear projection $\mathcal{F}((\Gamma/\Delta)(2); \mathbf{R}) \to \mathcal{F}(\mathcal{X}(2); \mathbf{R})$.

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Let $\rho : \Gamma \to \mathrm{O}(d)$ be an orthogonal representation. If X is a finite set we say that a metric μ on X is of type (Γ, d, ρ) when there exists a subgroup Δ of Γ and a subset $\mathcal{X} \subset \Gamma/\Delta$ such that for some bijection $\psi : \mathcal{X} \to X$,

$$\mu = \psi_* \circ r \circ \iota^{-1}(\eta)$$

where $\eta \in \mathrm{Im}(\nu(\Gamma, \rho)) \cap S(\Gamma, \Delta)$, where $\iota : \mathcal{M}_\Gamma(\Gamma/\Delta) \to S(\Gamma, \Delta)$ is the bijection constructed above, and where finally $\psi_* : \mathcal{M}(\mathcal{X}) \to \mathcal{M}(X)$ is the bijection on metrics induced by ψ.

PROPOSITION 4·3. *Let $\rho : \Gamma \to \mathrm{O}(d)$ be an orthogonal representation of a finite group Γ; then for any finite set X, the metrics of type (Γ, d, ρ) on X constitute a subset of dimension $\leqslant d$ in $\mathcal{M}(X)$.*

Proof. Since there are only finitely many subgroups Δ of Γ, and, for any subgroup Δ, only finitely many bijections $\psi : \mathcal{X} \to X$ where $\mathcal{X} \subset \Gamma/\Delta$, it suffices to show that for any subset $\mathcal{X} \subset \Gamma/\Delta$, $r \circ \iota^{-1}(\mathrm{Im}(\nu(\Gamma, \rho)) \cap S(\Gamma, \Delta))$ has dimension $\leqslant d$ in $\mathcal{M}(\mathcal{X})$.

However, by Proposition 4·1 $\iota^{-1}(\mathrm{Im}(\nu(\Gamma, \rho)) \cap S(\Gamma, \Delta))$ has dimension $\leqslant d$ in $\mathcal{M}_\Gamma(\Gamma/\Delta)$, so the result follows since r is the restriction of a linear map and so cannot raise dimension.

To consider the effect of varying ρ through orthogonal equivalence suppose that $\sigma : \Gamma \to \mathrm{O}(d)$ satisfies

$$\sigma(g) = T\rho(g)T^{-1}$$

for all $g \in \Gamma$. Then $\nu(\Gamma, \sigma)(T(\mathbf{x})) = \nu(\Gamma, \rho)(\mathbf{x})$ for all $\mathbf{x} \in \mathbf{R}^d$. In particular, the set $\mathrm{Im}(\nu(\Gamma, \rho))$ depends only upon the *orthogonal equivalence class* of the representation ρ.

If X is a finite set and $d \geqslant 1$ we denote by $\mathcal{M}(X, d)$ the subset of $\mathcal{M}(X)$ consisting of metrics which imbed subtransitively in an orthogonal Γ-representation $\rho : \Gamma \to \mathrm{O}(d)$, for some finite group Γ.

COROLLARY 4·4. *Let X be a set with $|X| = m$. If $d < m(m-1)/2$ then $\mathcal{M}(X, d)$ is a subset of measure zero in $\mathcal{M}(X)$.*

Proof. Let $\langle \Gamma, d, \rho \rangle$ denote the set of metrics on X of type (Γ, d, ρ). It is straightforward to see that a metric μ on X is finitely subtransitive, that is, arises from an imbedding into an orbit of an orthogonal representation of Γ, if and only if it is of type (Γ, d, ρ) for some (d, ρ). In the above description $\mu = \psi_* \circ r \circ \iota^{-1}(\eta)$, if $\eta = \nu(\Gamma, \rho)(\mathbf{x}) \in S(\Gamma, \Delta)$ then Δ is the isotropy subgroup of $\mathbf{x}$ under the representation ρ and ψ^{-1} parametrizes the subset of Γ/Δ which describes the imbedding of X into the orbit of $\mathbf{x} \in \mathbf{R}^d - \{0\}$. That is, $\mathcal{M}(X, d)$ is the union of all $\langle \Gamma, d, \rho \rangle$ where d is the representation degree of ρ. Now $\dim(\langle \Gamma, d, \rho \rangle) \leqslant d$ by Proposition 4·2 and hence has measure zero in $\mathcal{M}(X)$ since $d < m(m-1)/2 = \dim(\mathcal{M}(X))$. However, the set $\langle \Gamma, d, \rho \rangle$ depends only upon the orthogonal equivalence class of ρ. But for $d \geqslant 1$ and any finite group Γ there are only finitely many orthogonal equivalence classes of representations $\Gamma \to \mathrm{O}(d)$. Since there are only countably many finite groups, it follows that

$$\mathcal{M}(X, d) = \bigcup_{(\Gamma, \rho)} \langle \Gamma, d, \rho \rangle$$

is a countable union of sets of measure zero and so has measure zero.

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We denote by $\mathcal{E}(m,d)$ the subset of $\mathcal{E}(m)$ represented by metrics which imbed in an orbit of an orthogonal representation $\rho:\Gamma\to \mathrm{O}(d)$ where Γ is finite; that is,

$$\mathcal{E}(m,d) = \mathcal{M}(m,d)/\Sigma_m$$

where $\mathcal{M}(m,d) = \mathcal{M}(\{0,1,\dots,m-1\},d)$; then:

THEOREM B. *If $d < m(m-1)/2$ then $\mathcal{E}(m,d)$ has measure zero in $\mathcal{E}(m)$.*

Proof. Since the projection map $\mathcal{M}(m,d)\to\mathcal{M}(m,d)/\Sigma_m \subset \mathcal{I}(m)$ is finite-to-one, then under the hypothesis $d < m(m-1)/2 = \dim(\mathcal{I}(m))$ it follows from Corollary 4·4 that $\mathcal{E}(m,d) = \mathcal{M}(m,d)/\Sigma_m$ has measure zero in $\mathcal{I}(m)$.

However, $\mathcal{E}(m,d)\subset\mathcal{E}(m)$, and by Schoenberg's Theorem, $\mathcal{E}(m)$ is open in $\mathcal{I}(m)$, so that $\mathcal{E}(m,d)$ has measure zero in $\mathcal{E}(m)$.

5. *Subtransitivity of triangles and an observation of Leader*

In this section we give a proof of Leader's observation that, up to similarity, any Euclidean triangle is subtransitive in a 3-dimensional representation of a finite group. The corresponding result for isometry classes follows from the product structure, by suitably dilating the radius of the 2-sphere in the representation space. Since the space $\mathcal{E}(3)$ of isometry classes of triangles is three dimensional it follows by Theorem B that this is best possible.

Up to similarity, a Euclidean triangle may be described by a triple of angles $(\lambda_1,\lambda_2,\lambda_3)$ taken in decreasing size $0<\lambda_1\leqslant\lambda_2\leqslant\lambda_3$ subject to the condition $\lambda_1+\lambda_2+\lambda_3=\pi$.

Eliminating λ_3 as a dependent variable, we parametrize similarity classes of Euclidean triangles by the bounded region $\tilde{\mathcal{P}}$ of the (λ_1,λ_2) plane defined by the inequalities $0<\lambda_1;\ \lambda_1\leqslant\lambda_2;\ \lambda_1+2\lambda_2\leqslant\pi$. The region $\tilde{\mathcal{P}}$ is sketched in Figure 1, where it is understood that points on the vertical axis are excluded.

In this description, isoceles triangles are singular, being represented by the boundary lines, with the equilateral triangle located at the corner point. We denote by $\mathcal{P}$ the interior of $\tilde{\mathcal{P}}$; evidently $\mathcal{P}$ is the parameter space of scalene (i.e. generic) triangles. We deal with these first. As we shall then see, isoceles triangles can be treated in a similar though simpler manner.

We begin by constructing a family of triangles imbedded in $\mathbf{R}^3$ which, for convenience, we describe as $\mathbf{C}\times\mathbf{R}$. For $(\theta,\alpha)\in\mathcal{P}$ and $t\in\mathbf{R}$ we denote by $T(\theta,\alpha,t)$ the triangle in $\mathbf{C}\times\mathbf{R}$ given by

$$T(\theta,\alpha,t) = \{(e^{i(\theta+2\alpha)},-t),(e^{-i\theta},t),(e^{i\theta},t)\}.$$

We illustrate the situation in Figure 2 below:

Denoting by $\lambda_i=\lambda_i(\theta,\alpha,t)$ the angle at $\mathbf{v}_i$, and by S_i the side length opposite to $\mathbf{v}_i$, one calculates as indicated that

$$S_1 = 2\sin(\theta);\quad S_2 = 2\sqrt{\sin^2(\alpha)+t^2};\quad S_3 = 2\sqrt{\sin^2(\alpha+\theta)+t^2}.$$

From the identity $\sin(\alpha+\theta)-\sin(\alpha) = 2\sin(\theta/2)\cos(\alpha+\theta/2)$ and the definition of $\mathcal{P}$ we see easily that $0<\sin(\theta)<\sin(\alpha)<\sin(\alpha+\theta)$ for $(\theta,\alpha)\in\mathcal{P}$. From the above side length calculations one now concludes:

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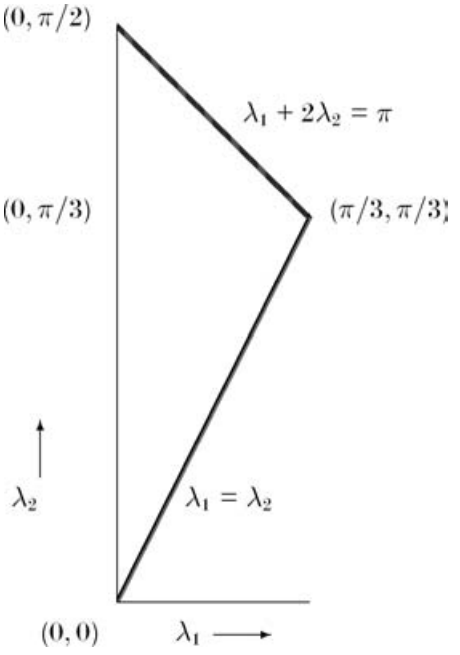


Fig. 1.

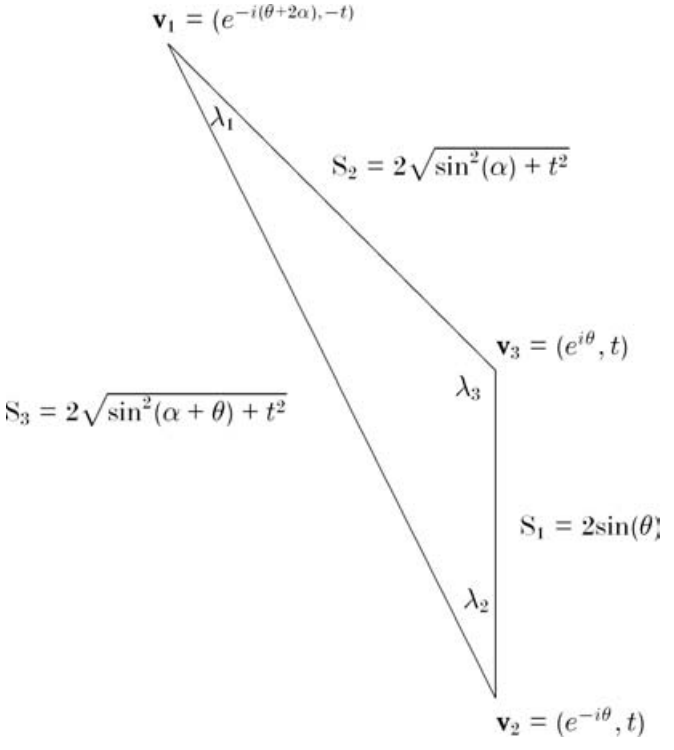


Fig. 2.

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PROPOSITION 5·1. *For $(\theta, \alpha) \in \mathcal{P}$ the triangle $T(\theta, \alpha, t)$ is scalene, with $S_1 < S_2 < S_3$ and $\lambda_1 < \lambda_2 < \lambda_3$.*

Consider the smooth mapping $\Lambda : \mathcal{P} \times \mathbf{R} \to \mathbf{R}^2$ defined by

$$\Lambda(\theta, \alpha, t) = \left(\frac{\sqrt{\sin^2(\alpha) + t^2}}{\sin(\theta)}, \qquad \frac{\sqrt{\sin^2(\alpha + \theta) + t^2}}{\sin(\theta)} \right).$$

The values of the component functions of Λ at (θ, α, t) are simply the ratios of the short side length to, respectively, the middle and longest side lengths of the triangle $T(\theta, \alpha, t)$. Let A be a triangle, represented up to similarity in $\mathcal{P}$ by the parameters (θ, α). At $(\theta, \alpha, 0)$ the Jacobian determinant of Λ takes the value

$$\left| \frac{\partial \Lambda}{\partial \theta \partial \alpha}(\theta, \alpha, 0) \right| = \frac{\sin(\alpha)\sin(\alpha + \theta)}{\sin^2(\theta)} \neq 0.$$

Since Λ is continously differentiable, there exists a neighbourhood $\mathcal{N}$ of $(\theta, \alpha, 0)$ in $\mathcal{P} \times \mathbf{R}$ such that for all $(\theta', \alpha', t) \in \mathcal{N}$

$$\left| \frac{\partial \Lambda}{\partial \theta \partial \alpha}(\theta', \alpha', t) \right| \neq 0.$$

By the Implicit Function Theorem there exists $d_A > 0$ and smooth mapping $\theta^A \times \alpha^A : (-d_A, d_A) \to \mathcal{P}$ such that $\theta^A(0) = \theta$, $\alpha^A(0) = \alpha$, and such that for all $t \in (-d_A, d_A)$,

$$\Lambda(\theta^A(t), \alpha^A(t), t) = \Lambda(\theta, \alpha, 0).$$

Since the side length ratios remain unchanged throughout the variation, then each $T(\theta^A(t), \alpha^A(t), t)$ is similar to A. Direct computation shows that for $(\theta', \alpha') \in \mathcal{P}$, the middle angle $\lambda_2(\theta', \alpha', t)$ of $T(\theta', \alpha', t)$ has cosine given by

$$\cos(\lambda_2) = \frac{\cos(\alpha')}{\sqrt{1 + \left(t/\sin(\alpha' + \theta')\right)^2}}.$$

We claim that $\alpha^A : [0, d^A) \to \mathbf{R}$ is not constant. Otherwise, the constant value would be the limiting value as $t \to 0$, namely α, the middle angle of A, and we would have $\alpha^A(t) = \alpha$ for all t. Since the similarity class of A is constant throughout the variation we would obtain an identity

$$\cos(\alpha) = \frac{\cos(\alpha)}{\sqrt{1 + \left(t/\sin(\alpha + \theta^A(t))\right)^2}},$$

which is a contradiction for $t \neq 0$ since $\cos(\alpha) \neq 0$.

COROLLARY 5·2. *Any scalene triangle is similar to a triangle of the form $T(\theta, \pi r/n, t)$ for some $t > 0$ and some integers $2 \leqslant n$, $1 \leqslant r < n$.*

Proof. Any such triangle A is similar to $T(\theta^A(t), \alpha^A(t), t)$ for all $t \in [0, d_A)$. It suffices to show there exists $t > 0$ such that $\alpha^A(t) = \pi r/n \in (0, \pi/2)$ for integers r, n. This now follows from the Intermediate Value Theorem since α^A is nonconstant and rational multiples of π are dense in $(0, \pi/2)$.

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For each $n \geqslant 2$, let $\Gamma(n)$ be the direct product $D_{2n} \times C_2$ where D_{2n} is the dihedral group of order $2n$ generated by ξ, η , and C_2 is the cyclic group of order two generated by σ; that is:

$$\Gamma(n) = \langle \xi, \eta, \sigma \mid \xi^n = \eta^2 = \sigma^2 = 1; \quad \eta\xi = \xi^{n-1}\eta; \quad \sigma\xi = \xi\sigma; \sigma\eta = \eta\sigma \rangle.$$

$\Gamma(n)$ acts orthogonally on $\mathbf{R}^3$ via the following action on $\mathbf{C} \times \mathbf{R}$:

$$\xi \cdot (\mathbf{z}, t) = (e^{i2\pi r/n}\mathbf{z}, t); \quad \eta \cdot (\mathbf{z}, t) = (\overline{\mathbf{z}}, t); \quad \sigma \cdot (\mathbf{z}, t) = (\mathbf{z}, -t),$$

where '–' is complex conjugation. The triangle $T(\theta, \alpha, t)$ is subtransitive under $\Gamma(n)$ if α is a rational multiple of π, $\alpha = \pi r/n$ say. Then putting $\mathbf{v}_3 = (e^{i\theta}, t)$, we see that $(\xi\sigma) \cdot \mathbf{v}_3 = (e^{i\theta + 2\pi r/n}, -t)$ and $\eta \cdot \mathbf{v}_3 = (e^{-i\theta}, t)$, showing that $T(\theta, \pi r/n, t)$ is contained in the orbit of $\mathbf{v}_3$ under $\Gamma(n)$. It now follows from Corollary(5·2) that:

COROLLARY 5·3. *Any scalene triangle A is similar to one which is subtransitive under the above action of $D_{2n} \times C_2$ on $\mathbf{R}^3 = \mathbf{C} \times \mathbf{R}$ for some positive integer n.*

In Figure 1, the equilateral triangle, represented by the corner point $(\pi/3, \pi/3)$, is evidently transitive under the natural action of C_3, regarded as the cube roots of unity, on $\mathbf{C}$, and hence subtransitive under $D_{2n} \times C_2$ whenever 3 divides n. The boundary line $\lambda_1 = \lambda_2$ in Figure 1 represents isoceles triangles with two smaller angles equal. A variant of the above argument, in which the flow α^A simply parametrizes the boundary line, shows that any such isoceles triangle is similar to one of the form $T(\pi r/n, \pi r/n, t)$, and hence is subtransitive under $D_{2n} \times C_2$. Finally the line $\lambda_1 + 2\lambda_2 = \pi$ represents isoceles triangles with two larger angles equal. Any such triangle is then similar to one of the form $T(\pi(1 - 2r/n), \pi r/n, t)$, and hence again subtransitive under $D_{2n} \times C_2$. Collecting these observations together with Corollary 5·3 we obtain the following re-formulation of Leader's original observation:

THEOREM 5·4 (Leader [**6**]). *For any Euclidean triangle A there exists a positive integer n such that A is similar to one which is subtransitive under the action of $D_{2n} \times C_2$ on $\mathbf{R}^3 = \mathbf{C} \times \mathbf{R}$.*

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