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On the parametric decay of a circularly polarized wave

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Abstract. The nonlinear decay of a circularly polarized wave is reconsidered. We deduce an explicit expression for the growth rate, and present it in a form which is significantly simpler than those of previous papers.

1. Introduction

Circularly polarized waves propagating along the external magnetic field in a magnetized plasma are of particular interest to plasma physicists. The reason is that they can appear as exact solutions of the coupled set of fluid and Maxwell equations. Thus it is possible to derive the nonlinear dispersion relations governing the decay of very large amplitude pump waves if the decay waves propagate in the same direction as the pump wave [1]. The growth rates of the excited waves can accordingly be deduced. Such results, which can be derived for any pump wave amplitude, are of interest in a variety of different contexts (e.g. [2]), and they can easily be generalized to include relativistic effects as well as quantum electrodynamic effects due to vacuum polarization [3]. However, if the decay waves propagate in arbitrary directions, it is algebraically very difficult to derive the correct nonlinear growth rates. Numerous recent papers (e.g. [4]) have thus failed in this respect. Here, our purpose is therefore to present the correct growth rates in forms that are much simpler than those of previous papers [5] and which accordingly can be significantly more useful for specific cases.

In Sec. 2 we shall first reconsider the linear theory for wave propagation in a cold uniform magnetized plasma. Using the general results for three-wave interactions [6] in Sec. 3 we then consider a pump wave which propagates along the external magnetic field. Our conclusions are finally stressed in Sec. 4.

2. Linear theory results

The dispersion function $D(\omega,\mathbf{k})$, where $D(\omega,\mathbf{k})=0$ is the dispersion relation for a cold magnetized multi-component plasma, can be written as [7]

$$D(\omega,\mathbf{k})=\left(1-\frac{k^2c^2}{\omega^2}-\sum\frac{\omega_p^2}{\omega^2-\omega_c^2}\right)\left[\left(\left(1-\frac{k_z^2c^2}{\omega^2}-\sum\frac{\omega_p^2}{\omega^2-\omega_c^2}\right)\left(1-\frac{k_\perp^2c^2}{\omega^2}-\sum\frac{\omega_p^2}{\omega^2}\right)-\frac{k_\perp^2k_z^2c^4}{\omega^4}\right]-\left(\sum\frac{\omega_p^2\omega_c}{\omega\left(\omega^2-\omega_c^2\right)}\right)^2\left(1-\frac{k_\perp^2c^2}{\omega^2}-\sum\frac{\omega_p^2}{\omega^2}\right).\quad(2.1)$$

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Here $\omega_p = (n_0 q^2/\varepsilon_0 m)^{1/2}$ is the plasma frequency, q is the particle charge, m is the mass, c is the velocity of light in vacuum, $\omega_c = qB_0/m$ is the gyrofrequency, B_0 is the unperturbed magnetic field strength (with $\mathbf{B}_0 = B_0\hat{\mathbf{z}}$), k_z is the wavenumber parallel to the magnetic field and $k_\perp$ is the wavenumber perpendicular to the magnetic field. The sum sign stands for summation over all particle species.

In the particular case of wave propagation parallel to $\hat{\mathbf{z}}$ we note that $D(\omega, \mathbf{k})$ directly reduces to $D_\parallel(\omega, \mathbf{k}) = -(1 - \sum \omega_p^2/\omega^2) D_T(\omega, \mathbf{k})$, where

$$D_T(\omega, \mathbf{k}) = \left(\sum \frac{\omega_p^2 \omega_c}{\omega(\omega^2 - \omega_c^2)} \right)^2 - \left(1 - \frac{k^2 c^2}{\omega^2} - \sum \frac{\omega_p^2}{\omega^2 - \omega_c^2} \right)^2 . \tag{2.2}$$

Furthermore, in the limit of short wavelength electrostatic modes (with $kc \gg \omega$), one has $D(\omega, \mathbf{k}) \approx D_{\mathrm{es}}(\omega, \mathbf{k})$, with $D_{\mathrm{es}}(\omega, \mathbf{k}) = (k^4 c^4/\omega^4)\varepsilon(\omega, \mathbf{k})$, where

$$\varepsilon(\omega, \mathbf{k}) = 1 - \frac{k_\perp^2}{k^2} \sum \frac{\omega_p^2}{(\omega^2 - \omega_c^2)} - \frac{k_z^2}{k^2} \sum \frac{\omega_p^2}{\omega^2} . \tag{2.3}$$

The expressions (2.1) and (2.2), and in particular (2.3), can easily be generalized to include thermal effects also [8]. Finally, for a one-component plasma we note that (2.1) reduces to the rather well-known expression [2]

$$D(\omega, \mathbf{k}) = \frac{1}{\omega^2(\omega^2 - \omega_c^2)} \left[\left(1 - \frac{\omega_p^2}{\omega^2} \right) \left(k^2 c^2 - \omega^2 + \omega_p^2 \right)^2 - \frac{\omega_c^2}{\omega^2} (k^2 c^2 - \omega^2) \left[(k^2 c^2 - \omega^2) \left(1 - \frac{\omega_p^2}{\omega^2} \right) + \frac{k_\perp^2 c^2 \omega_p^2}{\omega^2} \right] \right] . \tag{2.4}$$

3. A circularly polarized pump wave

We start our evaluation directly from the general results for the resonant interaction between three waves with frequencies ω_j $(j = 1, 2, 3)$ and wave vectors $\mathbf{k}_j$, with $\omega_3 = \omega_1 + \omega_2$ and $\mathbf{k}_3 = \mathbf{k}_1 + \mathbf{k}_2$, in a cold multi-component plasma (see the appendix). Considering wave three as the pump wave which propagates along the external magnetic field $B_0\hat{\mathbf{z}}$ with wave vector $\mathbf{k}_3 = k_3\hat{\mathbf{z}}$ and electric field $\mathbf{E}_3 = E_{3\perp}(\hat{\mathbf{x}} + i\hat{\mathbf{y}})$ we then evaluate (A 1)–(A 6) in the appendix to deduce the squared growth rate

$$\gamma^2 = \frac{k_\perp^6 c^8 |E|_{3\perp}^2 \beta_1 \beta_2}{\omega_1^6 \omega_2^6 [\partial D/\partial \omega_1][\partial D/\partial \omega_2]} \left| \sum \frac{q \omega_p^2 \alpha_2}{m(\omega_3 + \omega_c)} \times \left(\frac{\mathbf{k}_{1\perp} \cdot \mathbf{K}_{1\perp}}{\omega_1} + \frac{k_{1z}^2}{\omega_1} \left(1 + \frac{\omega_c}{\omega_3} \right) - \frac{k_{1z} k_{2z}}{\omega_2} \frac{\omega_c}{\omega_3} \right) - (1 \leftrightarrow 2) \right|^2 , \tag{3.1}$$

where $(1 \leftrightarrow 2)$ means a similar term with interchange of indicies 1 and 2, and where

$$\alpha = -\frac{\omega}{k_\perp^2 c^2 (\omega + \omega_c)\beta} \left[\omega^2 - k^2 c^2 - \sum \frac{\omega_p^2 \omega}{\omega - \omega_c} \right] , \tag{3.2}$$

$$\beta = \frac{1 - \dfrac{k^2 c^2}{\omega^2} - \sum \dfrac{\omega_p^2}{\omega^2 - \omega_c^2}}{1 - \dfrac{k_\perp^2 c^2}{\omega^2} - \sum \dfrac{\omega_p^2}{\omega^2}} \tag{3.3}$$

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and

$$\frac{\mathbf{k}_{1\perp}\cdot\mathbf{K}_{1\perp}}{\omega_1}=-\frac{\left(\omega_1^2-k_\perp^2c^2-\sum\omega_p^2\right)\omega_1}{c^2\left(\omega_1^2-\omega_c^2\right)}-\frac{\omega_1\omega_c\sum\dfrac{\omega_c\omega_p^2}{\omega_1^2-\omega_c^2}}{c^2\left(\omega_1^2-\omega_c^2\right)\beta_1}. \tag{3.4}$$

Further manipulations of (3.1) yield

$$\gamma^2=\frac{k_\perp^2\,|E|_{3\perp}^2\,\beta_1\beta_2}{\omega_1^4\omega_2^4[\partial D/\partial\omega_1][\partial D/\partial\omega_2]}\left[\sum\frac{q\omega_p^2}{m(\omega_3+\omega_c)(\omega_2+\omega_c)\beta_2}\left(\frac{\left(\omega_1^2-k_\perp^2c^2-\sum\omega_p^2\right)}{\left(\omega_1^2-\omega_c^2\right)}\right.\right.$$
$$\left.+\frac{\omega_c\sum\omega_p^2/\left(\omega_1^2-\omega_c^2\right)}{\left(\omega_1^2-\omega_c^2\right)\beta_1}-\frac{k_{1z}^2c^2}{\omega_1^2}\left(1+\frac{\omega_c}{\omega_3}\right)+\frac{k_{1z}k_{2z}c^2}{\omega_1\omega_2}\frac{\omega_c}{\omega_3}\right)$$
$$\left.\times\left(\omega_2^2-k_2^2c^2-\sum\frac{\omega_2\omega_p^2}{(\omega_2-\omega_c)}\right)-(1\leftrightarrow 2)\right]^2. \tag{3.5}$$

We note that the squared bracket $[\ldots]^2$ is always a positive quantity.

4. Conclusions

The general theory for resonant three-wave interactions in plasmas shows that the growth rate γ can be determined from [6]

$$\gamma^2=\frac{\omega_1\omega_2\,|V|^2}{W_1W_2}, \tag{4.1}$$

where the wave energies $W_{1,2}$ are given by $W=\varepsilon_0\mathbf{E}^*\cdot(1/\omega)\partial(\omega^2\boldsymbol{\varepsilon})\mathbf{E}$, $\boldsymbol{\varepsilon}$ is the usual textbook dielectric tensor, and the expression for V can be found in [6] or [7]. Considering positive frequencies $\omega_{1,2}>0$, it is then obvious that the sign of γ^2 depends only on the signs of W_1 and W_2. Evaluating the explicit expressions for $W_{1,2}$, one then finds that for the present plasma, W is proportional to $\partial D(\omega,\mathbf{k})/\partial\omega$, as well as to $1/\beta$. It should be noted that *only* those factors thus determine the sign of γ^2, and that the wave energies in the present plasma are always positive. But it is easy to generalize our problem to include particle drifts in the unperturbed plasma, and in that case it is possible that the wave energies are negative in particular situations, and that explosive instabilities thus can occur [2].

Our main achievement in the present paper is that we have deduced a comparatively very simple expression for the growth rate γ of the decay waves of a circularly polarized wave. As very strong laser fields are going to be adopted in the near future [9], it seems to us as if the theory for nonlinear wave interactions has to be further generalized in various ways, and that the present results then may serve as a prerequisite to more refined formulas. Such results can be of interest for the theory of stimulated electromagnetic emissions [10], as well as in various other contexts [11].

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Appendix

The general expression for the growth rate in a cold magnetized multi-component plasma is [5, 7]

$$\gamma^2 = \frac{M_1 M_2 |CE_{3z}|^2}{[\partial D(\omega_1, \mathbf{k}_1)/\partial\omega_1][\partial D(\omega_2, \mathbf{k}_2)/\partial\omega_2]}, \tag{A 1}$$

where

$$M = \frac{k_\perp^2 k_z^2 c^4}{\omega^4}\beta \tag{A 2}$$

and

$$C = \sum_\sigma \frac{q\omega_p^2}{m\omega_1\omega_2\omega_3 k_{1z}k_{2z}k_{3z}} \left[\frac{\mathbf{k}_1\cdot\mathbf{K}_1}{\omega_1}\mathbf{K}_2\cdot\mathbf{K}_3^* + \frac{\mathbf{k}_2\cdot\mathbf{K}_2}{\omega_2}\mathbf{K}_1\cdot\mathbf{K}_3^* + \frac{\mathbf{k}_3\cdot\mathbf{K}_3^*}{\omega_3}\mathbf{K}_1\cdot\mathbf{K}_2 \right.$$
$$\left. - \frac{i\omega_c}{\omega_3}\left(\frac{k_{2z}}{\omega_2} - \frac{k_{1z}}{\omega_1}\right)\mathbf{K}_3^*\cdot(\mathbf{K}_1\times\mathbf{K}_2)\right], \tag{A 3}$$

with

$$\mathbf{K} = -\left[\mathbf{k}_\perp + i\frac{\omega_c}{\omega}\mathbf{k}\times\hat{\mathbf{z}} + \left(\frac{\sum i\frac{\omega_c}{\omega}\frac{\omega_p^2}{\omega^2-\omega_c^2}}{1-\frac{k^2c^2}{\omega^2}-\sum\frac{\omega_p^2}{\omega^2-\omega_c^2}}\right)\left(\mathbf{k}\times\hat{\mathbf{z}} - i\frac{\omega_c}{\omega}\mathbf{k}_\perp\right)\right]$$
$$\times\frac{\left(\omega^2 - k_\perp^2c^2 - \sum\omega_p^2\right)\omega^2}{\left(\omega^2-\omega_c^2\right)k_\perp^2c^2} + k_z\hat{\mathbf{z}}. \tag{A 4}$$

The case, where the pump wave is a parallel propagating circularly polarized wave can be handled by taking the limit of $\mathbf{K}_3$ when $k_{3\perp}\to 0$, in which case we have [5]

$$\mathbf{K}_3 = -(\hat{\mathbf{x}} + i\hat{\mathbf{y}})\frac{k_{3z}\omega_3 E_{3\perp}}{(\omega_3+\omega_c)E_{3z}}, \tag{A 5}$$

which means that

$$\frac{\mathbf{k}_1\cdot\mathbf{K}_1}{\omega_1}\mathbf{K}_2\cdot\mathbf{K}_3^* + \frac{\mathbf{k}_2\cdot\mathbf{K}_2}{\omega_2}\mathbf{K}_1\cdot\mathbf{K}_3^* + \frac{\mathbf{k}_3\cdot\mathbf{K}_3^*}{\omega_3}\mathbf{K}_1\cdot\mathbf{K}_2 - \frac{i\omega_c}{\omega_3}\left(\frac{k_{2z}}{\omega_2} - \frac{k_{1z}}{\omega_1}\right)\mathbf{K}_3^*\cdot(\mathbf{K}_1\times\mathbf{K}_2)$$
$$= -\frac{k_{3z}\omega_3 E_{3\perp}^*(K_{2x} - iK_{2y})}{(\omega_3+\omega_c)E_{3z}^*}\left[\frac{\mathbf{k}_1\cdot\mathbf{K}_1}{\omega_1} - \frac{\omega_c}{\omega_3}\left(\frac{k_{2z}}{\omega_2} - \frac{k_{1z}}{\omega_1}\right)k_{1z}\right] + (1\leftrightarrow 2). \tag{A 6}$$

Straightforward algebra, using (A 4), then leads to $K_x - iK_y = \alpha(k_x - ik_y)$, where α is defined in (3.2).

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