THREE-DIMENSIONAL ELASTIC ANALYSIS OF TRANSVERSELY-ISOTROPIC COMPOSITES

Yu. V. Tokovyy*

Pidstryhach Institute for Applied Problems of Mechanics and Mathematics
National Academy of Sciences of Ukraine
Lviv, Ukraine

C. C. Ma

Mechanical Engineering Department
National Taiwan University
Taipei, Taiwan

ABSTRACT

An exact analytical solution to the three-dimensional elasticity problem for a transversely-isotropic composite layer is constructed by making use of the direct integration method along with the Fourier double-integral transform. The original problem is reduced to a system of governing partial-differential equations for separate stress-tensor components. The governing equations are accompanied with corresponding local and integral boundary conditions, obtained on the basis of the original local boundary conditions imposing the normal and shearing forces on the limiting planes of the layer. The numerical analysis of the obtained solution is presented for certain transversely-isotropic composite materials.

Keywords: Transversely-isotropic material, Composite material, Three-dimensional problem, Elasticity, Exact analytical solution.

1. INTRODUCTION

The rapid development of modern technologies implementing different kinds of composite materials, have heightened the need for efficient modeling and analysis of structure members exhibiting anisotropic mechanical properties [1]. Among different kinds of practical anisotropic materials, the transversely isotropic materials (TIM), whose physical properties are symmetric about an axis that is normal to a plane of isotropy, are of the special interest. The aforementioned kind of symmetry is typical for lattices of the hexagonal syngony [2, 3], various kinds of thin-layer laminates implemented in the way that the macroscopic material properties are likely the same in parallel to the layer orientation but differ from the ones in the transversal direction [4, 5] as well as for the fiber composites with hexagonal fiber packing [6-8].

The three-dimensional elastic analysis of TIM for both dynamic and static cases presents a challenge due to the complexity of the governing system of partial-differential equations involving dissimilar elastic moduli. Therefore, a considerable amount of literature has been published on particular classes of the elasticity problems implying certain simplifying assumptions, e.g., the theories of pates and shells, plane (stress/strain) problems, anti-plane problems, problems on the pure torsion, bending etc. Extensive reviews on solutions for plane, axisymmetric and spatial problems of the elasticity theory for TIM bodies have been reported, e.g., by Podil'chuk [9], Nemish [10], and Wang *et al*. [11]. In [12, 13], Lekhnitskii presented the solutions to the plane and axisymmetric elasticity problems by generalizing the Airy and Love biharmonic stress-functions, respectively, for anisotropic bodies. These solutions have later been extended to the three-dimensional case [14, 15]. An alternative representation of the solutions has been suggested by Elliot [16] and Lodge [17] through the three independent harmonic functions. The latter representation has been used in [18-21] for the solution of three-dimensional problems for TIM solids of canonical shape. Liao and Wang [22] presented a review of results obtained on the three-dimensional elasticity problems for a TIM half-space subjected to a point-force load over its limiting plane or in the interior of the half-space. By making use of the integral-transform method, they constructed a solution to the specified problem in a closed form which complies with solutions given by Lekhitskii [13]. This solution has latter been extended

* Corresponding author (tokovyy@gmail.com)

DOI : 10.1017/jmech.2017.91
Copyright © 2017 The Society of Theoretical and Applied Mechanics
https://doi.org/10.1017/jmech.2017.91 Published online by Cambridge University Press

to the case of complex loading types [23]. In [24], the Lekhnitskii representation has been employed alongside with the method of homogeneous solutions in order to solve three-dimensional problems for simply- and multiply-connected plates. Similarly to the isotropic materials, the complex variable method has been developed for the analysis of TIM by making use of an appropriate solution representation through the potential functions being analytical in the domain of consideration [3, 25]. In [26], the theory of *p*-analytical functions has been employed for the analysis of non-axisymmetric elasticity problems for TIM solids. A solution based on the implementation of a Green function has been obtained in [27] by making use of the Stroh formalism and two-dimensional Fourier transform. Various methods for the solution of one-, two-, and three-dimensional elasticity problems for the TIM solids are summarized by Ding *et al*. [28]. The exhaustive reviews on the numerical methods developed for the solution of TIM-related problems are given, e.g., in [29-31].

The implementation of methods involving potential functions for the TIM is often associated with both analytical and numerical complications concerning with *i*) the derivatives of the higher order (in comparison to the original equations either in terms of stresses or displacements) and *ii*) the fact that the coefficients or eigenvalues of the relevant auxiliary or spectral problems are expressed through the combinations of dissimilar material constants (see, e.g., the studies on the uniqueness and completeness of the generalized harmonic and biharmonic solutions in [32-34]).

In this paper, we implement the direct integration method [35-37] in order to construct a closed form analytical solution to a three-dimensional elasticity problem for a TIM composite layer subjected to external force loadings. The application of this method allows for the formulation of a closed system of governing equations in terms of stresses along with the sufficient number of local and integral boundary conditions. By making use of the Fourier double-integral transform with respect to the in-plane coordinates, the explicit solutions to the above-mentioned equations are obtained. The efficiency of solution is analyzed numerically for certain composite materials.

2. FORMULATION OF THE PROBLEM

Consider a TIM layer related to the dimensionless Cartesian coordinate system as $-\infty < x < \infty$, $-\infty < y < \infty$, $-1 \le z \le 1$ so that the plane of isotropy is parallel to the (xy)-plane. Here, $x = \overline{x}/h$, $y = \overline{y}/h$, $z = \overline{z}/h$, $\overline{x}$, $\overline{y}$, and $\overline{z}$ are the dimensional Cartesian coordinates, and h is the half-thickness of the layer. Assume the layer to be in the state of elastic equilibrium, which is governed by the equations [38, 39]

$$\frac{\partial \sigma_\xi}{\partial \xi} + \frac{\partial \tau_{\xi\eta}}{\partial \eta} + \frac{\partial \tau_{\xi\zeta}}{\partial \zeta} = 0, \quad \{\xi,\eta,\zeta\} = \{x,y,z\}, \quad \xi \ne \eta \ne \zeta, \tag{1}$$

under the normal and shearing force loadings

$$\sigma_z(x,y,1) = -p_1(x,y), \quad \sigma_z(x,y,-1) = -p_2(x,y),$$
$$\tau_{\xi z}(x,y,1) = q_{\xi 1}(x,y), \quad \tau_{\xi z}(x,y,-1) = q_{\xi 2}(x,y),$$
$$\xi = \{x,y\}, \tag{2}$$

applied to the limiting planes $z = \pm 1$. Here, σ_ξ and $\tau_{\xi\eta}$ ($\{\xi,\eta\} = \{x,y,z\}$, $\xi \ne \eta$, $\tau_{\xi\eta} = \tau_{\eta\xi}$) are the normal and shearing stress-tensor components, respectively, and p_j and $q_{\xi j}$ ($\xi = \{x,y\}$, $j = 1, 2$) are given functions vanishing at the points of infinity $x,y \to \pm\infty$. The continuity of the material is described by the strain-compatibility equations

$$\frac{\partial^2 \varepsilon_\xi}{\partial \eta^2} + \frac{\partial^2 \varepsilon_\eta}{\partial \xi^2} = \frac{\partial^2 \gamma_{\xi\eta}}{\partial \xi \partial \eta},$$
$$2\frac{\partial^2 \varepsilon_\xi}{\partial \eta \partial \zeta} = \frac{\partial}{\partial \xi}\left(\frac{\partial \gamma_{\xi\eta}}{\partial \zeta} + \frac{\partial \gamma_{\xi\zeta}}{\partial \eta} - \frac{\partial \gamma_{\eta\zeta}}{\partial \xi}\right),$$
$$\{\xi,\eta,\zeta\} = \{x,y,z\}, \quad \xi \ne \eta \ne \zeta, \tag{3}$$

where ε_ξ and $\gamma_{\xi\eta}$ ($\{\xi,\eta\} = \{x,y,z\}$, $\xi \ne \eta$, $\gamma_{\xi\eta} = \gamma_{\eta\xi}$) denote the normal and shearing strains, respectively. The stress- and strain-tensor components are related by means of the physical relations [28, 38]:

$$\varepsilon_x = \frac{1}{E}\left(\sigma_x - \nu\sigma_y\right) - \frac{\nu'}{E'}\sigma_z, \quad \varepsilon_y = \frac{1}{E}\left(-\nu\sigma_x + \sigma_y\right) - \frac{\nu'}{E'}\sigma_z,$$
$$\varepsilon_z = -\frac{\nu'}{E'}\left(\sigma_x + \sigma_y\right) + \frac{\sigma_z}{E'}, \quad \gamma_{xz} = \frac{\tau_{xz}}{G'},$$
$$\gamma_{yz} = \frac{\tau_{yz}}{G'}, \quad \gamma_{xy} = \frac{\tau_{xy}}{G}. \tag{4}$$

Here, ν is the Poisson ratio characterizing the transverse contraction in response to the tension applied in the plane of isotropy, ν' denotes the ratio characterizing the transverse contraction in response to the tension applied transversely to the plane of isotropy, E and E' are the Young moduli at tension or compression in the plane of isotropy and in the transversal direction, respectively, and $G = E/(2 + 2\nu)$ and G' stand for the shearing moduli in the plane of isotropy and in the transversal direction, respectively.

The objectives of this research are to: *i*) reduce Eqs. (1)-(4) to a system of governing equations in terms of stresses, accompanied with corresponding local and integral conditions, and *ii*) construct the explicit expressions for the stress-tensor components within the formulated problem.

3. GOVERNING EQUATIONS AND CONDITIONS

In order to derive the governing equation for the stresses determination, we employ the direct integration method [35-37]. First, we use Eq. (1) to express the shearing stresses in terms of the normal ones as follows:

https://doi.org/10.1017/jmech.2017.91 Published online by Cambridge University Press

$$2\frac{\partial^2\tau_{\xi\eta}}{\partial\xi\partial\eta}=\frac{\partial^2\sigma_\zeta}{\partial\zeta^2}-\frac{\partial^2\sigma_\xi}{\partial\xi^2}-\frac{\partial^2\sigma_\eta}{\partial\eta^2},\quad \{\xi,\eta,\zeta\}=\{x,y,z\},\quad \xi\neq\eta\neq\zeta. \tag{5}$$

Assuming $\xi = x$ and $\eta = y$ in the first Eq. (3) and making use of Eqs. (1) and (4), we can derive the following governing equation

$$\Delta^+\sigma_z=\mu^+\Delta_{xy}\sigma, \tag{6}$$

where $\Delta_{xy}=\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}$, $\Delta^\pm=\Delta_{xy}\pm(1\pm\nu)\mu^\pm\frac{\partial^2}{\partial z^2}$, $\mu^\pm=\frac{E'}{\nu' E\pm E'}$, and

$$\sigma=\sigma_x+\sigma_y+\sigma_z. \tag{7}$$

With Eqs. (1), (4), (6), and (7) in view, the first Eq. (3) for $\xi = y$ and $\eta = z$ takes the form

$$\left\langle\Delta_{xy}+\mu_1\frac{\partial^2}{\partial z^2}\right\rangle\sigma_y=\left(\frac{\mu_2}{\mu^+}-1\right)\Delta\sigma_z+(1-\mu_2)\frac{\partial^2\sigma}{\partial x^2}+\frac{\partial^2}{\partial y^2}\left(\frac{\mu_2}{\mu^-}\sigma+(1-\mu_3)\sigma_z\right)+(\mu_1-\mu_2)\frac{\partial^2\sigma}{\partial z^2}. \tag{8}$$

Here, $\mu_1=\frac{G'}{G}$, $\mu_2=2\frac{G'}{E}$, $\mu_3=2G'\frac{1+\nu'}{E'}$, $\Delta=\Delta_{xy}+\frac{\partial^2}{\partial z^2}$. In a similar fashion, the equation

$$\left\langle\Delta_{xy}+\mu_1\frac{\partial^2}{\partial z^2}\right\rangle\sigma_x=\left(\frac{\mu_2}{\mu^+}-1\right)\Delta\sigma_z+(1-\mu_2)\frac{\partial^2\sigma}{\partial y^2}+\frac{\partial^2}{\partial x^2}\left(\frac{\mu_2}{\mu^-}\sigma+(1-\mu_3)\sigma_z\right)+(\mu_1-\mu_2)\frac{\partial^2\sigma}{\partial z^2} \tag{9}$$

can be obtained for $\xi = x$ and $\eta = z$ in the first Eq. (3).

Equations (6), (8), (9) can be given in an alternative form

$$\frac{\Delta\sigma_z}{2G}+\left(\frac{\nu'}{E'}-\frac{\nu}{E}\right)\Delta_{xy}\sigma_z=\frac{\Delta_{xy}\sigma}{E}, \tag{10}$$

$$\frac{\Delta\sigma_y}{2G}+\frac{1}{2}\left(\frac{1}{G'}-\frac{1}{G}\right)\Delta_{xy}\sigma_y=\left(\frac{\nu'}{E'}+\frac{1}{E}-\frac{1}{2G'}\right)\left(\frac{\partial^2\sigma_z}{\partial x^2}+\frac{\partial^2\sigma_z}{\partial z^2}\right)+\left(\frac{1}{E}-\frac{1}{E'}\right)\frac{\partial^2\sigma_z}{\partial y^2}$$
$$+\left(\frac{1}{2G'}-\frac{1}{E}\right)\frac{\partial^2\sigma}{\partial x^2}+\left(\frac{\nu'}{E'}-\frac{1}{E}\right)\frac{\partial^2\sigma}{\partial y^2}+\frac{\nu}{E}\frac{\partial^2\sigma}{\partial z^2}, \tag{11}$$

$$\frac{\Delta\sigma_x}{2G}+\frac{1}{2}\left(\frac{1}{G'}-\frac{1}{G}\right)\Delta_{xy}\sigma_x=\left(\frac{\nu'}{E'}+\frac{1}{E}-\frac{1}{2G'}\right)\left(\frac{\partial^2\sigma_z}{\partial y^2}+\frac{\partial^2\sigma_z}{\partial z^2}\right)+\left(\frac{1}{E}-\frac{1}{E'}\right)\frac{\partial^2\sigma_z}{\partial x^2}$$
$$+\left(\frac{1}{2G'}-\frac{1}{E}\right)\frac{\partial^2\sigma}{\partial y^2}+\left(\frac{\nu'}{E'}-\frac{1}{E}\right)\frac{\partial^2\sigma}{\partial x^2}+\frac{\nu}{E}\frac{\partial^2\sigma}{\partial z^2}. \tag{12}$$

By summing up Eqs. (10)-(12) with Eq. (7) in mind, we can obtain the following equation

$$\Delta^-\sigma=\mu^-\Delta_1\sigma_z, \tag{13}$$

where $\Delta_1=(\mu_4-1)\Delta_{xy}+2\left(\frac{1}{\mu_2}-\frac{1}{\mu^+}\right)\frac{\partial^2}{\partial z^2}$, $\mu_4=\frac{E}{E'}$.

Having applied operator Δ^- to Eq. (6) and operator Δ_{xy} to Eq. (13), we derive the equation

$$\left\langle\Delta^+\Delta^--\mu^+\mu^-\Delta_{xy}\Delta_1\right\rangle\sigma_z=0, \tag{14}$$

which serves for the determination of stress σ_z. After this stress is found, the total stress σ can be easily determined from Eq. (6). Then, the in-plane stresses σ_x and σ_y can be found from Eq. (7) and either Eq. (8) or Eq. (9).

Due to the fact that Eq. (14) is a fourth-order partial- differential equation, it is necessary to have four boundary conditions for the stress σ_z. The two of them are given by Eq. (2). Two more conditions can be given in the form

$$\left.\frac{\partial\sigma_z}{\partial z}\right|_{z=1}=-\frac{\partial q_{x1}}{\partial x}-\frac{\partial q_{y1}}{\partial y},\quad \left.\frac{\partial\sigma_z}{\partial z}\right|_{z=-1}=-\frac{\partial q_{x2}}{\partial x}-\frac{\partial q_{y2}}{\partial y}, \tag{15}$$

https://doi.org/10.1017/jmech.2017.91 Published online by Cambridge University Press

by making use of Eq. (1) and the boundary conditions (2) for the shearing stresses.

Note that the satisfaction of the second Eq. (3) implies [36] the following integral condition

$$\int_{-\infty}^{x}\left(e_{xz}(\xi,y,-1)-\int_{-\infty}^{\xi}\frac{\partial e_x(\xi_1,y,-1)}{\partial z}d\xi_1\right)d\xi=\int_{-\infty}^{y}\left(e_{yz}(x,\eta,-1)-\int_{-\infty}^{\eta}\frac{\partial e_y(x,\eta_1,-1)}{\partial z}d\eta_1\right)d\eta. \tag{16}$$

4. CONSTRUCTION OF THE SOLUTION

Introduce the Fourier double-integral transformation [40]

$$\bar{f}(z)\equiv\bar{f}(s_x,s_y;z)=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}f(x,y,z)\exp\left(-i(s_xx+s_yy)\right)dxdy, \tag{17}$$

where s_x and s_y are the parameters of transformation with respect to x and y and $i^2 = -1$. In the mapping space, Eq. (14) and conditions (15) and (2), for the normal stress, take the following form

$$\left\langle\frac{d^4}{dz^4}-2\alpha_1k^2\frac{d^2}{dz^2}+\alpha_2k^4\right\rangle\bar{\sigma}_z=0, \tag{18}$$

and

$$\bar{\sigma}_z(1)=-\bar{p}_1,\quad\frac{d\bar{\sigma}_z(1)}{dz}=-is_x\bar{q}_{x1}-is_y\bar{q}_{y1},\quad\bar{\sigma}_z(-1)=-\bar{p}_2,\quad\frac{d\bar{\sigma}_z(-1)}{dz}=-is_x\bar{q}_{x2}-is_y\bar{q}_{y2}. \tag{19}$$

Here, $k^2=s_x^2+s_y^2$,

$$\alpha_1=\frac{E}{E'}\frac{E'-2(1+\nu)\nu'G'}{2(1-\nu^2)G'},\quad\alpha_2=\frac{E}{E'}\frac{E'-\nu'^2E}{(1-\nu^2)E'} \tag{20}$$

A solution to Eq. (18) can be found in the form

$$\bar{\sigma}_z=\sum_{j=1}^{4}\bar{C}_j\exp(\kappa_jz), \tag{21}$$

where $\bar{C}_j$, $j=1,2,...,4$, are arbitrary constants, $\kappa_{1,2}=\pm k\lambda_1$, $\kappa_{3,4}=\pm k\lambda_2$, and

$$\lambda_1=\sqrt{\alpha_1-\sqrt{\alpha_1^2-\alpha_2}},\quad\lambda_2=\sqrt{\alpha_1+\sqrt{\alpha_1^2-\alpha_2}}. \tag{22}$$

By taking into account Eq. (20) and the constraints for elastic properties [2], we can conclude that $\alpha_1^2-\alpha_2\geq0$, which makes the eigenvalues (22) to be real numbers for any physically permitted values of the material constants. In view of this fact, solution (21), which meets conditions (19), takes the form

$$\bar{\sigma}_z=A_1\cosh k\lambda_1z+A_2\sinh k\lambda_1z+B_1\cosh k\lambda_2z+B_2\sinh k\lambda_2z, \tag{23}$$

where the constants of integration A_j and B_j, j = 1,2, are given in Appendix by Eq. (A1).

Having applied transform (17) to Eq. (6), we arrive at the expression

$$\bar{\sigma}=\frac{\bar{\sigma}_z}{\mu^+}-\frac{1+\nu}{k^2}\frac{d^2\bar{\sigma}_z}{dz^2}. \tag{24}$$

In view of the latter expression, Eq. (8) takes the following form

$$\frac{d^2\bar{\sigma}_y}{dz^2}-\left(\frac{k}{\sqrt{\mu_1}}\right)^2\bar{\sigma}_y=-\frac{\nu}{k^2}\frac{d^4\bar{\sigma}_z}{dz^4}+b_1\frac{d^2\bar{\sigma}_z}{dz^2}+b_2\bar{\sigma}_z \tag{25}$$

in the mapping space of transform (17). Here, $b_1=\frac{E}{k^2}\left(\frac{\nu'}{E'}\left(k^2+s_y^2\right)+\frac{\nu s_x^2-s_y^2}{2(1+\nu)G'}\right)$, $b_2=\frac{G}{E'}\left(2s_y^2\frac{E'-\nu'^2E}{E'}-s_x^2E\frac{\nu'}{G'}\right)$.

Putting Eq. (23) into Eq. (25) yields

https://doi.org/10.1017/jmech.2017.91 Published online by Cambridge University Press

$$\frac{d^2\bar{\sigma}_y}{dz^2}-\left(\frac{k}{\sqrt{\mu_1}}\right)^2\bar{\sigma}_y=a_1\left(A_1\cosh k\lambda_1 z+A_2\sinh k\lambda_1 z\right)+a_2\left(B_1\cosh k\lambda_2 z+B_2\sinh k\lambda_2 z\right). \tag{26}$$

Here, $a_j=k^2\lambda_j^2\left(b_1-\nu\lambda_j^2\right)+b_2$, $j=1,2$. A solution to Eq. (26) can be given in the form

$$\bar{\sigma}_y=C_1\cosh\frac{kz}{\sqrt{\mu_1}}+C_2\sinh\frac{kz}{\sqrt{\mu_1}}+\frac{\mu_1}{k^2}\left(\frac{a_1}{\mu_1\lambda_1^2-1}\left(A_1\cosh k\lambda_1 z+A_2\sinh k\lambda_1 z\right)\right.$$
$$\left.+\frac{a_2}{\mu_1\lambda_2^2-1}\left(B_1\cosh k\lambda_2 z+B_2\sinh k\lambda_2 z\right)\right)+\varphi_1(z), \tag{27}$$

where function $\varphi_1(z)$ is given in Appendix by Eq. (A2) and the C_1 and C_2 are unknown constants of integration.

On the basis of Eq. (5) for $\xi=y$, $\eta=z$, $\zeta=x$ along with Eqs. (7), (17) and (24), we can obtain the following expression

$$\frac{d\bar{\tau}_{yz}}{dz}=\frac{i}{2s_y}\left(s_x^2\nu'\frac{E}{E'}\bar{\sigma}_z+\frac{s_y^2-\nu s_x^2}{k^2}\frac{d^2\bar{\sigma}_z}{dz^2}-k^2\bar{\sigma}_y\right)$$

for the shearing stress τ_{yz} in the mapping domain. Putting expressions (23) and (27) into the latter equation yields

$$\frac{d\bar{\tau}_{yz}}{dz}=-\frac{ik^2}{2s_y}\left(C_1\cosh\frac{kz}{\sqrt{\mu_1}}+C_2\sinh\frac{kz}{\sqrt{\mu_1}}+\varphi_1(z)\right)$$
$$+c_1\left(A_1\cosh k\lambda_1 z+A_2\sinh k\lambda_1 z\right)+c_2\left(B_1\cosh k\lambda_2 z+B_2\sinh k\lambda_2 z\right). \tag{28}$$

Here, $c_j=\frac{i}{2s_y}\left(s_x^2\nu'\frac{E}{E'}+\lambda_j^2\left(s_y^2-\nu s_x^2\right)+\frac{\mu_1 a_j}{1-\mu_1\lambda_j^2}\right)$, $j=1,2$. Having integrated Eq. (28) from -1 to z in view of the first boundary condition (2) for the shearing stress, we arrive at the expression

$$\bar{\tau}_{yz}=\bar{q}_{y2}-\frac{ik\sqrt{\mu_1}}{s_y}\sinh\frac{k(1+z)}{2\sqrt{\mu_1}}\left(C_1\cosh\frac{k(1-z)}{\sqrt{\mu_1}}-C_2\sinh\frac{k(1-z)}{\sqrt{\mu_1}}+\varphi_2(z)\right)$$
$$+\frac{2c_1}{k\lambda_1}\sinh\frac{k\lambda_1(1+z)}{2}\left(A_1\cosh\frac{k\lambda_1(1-z)}{2}-A_2\sinh\frac{k\lambda_1(1-z)}{2}\right)$$
$$+\frac{2c_2}{k\lambda_2}\sinh\frac{k\lambda_2(1+z)}{2}\left(B_1\cosh\frac{k\lambda_2(1-z)}{2}-B_2\sinh\frac{k\lambda_2(1-z)}{2}\right). \tag{29}$$

Here, function $\varphi_2(z)$ is given in Appendix by Eq. (A2).

Putting formula (29) into the second boundary condition (2) for the searing stress allows for the determination of constant C_1. In order to determine constant C_2, we use Eq. (16), which, in view of Eq. (17), takes the form

$$\frac{d}{dz}\left(s_y^2\bar{\varepsilon}_x-s_x^2\bar{\varepsilon}_y\right)=is_xs_y\left(s_y\bar{\gamma}_{xz}-s_x\bar{\gamma}_{yz}\right),\quad z=-1.$$

Making use of Eqs. (2), (4), (7), and (24), allows for the representation of the latter condition as

$$k^2\frac{d\bar{\sigma}_y}{dz}+\frac{s_y^2+\nu s_x^2}{k^2}\frac{d^3\bar{\sigma}_z}{dz^3}=2is_x\left(s_xs_y\left(\frac{G}{G'}-\frac{\nu'E}{2E'}\right)\bar{q}_{y2}-\left(s_y^2\frac{G}{G'}+s_x^2\frac{\nu'E}{2E'}\right)\bar{q}_{x2}\right),\quad z=-1. \tag{30}$$

Putting Eqs. (23) and (27) into Eq. (30) yields an algebraic equation for the determination of constant C_2. The expressions for C_1 and C_2 are given in Appendix by Eq. (A3).

In such manner the stresses $\bar{\sigma}_z$, $\bar{\sigma}$, $\bar{\sigma}_y$, and $\bar{\tau}_{yz}$ are found in the mapping domain in the form (23), (24), (27), and (29), respectively. The normal stress $\bar{\sigma}_x$ can now be easily determined by Eqs. (7) and (17) as

$$\bar{\sigma}_x=\bar{\sigma}-\bar{\sigma}_y-\bar{\sigma}_z.$$

Then, the shearing stress $\bar{\tau}_{xz}$ can be determined in the mapping domain by making use of Eq. (5) at $\xi=x$, $\eta=z$, $\zeta=y$ along with Eqs. (7), (17), and (24).

https://doi.org/10.1017/jmech.2017.91 Published online by Cambridge University Press

As the stress-tensor components are determined in the mapping domain, we can employ the inverse transformation [40]

$$f(x,y,z)=\frac{1}{4\pi^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\bar{f}(z)\exp\big(i(s_x x+s_y y)\big)ds_x ds_y \quad (31)$$

in order to determine the stresses in the physical domain.

5. NUMERICAL EXAMPLES AND DISCUSSION

Consider the computation of stresses induced in a TIM-composite layer by the symmetrical normal loading (2), where

$$p_1 = p_2 = p\exp(-c_x x^2 - c_y y^2),$$
$$q_{\xi j} = 0, \quad \xi = \{x, y\}, \quad j = 1,2, \qquad (32)$$

and p is an arbitrary constant in the dimension of stresses. The exhaustive analysis of the stress-tensor components in an isotropic layer due to the force loading (32) has been presented in [36] for $c_x = 3$ and $c_y = 5$ (we keep the same values in what follows). Note that for $E' \to E$, $G' \to G$, and $\nu' \to \nu = 0.3$, the distributions of stresses computed by means of the obtained solution coincide with the ones presented in [36] for the case of homogeneous isotropic material. By taking into consideration the symmetry of loading (32), the computation of stress-tensor components in the physical domain has been implemented by means of the formulae (A4), presented in Appendix, which are obtained from Eq. (31), where the integration over s_x and s_y is narrowed from the infinite range to the finite domain $[0,L] \times [0,L]$, beyond which the integrands vanish. It can be shown that for the considered loading (32), we can sufficiently impose $L = 40$. For the practical computations of the integrals over s_x and s_y, we employed the sixth-order quadrature formula.

In Figs. 1-3, the normal, σ_z and σ_y, and shearing τ_{yz} stress-tensor components are presented in the physical domain versus the half-thickness of the layer for the cases of TIM composites, whose properties are given in Table 1 (the curve numbers in figures correspond to those given in the first column of the table). Note that for the loading (32), the normal components are even functions of z, while the shearing stress τ_{yz} is an odd function of z and y and an even function of x.

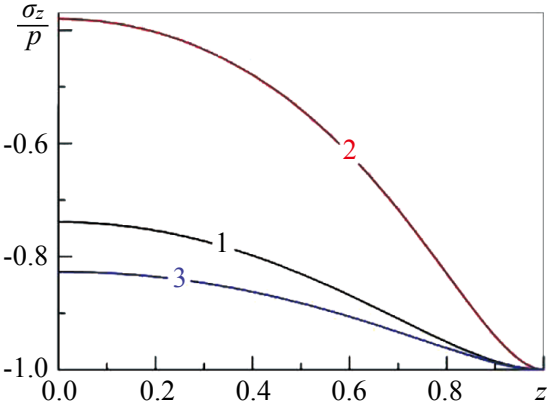


Fig. 1 Distribution of the normal stress σ_z versus z at $x = y = 0$.

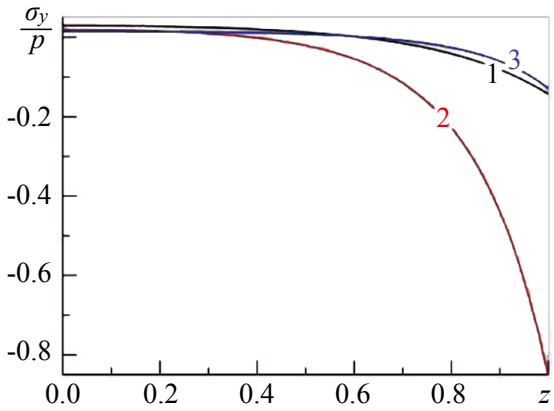


Fig. 2 Distribution of the normal stress σ_y versus z at $x = y = 0$.

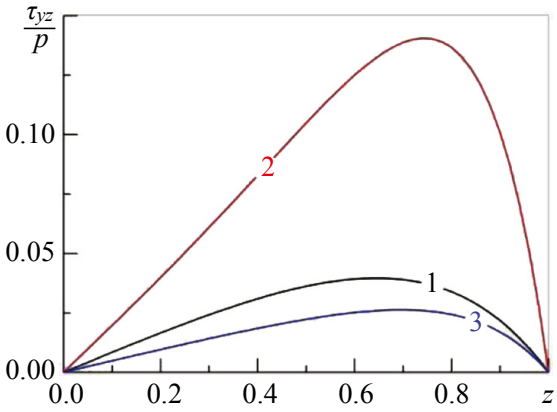


Fig. 3 z-Distribution of the shearing stress τ_{yz} for $x = 0$, $y = 0.5$.

The distribution of the stress σ_z is presented in Fig. 1 for the coordinates $x = y = 0$ those correspond to the maximum of the applied loading (32). Being the same on the limiting planes of the layer $z = \pm 1$ (due to the boundary conditions), this stress is, however, significantly affected by the anisotropy in elastic properties when moving away from the boundaries and approaching the mid-plane of the layer, where it makes the minimum compression. For the considered cases 1-3, the stress σ_z, computed for the Ceramic PZT-4 (curve 2), exhibits the highest decrement when approaching the mid-plane in-parallel to the z-axis (the value of compression on the mid-plane drops over 62% from the maximum observed on the boundary plane), while the one computed for the 60% fiber composite (curve 3) decreases much slower so that the difference between the maximum (on the boundary) and the minimum (on the mid-plane) compressive values falls within 18%.

In Fig. 2, the normal stress σ_y is depicted for the same

https://doi.org/10.1017/jmech.2017.91 Published online by Cambridge University Press

Table 1 Material properties of the transversally isotropic composites (computed by the compliance coefficients given in [28]).

	Material	E (GPa)	E' (GPa)	ν	ν'	G (GPa)	G' (GPa)
1	Carbon fiber	15	232	0.49	0.28	5.03	24
2	Ceramic PZT-4	81.28	64.53	0.33	0.34	30.56	25.6
3	Composite 60% fiber	9.95	141.1	0.5	0.27	3.32	6

values of x and y. In contrast to the previously analyzed stress, the maximum difference in the stress distribution for the three considered cases is observed when approaching the limiting planes of the TIM composite layer. Again, the stress computed for the Ceramic PZT-4 (curve 2) exhibits the significant (comparing to the other cases) variation along the z-axis, while the stress for the 60% fiber composite (curve 3) varies lesser. Similar conclusion can be drawn for the shearing stress τ_{yz} presented in Fig. 3.

6. CONCLUSIONS

An analytical solution is constructed to the general three-dimensional elasticity problem for a TIM composite layer subjected to external force loadings on its limiting planes. By making use of the direct integration method, the problem is reduced to solution of the governing equations for the separate stress-tensor components, which were obtained on the basis of the compatibility equations in terms of strains. Analogously, the local and integral boundary conditions were derived for the sufficient determination of the stress-tensor components. The obtained boundary-value problems for the governing equations have been solved by making use of the Fourier double-integral transformation. The solution is convenient for the analysis as well as the practical computation of the stress state. It is also potentially attractive for the verification of the analytical or numerical methods developed for the analysis of TIM.

ACKNOWLEDGEMENTS

The authors gratefully acknowledge the financial support of this research by the Ministry of Science and Technology (Republic of China) under grants MOST 104-2811-E-002-046 and MOST 105-2923-E-002-004- MY3.

APPENDIX

The constants of integration A_j and B_j, $j = 1,2$, appearing in Eq. (23):

$$A_1 = \frac{1}{a}\left(\frac{\overline{p}_1+\overline{p}_2}{2}\lambda_2 \sinh k\lambda_2 - \left(s_x\left(\overline{q}_{x1}-\overline{q}_{x2}\right)+s_y\left(\overline{q}_{y1}-\overline{q}_{y2}\right)\right)\frac{i\cosh k\lambda_2}{2k}\right),$$

$$A_2 = \frac{1}{b}\left(\frac{\overline{p}_1-\overline{p}_2}{2}\lambda_2 \cosh k\lambda_2 - \left(s_x\left(\overline{q}_{x1}+\overline{q}_{x2}\right)+s_y\left(\overline{q}_{y1}+\overline{q}_{y2}\right)\right)\frac{i\sinh k\lambda_2}{2k}\right),$$

$$B_1 = \frac{1}{a}\left(-\frac{\overline{p}_1+\overline{p}_2}{2}\lambda_1 \sinh k\lambda_1 + \left(s_x\left(\overline{q}_{x1}-\overline{q}_{x2}\right)+s_y\left(\overline{q}_{y1}-\overline{q}_{y2}\right)\right)\frac{i\cosh k\lambda_1}{2k}\right),$$

$$B_2 = \frac{1}{b}\left(-\frac{\overline{p}_1-\overline{p}_2}{2}\lambda_1 \cosh k\lambda_1 + \left(s_x\left(\overline{q}_{x1}+\overline{q}_{x2}\right)+s_y\left(\overline{q}_{y1}+\overline{q}_{y2}\right)\right)\frac{i\sinh k\lambda_1}{2k}\right), \tag{A1}$$

where $a = \lambda_1 \sinh k\lambda_1 \cosh k\lambda_2 - \lambda_2 \cosh k\lambda_1 \sinh k\lambda_2$, $b = \lambda_1 \cosh k\lambda_1 \sinh k\lambda_2 - \lambda_2 \sinh k\lambda_1 \cosh k\lambda_2$.

Functions $\varphi_j(z)$, $j = 1,2$, appearing in Eqs. (27) and (29):

https://doi.org/10.1017/jmech.2017.91 Published online by Cambridge University Press

$$\varphi_j(z)=\frac{\mu_1}{2k^2}\left(\frac{a_1}{1+\lambda_1\sqrt{\mu_1}}\left(A_1\cosh k\left(\lambda_1+\frac{1+z}{j\sqrt{\mu_1}}\right)-A_2\sinh k\left(\lambda_1+\frac{1+z}{j\sqrt{\mu_1}}\right)\right)\right.$$
$$+\frac{a_1}{1-\lambda_1\sqrt{\mu_1}}\left(A_1\cosh k\left(\lambda_1-\frac{1+z}{j\sqrt{\mu_1}}\right)-A_2\sinh k\left(\lambda_1-\frac{1+z}{j\sqrt{\mu_1}}\right)\right)$$
$$+\frac{a_2}{1+\lambda_2\sqrt{\mu_1}}\left(B_1\cosh k\left(\lambda_2+\frac{1+z}{j\sqrt{\mu_1}}\right)-B_2\sinh k\left(\lambda_2+\frac{1+z}{j\sqrt{\mu_1}}\right)\right)$$
$$\left.+\frac{a_2}{1-\lambda_2\sqrt{\mu_1}}\left(B_1\cosh k\left(\lambda_2-\frac{1+z}{j\sqrt{\mu_1}}\right)-B_2\sinh k\left(\lambda_2-\frac{1+z}{j\sqrt{\mu_1}}\right)\right)\right). \tag{A2}$$

Constants $C_j, j = 1,2$, appearing in Eq. (29):

$$C_1=\frac{is_y}{k\sqrt{\mu_1}\sinh k/\sqrt{\mu_1}}\left(\overline{q}_{y1}-\overline{q}_{y2}-\frac{2}{k}\left(\frac{c_1A_1}{\lambda_1}\sinh k\lambda_1+\frac{c_2B_1}{\lambda_2}\sinh k\lambda_2\right)\right)-\varphi_2(1),$$

$$C_2=C_1\tanh\frac{k}{\sqrt{\mu_1}}+\frac{\sqrt{\mu_1}\left(s_y^2+\nu s_x^2\right)\lambda_1^3}{k^2\cosh k/\sqrt{\mu_1}}\left(A_1\sinh k\lambda_1-A_2\cosh k\lambda_1\right)$$
$$+\frac{\sqrt{\mu_1}\left(s_y^2+\nu s_x^2\right)\lambda_2^3}{k^2\cosh k/\sqrt{\mu_1}}\left(B_1\sinh k\lambda_2-B_2\cosh k\lambda_2\right)$$
$$-\frac{2i\sqrt{\mu_1}s_x}{k^3\cosh k/\sqrt{\mu_1}}\left(s_x^2\frac{\nu' E}{2E'}+s_y^2\frac{G}{G'}\right)\overline{q}_{x2}-\frac{2i\sqrt{\mu_1}s_x^2s_y}{k^3\cosh k/\sqrt{\mu_1}}\left(\frac{\nu' E}{2E'}-\frac{G}{G'}\right)\overline{q}_{y2}. \tag{A3}$$

The formulae for the computation of stresses obtained by simplification of Eq. (31) for the loading (32):

$$\begin{Bmatrix}\sigma_z(x,y,z)\\ \sigma_y(x,y,z)\end{Bmatrix}=\frac{1}{\pi^2}\int_0^L\int_0^L\begin{Bmatrix}\overline{\sigma}_z(z)\\ \overline{\sigma}_y(z)\end{Bmatrix}\cos(s_xx)\cos(s_yy)ds_xds_y;$$

$$\tau_{yz}(x,y,z)=\frac{i}{\pi^2}\int_0^L\int_0^L\overline{\tau}_{yz}(z)\cos(s_xx)\sin(s_yy)ds_xds_y. \tag{A4}$$

REFERENCES

1. Mantič, V. (Ed.), *Mathematical Methods and Models in Composites*, Imperial College Press, London, p. 506 (2014).
2. Chernykh, K. F., *An Introduction to Modern Anisotropic Elasticity*, Begell House, New York, p. 248 (1998).
3. Rand, O. and Rovenski, V., *Analytical Methods in Anisotropic Elasticity (with Symbolic Computational Tools)*, Birkhäuser, Boston-Basel-Berlin, p. 454 (2005).
4. Nemat-Nasser, S. and Yamada, M. "Harmonic Waves in Layered Transversely Isotropic Composites," *Journal of Sound and Vibration*, **79**, pp. 161-170 (1981).
5. Liu, G. R., Tani, J., Watanabe, K. and Ohyoshi, T., "Lamb Wave Propagation in Anisotropic Laminates," *Journal of Applied Mechanics*, **57**, pp. 923-929 (1990).
6. Christensen, R. M. and Zywicz, E., "A Three-Dimensional Constitutive Theory for Fiber Composite Laminated Media," *Journal of Applied Mechanics*, **57**, pp. 948-955 (1990).
7. Spencer, A. J. M. *Deformations of Fibre Reinforced Materials*, Oxford University Press, Oxford, p. 134 (1972)
8. Dong, X. N., Zhang, X., Huang, Y. and Guo, X. E., "A Generalized Self-Consistent Estimate for the Effective Elastic Moduli of Fiber-Reinforced Composite Materials with Multiple Transversely Isotropic Inclusions," *International Journal of Mechanical Sciences*, **47**, pp. 922-940 (2005).
9. Podil'chuk, Y. N., "Exact Analytic Solutions of Three-Dimensional Boundary-Value Problems of the Statics of a Transversely Isotropic Body of Canonical Form (Survey)," *International Applied Me-*

https://doi.org/10.1017/jmech.2017.91 Published online by Cambridge University Press

chanics, **33**, pp. 763-787 (1997).

10. Nemish, Yu. N., "Development of Analytical Methods in Three-Dimensional Problems of the Statics of Anisotropic Bodies (Review)," *International Applied Mechanics*, **36**, pp. 135-172 (2000).
11. Wang, M. Z., Xu, B. X. and Gao, C. F., "Recent General Solutions in Linear Elasticity and Their Applications," *Applied Mechanics Reviews*, **61**, pp. 030803-1 – 030803-20 (2008).
12. Lekhnitskii, S. G., "On the Effect of Concentrated Forces on the Stress Distribution in an Aelotropic Elastic Solid," *Prikladnaya Matematika i Mekhanika,* **3**, pp. 66-69 (1936).
13. Lekhnitskii, S. G., "Symmetrical Deformation and Torsion of a Body of Revolution with a Special Kind of Anisotropy," *Prikladnaya Matematika i Mekhanika*, **4**, pp. 43-60 (1940).
14. Hu, H.-C., "On The Three-Dimensional Problems of the Theory of Elasticity of a Transversely Isotropic Body," *Acta Physica Sinica*, **9**, pp. 130-148 (1953).
15. Nowacki, W., "The Stress Function in Three-Dimensional Problems Concerning an Elastic Body Characterized by Transverse Isotropy," *Bulletin de l'Academie Polonaise des Sciences*, **2**, pp. 21-25 (1954).
16. Elliott, H. A., "Three-Dimensional Stress Distributions in Hexagonal Aeolotropic Crystals," *Mathematical Proceedings of the Cambridge Philosophical Society*, **44**, pp. 522-533 (1948).
17. Lodge, A. S., "The Transformation to Isotropic Form of the Equilibrium Equations for a Class of Anisotropic Elastic Solids," *The Quarterly Journal of Mechanics and Applied Mathematics*, **8**, pp. 211-225 (1955).
18. Podil'chuk, Y. N. and Tkachenko, V. F., "Stress State of a Transversely Isotropic Medium with a Parabolic Crack when Linearly Changing Pressure is Applied to its Surface," *International Applied Mechanics*, **30**, pp. 927-932 (1994).
19. Podil'chuk, Y. N. and Tkachenko, V. F., "Stress State of a Transversally Isotropic Medium with a Parabolic Crack under a Linearly Varying Shear Load," *International Applied Mechanics*, **31**, pp. 338-345 (1995).
20. Podil'chuk, Y. N. and Tkachenko, V. F., "Stressed State of a Transversally Isotropic Half-Space in the Case of a Flat Elliptical Die with Forces and Moments Acting on It," *International Applied Mechanics*, **31**, pp. 806-811 (1995).
21. Chen, T. W., "On Some Problems in Transversely Isotropic Elastic Materials," *Journal of Applied Mechanics*, **33**, pp. 347-355 (1966).
22. Liao, J. J. and Wang, C. D., "Elastic Solutions for a Transversely Isotropic Half-Space Subjected to a Point Load," *International Journal for Numerical and Analytical Methods in Geomechanics* **22**, pp. 425-447 (1998).
23. Wang, C.-D. and Liao, J.-J., "Elastic Solutions of Displacements for a Transversely Isotropic Half-Space Subjected to Three-Dimensional Buried Parabolic Rectangular Loads," *International Journal of Solids and Structures*, **39**, pp. 4805-4824 (2002).
24. Shaldyrvan, V. A., "Three-Dimensional Thermoelasticity Problems for Transversally Isotropic Plates," *Soviet Applied Mechanics*, **16**, pp. 370-375 (1980).
25. Aleksandrov, A. Ya. and Vol'pert, V. S., "Solution of Three-Dimensional Problems of the Theory of Elasticity for a Transversally Isotropic Body Using Analytical Functions," *Izvestiya Academii Nauk SSSR: Mekhanika Tverdogo Tela*, **5**, pp. 82-91 (1967).
26. Goman, O. G., "Representation in Terms of p-Analytic Functions of the General Solution of Equations of the Theory of Elasticity of a Transversely Isotropic Body," *Journal of Applied Mathematics and Mechanics*, **48**, pp. 62-67 (1984).
27. Pan, E. and Yuan, F. G., "Three-Dimensional Green's Functions in Anisotropic Bimaterials," *International Journal of Solids and Structures,* **37**, pp. 5329-5351 (2000).
28. Ding, H., Chen, W. and Zhang, L., *Elasticity of Transversely Isotropic Materials*, Springer, Dordrecht, 435 (2006).
29. Xie, L., Zhang, C., Sladek, J. and Sladek, V., "Unified Analytical Expressions of the Three-Dimensional Fundamental Solutions and Their Derivatives for Linear Elastic Anisotropic Materials," *Proceedings of the Royal Society A. Mathematical, Physical and Engineering Sciences*, **472**, 20150272 (2016).
30. Távara, L., Ortiz, J. E., Mantič, V. and París, F., "Unique Real-Variable Expressions of Displacement and Traction Fundamental Solutions Covering All Transversely Isotropic Elastic Materials For 3D BEM," *International Journal for Numerical Methods in Engineering*, **74**, pp. 776-798 (2008).
31. Távara, L., Mantič, V., Ortiz, J. E. and París, F., "Unique Real-Variable Expressions of the Integral Kernels in the Somigliana Stress Identity Covering All Transversely Isotropic Elastic Materials for 3D BEM," *Computer Methods in Applied Mechanics and Engineering*, **225**, pp. 128-141 (2012).
32. Wang, M. Z. and Wang, W., "Completeness and Nonuniqueness of General Solutions of Transversely Isotropic Elasticity," *International Journal of Solids and Structures*, **32**, pp. 501-513 (1995).
33. Wang, W. and Shi, M. X., "On the General Solutions of Transversely Isotropic Elasticity," *International Journal of Solids and Structures*, **35**, pp. 3283-3297 (1998).
34. Eubanks, R. A. and Sternberg, E., "On the Axisymmetric Problems Elasticity Theory for a Medium with Transverse Isotropy," *Journal of Rational Mechanics and Analysis*, **3**, pp. 89-101 (1954).

https://doi.org/10.1017/jmech.2017.91 Published online by Cambridge University Press

35. Tokovyy, Y. V., "Direct Integration Method," *Encyclopedia of Thermal Stresses*, B. Hetnarski (ed.), Springer, Dordrecht, **2**, pp. 951-960 (2014).
36. Tokovyy, Y. and Ma, C.-C., "An Analytical Solution to the Three-Dimensional Problem on Elastic Equilibrium of an Exponentially-Inhomogeneous Layer," *Journal of Mechanics,* **31**, pp. 545-555 (2015).
37. Tokovyy, Y. and Ma, C.-C., "Three-Dimensional Temperature and Thermal Stress Analysis of An Inhomogeneous Layer," *Journal of Thermal Stresses*, **36**, pp. 790-808 (2013).
38. Lekhnitskii, S. G., *Theory of Elasticity of an Anisotropic Body*, Mir Publisher, Moscow, p. 340 (1981).
39. Lur'e, A. I., *Three-Dimensional Problems of the Theory of Elasticity*, Interscience Publisher, New York, p. 493 (1964).
40. Brigham, E. O., *The Fast Fourier Transform and Its Applications*, Prentice-Hall Inc., New Jersey, p. 448 (1988).

(Manuscript received July 5, 2017,
accepted for publication August 17, 2017.)

https://doi.org/10.1017/jmech.2017.91 Published online by Cambridge University Press