Probability in the Engineering and Informational Sciences, **32**, 2018, 144–162.
doi:10.1017/S0269964816000462

ON CONSISTENCY OF LS ESTIMATORS IN THE ERRORS-IN-VARIABLE REGRESSION MODEL

XUEJUN WANG AND MENGMEI XI
School of Mathematical Sciences,
Anhui University,
Hefei 230601,
People's Republic of China
E-mail: wxjahdx2000@126.com

HONGXIA WANG
Department of Statistics,
Nanjing Audit University,
Nanjing 211815,
People's Republic of China

SHUHE HU
School of Mathematical Sciences,
Anhui University,
Hefei 230601,
People's Republic of China

Under some mild conditions, the strong consistency and complete consistency of the LS estimators in the errors-in-variable regression model with weakly negative dependent errors are obtained, which generalize the corresponding ones for negatively associated random variables. In addition, the simulation study shows that the biases of our method are small, and our method performs well.

Keywords: complete consistency, EV regression model, strong consistency, weakly negative dependent random variables

1. INTRODUCTION

Consider the following simple linear EV (errors in variables) regression model:

$$\eta_i = \theta + \beta x_i + \varepsilon_i, \quad \xi_i = x_i + \delta_i, \quad 1 \le i \le n, \tag{1.1}$$

where $\theta, \beta, x_1, x_2, \ldots$ are unknown constants (parameters), $(\varepsilon_1, \delta_1)$, $(\varepsilon_2, \delta_2)$, $\ldots$ are random vectors and ξ_i, η_i, $i = 1, 2, \ldots$ are observable. From (1.1) we have

$$\eta_i = \theta + \beta \xi_i + \nu_i, \quad \nu_i = \varepsilon_i - \beta \delta_i, \quad 1 \le i \le n. \tag{1.2}$$

© Cambridge University Press 2016 0269-9648/16 $25.00

https://doi.org/10.1017/S0269964816000462 Published online by Cambridge University Press

Consider (1.2) as a usual regression model of η_i on ξ_i, we can get the LS estimators of θ and β as follows:

$$\hat{\beta}_n = \frac{\sum_{i=1}^n (\xi_i - \bar{\xi}_n)(\eta_i - \bar{\eta}_n)}{\sum_{i=1}^n (\xi_i - \bar{\xi}_n)^2}, \qquad \hat{\theta}_n = \bar{\eta}_n - \hat{\beta}_n \bar{\xi}_n, \tag{1.3}$$

where $\bar{\xi}_n = n^{-1} \sum_{i=1}^n \xi_i$, and other similar notations, such as $\bar{\eta}_n$, $\bar{\delta}_n$ and $\bar{x}_n$, are defined in the same way.

The simple EV regression model was proposed by Deaton [6]. Since then, many authors studied the limiting behaviors for the LS estimators $\hat{\beta}_n$ and $\hat{\theta}_n$ in the EV model. For the case that the errors are independent random variables, one can refer to Amemiya and Fuller [1], Cui and Chen [4], Gleser [8], Hslao, Wang, and Wang [9], Lai, Robbins, and Wei [12], Liu and Chen [14] among others. Under the case that the errors are dependent random variables, Fan et al. [7] and Miao, Wang, and Zheng [16] Miao et al. [17] studied the consistency of LS estimators in the EV regression model with stationary α-mixing errors, negatively associated (NA, in short) errors and martingale difference errors, respectively; Wang et al. [29] established the complete convergence for weighted sums of negatively superadditive dependent (NSD) random variables and gave its application in the EV regression model, and so on. For more details about the EV regression model and applications, one can refer to Carroll et al. [2], Dagenais and Dagenais [5], Jung [11], Li [13] and Taupin [26] among others. The main purpose of the paper is to further investigate the complete consistency and strong consistency of LS estimators in the EV regression model with weakly negative dependent errors.

Now, let us recall the concept of weakly negative dependence.

Definition 1.1: *A finite collection of random variables $X_1, X_2, \ldots, X_n$ is said to be weakly negative dependent (WND, in short) if there exist a constant $M_1 \geq 1$ such that*

$$f(x_1, x_2, \ldots, x_n) \leq M_1 \prod_{i=1}^n f_i(x_i)$$

for all $x_1, x_2, \ldots, x_n \in \mathbb{R}$, where $f(x_1, x_2, \ldots, x_n)$ and $f_i(x_i)'s$ are joint density and marginal densities of random variables $X_1, X_2, \ldots, X_n$, respectively.

The sequence of WND random variables and arrays of rowwise WND random variables are defined as follows.

Definition 1.2: *An infinite sequence $\{X_n, n \geq 1\}$ is said to be WND if every finite subcollection is WND.*

An array of random variables $\{X_{ni}, i \geq 1, n \geq 1\}$ is called rowwise WND if for every $n \geq 1$, $\{X_{ni}, i \geq 1\}$ is WND.

The concept of WND random variables was introduced by Ranjbar, Amini, and Bozorgnia [19] in bivariate case, which contains a large class of multinomial distribution functions. Since then some interesting application for WND sequence have been found. See for example, Ranjbar et al. [19] studied the asymptotic behavior of convolution of WND random variables with heavy-tail distribution, Ranjbar et al. [20] obtained the asymptotic behavior of product of two heavy-tails WND random variables, and so on.

https://doi.org/10.1017/S0269964816000462 Published online by Cambridge University Press

The next concept of negative dependence is extended negatively dependence, which was introduced by Liu [15] as follows.

DEFINITION 1.3: *A finite collection of random variables $X_1, X_2, \ldots, X_n$ is said to be extended negatively dependent (END, in short) if there exists a constant $M_2 > 0$ such that both*

$$P(X_1 > x_1, X_2 > x_2, \ldots, X_n > x_n) \leq M_2 \prod_{i=1}^{n} P(X_i > x_i)$$

and

$$P(X_1 \leq x_1, X_2 \leq x_2, \ldots, X_n \leq x_n) \leq M_2 \prod_{i=1}^{n} P(X_i \leq x_i)$$

hold for all real numbers $x_1, x_2, \ldots, x_n$. An infinite sequence $\{X_n, n \geq 1\}$ is said to be END if every finite subcollection is END.

For more details about the limiting behaviors of END random variables and their applications, one can refer to Chen, Chen, and Ng [3], Qiu et al. [18], Shen [21,23], Wang and Wang [27], Wang et al. [28,30,31], Wu and Guan [32] for instance.

Remark 1.1: It is easily checked that WND implies END following by the definitions of WND and END random variables.

The paper is organized as follows: some important lemmas are provided in Section 2. The main results of the paper including the strong consistency and complete consistency of the LS estimators in the EV regression model with WND errors will be given in Section 3. Simulation study and the proofs of the main results are presented in Sections 4 and 5, respectively.

Throughout the paper, let C, M_1 and M_2 be positive constants not depending on n, which may be different in various places. Denote $\log x = \ln \max(x, e)$, $x^+ = xI(x \geq 0)$ and $x^- = -xI(x < 0)$.

2. PRELIMINARIES

To prove the main results of the paper, we need the following important lemmas. The first one is a basic property for END random variables, which can be found in Liu [15].

LEMMA 2.1: *Let $X_1, X_2, \ldots, X_n$ be END random variables. If $f_1, f_2, \ldots, f_n$ are all non-decreasing (or all non-increasing) functions, then random variables $f_1(X_1), f_2(X_2), \ldots, f_n(X_n)$ are also END.*

LEMMA 2.2: *Let $X_1, X_2, \ldots, X_n$ and $Y_1, Y_2, \ldots, Y_n$ be both WND random variables. If random variables $X_1, X_2, \ldots, X_n$ and $Y_1, Y_2, \ldots, Y_n$ are independent, then $X_1 + Y_1, X_2 + Y_2, \ldots, X_n + Y_n$ are END random variables.*

https://doi.org/10.1017/S0269964816000462 Published online by Cambridge University Press

PROOF: For any $z_1, z_2, \ldots, z_n \in \mathbb{R}$, denote $A = (x_1 + y_1 > z_1, x_2 + y_2 > z_2, \ldots, x_n + y_n > z_n)$. It follows from the definition of WND random variables that

$$\begin{aligned}
&P(X_1 + Y_1 > z_1, X_2 + Y_2 > z_2, \ldots, X_n + Y_n > z_n) \\
&\quad = \int_A f(x_1, \ldots, x_n, y_1, \ldots, y_n) dx_1 \cdots dx_n dy_1 \cdots dy_n \\
&\quad = \int_A f_1(x_1, \ldots, x_n) f_2(y_1, \ldots, y_n) dx_1 \cdots dx_n dy_1 \cdots dy_n \\
&\quad \le M_1 M_2 \int_A f_{X_1}(x_1) \cdots f_{X_n}(x_n) f_{Y_1}(y_1) \cdots f_{Y_n}(y_n) dx_1 \cdots dx_n dy_1 \cdots dy_n \\
&\quad = M_1 M_2 \int_{x_1 + y_1 > z_1} f_{X_1}(x_1) f_{Y_1}(y_1) dx_1 dy_1 \cdots \int_{x_n + y_n > z_n} f_{X_n}(x_n) f_{Y_n}(y_n) dx_n dy_n \\
&\quad = M_1 M_2 P(X_1 + Y_1 > z_1) P(X_2 + Y_2 > z_2) \cdots P(X_n + Y_n > z_n).
\end{aligned}$$

Similarly, we have

$$\begin{aligned}
&P(X_1 + Y_1 \le z_1, X_2 + Y_2 \le z_2, \ldots, X_n + Y_n \le z_n) \\
&\quad \le M_1 M_2 P(X_1 + Y_1 \le z_1) P(X_2 + Y_2 \le z_2) \cdots P(X_n + Y_n \le z_n).
\end{aligned}$$

Hence, by the definition of END random variables, we can see that $X_1 + Y_1, X_2 + Y_2, \ldots, X_n + Y_n$ are END random variables. The proof is completed. ■

Combining Lemmas 2.1 and 2.2, we can get the following important property for WND and END random variables, which will be used to prove the main results of the paper.

COROLLARY 2.1: *Let $X_1, X_2, \ldots, X_n$ and $Y_1, Y_2, \ldots, Y_n$ be both WND random variables. If random variables $X_1, X_2, \ldots, X_n$ and $Y_1, Y_2, \ldots, Y_n$ are independent, then for any $\beta \in \mathbb{R}$, $X_1 + \beta Y_1, X_2 + \beta Y_2, \ldots, X_n + \beta Y_n$ are END random variables.*

The next lemma is the Rosenthal-type inequality for END random variables. The first inequality can be found in Shen [21], and the second one could be proved by the first inequality and the same argument as that of Theorem 2.3.1 in Stout [25].

LEMMA 2.3: *Let $p \ge 2$ and $\{X_n, n \ge 1\}$ be a sequence of END random variables with $EX_n = 0$ and $E|X_n|^p < \infty$ for every $n \ge 1$. Then there exists a positive constant C_p depending only on p such that for every $n \ge 1$,*

$$E\left|\sum_{i=1}^{n} X_i\right|^p \le C_p \left\{\sum_{i=1}^{n} E|X_i|^p + \left(\sum_{i=1}^{n} EX_i^2\right)^{p/2}\right\}$$

and

$$E\left(\max_{1\le j\le n}\left|\sum_{i=1}^{j} X_i\right|^p\right) \le C \log^p n \left\{\sum_{i=1}^{n} E|X_i|^p + \left(\sum_{i=1}^{n} EX_i^2\right)^{p/2}\right\}.$$

The following one is the Kolmogorov-type exponential inequality for END random variables. The proof is similar to that of Lemma 2.4 in Shen [22]. So we omit the details.

https://doi.org/10.1017/S0269964816000462 Published online by Cambridge University Press

LEMMA 2.4: *Let $\{X_n, n \geq 1\}$ be a sequence of END random variables with zero means and finite second moments. Denote $S_n = \sum_{i=1}^n X_i$ and $B_n^2 = \sum_{i=1}^n EX_i^2$ for each $n \geq 1$. Then there exists a positive constant M_2 such that for all $x > 0$ and $y > 0$,*

$$P(|S_n| \geq x) \leq 2P\left(\max_{1\leq i\leq n} |X_i| \geq y\right) + 2M_2 \exp\left\{-\frac{x^2}{2(xy + B_n^2)}\left[1 + \frac{2}{3}\log\left(1 + \frac{xy}{B_n^2}\right)\right]\right\}.$$

With the above lemmas accounted for, we can get the following results on strong convergence and complete convergence for weighted sums of END random variables, which will be applied to prove the main results of the paper. The proof of Lemma 2.5 is similar to that of Jing and Liang [10], and the proof of Lemma 2.6 will be given in Appendix 5.

LEMMA 2.5: *Let $\{X_n, n \geq 1\}$ be a sequence of END random variables, which is stochastically dominated by a random variable X such that $E|X|^p < \infty$ for some $p > 0$. Assume further that $EX_n = 0$ if $p > 1$. Let $\{b_{ni}, i \geq 1, n \geq 1\}$ be an array of constants satisfying*

$$\max_{1\leq i\leq n} |b_{ni}| = O(n^{-1/p}) \tag{2.1}$$

and

$$\begin{cases} \sum_{i=1}^n |b_{ni}|^p = O\left(n^{-\delta}\right) \quad \text{for some } \delta > 0, & \text{if } 0 < p \leq 2, \\ \sum_{i=1}^n b_{ni}^2 = o\left((\log n)^{-1}\right), & \text{if } p > 2. \end{cases} \tag{2.2}$$

Then

$$T_n \doteq \sum_{i=1}^n b_{ni} X_i \to 0 \quad \text{a.s.} \tag{2.3}$$

LEMMA 2.6: *Let $\{X_n, n \geq 1\}$ be a sequence of END random variables, which is stochastically dominated by a random variable X such that $E|X|^{2p} < \infty$ for some $p > 0$. Assume further that $EX_n = 0$ if $p > 1$. Let $\{b_{ni}, i \geq 1, n \geq 1\}$ be an array of constants satisfying*

$$\max_{1\leq i\leq n} |b_{ni}| = O(n^{-1/p}) \tag{2.4}$$

and

$$\begin{cases} \sum_{i=1}^n |b_{ni}|^p = O(n^{-\delta}) \quad \text{for some } \delta > 0, & \text{if } 0 < p \leq 1, \\ \sum_{i=1}^n b_{ni}^2 = o((\log n)^{-1}), & \text{if } p > 1. \end{cases} \tag{2.5}$$

Then

$$\sum_{n=1}^\infty P\left(\left|\sum_{i=1}^n b_{ni} X_i\right| > \varepsilon\right) < \infty \quad \text{for all } \varepsilon > 0. \tag{2.6}$$

The following concept of stochastic domination will be used in the paper.

DEFINITION 2.1: *A sequence $\{X_n, n \geq 1\}$ of random variables is said to be stochastically dominated by a random variable X if there exists a positive constant C such that*

$$P(|X_n| > x) \leq CP(|X| > x)$$

for all $x \geq 0$ and $n \geq 1$.

https://doi.org/10.1017/S0269964816000462 Published online by Cambridge University Press

By the integration by parts, we can get the following property for stochastic domination. For the proof, one can refer to Shen, Zhang, and Volodin [24] for instance.

LEMMA 2.7: *Let $\{X_n, n \geq 1\}$ be a sequence of random variables which is stochastically dominated by a random variable X. For any $\alpha > 0$ and $b > 0$, the following two statements hold:*

$$E|X_n|^\alpha I(|X_n| \leq b) \leq C_1 [E|X|^\alpha I(|X| \leq b) + b^\alpha P(|X| > b)], \tag{2.7}$$

$$E|X_n|^\alpha I(|X_n| > b) \leq C_2 E|X|^\alpha I(|X| > b), \tag{2.8}$$

where C_1 and C_2 are positive constants. Consequently, $E|X_n|^\alpha \leq CE|X|^\alpha$, where C is a positive constant.

3. MAIN RESULTS

The model (1.1) to be studied can be exactly described as follows:

$$\begin{cases} \eta_i = \theta + \beta x_i + \varepsilon_i, & \xi_i = x_i + \delta_i, \quad 1 \leq i \leq n; \\ E\varepsilon_i = E\delta_i = 0, & 1 \leq i \leq n. \end{cases} \tag{3.1}$$

Here (ξ_i, η_i), $1 \leq i \leq n$ are observable, while $x_i, 1 \leq i \leq n, \theta, \beta$ are unknown constants (parameters). In what follows, we assume that the two error sequences $\{\varepsilon_i, i \geq 1\}$, $\{\delta_i, i \geq 1\}$ are independent with each other, where $\{\varepsilon_i, i \geq 1\}$ and $\{\delta_i, i \geq 1\}$ are both sequences of WND random variables with identical distribution. Denote $S_n = \sum_{i=1}^n (x_i - \bar{x}_n)^2$ for each $n \geq 1$.

Based on the notations above, we can get that

$$\hat{\beta}_n - \beta = \frac{\sum_{i=1}^n (\delta_i - \bar{\delta}_n)\varepsilon_i + \sum_{i=1}^n (x_i - \bar{x}_n)(\varepsilon_i - \beta\delta_i) - \beta \sum_{i=1}^n (\delta_i - \bar{\delta}_n)^2}{\sum_{i=1}^n (\xi_i - \bar{\xi}_n)^2} \tag{3.2}$$

and

$$\hat{\theta}_n - \theta = (\beta - \hat{\beta}_n)\bar{x}_n + (\beta - \hat{\beta}_n)\bar{\delta}_n + \bar{\varepsilon}_n - \beta\bar{\delta}_n. \tag{3.3}$$

These relations above will play important roles to prove the main results of this paper.

Our main results are as follows, including the strong consistency and complete consistency of LS estimators in the EV regression model with WND errors.

3.1. Strong Consistency

THEOREM 3.1: *Under the model (3.1), let $E|\varepsilon_1|^{2p} < \infty$ and $E|\delta_1|^{2p} < \infty$ for some $p > 1$. Suppose that there exist some $\tau > 0$ such that*

$$\max_{1 \leq i \leq n} \frac{|x_i - \bar{x}_n|}{\sqrt{S_n} n^{\tau - 1/p}} = O(1), \tag{3.4}$$

$$\frac{n^\tau}{\sqrt{S_n}} = O(1) \tag{3.5}$$

and

$$\frac{n^{1-\tau}}{\sqrt{S_n}} \to 0. \tag{3.6}$$

Assume further that $\tau > 1/p - 1/2$ if $1 < p \leq 2$. Then

$$\frac{\sqrt{S_n}}{n^\tau}(\hat{\beta}_n - \beta) \to 0 \quad \text{a.s.} \tag{3.7}$$

https://doi.org/10.1017/S0269964816000462 Published online by Cambridge University Press

THEOREM 3.2: *Suppose that the conditions of Theorem 3.1 are satisfied. Assume further that there exist some* $0 < \nu < \min(1 - 1/p, 1/2)$ *such that*

$$\frac{n^{\tau+\nu}}{\sqrt{S_n}}|\bar{x}_n| = O(1). \tag{3.8}$$

Then

$$n^{\nu}(\hat{\theta}_n - \theta) \to 0 \quad \text{a.s.} \tag{3.9}$$

3.2. Complete Consistency

THEOREM 3.3: *Under the model (3.1), let* $E|\varepsilon_1|^{4p} < \infty$ *and* $E|\delta_1|^{4p} < \infty$ *for some* $p > 1$. *Suppose that there exist some* $\tau > 0$ *such that (3.4)–(3.6) hold. Then for any* $\varepsilon > 0$,

$$\sum_{n=1}^{\infty} P\left(\left|\frac{\sqrt{S_n}}{n^{\tau}}(\hat{\beta}_n - \beta)\right| > \varepsilon\right) < \infty,$$

that is,

$$\frac{\sqrt{S_n}}{n^{\tau}}(\hat{\beta}_n - \beta) \to 0 \quad \textit{completely.} \tag{3.10}$$

THEOREM 3.4: *Suppose that the conditions of Theorem 3.3 are satisfied. Assume further that there exist some* $0 < \nu < \min(1 - 1/p, 1/2)$ *such that (3.8) holds. Then*

$$n^{\nu}(\hat{\theta}_n - \theta) \to 0 \quad \textit{completely.} \tag{3.11}$$

Remark 3.1: Comparing Theorem 3.1 with Theorem 3.3, and Theorem 3.2 with Theorem 3.4, respectively, the strong consistency is improved to complete consistency.

4. SIMULATION

In this section, we carry out three models with different θ and β to show how our method performs. In the EV regression model (1.1), we set $x_i, i = 1, \ldots, n$ are independent and identically distributed (i.i.d.) from a standard normal distribution. $\delta_i, i = 1, \ldots, n$ are i.i.d. from normal distribution with mean 0 and variance 0.01, 0.001. $\varepsilon_i, i = 1, \ldots, n$ are i.i.d. from normal distribution with mean 0 and variance 0.05, 0.005.

For the three models, we conduct 100 simulations with sample size $n = 100, 500, 1000$. Tables 1–3 display the simulation results. Form Tables 1–3, we can see that the biases of our method are small, our method performs well.

TABLE 1. Estimators and Biases with different variances for $\theta = 0.5$ and $\beta = 2$ in 100 simulations

$\theta = 0.5, \beta = 2$						
n	100		500		1000	
$\mathrm{Var}(\delta)$	0.01	0.001	0.01	0.001	0.01	0.001
$\mathrm{Var}(\varepsilon)$	0.05	0.005	0.05	0.005	0.05	0.005
$\widehat{\beta}$	1.99971	2.000003	1.999762	1.999995	1.999682	2.000008
$\widehat{\beta} - \beta$	-0.0003	3.35×10^{-6}	-0.0002	-4.92×10^{-6}	-0.0003	8.36×10^{-6}
$\widehat{\theta}$	0.499965	0.5000103	0.5000373	0.5000039	0.5000024	0.5000002
$\widehat{\theta} - \theta$	-3.50×10^{-5}	1.03×10^{-5}	3.73×10^{-5}	3.93×10^{-6}	2.37×10^{-6}	2.22×10^{-7}

https://doi.org/10.1017/S0269964816000462 Published online by Cambridge University Press

TABLE 2. Estimators and Biases with different variances for $\theta = 0.8$ and $\beta = 3.5$ in 100 simulations

$\theta = 0.8, \beta = 3.5$						
n	100		500		1000	
$\mathrm{Var}(\delta)$	0.01	0.001	0.01	0.001	0.01	0.001
$\mathrm{Var}(\varepsilon)$	0.05	0.005	0.05	0.005	0.05	0.005
$\widehat{\beta}$	3.499652	3.499966	3.499655	3.499988	3.499588	3.499996
$\widehat{\beta} - \beta$	-0.0003	-3.37×10^{-5}	-0.0003	-1.20×10^{-5}	-0.0004	-4.23×10^{-6}
$\widehat{\theta}$	0.8001787	0.7999784	0.8000234	0.8000033	0.800054	0.8000119
$\widehat{\theta} - \theta$	0.0002	-2.16×10^{-5}	2.43×10^{-5}	3.33×10^{-6}	5.40×10^{-5}	1.19×10^{-5}

TABLE 3. Estimators and Biases with different variances for $\theta = 1.5$ and $\beta = 5$ in 100 simulations

$\theta = 1.5, \beta = 5$						
n	100		500		1000	
$\mathrm{Var}(\delta)$	0.01	0.001	0.01	0.001	0.01	0.001
$\mathrm{Var}(\varepsilon)$	0.05	0.005	0.05	0.005	0.05	0.005
$\widehat{\beta}$	4.999896	5.000022	4.999414	4.999972	4.999533	4.999997
$\widehat{\beta} - \beta$	-0.0001	2.16×10^{-5}	-0.0006	-2.83×10^{-5}	-0.0005	-3.29×10^{-6}
$\widehat{\theta}$	1.500583	1.499996	1.500177	1.499995	1.500045	1.499999
$\widehat{\theta} - \theta$	0.0006	-4.13×10^{-6}	0.0002	-4.80×10^{-6}	4.53×10^{-5}	-5.81×10^{-7}

5. PROOFS OF THE MAIN RESULTS

PROOF OF THEOREM 3.1: In view of (3.2), to prove the main result (3.7), it suffices to show that

$$\frac{1}{\sqrt{S_n} n^{\tau}} \sum_{i=1}^{n} (\delta_i - \bar{\delta}_n)^2 \to 0 \quad \text{a.s.}, \tag{5.1}$$

$$\frac{1}{\sqrt{S_n} n^{\tau}} \sum_{i=1}^{n} (\delta_i - \bar{\delta}_n)\varepsilon_i \to 0 \quad \text{a.s.}, \tag{5.2}$$

$$\frac{1}{\sqrt{S_n} n^{\tau}} \sum_{i=1}^{n} (x_i - \bar{x}_n)(\varepsilon_i - \beta\delta_i) \to 0 \quad \text{a.s.} \tag{5.3}$$

and

$$\frac{1}{S_n} \sum_{i=1}^{n} (\xi_i - \bar{\xi}_n)^2 \to 1 \quad \text{a.s.} \tag{5.4}$$

To prove (5.1), we first prove that

$$J_{1n} \doteq \frac{1}{\sqrt{S_n} n^{\tau}} \sum_{i=1}^{n} \left[\delta_i^2 I(\delta_i \geq 0) - E\delta_i^2 I(\delta_i \geq 0)\right] \to 0 \quad \text{a.s.} \tag{5.5}$$

https://doi.org/10.1017/S0269964816000462 Published online by Cambridge University Press

and

$$J_{2n} \doteq \frac{1}{\sqrt{S_n}n^{\tau}} \sum_{i=1}^{n} \left[\delta_i^2 I(\delta_i < 0) - E\delta_i^2 I(\delta_i < 0)\right] \to 0 \quad \text{a.s.} \tag{5.6}$$

It is easily seen that $\{\delta_i^2 I(\delta_i \geq 0) - E\delta_i^2 I(\delta_i \geq 0), i \geq 1\}$ and $\{\delta_i^2 I(\delta_i < 0) - E\delta_i^2 I(\delta_k < 0), i \geq 1\}$ are END random variables by Remark 1.1 and Lemma 2.1. Applying Lemma 2.5 with $X_i = \delta_i^2 I(\delta_i \geq 0) - E\delta_i^2 I(\delta_i \geq 0)$ and $b_{ni} = \frac{1}{\sqrt{S_n}n^{\tau}}$, we have by $E|\delta_1|^{2p} < \infty$ and (3.6) that:

(i) $E|X_1|^p \leq CE|\delta_1|^{2p} < \infty$;

(ii) $\max_{1\leq i\leq n} b_{ni} = \frac{1}{\sqrt{S_n}n^{\tau}} \leq \frac{n^{1-\tau}}{\sqrt{S_n}} \cdot \frac{1}{n^{1/p}} = O(n^{-1/p})$;

(iii) if $p > 2$, then

$$\sum_{i=1}^{n} b_{ni}^2 = \frac{n^{2-2\tau}}{S_n} \cdot \frac{1}{n} = o((\log n)^{-1});$$

if $1 < p \leq 2$, then

$$\sum_{i=1}^{n} |b_{ni}|^p = \frac{n}{S_n^{p/2} n^{p\tau}} = n^{1-p} \cdot \left(\frac{n^{1-\tau}}{\sqrt{S_n}}\right)^p = O\left(n^{-(p-1)}\right).$$

That is to say, the conditions of Lemma 2.5 are satisfied. Hence, (5.5) follows from Lemma 2.5 immediately. Similarly, we can also get (5.6).
Noting that $E\delta_1^2 < \infty$ by $E|\delta_1|^{2p} < \infty$ for $p > 1$, we have

$$\begin{aligned} \frac{1}{\sqrt{S_n}n^{\tau}} \sum_{i=1}^{n} \left(\delta_i - \bar{\delta}_n\right)^2 &\leq \frac{1}{\sqrt{S_n}n^{\tau}} \sum_{i=1}^{n} \left(\delta_i^2 - E\delta_i^2\right) + \frac{1}{\sqrt{S_n}n^{\tau}} \sum_{i=1}^{n} E\delta_i^2 \\ &= J_{1n} + J_{2n} + \frac{n^{1-\tau}}{\sqrt{S_n}} E\delta_1^2. \end{aligned} \tag{5.7}$$

Hence (5.1) follows by (5.5)–(5.7) and (3.6) immediately.

Similar to the proof of (5.1), we can get that

$$\frac{1}{\sqrt{S_n}n^{\tau}} \sum_{i=1}^{n} (\varepsilon_i - \bar{\varepsilon}_n)^2 \to 0 \quad \text{a.s.} \tag{5.8}$$

Note that

$$\begin{aligned} \left|\sum_{i=1}^{n} (\delta_i - \bar{\delta}_n)\varepsilon_i\right| &= \left|\sum_{i=1}^{n} (\delta_i - \bar{\delta}_n)(\varepsilon_i - \bar{\varepsilon}_n)\right| \\ &\leq \frac{1}{2}\left[\sum_{i=1}^{n} (\delta_i - \bar{\delta}_n)^2 + \sum_{i=1}^{n} (\varepsilon_i - \bar{\varepsilon}_n)^2\right], \end{aligned} \tag{5.9}$$

which together with (5.1) and (5.8) yields (5.2).

https://doi.org/10.1017/S0269964816000462 Published online by Cambridge University Press

It is easily checked that for any $r > 0$,

$$\begin{aligned}\sum_{i=1}^{n} |x_i - \bar{x}_n| \cdot \left|\delta_i - \bar{\delta}_n\right| &\le \sqrt{\sum_{i=1}^{n}(x_i - \bar{x}_n)^2} \cdot \sqrt{\sum_{i=1}^{n}(\delta_i - \bar{\delta}_n)^2} \\ &= \sqrt{r\sum_{i=1}^{n}(x_i - \bar{x}_n)^2} \cdot \sqrt{\frac{1}{r}\sum_{i=1}^{n}(\delta_i - \bar{\delta}_n)^2} \\ &\le \frac{r\sum_{i=1}^{n}(x_i - \bar{x}_n)^2 + \frac{1}{r}\sum_{i=1}^{n}(\delta_i - \bar{\delta}_n)^2}{2} \\ &= \frac{r}{2}S_n + \frac{1}{2r}\sum_{i=1}^{n}(\delta_i - \bar{\delta}_n)^2,\end{aligned}$$

which implies that

$$\begin{aligned}\left|\sum_{i=1}^{n}(\xi_i - \bar{\xi}_n)^2 - S_n\right| &= \left|2\sum_{i=1}^{n}(x_i - \bar{x}_n)(\delta_i - \bar{\delta}_n) + \sum_{i=1}^{n}(\delta_i - \bar{\delta}_n)^2\right| \\ &\le 2\sum_{i=1}^{n}|x_i - \bar{x}_n| \cdot \left|\delta_i - \bar{\delta}_n\right| + \sum_{i=1}^{n}(\delta_i - \bar{\delta}_n)^2 \\ &\le rS_n + \frac{1+r}{r}\sum_{i=1}^{n}(\delta_i - \bar{\delta}_n)^2. \qquad \textbf{(5.10)}\end{aligned}$$

Hence, we have by (5.10), (3.5) and (5.1) that

$$\begin{aligned}\left|\frac{1}{S_n}\sum_{i=1}^{n}(\xi_i - \bar{\xi}_n)^2 - 1\right| &\le r + \frac{1+r}{r}\frac{1}{S_n}\sum_{i=1}^{n}(\delta_i - \bar{\delta}_n)^2 \\ &= r + \frac{1+r}{r} \cdot \frac{n^\tau}{\sqrt{S_n}} \cdot \frac{1}{\sqrt{S_n}n^\tau}\sum_{i=1}^{n}(\delta_i - \bar{\delta}_n)^2 \to r \quad \text{a.s.} \qquad \textbf{(5.11)}\end{aligned}$$

Since $r > 0$ is arbitrary, it follows from (5.11) that

$$\frac{1}{S_n}\sum_{i=1}^{n}(\xi_i - \bar{\xi}_n)^2 - 1 \to 0 \quad \text{a.s.},$$

which implies (5.4).

Finally, we will prove (5.3). Denote

$$b_{ni} = \frac{x_i - \bar{x}_n}{\sqrt{S_n}n^\tau}, \quad X_i = \varepsilon_i - \beta\delta_i.$$

It follows from Corollary 2.1 that $\{X_i, i \ge 1\}$ is a sequence of END random variables. In addition, we have by $E|\varepsilon_1|^{2p} < \infty$, $E|\delta_1|^{2p} < \infty$ and (3.4) that:

https://doi.org/10.1017/S0269964816000462 Published online by Cambridge University Press

(i) $E|X_1|^p = E|\varepsilon_1 - \beta\delta_1|^p \le CE|\varepsilon_1|^p + CE|\delta_1|^p < \infty$;

(ii) $\max\limits_{1\le i\le n} |b_{ni}| = O\left(n^{-1/p}\right)$;

(iii) if $p > 2$, then

$$\sum_{i=1}^{n} b_{ni}^2 = \frac{1}{n^{2\tau}} = o\left((\log n)^{-1}\right);$$

if $1 < p \le 2$, noting that $\tau > 1/p - 1/2$, we have

$$\begin{aligned}\sum_{i=1}^{n} |a_{ni}|^p &= \frac{1}{S_n^{p/2} n^{p\tau}} \sum_{i=1}^{n} |x_i - \bar{x}_n|^p \\ &\le \frac{1}{S_n^{p/2} n^{p\tau}} \cdot n \left(\frac{1}{n}\sum_{i=1}^{n} |x_i - \bar{x}_n|^2\right)^{p/2} \\ &= n^{1-p/2-p\tau} = O\left(n^{-(p\tau+p/2-1)}\right).\end{aligned}$$

That is to say, the conditions of Lemma 2.5 are satisfied. Hence, (5.3) follows from Lemma 2.5 immediately. This completes the proof of the theorem.

■

Proof of Theorem 3.2: According to the relation (3.3), to prove (3.9), it suffices to show that

$$n^{\nu}(\bar{\varepsilon}_n - \beta\bar{\delta}_n) \to 0 \quad \text{a.s.}, \tag{5.12}$$

$$n^{\nu}(\beta - \hat{\beta}_n)\bar{x}_n \to 0 \quad \text{a.s.} \tag{5.13}$$

and

$$n^{\nu}(\beta - \hat{\beta}_n)\bar{\delta}_n \to 0 \quad \text{a.s.} \tag{5.14}$$

Firstly, we will prove (5.12). It is easily seen that

$$n^{\nu}(\bar{\varepsilon}_n - \beta\bar{\delta}_n) = \frac{1}{n^{1-\nu}} \sum_{i=1}^{n} (\varepsilon_i - \beta\delta_i).$$

Applying Lemma 2.5 with $b_{ni} = \frac{1}{n^{1-\nu}}$, $X_i = \varepsilon_i - \beta\delta_i$, and noting that $0 < \nu < \min(1 - 1/p, 1/2)$, $p > 1$, we can see that:

(i) $E|X_1|^p = E|\varepsilon_1 - \beta\delta_1|^p \le CE|\varepsilon_1|^p + CE|\delta_1|^p < \infty$;

(ii) $\max_{1\le i\le n} |b_{ni}| = n^{-(1-\nu)} = O\left(n^{-1/p}\right)$;

(iii) if $p > 2$, then

$$\sum_{i=1}^{n} b_{ni}^2 = \frac{1}{n^{1-2\nu}} = o\left((\log n)^{-1}\right);$$

if $1 < p \le 2$, noting that $p - 1 - p\nu > 0$, we have

$$\sum_{i=1}^{n} |b_{ni}|^p = \frac{n}{n^{p(1-\nu)}} = O\left(n^{-(p-1-p\nu)}\right).$$

That is to say, the conditions of Lemma 2.5 are satisfied. Hence, (5.12) follows from Lemma 2.5 immediately.

https://doi.org/10.1017/S0269964816000462 Published online by Cambridge University Press

Next, we will prove (5.13). It can be checked that

$$n^{\nu}(\beta-\hat{\beta}_n)\bar{x}_n=\frac{\sqrt{S_n}}{n^{\tau}}(\beta-\hat{\beta}_n)\cdot\frac{n^{\tau+\nu}}{\sqrt{S_n}}\bar{x}_n. \tag{5.15}$$

Hence, (5.13) follows from (5.15), (3.8) and Theorem 3.1 immediately.

Finally, we will prove (5.14). Similarly to the proof of (5.12), we can get that

$$\frac{1}{n^{1-\nu}}\sum_{i=1}^{n}\delta_i\to 0\quad\text{a.s.} \tag{5.16}$$

Note that

$$n^{\nu}(\beta-\hat{\beta}_n)\bar{\delta}_n=\frac{\sqrt{S_n}}{n^{\tau}}(\beta-\hat{\beta}_n)\cdot\frac{n^{\tau}}{\sqrt{S_n}}\cdot\frac{1}{n^{1-\nu}}\sum_{i=1}^{n}\delta_i. \tag{5.17}$$

It follows from (5.17), (3.5), Theorem 3.1 and (5.16) that (5.14) holds. This completes the proof of the theorem. ■

Proof of Theorem 3.3: The proof is similar to that of Theorem 3.1. In view of (3.2), to prove the main result (3.10), it suffices to show that

$$\frac{1}{\sqrt{S_n}n^{\tau}}\sum_{i=1}^{n}(\delta_i-\bar{\delta}_n)^2\to 0\quad\text{completely,} \tag{5.18}$$

$$\frac{1}{\sqrt{S_n}n^{\tau}}\sum_{i=1}^{n}(\delta_i-\bar{\delta}_n)\varepsilon_i\to 0\quad\text{completely,} \tag{5.19}$$

$$\frac{1}{\sqrt{S_n}n^{\tau}}\sum_{i=1}^{n}(x_i-\bar{x}_n)(\varepsilon_i-\beta\delta_i)\to 0\quad\text{completely} \tag{5.20}$$

and

$$\frac{1}{S_n}\sum_{i=1}^{n}(\xi_i-\bar{\xi}_n)^2\to 1\quad\text{a.s.} \tag{5.21}$$

To prove (5.18), we firstly prove that

$$J_{1n}\doteq\frac{1}{\sqrt{S_n}n^{\tau}}\sum_{i=1}^{n}\left(\delta_i^2I(\delta_i\ge 0)-E\delta_i^2I(\delta_i\ge 0)\right)\to 0\quad\text{completely} \tag{5.22}$$

and

$$J_{2n}\doteq\frac{1}{\sqrt{S_n}n^{\tau}}\sum_{i=1}^{n}\left(\delta_i^2I(\delta_i<0)-E\delta_i^2I(\delta_i<0)\right)\to 0\quad\text{completely.} \tag{5.23}$$

It is easily seen that $\{\delta_i^2I(\delta_i\ge 0)-E\delta_i^2I(\delta_i\ge 0),i\ge 1\}$ and $\{\delta_i^2I(\delta_i<0)-E\delta_i^2I(\delta_k<0),i\ge 1\}$ are END random variables by Remark 1.1 and Lemma 2.1. Applying Lemma 2.6 with $X_i=\delta_i^2I(\delta_i\ge 0)-E\delta_i^2I(\delta_i\ge 0)$ and $b_{ni}=\frac{1}{\sqrt{S_n}n^{\tau}}$, we have by $E|\delta_1|^{4p}<\infty$ and (3.6) that:

https://doi.org/10.1017/S0269964816000462 Published online by Cambridge University Press

(i) $E|X_1|^{2p} \le CE|\delta_1|^{4p} < \infty$;

(ii) $\max_{1\le i\le n} b_{ni} = \frac{1}{\sqrt{S_n}n^\tau} \le \frac{n^{1-\tau}}{\sqrt{S_n}} \cdot \frac{1}{n^{1/p}} = O(n^{-1/p})$;

(iii) $\sum_{i=1}^n b_{ni}^2 = \frac{n^{2-2\tau}}{S_n} \cdot \frac{1}{n} = o((\log n)^{-1})$.

That is to say, the conditions of Lemma 2.6 are satisfied. Hence, (5.22) follows from Lemma 2.6 immediately. Similarly, we can also get (5.23).

Noting that $E\delta_1^2 < \infty$ by $E|\delta_1|^{4p} < \infty$ for $p > 1$, we have

$$\begin{aligned}\frac{1}{\sqrt{S_n}n^\tau}\sum_{i=1}^n \left(\delta_i - \bar{\delta}_n\right)^2 &\le \frac{1}{\sqrt{S_n}n^\tau}\sum_{i=1}^n \left(\delta_i^2 - E\delta_i^2\right) + \frac{1}{\sqrt{S_n}n^\tau}\sum_{i=1}^n E\delta_i^2 \\ &= J_{1n} + J_{2n} + \frac{n^{1-\tau}}{\sqrt{S_n}}E\delta_1^2. \end{aligned} \tag{5.24}$$

It follows from (3.6) again that for any $\varepsilon > 0$,

$$\frac{n^{1-\tau}}{\sqrt{S_n}}E\delta_1^2 < \frac{\varepsilon}{2} \quad \text{for all } n \text{ large enough.} \tag{5.25}$$

By (5.22)–(5.25), we can get that for any $\varepsilon > 0$,

$$\sum_{n=1}^\infty P\left(\frac{1}{\sqrt{S_n}n^\tau}\sum_{i=1}^n \left(\delta_i - \bar{\delta}_n\right)^2 > \varepsilon\right) < \infty,$$

which implies (5.18).

Similarly to the proof of (5.18), we can get that

$$\frac{1}{\sqrt{S_n}}n^\tau \sum_{i=1}^n (\varepsilon_i - \bar{\varepsilon}_n)^2 \to 0 \quad \text{completely.} \tag{5.26}$$

Note that

$$\left|\sum_{i=1}^n (\delta_i - \bar{\delta}_n)\varepsilon_i\right| \le \frac{1}{2}\left[\sum_{i=1}^n (\delta_i - \bar{\delta}_n)^2 + \sum_{i=1}^n (\varepsilon_i - \bar{\varepsilon}_n)^2\right],$$

which together with (5.18) and (5.26) yield (5.19).

It is easily checked that for any $r > 0$,

$$\begin{aligned}\sum_{i=1}^n |x_i - \bar{x}_n| \cdot \left|\delta_i - \bar{\delta}_n\right| &\le \sqrt{\sum_{i=1}^n (x_i - \bar{x}_n)^2} \cdot \sqrt{\sum_{i=1}^n (\delta_i - \bar{\delta}_n)^2} \\ &\le \frac{r}{2}S_n + \frac{1}{2r}\sum_{i=1}^n (\delta_i - \bar{\delta}_n)^2, \end{aligned}$$

which implies that

$$\begin{aligned}\left|\sum_{i=1}^n (\xi_i - \bar{\xi}_n)^2 - S_n\right| &= \left|2\sum_{i=1}^n (x_i - \bar{x}_n)(\delta_i - \bar{\delta}_n) + \sum_{i=1}^n (\delta_i - \bar{\delta}_n)^2\right| \\ &\le rS_n + \frac{1+r}{r}\sum_{i=1}^n (\delta_i - \bar{\delta}_n)^2. \end{aligned} \tag{5.27}$$

https://doi.org/10.1017/S0269964816000462 Published online by Cambridge University Press

It follows from (5.18) and Borel–Cantelli lemma that

$$\frac{1}{\sqrt{S_n}n^{\tau}}\sum_{i=1}^{n}(\delta_i-\bar{\delta}_n)^2\to 0 \quad \text{a.s.} \tag{5.28}$$

Hence, we have by (3.5), (5.27) and (5.28) that

$$\left|\frac{1}{S_n}\sum_{i=1}^{n}(\xi_i-\bar{\xi}_n)^2-1\right|\le r+\frac{1+r}{r}\cdot\frac{n^{\tau}}{\sqrt{S_n}}\cdot\frac{1}{\sqrt{S_n}n^{\tau}}\sum_{i=1}^{n}(\delta_i-\bar{\delta}_n)^2\to r \quad \text{a.s.} \tag{5.29}$$

Since $r>0$ is arbitrary, we can get (5.21) immediately by (5.29).

Finally, we will prove (5.20). Denote

$$b_{ni}=\frac{x_i-\bar{x}_n}{\sqrt{S_n}n^{\tau}},\quad X_i=\varepsilon_i-\beta\delta_i.$$

It follows from Corollary 2.1 that $\{X_i, i\ge 1\}$ is a sequence of END random variables. In addition, we have by $E|\varepsilon_1|^{4p}<\infty$, $E|\delta_1|^{4p}<\infty$ and (3.4) that

$$E|X_1|^{2p}=E|\varepsilon_1-\beta\delta_1|^{2p}\le CE|\varepsilon_1|^{2p}+CE|\delta_1|^{2p}<\infty,$$

$$\max_{1\le i\le n}|b_{ni}|=O\left(n^{-1/p}\right),\quad \sum_{i=1}^{n}b_{ni}^2=\frac{1}{n^{2\tau}}=o\left((\log n)^{-1}\right).$$

That is to say, the conditions of Lemma 2.6 are satisfied. Hence, (5.20) follows from Lemma 2.6 immediately. This completes the proof of the theorem. ■

Proof of Theorem 3.4: According to the relation (3.3), to prove (3.11), it suffices to show that

$$n^{\nu}(\bar{\varepsilon}_n-\beta\bar{\delta}_n)\to 0 \quad \text{completely}, \tag{5.30}$$

$$n^{\nu}(\beta-\hat{\beta}_n)\bar{x}_n\to 0 \quad \text{completely} \tag{5.31}$$

and

$$n^{\nu}(\beta-\hat{\beta}_n)\bar{\delta}_n\to 0 \quad \text{completely}. \tag{5.32}$$

Firstly, we will prove (5.30). It is easily seen that

$$n^{\nu}(\bar{\varepsilon}_n-\beta\bar{\delta}_n)=\frac{1}{n^{1-\nu}}\sum_{i=1}^{n}(\varepsilon_i-\beta\delta_i).$$

Applying Lemma 2.6 with $b_{ni}=\frac{1}{n^{1-\nu}}$, $X_i=\varepsilon_i-\beta\delta_i$, and noting that $0<\nu<\min(1-1/p,1/2)$, $p>1$, we can see that the conditions of Lemma 2.6 are satisfied. Hence, (5.30) follows from Lemma 2.6 immediately.

Next, we will prove (5.31). It can be checked that

$$n^{\nu}(\beta-\hat{\beta}_n)\bar{x}_n=\frac{\sqrt{S_n}}{n^{\tau}}(\beta-\hat{\beta}_n)\cdot\frac{n^{\tau+\nu}}{\sqrt{S_n}}\bar{x}_n. \tag{5.33}$$

We have by (5.33), (3.8) and Theorem 3.3 that for any $\varepsilon>0$,

$$\sum_{n=1}^{\infty}P\left(n^{\nu}|(\beta-\hat{\beta}_n)\bar{x}_n|>\varepsilon\right)\le\sum_{n=1}^{\infty}P\left(\frac{\sqrt{S_n}}{n^{\tau}}|\beta-\hat{\beta}_n|>C\right)<\infty,$$

which implies (5.31).

https://doi.org/10.1017/S0269964816000462 Published online by Cambridge University Press

Finally, we will prove (5.32). Similarly to the proof of (5.30), we can get that

$$\frac{1}{n^{1-\nu}}\sum_{i=1}^{n}\delta_i \to 0 \quad \text{completely.} \tag{5.34}$$

Noting that

$$n^{\nu}(\beta-\hat{\beta}_n)\bar{\delta}_n = \frac{\sqrt{S_n}}{n^{\tau}}(\beta-\hat{\beta}_n)\cdot\frac{n^{\tau}}{\sqrt{S_n}}\cdot\frac{1}{n^{1-\nu}}\sum_{i=1}^{n}\delta_i, \tag{5.35}$$

we have by (5.35), (3.5), Theorem 3.3 and (5.34) that for any $\varepsilon > 0$,

$$\begin{aligned}\sum_{n=1}^{\infty}P\left(n^{\nu}|(\beta-\hat{\beta}_n)\bar{\delta}_n| > \varepsilon\right) &\le \sum_{n=1}^{\infty}P\left(\left|\frac{\sqrt{S_n}}{n^{\tau}}(\beta-\hat{\beta}_n)\right| > C\right)\\ &\quad + \sum_{n=1}^{\infty}P\left(\left|\frac{1}{n^{1-\nu}}\sum_{i=1}^{n}\delta_i\right| > C\right)\\ &< \infty,\end{aligned}$$

which implies (5.32). This completes the proof of the theorem. ■

Acknowledgement

The authors are most grateful to the Editor Sheldon M. Ross and anonymous referee for careful reading of the manuscript and valuable suggestions, which helped in improving an earlier version of this paper. This work was supported by the National Natural Science Foundation of China (grant numbers 11501004, 11501005, 11671012 and 11301073), the Natural Science Foundation of Anhui Province (grant numbers 1508085J06 and 1608085QA02) and the Key Projects for Academic Talent of Anhui Province (grant number gxbjZD2016005).

References

1. Amemiya, Y. & Fuller, W.A. (1984). Estimation for the multivariate errors-in-variables model with estimated error covariance matrix. *The Annals of Statistics* 12: 497–509.
2. Carroll, R.J., Ruppert, D., Crainiceanu, C., Tosteson, T.D., & Karagas, M.R. (2004). Nonlinear and nonparametric regression and instrumental Variables. *Journal of the American Statistical Association* 99: 736–750.
3. Chen, Y., Chen, A., & Ng, K.W. (2010). The strong law of large numbers for extend negatively dependent random variables. *Journal of Applied Probability* 47: 908–922.
4. Cui, H.J. & Chen, S.X. (2003). Empirical likelihood confidence region for parameter in the errors-in-variables models. *Journal of Multivariate Analysis* 84: 101–115.
5. Dagenais, M.G. & Dagenais, D.L. (1997). Higher moment estimators for linear regression models with errors in the variables. *Journal of Econometrics* 76: 193–221.
6. Deaton, A. (1985). Panel data from a time series of cross-sections. *Journal of Econometrics* 30: 109–126.
7. Fan, G.L., Liang, H.Y., Wang, J.F., & Xu, H.X. (2010). Asymptotic properties for LS estimators in EV regression model with dependent errors. *AStA-Advances in Statistical Analysis* 94: 89–103.
8. Gleser, L.J. (1981). Estimation in a multivariate 'errors in variables' regression model: large sample results. *The Annals of Statistics* 9: 24–44.
9. Hslao, C., Wang, L., & Wang, Q. (1997). Estimation of nonlinear errors-in-variables models: an approximate solution. *Statistical Papers* 38: 1–25.
10. Jing, B.Y. & Liang, H.Y. (2008). Strong limit theorems for weighted sums of negatively associated random variables. *Journal of Theoretical Probability* 21: 890–909.
11. Jung, K.-M. (2007). Least trimmed squares estimator in the errors-in-variables model. *Journal of Applied Statistics* 34: 331–338.
12. Lai, T.L., Robbins, H., & Wei, C.Z. (1979). Strong consistency of least squares estimates in multiple regression. *Journal of Multivariate Analysis* 9: 343–362.

https://doi.org/10.1017/S0269964816000462 Published online by Cambridge University Press

13. Li, T. (2002). Robust and consistent estimation of nonlinear errors-in-variables models. *Journal of Econometrics* 110: 1–26.
14. Liu, J.X. & Chen, X.R. (2005). Consistency of LS estimator in simple linear EV regression models. *Acta Mathematica Scientia B* 25: 50–58.
15. Liu, L. (2009). Precise large deviations for dependent random variables with heavy tails. *Statistics and Probability Letters* 79: 1290–1298.
16. Miao, Y., Wang, Y.L. & Zheng, H.J. (2015). Consistency of LS estimators in the EV regression model with martingale difference errors. *Statistics: A Journal of Theoretical and Applied Statistics* 49: 104–118.
17. Miao, Y., Zhao, F.F., Wang, K., & Chen, Y.P. (2013). Asymptotic normality and strong consistency of LS estimators in the EV regression model with NA errors. *Statistical Papers* 54: 193–206.
18. Qiu, D.H., Chen, P.Y., Antonini, R.G., & Volodin, A. (2013). On the complete convergence for arrays of rowwise extended negatively dependent random variables. *Journal of the Korean Mathematical Society* 50: 379–392.
19. Ranjbar, V., Amini, M., & Bozorgnia, A. (2009). Asympotic behavior of convolution of dependent random variables with heavy-tail distribution. *Thai Journal of Mathematics* 7: 21–34.
20. Ranjbar, V., Amini, M., Geluk, J., & Bozorgnia, A. (2013). Asympotic behavior of product of two heavy-tail dependent random variables. *Acta Mathematica Sinica, English Series* 29: 355–364.
21. Shen, A.T. (2011). Probability inequalities for END sequence and their applications. *Journal of Inequalities and Applications* 2011: 12 pages, Article ID 98.
22. Shen, A.T. (2013). On the strong convergence rate for weighted sums of arrays of rowwise negatively orthant dependent random variables. *RACSAM* 107: 257–271.
23. Shen, A.T. (2014). On asymptotic approximation of inverse moments for a class of nonnegative random variables. *Statistics: A Journal of Theoretical and Applied Statistics* 48: 1371–1379.
24. Shen, A.T., Zhang, Y., & Volodin, A. (2015). Applications of the Rosenthal-type inequality for negatively superadditive dependent random variables. *Metrika* 78: 295–311.
25. Stout, W.F. (1974). *Almost sure convergence*. New York: Academic Press.
26. Taupin, M.L. (2001). Semiparametric estimation in the nonlinear structural errors-in-variables model. *The Annals of Statistics* 29: 66–93.
27. Wang, S.J. & Wang, X.J. (2013). Precise large deviations for random sums of END real-valued random variables with consistent variation. *Journal of Mathematical Analysis and Applications* 402: 660–667.
28. Wang, X.J., Li, X.Q., Hu, S.H., & Wang, X.H. (2014). On complete convergence for an extended negatively dependent sequence. *Communications in Statistics – Theory and Methods* 43: 2923–2937.
29. Wang, X.J., Shen, A.T., Chen, Z.Y., & Hu, S.H. (2015). Complete convergence for weighted sums of NSD random variables and its application in the EV regression model. *TEST* 24: 166–184.
30. Wang, X.J., Wang, S.J., Hu, S.H., Ling, J.M., & Wei, Y.F. (2013). On complete convergence of weighted sums for arrays of rowwise extended negatively dependent random variables. *Stochastics: An International Journal of Probability and Stochastic Processes* 85: 1060–1072.
31. Wang, X.J., Zheng, L.L., Xu, C., & Hu, S.H. (2015). Complete consistency for the estimator of nonparametric regression models based on extended negatively dependent errors. *Statistics: A Journal of Theoretical and Applied Statistics* 49: 396–407.
32. Wu, Y.F. & Guan, M. (2012). Convergence properties of the partial sums for sequences of END random variables. *Journal of the Korean Mathematical Society* 49: 1097–1110.

APPENDIX A

Proof of Lemma 2.6: Without loss of generality, we may assume that $b_{ni} > 0$. For any $\varepsilon > 0$, we choose positive integer N (to be specified later) and small positive constant q such that $q < \frac{1-1/p}{p}$ if $p > 1$ and $q < \delta/p$ if $0 < p \le 1$.

Denote for $1 \le i \le n$ and $n \ge 1$ that

$$X_{ni}(1) = X_i I(|b_{ni}X_i| \le n^{-q}) - b_{ni}^{-1}n^{-q}I(b_{ni}X_i < -n^{-q}) + b_{ni}^{-1}n^{-q}I(b_{ni}X_i > n^{-q}),$$

$$X_{ni}(2) = (X_i - b_{ni}^{-1}n^{-q})I(n^{-q} < b_{ni}X_i \le \varepsilon/N),$$

$$X_{ni}(3) = (X_i + b_{ni}^{-1}n^{-q})I(-\varepsilon/N \le b_{ni}X_i < -n^{-q}),$$

$$X_{ni}(4) = (X_i + b_{ni}^{-1}n^{-q})I(b_{ni}X_i < -\varepsilon/N) + (X_i - b_{ni}^{-1}n^{-q})I(b_{ni}X_i > \varepsilon/N).$$

https://doi.org/10.1017/S0269964816000462 Published online by Cambridge University Press

It is easy to check that $X_{ni}(1)+X_{ni}(2)+X_{ni}(3)+X_{ni}(4)=X_i$, which implies that

$$\begin{aligned}\sum_{n=1}^{\infty} P\left(\left|\sum_{i=1}^{n} b_{ni}X_i\right|>4\varepsilon\right) &\le \sum_{n=1}^{\infty} P\left(\left|\sum_{i=1}^{n} b_{ni}X_{ni}(1)\right|>\varepsilon\right)\\ &\quad+\sum_{n=1}^{\infty} P\left(\left|\sum_{i=1}^{n} b_{ni}X_{ni}(2)\right|>\varepsilon\right)\\ &\quad+\sum_{n=1}^{\infty} P\left(\left|\sum_{i=1}^{n} b_{ni}X_{ni}(3)\right|>\varepsilon\right)\\ &\quad+\sum_{n=1}^{\infty} P\left(\left|\sum_{i=1}^{n} b_{ni}X_{ni}(4)\right|>\varepsilon\right)\\ &\doteq I_1+I_2+I_3+I_4.\end{aligned}$$

Hence, to prove (2.6), it suffices to show that $I_1<\infty$, $I_2<\infty$, $I_3<\infty$ and $I_4<\infty$.

Noting that $\max_{1\le i\le n}|b_{ni}(X_{ni}(1)-EX_{ni}(1))|\le 2n^{-q}$ for every $n\ge 1$, we have

$$P\left(\max_{1\le i\le n}|b_{ni}(X_{ni}(1)-EX_{ni}(1))|\le 2n^{-q}\right)=1,\quad \text{for every } n\ge 1. \tag{A.1}$$

If $0<p\le 1$, we have by Markov's inequality, Lemma 2.7 and (2.5) that

$$\begin{aligned}B_n^2 &\doteq \sum_{i=1}^{n} E[b_{ni}(X_{ni}(1)-EX_{ni}(1))]^2 \le \sum_{i=1}^{n} E(b_{ni}X_{ni}(1))^2\\ &\le Cn^{-(2-p)q}\sum_{i=1}^{n}|b_{ni}|^pE|X|^p \le Cn^{-(2-p)q-\delta}=o((\log n)^{-1}).\end{aligned} \tag{A.2}$$

If $p>1$, noting that $|X_{ni}(1)|\le|X_i|$, we have by Lemma 2.7 and (2.5) again that

$$B_n^2 \doteq \sum_{i=1}^{n} E[b_{ni}(X_{ni}(1)-EX_{ni}(1))]^2 \le C\sum_{i=1}^{n} b_{ni}^2EX^2=o((\log n)^{-1}). \tag{A.3}$$

Noting that for fixed $n\ge 1$, $\{b_{ni}(X_{ni}(1)-EX_{ni}(1)),1\le i\le n\}$ are END by Remark 1.1 and Lemma 2.1. Applying Lemma 2.4 with $x=\varepsilon$ and $y=2n^{-q}$, we have by (A.1)–(A.3) that

$$\begin{aligned}&\sum_{n=1}^{\infty} P\left(\left|\sum_{i=1}^{n} b_{ni}(X_{ni}(1)-EX_{ni}(1))\right|>\varepsilon\right)\\ &\le 2\sum_{n=1}^{\infty} P\left(\max_{1\le i\le n}|b_{ni}(X_{ni}(1)-EX_{ni}(1))|>2n^{-q}\right)+C\sum_{n=1}^{\infty}\exp\left\{-\frac{\varepsilon^2}{2(2\varepsilon n^{-q}+o((\log n)^{-1}))}\right\}\\ &\le C\sum_{n=1}^{\infty}\exp\{-2\log n\}<\infty.\end{aligned} \tag{A.4}$$

To show that $I_1<\infty$, it remains to show

$$\left|\sum_{i=1}^{n} b_{ni}EX_{ni}(1)\right|\to 0 \quad \text{as} \quad n\to\infty. \tag{A.5}$$

https://doi.org/10.1017/S0269964816000462 Published online by Cambridge University Press

If $0 < p \le 1$, we have by Markov's inequality, Lemma 2.7 and (2.5) that

$$\left|\sum_{i=1}^{n} b_{ni} EX_{ni}(1)\right| \le n^{-(1-p)q} \sum_{i=1}^{n} E|b_{ni}X_i|^p + \sum_{i=1}^{n} E|b_{ni}X_i|^p |b_{ni}X_i|^{1-p} I(|b_{ni}X_i| \le n^{-q})$$
$$\le Cn^{-(1-p)q} \sum_{i=1}^{n} |b_{ni}|^p E|X|^p \le Cn^{-(1-p)q-\delta} \to 0 \quad \text{as} \quad n \to \infty. \tag{A.6}$$

If $p > 1$, noting that $EX_i = 0$ and $q < \frac{1-1/p}{p} < \frac{2-2/p}{2p-1}$, we have by Markov's inequality, Lemma 2.7 and (2.4) and (2.5) that

$$\left|\sum_{i=1}^{n} b_{ni} EX_{ni}(1)\right| \le n^{(2p-1)q} \sum_{i=1}^{n} E|b_{ni}X_i|^{2p} + n^{(2p-1)q} \sum_{i=1}^{n} E|b_{ni}X_i|^{2p}$$
$$\le Cn^{(2p-1)q} \left(\max_{1\le i\le n} |b_{ni}|\right)^{2p-2} \sum_{i=1}^{n} b_{ni}^2$$
$$\le Cn^{(2p-1)q-(2-2/p)} (\log n)^{-1} \to 0 \quad \text{as } n \to \infty. \tag{A.7}$$

From (A.5) to (A.7), we have proved that $I_1 < \infty$.

Next, we will prove that $I_2 < \infty$. Since $0 < b_{ni}X_{ni}(2) \le \varepsilon/N$, we can see that

$$\left|\sum_{i=1}^{n} b_{ni} X_{ni}(2)\right| = \sum_{i=1}^{n} b_{ni} X_{ni}(2) > \varepsilon$$

implies that there are at least N integers such that $b_{ni}X_{ni}(2) \ne 0$. Hence, by conditions (2.4) and (2.5), we can see that

$$P\left(\left|\sum_{i=1}^{n} b_{ni} X_{ni}(2)\right| > \varepsilon\right)$$
$$\le \sum_{1\le i_1<i_2<\ldots<i_N\le n} P\left(b_{n,i_1}X_{n,i_1}(2) \ne 0, b_{n,i_2}X_{n,i_2}(2) \ne 0, \ldots, b_{n,i_N}X_{n,i_N}(2) \ne 0\right)$$
$$\le C \sum_{1\le i_1<i_2<\ldots<i_N\le n} P(b_{n,i_1}X_{i_1} > n^{-q}) P(b_{n,i_2}X_{i_2} > n^{-q}) \cdots P(b_{n,i_N}X_{i_N} > n^{-q})$$
$$\le C\left(\sum_{i=1}^{n} P(b_{ni}X_i > n^{-q})\right)^N \le C\left(\sum_{i=1}^{n} P(|b_{ni}X| > n^{-q})\right)^N$$
$$\le \begin{cases} C\left(n^{pq} \sum\limits_{i=1}^{n} E|b_{ni}X|^p\right)^N, & \text{if } 0 < p \le 1, \\ C\left(n^{2pq} \sum\limits_{i=1}^{n} E|b_{ni}X|^{2p}\right)^N, & \text{if } p > 1, \end{cases}$$
$$\le \begin{cases} Cn^{-(\delta-pq)N}, & \text{if } 0 < p \le 1, \\ C\left[n^{2pq}\left(\max\limits_{1\le i\le n} |b_{ni}|\right)^{2p-2} \sum\limits_{i=1}^{n} b_{ni}^2\right]^N, & \text{if } p > 1, \end{cases}$$
$$\le \begin{cases} Cn^{-(\delta-pq)N}, & \text{if } 0 < p \le 1, \\ Cn^{-2(1-1/p-pq)N} (\log n)^{-N}, & \text{if } p > 1. \end{cases}$$

Noting that $q < \frac{1-1/p}{p}$ if $p > 1$ and $q < \delta/p$ if $0 < p \le 1$, we choose some large integer N such that $(\delta - pq)N > 1$ and $2(1 - 1/p - pq)N > 1$. Hence, $I_2 < \infty$.

https://doi.org/10.1017/S0269964816000462 Published online by Cambridge University Press

Since $-\varepsilon/N \le b_{ni}X_{ni}(3) < 0$, we can see that

$$\left|\sum_{i=1}^{n} b_{ni}X_{ni}(3)\right| = -\sum_{i=1}^{n} b_{ni}X_{ni}(3) > \varepsilon$$

implies that there are at least N integers such that $b_{ni}X_{ni}(3) \neq 0$. Hence, similarly to the proof of $I_2 < \infty$, we can get $I_3 < \infty$.

Finally, we will prove that $I_4 < \infty$. By Lemma 2.7 and $E|X|^{2p} < \infty$, we have

$$\begin{aligned} I_4 &\le \sum_{n=1}^{\infty}\sum_{i=1}^{n} P(|b_{ni}X_i| > \varepsilon/N) \le C\sum_{n=1}^{\infty} nP(|X| > Cn^{1/p}) \\ &\le C\sum_{k=1}^{\infty} k^2 P(Ck^{1/p} < |X| \le C(k+1)^{1/p}) \\ &\le CE|X|^{2p} < \infty. \end{aligned}$$

This completes the proof the lemma. ■

https://doi.org/10.1017/S0269964816000462 Published online by Cambridge University Press