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Chaotic period doubling

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Abstract. The period doubling renormalization operator was introduced by Feigenbaum and by Coullet and Tresser in the 1970s to study the asymptotic small-scale geometry of the attractor of one-dimensional systems that are at the transition from simple to chaotic dynamics. This geometry turns out not to depend on the choice of the map under rather mild smoothness conditions. The existence of a unique renormalization fixed point that is also hyperbolic among generic smooth-enough maps plays a crucial role in the corresponding renormalization theory. The uniqueness and hyperbolicity of the renormalization fixed point were first shown in the holomorphic context, by means that generalize to other renormalization operators. It was then proved that, in the space of $C^{2+\alpha}$ unimodal maps, for $\alpha > 0$, the period doubling renormalization fixed point is hyperbolic as well. In this paper we study what happens when one approaches from below the minimal smoothness thresholds for the uniqueness and for the hyperbolicity of the period doubling renormalization generic fixed point. Indeed, our main result states that in the space of C^2 unimodal maps the analytic fixed point is not hyperbolic and that the same remains true when adding enough smoothness to get *a priori* bounds. In this smoother class, called $C^{2+|\cdot|}$, the failure of hyperbolicity is tamer than in C^2. Things get much worse with just a bit less smoothness than C^2, as then even the uniqueness is lost and other asymptotic behavior becomes possible. We show that the period doubling renormalization operator acting on the space of $C^{1+\mathrm{Lip}}$ unimodal maps has infinite topological entropy.

1. *Introduction*

The period doubling renormalization operator was introduced by Feigenbaum [**Fe, Fe2**] and by Coullet and Tresser [**CT, TC**] to study the asymptotic small-scale geometry of the attractor of one-dimensional systems that are at the transition from simple to chaotic

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dynamics. In 1978, they published certain rigidity properties of such systems, the small-scale geometry of the invariant Cantor set of generic smooth maps at the boundary of chaos being independent of the particular map under consideration. Coullet and Tresser treated this phenomenon as similar to *universality*, which has been observed in critical phenomena for a long time and explained since the early 1970s by Kenneth Wilson (see, e.g., [**Ma**]). In an attempt to explain universality at the transition to chaos, both groups formulated the following conjectures, which are similar to what was conjectured in statistical mechanics.

Renormalization conjectures: In the proper class of maps, the period doubling renormalization operator has a unique fixed point that is hyperbolic with a one-dimensional unstable manifold and a codimension-one stable manifold consisting of the systems at the transition to chaos.

These conjectures were extended to other types of dynamics on the interval and on other manifolds, but we will not be concerned here with such generalizations. During the last 30 years many authors have contributed to the development of a rigorous theory proving the renormalization conjectures and explaining the phenomenology. The ultimate goal may still be far off, since the universality class of smooth maps at the boundary of chaos contains many sorts of dynamical systems, including useful differential models of natural phenomena. There are even predictions about natural phenomena in [**CT**], which turned out to be experimentally corroborated. A historical review of the mathematics that have been developed can be found in [**FMP**], so that here we recall only a few milestones that will serve to better understand the contribution to the overall picture brought by the present paper.

The type of differentiability of the systems under consideration has a crucial influence on the actual small-scale geometrical behavior (as is the case in the related problem of smooth conjugacy of circle diffeomorphisms to rotations: compare [**He**] to [**KO**] and [**KS**]). The first result dealt with holomorphic systems and were first local [**La**], and later global [**Su, McM, Ly**] (a progression similar to what had been seen in the problem of smooth conjugacy to rotations: compare [**Ar**] to [**He**] and [**Yo**]). With global methods came also means to consider other renormalizations. Indeed, the hyperbolicity of the unique renormalization fixed point was shown in [**La**] for period doubling, and later in [**Ly**] by means that generalize to other sorts of dynamics. Then it was shown in [**Da**] that the renormalization fixed point is also hyperbolic in the space of $C^{2+\alpha}$ unimodal maps with $\alpha > 0$ (using [**La**]). These results were later extended in [**FMP**] to more general types of renormalization (using [**Ly**]). After the results of Lanford [**La**], the existence of renormalization fixed points has been proved in more generality. First, Epstein [**Ep**] constructed period doubling fixed points with arbitrary critical behavior. Renormalization fixed points do exist for any given combinatorics and arbitrary critical behavior, see [**M**].

We are interested in exploring from below the limit of smoothness that permits hyperbolicity of the fixed point of renormalization. Our main result concern a new smoothness class, $C^{2+|\cdot|}$, which is bigger than $C^{2+\alpha}$ for any positive $\alpha \leq 1$, and is in fact wider than C^2 in ways that are rather technical, as we shall describe later (this is the bigger class where the usual method to get *a priori* bounds for the geometry of the Cantor set works). We are interested here in the part of the hyperbolicity that consists in the attraction in the stable manifold made of infinitely renormalizable maps. We show that in the space of $C^{2+|\cdot|}$ unimodal maps the analytic fixed point is not hyperbolic for the action of the period

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doubling renormalization operator. We also show that, nevertheless, the renormalization converges to the analytic generic fixed point (here generic means that the second derivative at the critical point is not zero), proving it to be globally unique, a uniqueness that was formerly known in classes smaller than $C^{2+|\cdot|}$ (that is, assuming more smoothness). The convergence might only be polynomial as a concrete sign of non-hyperbolicity. The failure of hyperbolicity happens in a more serious way in the space of C^2 unimodal maps since there the convergence can be arbitrarily slow. The uniqueness of the fixed point in this case remains an open question. The uniqueness was known to be wrong in a serious way among $C^{1+\mathrm{Lip}}$ unimodal maps since a continuum of fixed points of renormalization could be produced [**Tr**]. Here we show that the period doubling renormalization operator acting on the space of $C^{1+\mathrm{Lip}}$ unimodal maps has infinite topological entropy.

After this informal discussion of what will be done here and how it relates to universality theory, we now give some definitions, which allow us next to turn to the precise formulation of our main results.

A *unimodal* map $f : [0, 1] \to [0, 1]$ is a C^1 mapping with the following properties:

- $f(1) = 0$;
- there is a unique point $c \in (0, 1)$, the *critical point*, where $Df(c) = 0$; and
- $f(c) = 1$.

A map is a C^r unimodal map if f is C^r. We will concentrate on unimodal maps of the type $C^{1+\mathrm{Lip}}$, C^2, and $C^{2+|\cdot|}$. This last type of differentiability will be introduced in §5.

The critical point c of a C^2 unimodal map f is called *non-flat* if $D^2 f(c) \neq 0$. A critical point c of a unimodal map f has a *quadratic tip* if there exists a sequence of points $x_n \to c$ and constant $A > 0$ such that

$$\lim_{n\to\infty} \frac{f(x_n) - f(c)}{(x_n - c)^2} = -A.$$

The set of C^r unimodal maps with a quadratic tip is denoted by $\mathcal{U}^r$. We will consider different metrics on this set denoted by dist_k with $k = 0, 1, 2$ (in fact, the usual C^k metrics).

A unimodal map $f : [0, 1] \to [0, 1]$ with quadratic tip c is *renormalizable* if:

- $c \in [f^2(c), f^4(c)] \equiv I_0^1$;
- $f(I_0^1) = [f^3(c), f(c)] \equiv I_1^1$; and
- $I_0^1 \cap I_1^1 = \emptyset$.

The set of renormalizable C^r unimodal maps is denoted by $\mathcal{U}_0^r \subset \mathcal{U}^r$. Let $f \in \mathcal{U}_0^r$ be a renormalizable map. The renormalization of f is defined by

$$Rf(x) = h^{-1} \circ f^2 \circ h(x),$$

where $h : [0, 1] \to I_0^1$ is the orientation-reversing affine homeomorphism. This map Rf is again a unimodal map. The nonlinear operator $R : \mathcal{U}_0^r \to \mathcal{U}^r$ defined by

$$R : f \mapsto Rf$$

is called the *renormalization operator*. The set of *infinitely* renormalizable maps is denoted by

$$W^r = \bigcap_{n\geq 1} R^{-n}(\mathcal{U}_0^r).$$

There are many fundamental steps needed to reach the following result by Davie [**Da**]. For a brief history see [**FMP**] and references therein.

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THEOREM 1.1. (Davie) *There exists $\alpha < 1$ such that the following holds. In the space of $\mathcal{U}^{2+\alpha}$, there exists a unique renormalization fixed point f_*^ω, with the following properties:*

- *f_*^ω is analytic;*
- *f_*^ω is a hyperbolic fixed point of $R : \mathcal{U}_0^{2+\alpha} \to \mathcal{U}^{2+\alpha}$;*
- *the codimension-one stable manifold of f_*^ω coincides with $W^{2+\alpha}$; and*
- *f_*^ω has a one-dimensional unstable manifold that consists of analytic maps.*

In our discussion we only deal with period doubling renormalization. However, there are other renormalization schemes. The hyperbolicity for the corresponding generalized renormalization operator has been established in [**FMP**].

Our main results deal with $R : \mathcal{U}_0^r \to \mathcal{U}^r$, where $r \in \{1 + \text{Lip}, 2, 2 + |\cdot|\}$.

THEOREM 1.2. *Let $d_n > 0$ be any sequence with $d_n \to 0$. There exists an infinitely renormalizable C^2 unimodal map f with quadratic tip such that*

$$\text{dist}_0(R^n f,\ f_*^\omega) \geq d_n.$$

COROLLARY 1.3. *The analytic unimodal map f_*^ω is not a hyperbolic fixed point of $R : \mathcal{U}_0^2 \to \mathcal{U}^2$.*

In §5 we will introduce a type of differentiability of a unimodal map, called $C^{2+|\cdot|}$, which is the minimal needed to be able to apply the classical proofs of *a priori* bounds for the invariant Cantor sets of infinitely renormalizable maps; see for example [**M2**], [**MMSS**], and [**MS**]. This type of differentiability will allow us to represent any $C^{2+|\cdot|}$ unimodal map as

$$f = \phi \circ q,$$

where q is a quadratic polynomial and ϕ has still enough differentiability to control cross-ratio distortion. The precise description of this decomposition is given in Proposition 5.6. For completeness, we include the proof of the *a priori* bounds in §7.

THEOREM 1.4. *If f is an infinitely renormalizable $C^{2+|\cdot|}$ unimodal map, then*

$$\lim_{n\to\infty} \text{dist}_0(R^n f,\ f_*^\omega) = 0.$$

A construction similar to the one provided for C^2 unimodal maps leads to the following result.

THEOREM 1.5. *Let $d_n > 0$ be any sequence with $\sum_{n\geq 1} d_n < \infty$. There exists an infinitely renormalizable $C^{2+|\cdot|}$ unimodal map f with a quadratic tip such that*

$$\text{dist}_0(R^n f,\ f_*^\omega) \geq d_n.$$

The analytic unimodal map f_*^ω is not a hyperbolic fixed point of $R : \mathcal{U}_0^{2+|\cdot|} \to \mathcal{U}^{2+|\cdot|}$.

Our second set of theorems deals with renormalization of $C^{1+\text{Lip}}$ unimodal maps with a quadratic tip.

THEOREM 1.6. *There exists an infinitely renormalizable $C^{1+\text{Lip}}$ unimodal map f with a quadratic tip that is not C^2 but*

$$Rf = f.$$

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The *topological entropy* of a system defined on a non-compact space is defined to be the supremum of the topological entropies contained in compact invariant subsets: we will always mean topological entropy when the type of entropy is not specified. As a consequence of Theorem 1.1 we get that renormalization on $\mathcal{U}_0^{2+\alpha}$ has entropy zero.

THEOREM 1.7. *The renormalization operator acting on the space of $C^{1+\mathrm{Lip}}$ unimodal maps with quadratic tip has infinite entropy.*

The last theorem illustrates a specific aspect of the chaotic behavior of the renormalization operator on $\mathcal{U}_0^{1+\mathrm{Lip}}$.

THEOREM 1.8. *There exists an infinitely renormalizable $C^{1+\mathrm{Lip}}$ unimodal map f with quadratic tip such that $\{c_n\}_{n\geq 0}$ is dense in a Cantor set. Here c_n is the critical point of $R^n f$.*

2. *Notation*

Let $I, J \subset \mathbb{R}^n$, with $n \geq 1$. We will use the following notation.

(i) We let $\mathrm{cl}(I)$, $\mathrm{int}(J)$, and ∂I stand for respectively the closure, the interior, and the boundary of I.

(ii) We let $|I|$ stand for the Lebesgue measure of I.

(iii) If $n = 1$ then $[I, J]$ is the smallest interval that contains I and J.

(iv) The Euclidean distance between x and y is denoted by $\mathrm{dist}(x, y)$ and

$$\mathrm{dist}(I, J) = \inf_{x\in I,\, y\in J} \mathrm{dist}(x, y).$$

(v) If F is a map between two sets, then $\mathrm{image}(F)$ stands for the image of F.

(vi) Define $\mathrm{Diff}_+^k([0, 1])$, $k \geq 1$, to be the set of orientation-preserving C^k diffeomorphisms.

(vii) We let $|\cdot|_k$, $k \geq 0$, stand for the C^k norm of the functions under consideration.

(viii) We let dist_k, $k \geq 0$, stand for the C^k distance in the function spaces under consideration.

(ix) There is a constant $K > 0$, held fixed throughout the paper, that lets us write $Q_1 \asymp Q_2$ if and only if

$$\frac{1}{K} \leq \frac{Q_1}{Q_2} \leq K.$$

There are two rather independent discussions, one on $C^{1+\mathrm{Lip}}$ maps and the other on C^2 maps. There is a slight conflict in the notation used for these two discussions. In particular, the notation I_1^n stands for different intervals in the two parts, but the context will make the meaning of the symbols unambiguous.

3. *Renormalization of $C^{1+\mathrm{Lip}}$ unimodal maps*

3.1. *Piecewise affine infinitely renormalizable maps.* Consider the open triangle $\Delta = \{(x, y) \mid x, y > 0 \text{ and } x + y < 1\}$. A point $(\sigma_0, \sigma_1) \in \Delta$ is called a *scaling bi-factor*. A scaling bi-factor induces a pair of affine maps

$$\tilde{\sigma}_0 : [0, 1] \to [0, 1],$$
$$\tilde{\sigma}_1 : [0, 1] \to [0, 1],$$

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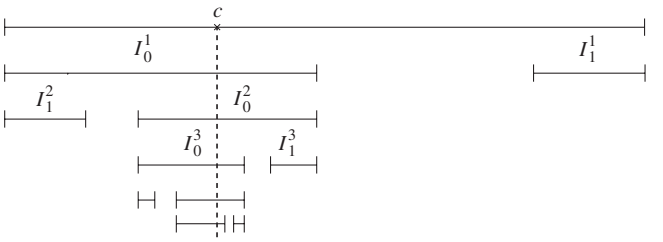


FIGURE 1.

defined by

$$\tilde{\sigma}_0(t) = -\sigma_0 t + \sigma_0 = \sigma_0(1-t),$$
$$\tilde{\sigma}_1(t) = \sigma_1 t + 1 - \sigma_1 = 1 - \sigma_1(1-t).$$

A function $\sigma : \mathbb{N} \to \Delta$ is called a *scaling data*. For each $n \in \mathbb{N}$ we set $\sigma(n) = (\sigma_0(n), \sigma_1(n))$, so that the point $(\sigma_0(n), \sigma_1(n)) \in \Delta$ induces a pair of maps $(\tilde{\sigma}_0(n), \tilde{\sigma}_1(n))$ as we have just described. For each $n \in \mathbb{N}$ we can now define the pair of intervals

$$I_0^n = \tilde{\sigma}_0(1) \circ \tilde{\sigma}_0(2) \circ \cdots \circ \tilde{\sigma}_0(n)([0,1]),$$
$$I_1^n = \tilde{\sigma}_0(1) \circ \tilde{\sigma}_0(2) \circ \cdots \circ \tilde{\sigma}_0(n-1) \circ \tilde{\sigma}_1(n)([0,1]).$$

A scaling data with the property

$$\operatorname{dist}(\sigma(n), \partial\Delta) \geq \epsilon > 0$$

is called *ϵ-proper*, and *proper* if it is ϵ-proper for some $\epsilon > 0$. For *ϵ-proper* scaling data we have

$$|I_j^n| \leq (1-\epsilon)^n$$

with $n \geq 1$ and $j = 0, 1$. Given proper scaling data define

$$\{c\} = \bigcap_{n \geq 1} I_0^n.$$

The point c, called the *critical* point, is shown in Figure 1. Consider the quadratic map $q_c : [0,1] \to [0,1]$ defined as

$$q_c(x) = 1 - \left(\frac{x-c}{1-c}\right)^2.$$

Given a proper scaling data $\sigma : \mathbb{N} \to \Delta$ and the set $D_\sigma = \bigcup_{n \geq 1} I_1^n$ induced by σ, we define a map

$$f_\sigma : D_\sigma \to [0,1]$$

by letting $f_\sigma|_{I_1^n}$ be the affine extension of $q_c|_{\partial I_1^n}$. The graph of f_σ is shown in Figure 2†.

† We apologize for the large size of Figure 2. In a smaller size picture it would be impossible to see the graphs of q_c and f_σ apart.

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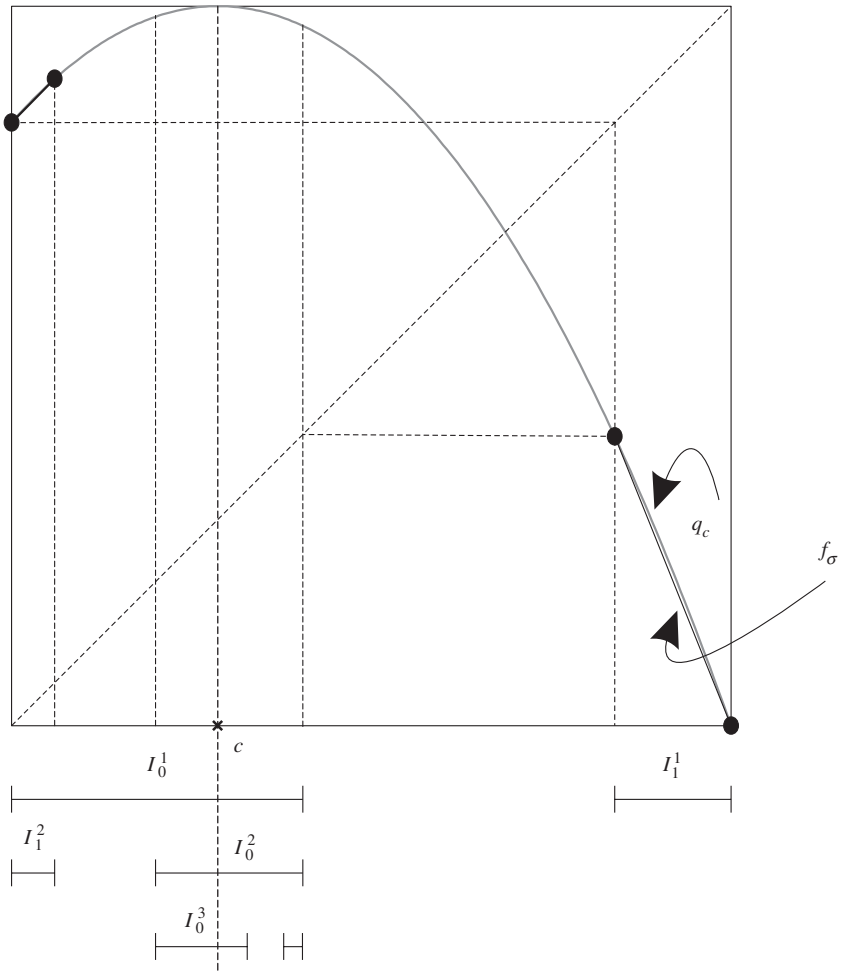


FIGURE 2.

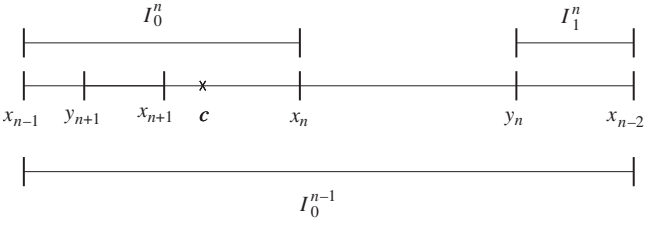


FIGURE 3.

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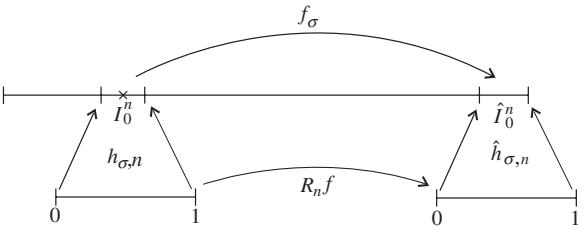


FIGURE 4.

Define $x_0 = 0$, $x_{-1} = 1$ and, for $n \geq 1$,

$$x_n = \partial I_0^n \setminus \partial I_0^{n-1},$$
$$y_n = \partial I_1^n \setminus \partial I_0^{n-1}.$$

These points are illustrated in Figure 3.

Definition 1. A map f_σ corresponding to proper scaling data $\sigma : \mathbb{N} \to \Delta$ is called infinitely renormalizable if, for $n \geq 1$:

(i) $[f_\sigma(x_{n-1}), 1]$ is the maximal domain containing 1 on which $f_\sigma^{2^n-1}$ is defined affinely; and

(ii) $f_\sigma^{2^n-1}([f_\sigma(x_{n-1}), 1]) = I_0^n$.

Define $W = \{f_\sigma : f_\sigma$ is infinitely renormalizable$\}$. Let $f \in W$ be given by the proper scaling data $\sigma : \mathbb{N} \to \Delta$ and define

$$\hat{I}_0^n = [q_c(x_{n-1}), 1] = [f(x_{n-1}), 1].$$

Let

$$h_{\sigma,n} : [0, 1] \to [0, 1]$$

be defined by

$$h_{\sigma,n} = \sigma_0(1) \circ \sigma_0(2) \circ \cdots \circ \sigma_0(n).$$

Furthermore let

$$\hat{h}_{\sigma,n} : [0, 1] \to \hat{I}_0^n$$

be the affine orientation-preserving homeomorphism. Then define

$$R_n f_\sigma : h_{\sigma,n}^{-1}(D_\sigma) \to [0, 1]$$

by

$$R_n f_\sigma = \hat{h}_{\sigma,n}^{-1} \circ f_\sigma \circ h_{\sigma,n}.$$

It is shown in Figure 4. Let $s : \Delta^{\mathbb{N}} \to \Delta^{\mathbb{N}}$ be the shift

$$s(\sigma)(k) = \sigma(k + 1).$$

The construction implies the following result.

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LEMMA 3.1. *Let $\sigma : \mathbb{N} \to \Delta$ be proper scaling data such that f_σ is infinitely renormalizable. Then*

$$R_n f_\sigma = f_{s^n(\sigma)}.$$

Next let f_σ be infinitely renormalizable. Then for $n \geq 0$ we have that

$$f_\sigma^{2^n} : \mathrm{D}_\sigma \cap I_0^n \to I_0^n$$

is well defined. Define the renormalization $R : W \to W$ by

$$Rf_\sigma = h_{\sigma,1}^{-1} \circ f_\sigma^2 \circ h_{\sigma,1}.$$

The map $f_\sigma^{2^n-1} : \hat{I}_0^n \to I_0^n$ is an affine homeomorphism whenever $f_\sigma \in W$. This implies immediately the following lemma.

LEMMA 3.2. *One has $R^n f_\sigma : \mathrm{D}_{s^n(\sigma)} \to [0, 1]$ and $R^n f_\sigma = R_n f_\sigma$.*

PROPOSITION 3.3. *One has $W = \{f_{\sigma^*}\}$ where σ^* is characterized by $Rf_{\sigma^*} = f_{\sigma^*}$.*

Proof. Let $\sigma : \mathbb{N} \to \Delta$ be proper scaling data such that f_σ is infinitely renormalizable. Let c_n be the critical point of $f_{s^n(\sigma)}$. Then

$$q_{c_n}(0) = 1 - \sigma_1(n), \tag{1}$$

$$q_{c_n}(1 - \sigma_1(n)) = \sigma_0(n), \tag{2}$$

$$c_{n+1} = \frac{\sigma_0(n) - c_n}{\sigma_0(n)}. \tag{3}$$

We also have the conditions

$$\sigma_0(n), \sigma_1(n) > 0, \tag{4}$$

$$\sigma_0(n) + \sigma_1(n) < 1, \tag{5}$$

$$0 < c_n < \tfrac{1}{2}. \tag{6}$$

From conditions (1), (2) and (3) we obtain

$$\sigma_0(n) = \frac{2c_n^2 - 6c_n^3 + 5c_n^4 - 2c_n^5}{(c_n - 1)^6} \equiv A_0(c_n), \tag{7}$$

$$\sigma_1(n) = \frac{c_n^2}{(c_n - 1)^2} \equiv A_1(c_n), \tag{8}$$

$$c_{n+1} = \frac{c_n^6 - 6c_n^5 + 17c_n^4 - 25c_n^3 + 21c_n^2 - 8c_n + 1}{2c_n^4 - 5c_n^3 + 6c_n^2 - 2c_n} \equiv R(c_n). \tag{9}$$

The conditions (4), (5) and (6) reduce to $c \in (0, 1/2)$ and $A_0(c_n) + A_1(c_n) < 1$. In particular, using Figure 5, this defines the feasible domain to be

$$C = \left\{ c \in (0, 1/2) \;\middle|\; 0 \leq \frac{c^2(3 - 10c + 11c^2 - 6c^3 + c^4)}{(c - 1)^6} < 1 \right\}$$
$$= [0, 0.35\ldots].$$

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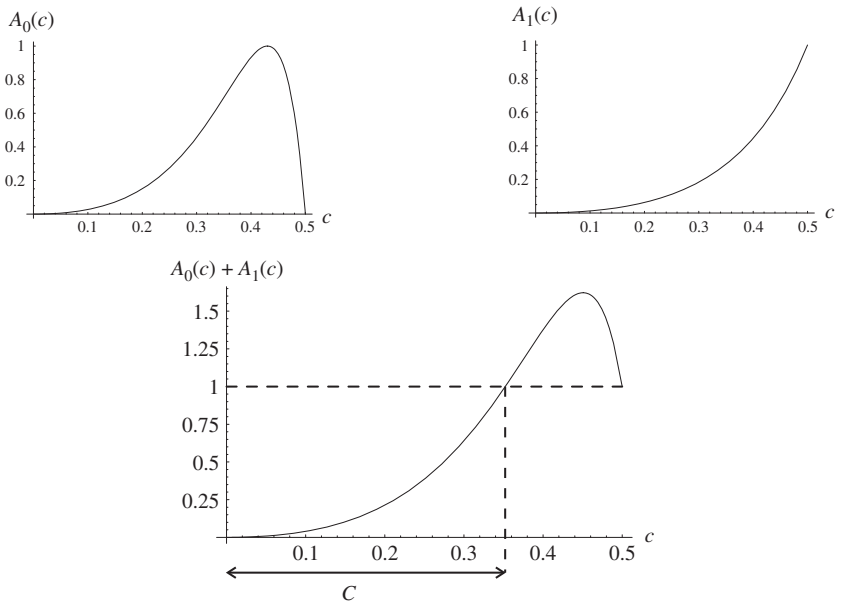


FIGURE 5. The graphs of A_0, A_1 and $A_0 + A_1$.

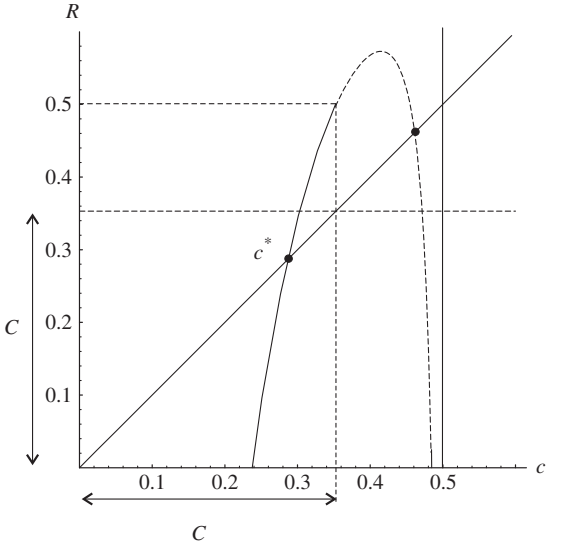


FIGURE 6. $R : C \to \mathbb{R}$.

Notice that the map $R : C \to \mathbb{R}$ (see Figure 6) is expanding. It follows readily that only the fixed point $c^* \in C$ and $R(c^*) = c^*$ corresponds to an infinitely renormalizable f_{σ^*}. Otherwise speaking, consider the scaling data $\sigma^* : \mathbb{N} \to \Delta$ with

$$\sigma^*(n) = (q_{c^*}^2(0),\ 1 - q_{c^*}(0)), \quad n \geq 1.$$

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Then $s(\sigma^*) = \sigma^*$ and Lemma 3.1 implies that

$$Rf_{\sigma^*} = f_{\sigma^*}. \qquad \square$$

Remark 3.4. Let $I_0^n = [x_{n-1}, x_n]$ be the interval corresponding to σ^*, then

$$f_{\sigma^*}(x_{n-1}) = q_{c^*}(x_{n-1}).$$

Hence f_{σ^*} has a quadratic tip.

Remark 3.5. The invariant Cantor set of the map f_{σ^*} is next in complexity to the well-known middle third Cantor set in the following sense:

(i) like in the middle third Cantor set, on each scale and everywhere the same scaling ratios are used; but

(ii) unlike in the middle third Cantor set, there are now two ratios (a small one and a bigger one) at each scale.

This situation of rather extreme tameness of the scaling data is very different from the geometry of the Cantor attractor of the analytic renormalization fixed point in which there are no two places where the same scaling ratios are used at all scales, and where the closure of the set of ratios is itself a Cantor set [**BMT**].

LEMMA 3.6. *Let $f_* = f_{\sigma^*}$ where $\sigma^* : \mathbb{N} \to \Delta$ is the scaling data with $\sigma^*(n)(\sigma_0^*, \sigma_1^*)$. Then*

$$(\sigma_0^*)^2 = \sigma_1^*.$$

Proof. Let $\hat{I}_0^n = f_*(I_0^n) = [f_*(x_{n-1}), 1]$ and $\hat{I}_1^{n+1} = f_*(I_1^{n+1})$. Then $f_*^{2^n-1} : \hat{I}_0^n \to I_0^n$ is affine, monotone and onto. Further, by construction

$$f^{2^n-1}(\hat{I}_0^{n+1}) = I_1^{n+1}.$$

Hence,

$$\frac{|\hat{I}_0^{n+1}|}{|\hat{I}_0^n|} = \sigma_1^*.$$

So $|I_0^n| = (\sigma_0^*)^n$ and $|\hat{I}_0^n| = (\sigma_1^*)^n$. Now f_{σ^*} has a quadratic tip with

$$f_{\sigma^*}(x_n) = q_{c_*}(x_n).$$

Hence,

$$\sigma_1^* = \frac{|\hat{I}_0^{n+1}|}{|\hat{I}_0^n|} = \left(\frac{x_n - c}{x_{n-1} - c}\right)^2 = \left(\frac{|I_0^{n+1}|}{|I_0^n|}\right)^2 = (\sigma_0^*)^2.$$

This completes the proof. $\square$

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3.2. $C^{1+\mathrm{Lip}}$ *extension.* In this subsection we will extend the piecewise affine map f_* to a $C^{1+\mathrm{Lip}}$ unimodal map. Let $S : [0, 1]^2 \to [0, 1]^2$ be the scaling function defined by

$$S\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -\sigma_0^* x + \sigma_0^* \\ \sigma_1^* y + 1 - \sigma_1^* \end{pmatrix} \equiv \begin{pmatrix} S_1(x) \\ S_2(y) \end{pmatrix}$$

and let F be the graph of $f_* = f_{\sigma^*}$, where $f_{\sigma^*} : D_{\sigma^*} \to [0, 1]$, $D_{\sigma^*} = \bigcup_{n\geq 1} I_1^n$. Then the idea of how to construct an extension g of f_* is contained in the following lemma.

LEMMA 3.7. *One has* $F \cap \mathrm{image}(S) = S(F)$.

Proof. Let $\hat{h} = \hat{h}_{\sigma^*,1}$ and $h = h_{\sigma^*,1}$. Let $(x, y) \in \mathrm{graph}(f_*) \cap \mathrm{image}(S)$. Say $(x, y) = (S_1(x'), S_2(y'))$ with $S_2(y') = f_*(S_1(x'))$. Since $S_1(x') = h(x')$ and $S_2(y') = \hat{h}(y')$, we can write $y' = \hat{h}^{-1} \circ f_* \circ h(x')$. By Lemma 3.1,

$$y' = R_1 f_*(x') = f_*(x'),$$

which gives $(x', y') \in \mathrm{graph}(f_*)$. This in turn implies that $(x, y) \in S(\mathrm{graph}\, f_*)$. By reading the previous argument backward, we prove that $S(\mathrm{graph}\, f_*) \subset F \cap \mathrm{image}(S)$. □

LEMMA 3.8. *One has* $S(\mathrm{graph} q_{c^*}) \subset \mathrm{graph}(q_{c^*})$.

Proof. Let $S(\mathrm{graph}(q_{c^*}))$ be the graph of the function q. Since S is linear and q_c is quadratic, we get that q is also a quadratic function. Then both $q_{c^*}(c^*) = 1$ and $q(c^*) = 1$, because $S(c^*, 1) = (c^*, 1)$. Furthermore, by construction,

$$S(1, 0) = (0, q_{c^*}(0)) = (0, q(0)).$$

Hence $q_{c^*}(0) = q(0)$. Differentiate twice $S_2(y) = q(S_1(x))$ and use $(\sigma_0^*)^2 = \sigma_1^*$ from Lemma 3.6, which proves that $q''(c^*) = q''_{c^*}(c^*)$. Now we conclude that the quadratic maps q and q_{c^*} are equal. □

Let F_0 be the graph of $f_*|_{I_1^1}$. Then by Lemma 3.7, $F = \bigcup_{k\geq 0} S^k(F_0)$. Let g be a $C^{1+\mathrm{Lip}}$ extension of f_* on $D_{\sigma_*} \cup [x_1, 1]$ and $G_0 = \mathrm{graph}(g|_{[x_1,1]})$. Then $G = \bigcup_{k\geq 0} S^k(G_0)$ is the graph of an extension of f_*. We prove that g is $C^{1+\mathrm{Lip}}$ and also has a quadratic tip. Let $B^k = S^k([0, 1]^2)$, where

$$B^k = [x_{k-1}, x_k] \times [\hat{x}_{k-1}, 1] \quad \text{for } k = 1, 3, 5, \ldots,$$
$$B^k = [x_k, x_{k-1}] \times [\hat{x}_{k-1}, 1] \quad \text{for } k = 2, 4, \ldots,$$

where $\hat{x}_{k-1} = q_c(x_{k-1}) = 1 - (\sigma_1^*)^k$. Let $b_n = (x_{n-1}, \hat{x}_{n-1}) = S^n(1, 0)$.

Remark 3.9. Notice that the points b_n lie on the graph of q_{c^*}. This follows from Lemma 3.8.

LEMMA 3.10. *One has that* G *is the graph of a* C^1 *extension of* f_*.

Proof. Note that $G_k = S^k(G_0)$ is the graph of a C^1 function on $[x_{k-1}, x_{k+1}]$ for k odd and on $[x_{k+1}, x_{k-1}]$ for k even. To prove the lemma we need to show continuous differentiability at the points b_n, where these graphs intersect (see Figure 7).

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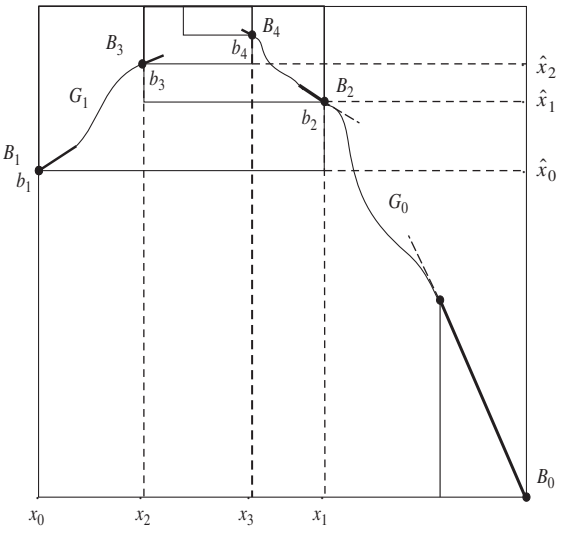


FIGURE 7. Extension of f_{σ_*}.

By construction G_0 is C^1 at b_2. Namely, consider a small interval $(x_1 - \delta, x_1 + \delta)$. Then on the interval $(x_1 - \delta, x_1)$ the slope is given by an affine piece of f_*, and on $(x_1, x_1 + \delta)$ the slope is given by the chosen $C^{1+\mathrm{Lip}}$ extension. Let $\Gamma \subset G$ be the graph over this interval $(x_1 - \delta, x_1 + \delta)$. Then locally around b_n the graph G equals $S^{n-1}(\Gamma)$. Hence G is C^1 on $[0, 1] \setminus \{c^*\}$. From Lemma 3.6, notice that the vertical contraction of S is stronger than the horizontal contraction. This implies that the slope of G_n tends to zero. Indeed, G is the graph of a C^1 function on $[0, 1]$. □

PROPOSITION 3.11. *Let g be the function whose graph is G; then g is $C^{1+\mathrm{Lip}}$ with a quadratic tip.*

Proof. Since $f_*|_{D_\sigma}$ has a quadratic tip, the extension g has a quadratic tip. Because g is C^1, we only need to show that G_n is the graph of a $C^{1+\mathrm{Lip}}$ function

$$g_n : [x_{n-1}, x_{n+1}] \to [0, 1]$$

with a uniform Lipschitz bound. That is, for $n \geq 1$,

$$\mathrm{Lip}(g'_{n+1}) \leq \mathrm{Lip}(g'_n).$$

Assume that g_n is $C^{1+\mathrm{Lip}}$ with Lipschitz constant Lip_n for its derivative. We prove that $\mathrm{Lip}_{n+1} \leq \mathrm{Lip}_n$, and in particular $\mathrm{Lip}_n \leq \mathrm{Lip}_0$. For, given (x, y) on the graph of g_n, there is $(x', y') = S(x, y)$, on the graph of g_{n+1}. Therefore, we can write

$$g_{n+1}(x') = \sigma_1^* g_n(x) + 1 - \sigma_1^*.$$

Since $x = 1 - (x'/\sigma_0^*)$, we have

$$g_{n+1}(x') = \sigma_1^* g_n\left(1 - \frac{x'}{\sigma_0^*}\right) + 1 - \sigma_1^*.$$

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Differentiate, giving

$$g'_{n+1}(x') = \frac{-\sigma_1^*}{\sigma_0^*} g'_n\left(1 - \frac{x'}{\sigma_0^*}\right).$$

Therefore, we obtain

$$\begin{aligned}|g'_{n+1}(x'_1) - g'_{n+1}(x'_2)| &= \left|\frac{-\sigma_1^*}{\sigma_0^*}\right| \left|g'_n\left(1 - \frac{x'_1}{\sigma_0^*}\right) - g'_n\left(1 - \frac{x'_2}{\sigma_0^*}\right)\right| \\ &\leq \frac{\sigma_1^*}{(\sigma_0^*)^2}\mathrm{Lip}(g'_n)|x'_1 - x'_2|.\end{aligned}$$

From Lemma 3.6 we have $\sigma_1^*/(\sigma_0^*)^2 = 1$. Hence

$$\mathrm{Lip}(g'_{n+1}) \leq \mathrm{Lip}(g'_n) \leq \mathrm{Lip}(g'_1),$$

which completes the proof. □

If f_σ is infinitely renormalizable, then every extension g of f_σ maps I_0^1 onto I_1^1, and I_1^1 monotonically onto I_0^1. Hence, g is renormalizable in the classical sense. Observe that Rg is an extension of Rf_σ. Hence, Rg is renormalizable in the classical sense. In fact, g is infinitely renormalizable.

THEOREM 3.12. *There exists an infinitely renormalizable $C^{1+\mathrm{Lip}}$ unimodal map f with a quadratic tip that is not C^2 but*

$$Rf = f.$$

3.3. *Entropy of renormalization.* For all $\phi \in C^{1+\mathrm{Lip}}$, $\phi : [x_1, 1] \to [0, 1]$, which extends f_*, we constructed $f_\phi \in C^{1+\mathrm{Lip}}$ in such a way that:

(i) $Rf_\phi = f_\phi$; and

(ii) f_ϕ has a quadratic tip.

Now choose two $C^{1+\mathrm{Lip}}$ functions that extend f_*, say $\phi_0 : [x_1, 1] \to [0, 1]$ and $\phi_1 : [x_1, 1] \to [0, 1]$. For $\omega = (\omega_k)_{k\geq 1} \in \{0, 1\}^{\mathbb{N}}$, define

$$F_n(\omega) = S^n(\mathrm{graph}\phi_{\omega_n})$$

and

$$F(\omega) = \bigcup_{k\geq 1} F_k(\omega).$$

Use the same argument as was given before to show that the set $F(\omega)$ is the graph of a $C^{1+\mathrm{Lip}}$ map with a quadratic tip. Now let

$$\tau : \{0, 1\}^{\mathbb{N}} \to \{0, 1\}^{\mathbb{N}}$$

be the shift map defined by

$$\tau(\omega)_n = \omega_{n+1}$$

(so that the map τ acting on the set $\{0, 1\}^{\mathbb{N}}$ is the full 2-shift).

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PROPOSITION 3.13. *For all* $\omega \in \{0, 1\}^{\mathbb{N}}$,

$$f_\omega^2 : [0, x_1] \to [0, x_1]$$

is a unimodal map. In particular, f_ω *is renormalizable and*

$$Rf_\omega = f_{\tau(\omega)}.$$

Proof. Note that $f_\omega : [0, x_1] \to I_1^1$ is unimodal and onto. Furthermore, $f_\omega : I_1^1 \to [0, x_1]$ is affine and onto. Hence f_ω is renormalizable. The construction also gives

$$Rf_\omega = f_{\tau(\omega)}. \qquad \square$$

THEOREM 3.14. *Renormalization acting on the space of* $C^{1+\mathrm{Lip}}$ *unimodal maps has positive entropy.*

Proof. Note that $\omega \to f_\omega \in C^{1+\mathrm{Lip}}$ is injective. Hence the domain of R contains a copy of the full 2-shift (i.e., contains a subset on which the restriction of R is topologically conjugate to the full 2-shift). $\square$

Remark 3.15. We can also embed a full k-shift in the domain of R by choosing $\phi_0, \phi_1, \ldots, \phi_{k-1}$ and repeat the construction. The entropy of R on $C^{1+\mathrm{Lip}}$ is actually unbounded.

4. *Chaotic scaling data*

In this section we will use a variation on the construction of scaling data as presented in §3 to obtain the following theorem.

THEOREM 4.1. *There exists an infinitely renormalizable* $C^{1+\mathrm{Lip}}$ *unimodal map* g *with quadratic tip such that* $\{c_n\}_{n\geq 0}$*, where* c_n *is the critical point of* $R^n g$*, is dense in a Cantor set.*

The proof needs some preparation. For $\epsilon > 0$ we will modify the construction as described in §3. This modification is illustrated in Figure 8. For $c \in (0, 1/2)$ let

$$\sigma_1(c, \epsilon) = 1 - q_c(0),$$
$$\sigma_0(c, \epsilon) = \epsilon q_c^2(0),$$

where $\epsilon > 0$ and close to 1. Also let

$$R(c, \epsilon) = \frac{\sigma_0(c, \epsilon) - c}{\sigma_0(c, \epsilon)} = 1 - \frac{c}{q_c^2(0)} \frac{1}{\epsilon}.$$

In §3 we observed that $R(c, 1)$ has a unique fixed point $c^* \in (0, 1/2)$ with feasible $\sigma_0(c^*, 1)$ and $\sigma_1(c^*, 1)$. This fixed point is expanding. Although we will not use this, a numerical computation gives

$$\frac{\partial R}{\partial c}(c^*, 1) > 2.$$

Now choose $\epsilon_0 > \epsilon_1$ close to 1. Then $R(\cdot, \epsilon_0)$ will have an expanding fixed point c_0^* and $R(\cdot, \epsilon_1)$ a fixed point c_1^*. In particular, by choosing $\epsilon_0 > \epsilon_1$ close enough to 1 we will

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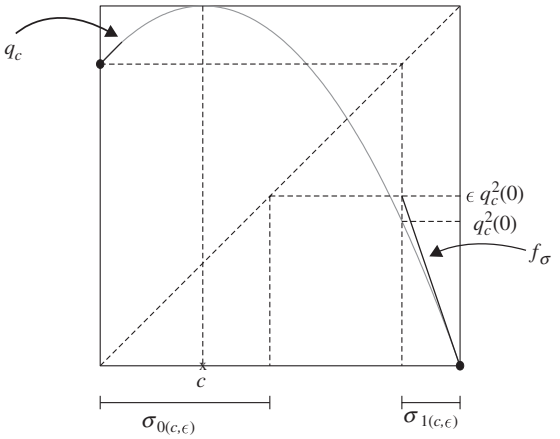


FIGURE 8.

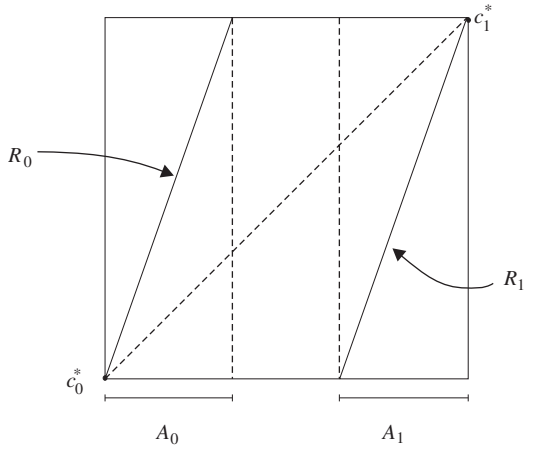


FIGURE 9.

get the following horseshoe as shown in Figure 9. More precisely, there exists an interval $A_0 = [c_0^*, a_0]$ and $A_1 = [a_1, c_1^*]$ such that

$$R_0 : A_0 \to [c_0^*, c_1^*] \supset A_0$$

and

$$R_1 : A_1 \to [c_0^*, c_1^*] \supset A_1$$

are expanding diffeomorphisms (with derivative larger than 2, but larger than 1 would suffice to get a horseshoe). Here

$$R_0(c) = R(c, \epsilon_0) \quad \text{and} \quad R_1(c) = R(c, \epsilon_1).$$

Use the following coding for the invariant Cantor set of the horseshoe map

$$c : \{0, 1\}^{\mathbb{N}} \to [c_0^*, c_1^*]$$

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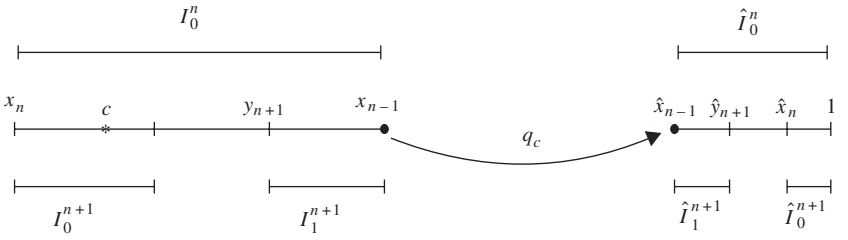


FIGURE 10.

with

$$c(\tau\omega) = R(c(\omega), \epsilon_{\omega_0}),$$

where $\tau : \{0, 1\}^{\mathbb{N}} \to \{0, 1\}^{\mathbb{N}}$ is the shift. Given $\omega \in \{0, 1\}^{\mathbb{N}}$, define the following scaling data $\sigma : \mathbb{N} \to \Delta$:

$$\sigma(n) = (\sigma_0(c(\tau^n\omega), \epsilon_{\omega_n}), \sigma_1(c(\tau^n\omega), \epsilon_{\omega_n})).$$

Again, by taking ϵ_0, ϵ_1 close enough to 1, we can assume that $\sigma(n)$ is proper scaling data for any chosen $\omega \in \{0, 1\}^{\mathbb{N}}$. As in §3 we will define a piecewise affine map

$$f_\omega : D_\omega = \bigcup_{n\geq 1} I_1^n \to [0, 1].$$

The precise definition needs some preparation. Use the notation as illustrated in Figure 10. For $n \geq 0$ let

$$I_0^n = [x_n, x_{n-1}],$$

where $x_n = \partial I_0^n \setminus \partial I_0^{n-1}$, $n \geq 1$, and

$$I_1^n = [y_n, x_{n-2}],$$

where $y_n = \partial I_1^n \setminus \partial I_0^{n-1}$, $n \geq 1$.

Let

$$\hat{I}_0^n = q_c([x_{n-1}, 1]) = q_c(I_0^n) = [\hat{x}_{n-1}, 1],$$

where $\hat{x}_{n-1} = q_c(x_{n-1})$. Finally, let $\hat{I}_1^{n+1} = [\hat{x}_{n-1}, \hat{y}_{n+1}] \subset \hat{I}_0^n$ such that

$$|\hat{I}_1^{n+1}| = \sigma_0(n)|\hat{I}_0^n|.$$

Now define $f_\omega : I_1^{n+1} \to \hat{I}_1^{n+1}$ to be the affine homeomorphism such that

$$f_\omega(x_{n-1}) = q_c(x_{n-1}) = \hat{x}_{n-1}.$$

LEMMA 4.2. *There exists $K > 0$ such that*

$$\frac{1}{K} \leq \frac{|\hat{I}_0^n|}{|I_0^n|^2} \leq K.$$

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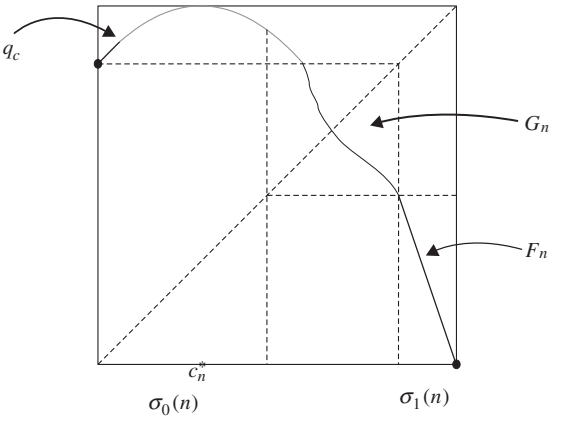


FIGURE 11.

Proof. Observe that $c(n) = c(\tau^n \omega) \in [c_0^*, c_1^*]$, which is a small interval around c^*. This implies that, for some $K > 0$,

$$\frac{1}{K} \leq \frac{|c - x_{n-1}|}{|I_0^n|} \leq K.$$

Then

$$\frac{|\hat{I}_0^n|}{|I_0^n|^2} = \frac{|q_c([c, x_{n-1}])|}{|I_0^n|^2} = \frac{(c - x_{n-1})^2}{(1-c)^2} \frac{1}{(I_0^n)^2},$$

which implies the bound. □

Let $S_2^n : [0, 1] \to \hat{I}_0^n$ be the affine orientation-preserving homeomorphism and $S_1^n : [0, 1] \to I_0^n$ be the affine homeomorphism with $S_1^n(1) = x_{n-1}$. Define

$$S^n : [0, 1]^2 \to [0, 1]^2$$

by

$$S^n \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} S_1^n(x) \\ S_2^n(y) \end{pmatrix}.$$

The image of S^n is B_n.

Let $F_n = (S^n)^{-1}(\text{graph} f_\omega)$. This is the graph of a function f_n. We will extend this function (and its graph) on the gap $[\sigma_0(n), 1 - \sigma_1(n)]$. See Figure 11. Notice that $\sigma_0(n)$, $1 - \sigma_1(n)$, $Df_n(\sigma_0(n))$, and $Df_n(1 - \sigma_1(n))$ vary within a compact family. This allows us to choose from a compact family of $C^{1+\text{Lip}}$ diffeomorphisms an extension

$$g_n : [\sigma_0(n), 1] \to [0, f_n(\sigma_0(n))]$$

of the map f_n. The Lipschitz constant of Dg_n is bounded by $K_0 > 0$. Let G_n be the graph of g_n and

$$G = \bigcup_{n \geq 0} S^n(G_n).$$

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Then G is the graph of a unimodal map

$$g : [0, 1] \to [0, 1]$$

that extends f_ω. Notice that g is C^1. It has a quadratic tip because f_ω has a quadratic tip. Also notice that $S^n(G_n)$ is the graph of a $C^{1+\mathrm{Lip}}$ diffeomorphism. The Lipschitz bound L_n of its derivative satisfies, for a similar reason as in §3,

$$L_n \leq \frac{|\hat{I}_0^n|}{|(I_0^n)|^2} K_0.$$

This is bounded, by Lemma 4.2. Thus g_ω is a $C^{1+\mathrm{Lip}}$ unimodal map with quadratic tip. The construction implies that g is infinitely renormalizable and

$$\mathrm{graph}(R^n g_\omega) \supset F_n.$$

One can prove Theorem 4.1 by choosing $\omega \in \{0, 1\}^{\mathbb{N}}$ such that the orbit under the shift τ is dense in the invariant Cantor set of the horseshoe map.

Remark 4.3. Let $\omega = \{0, 0, \ldots\}$, then we will get another renormalization fixed point that is a modification of the one constructed in §3.

5. $C^{2+|\cdot|}$ *unimodal maps*

Let $f : [0, 1] \to [0, 1]$ be a C^2 unimodal map with critical point $c \in (0, 1)$. Say, $D^2 f(x) = E(1 + \varepsilon(x))$, where

$$\varepsilon : [0, 1] \to \mathbb{R}$$

is continuous with $\varepsilon(c) = 0$ and $E = D^2 f(c) \neq 0$. Then let

$$\bar{\varepsilon} : [0, 1] \to \mathbb{R}$$

be defined by

$$\bar{\varepsilon}(x) = \frac{1}{x - c} \int_c^x \varepsilon(t)\, dt.$$

Notice that $\bar{\varepsilon}$ is continuous with $\bar{\varepsilon}(c) = 0$. Furthermore, $1 + \bar{\varepsilon}(x) \neq 0$ for all $x \in [0, 1]$. Since

$$Df(x) = E(x - c)(1 + \bar{\varepsilon}(x)),$$

$Df(x)$ equals zero only when $x = c$. Let the map

$$\delta : [0, 1] \to \mathbb{R}$$

be defined by

$$\delta(x) = \varepsilon(x) - \bar{\varepsilon}(x).$$

Notice that δ is continuous and $\delta(c) = 0$. Finally, define

$$\beta : [0, 1] \to \mathbb{R}$$

by

$$\beta(x) = \int_c^x \frac{1}{t - c} \delta(t)\, dt.$$

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LEMMA 5.1. *The function β is continuous and $\varepsilon = \delta + \beta$.*

Proof. The definition of δ gives $\bar{\varepsilon} = \varepsilon - \delta$, which is differentiable on $[0, 1] \setminus \{c\}$, and

$$\begin{aligned}\varepsilon(x) &= ((x-c)(\varepsilon-\delta)(x))' \\ &= \varepsilon(x) - \delta(x) + (x-c)(\varepsilon-\delta)'(x).\end{aligned}$$

Hence,

$$\delta(x) = (x-c)(\varepsilon-\delta)'(x).$$

This implies that

$$\varepsilon(x) = \delta(x) + \int_c^x \frac{1}{t-c}\delta(t)\,dt = \delta(x) + \beta(x). \qquad \square$$

Definition 2. Let $f : [0, 1] \to [0, 1]$ be a unimodal map with critical point $c \in (0, 1)$. We say that f is $C^{2+|\cdot|}$ if and only if

$$\hat{\beta} : x \longmapsto \int_c^x \frac{1}{|t-c|}|\delta(t)|\,dt$$

is continuous.

Remark 5.2. Every $C^{2+\alpha}$ Hölder unimodal map, $\alpha > 0$, is $C^{2+|\cdot|}$.

Remark 5.3. If $D^2 f$ is monotone, then $\hat{\beta} = \beta$ or $\hat{\beta} = -\beta$. So $\hat{\beta}$ is continuous according to Lemma 5.1. Hence, the very weak condition of local monotonicity of $D^2 f$ is sufficient for f to be $C^{2+|\cdot|}$.

Remark 5.4. The $C^{2+|\cdot|}$ unimodal maps are dense in C^2.

Remark 5.5. There exist C^2 unimodal maps that are not $C^{2+|\cdot|}$. See also Remark 11.2.

The *nonlinearity* $\eta_\phi : [0, 1] \to \mathbb{R}$ of a C^1 diffeomorphism $\phi : [0, 1] \to [0, 1]$ is given by

$$\eta_\phi(x) = D \ln D\phi(x),$$

wherever it is defined.

PROPOSITION 5.6. *Let f be a $C^{2+|\cdot|}$ unimodal map with critical point $c \in (0, 1)$. There exist diffeomorphisms*

$$\phi_\pm : [0, 1] \to [0, 1]$$

such that

$$f(x) = \begin{cases} \phi_+(q_c(x)), & x \in [c, 1], \\ \phi_-(q_c(x)), & x \in [0, c], \end{cases}$$

with

$$\eta_{\phi_\pm} \in L^1([0, 1]).$$

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Proof. It is plain that there exists a C^1 diffeomorphism

$$\phi_+ : [0, 1] \to [0, 1]$$

such that, for $x \in [c, 1]$,

$$f(x) = \phi_+(q_c(x)).$$

We will analyze the nonlinearity of ϕ_+. Observe that

$$Df(x) = -2\frac{(x-c)}{(1-c)^2} D\phi_+(q_c(x))$$

and

$$\begin{aligned} D^2 f(x) &= 4\frac{(x-c)^2}{(1-c)^4} D^2\phi_+(q_c(x)) - 2\frac{1}{(1-c)^2} D\phi_+(q_c(x)) \\ &= E(1 + \varepsilon(x)). \end{aligned} \tag{10}$$

As we have seen before, we also have

$$Df(x) = E(x-c)(1 + \bar{\varepsilon}(x)).$$

This implies that

$$\eta_{\phi_+}(q_c(x)) = \frac{-(1-c)^2}{2} \frac{\varepsilon(x) - \bar{\varepsilon}(x)}{1 + \bar{\varepsilon}(x)} \frac{1}{(x-c)^2}. \tag{11}$$

Therefore, by performing the substitution $u = q_c(x)$, we get

$$\int_0^1 |\eta_\phi(u)|\, du = \int_1^c -2|\eta_{\phi_+}(q_c(x))| \frac{x-c}{(1-c)^2}\, dx \tag{12}$$

$$= \int_c^1 \frac{|\varepsilon(x) - \bar{\varepsilon}(x)|}{1 + \bar{\varepsilon}(x)} \frac{1}{x-c}\, dx \tag{13}$$

$$\leq \frac{1}{\min(1 + \bar{\varepsilon})} \int_c^1 \frac{|\delta(x)|}{|x-c|}\, dx < \infty. \tag{14}$$

We have proved that $\eta_{\phi_+} \in L^1([0, 1])$. Similarly, one can prove the existence of a C^1 diffeomorphism

$$\phi_- : [0, 1] \to [0, 1]$$

such that, for $x \in [0, c]$,

$$f(x) = \phi_-(q_c(x))$$

and

$$\eta_{\phi_-} \in L^1([0, 1]). \qquad \square$$

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6. *Distortion of cross-ratios*

Definition 3. Let $J \subset T \subset [0, 1]$ be open and bounded intervals such that $T \setminus J$ consists of two components L and R. Define the cross-ratios of these intervals as

$$D(T, J) = \frac{|J||T|}{|L||R|}.$$

If f is continuous and monotone on T, then define the cross-ratio distortion of f as

$$B(f, T, J) = \frac{D(f(T), f(J))}{D(T, J)}.$$

If $f^n|_T$ is monotone and continuous, then

$$B(f^n, T, J) = \prod_{i=0}^{n-1} B(f, f^i(T), f^i(J)).$$

Definition 4. Let $f : [0, 1] \to [0, 1]$ be a unimodal map and $T \subset [0, 1]$. We say that

$$\{f^i(T) : 0 \le i \le n\}$$

has intersection multiplicity $m \in \mathbb{N}$ if and only if, for every $x \in [0, 1]$,

$$\#\{i \le n \mid x \in f^i(T)\} \le m$$

and m is minimal with this property.

THEOREM 6.1. *Let $f : [0, 1] \to [0, 1]$ be a $C^{2+|\cdot|}$ unimodal map with critical point $c \in (0, 1)$. Then there exists $K > 0$, such that the following holds. If T is an interval such that $f^n|_T$ is a diffeomorphism then, for any interval $J \subset T$ with* $\mathrm{cl}(J) \subset \mathrm{int}(T)$*, we have*

$$B(f^n, T, J) \ge \exp\{-Km\},$$

where m is the intersection multiplicity of $\{f^i(T) : 0 \le i \le n\}$.

Proof. Observe that q_c expands cross-ratios. Then Proposition 5.6 implies that

$$B(f, f^i(T), f^i(J)) > \frac{D\phi_i(j_i)D\phi_i(t_i)}{D\phi_i(l_i)D\phi_i(r_i)},$$

where $\phi_i = \phi_+$ or ϕ_- depending whether $f^i(T) \subset [c, 1]$ or $[0, c]$, and

$$\begin{aligned} j_i &\in q_c(f^i(J)), \\ t_i &\in q_c(f^i(T)), \\ l_i &\in q_c(f^i(L)), \\ r_i &\in q_c(f^i(R)). \end{aligned}$$

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Thus

$$\begin{aligned}\ln B(f^n, T, J) &= \sum_{i=0}^{n-1} \ln B(f, f^i(T), f^i(J)) \\ &\geq \sum_{i=0}^{n-1} (\ln D\phi_i(j_i) - \ln D\phi_i(l_i)) + (\ln D\phi_i(t_i) - \ln D\phi_i(r_i)) \\ &\geq -\sum_{i=0}^{n-1} |\eta_{\phi_i}(\xi_i^1)|\,|j_i - l_i| + |\eta_{\phi_i}(\xi_i^2)|\,|t_i - r_i| \\ &\geq -2m\left(\int |\eta_{\phi_+}| + \int |\eta_{\phi_-}|\right) \\ &= -Km.\end{aligned}$$

Therefore

$$B(f^n, T, J) \geq \exp\{-Km\}. \qquad \square$$

The previous theorem allows us to apply the real Koebe lemma. See [**MS**] for a proof.

LEMMA 6.2. (Real Koebe lemma) *For each $K_1 > 0$, $0 < \tau < 1/4$, there exists $K < \infty$ with the following property.*

Let $g : T \to g(T) \subset [0, 1]$ be a C^1 diffeomorphism on some interval T. Assume that for any intervals J^* and T^* with $J^* \subset T^* \subset T$ one has

$$B(g, T^*, J^*) \geq K_1 > 0$$

for an interval $M \subset T$ such that $\mathrm{cl}(M) \subset \mathrm{int}(T)$. Let L, R be the components of $T \setminus M$. Then, if

$$\frac{|g(L)|}{|g(M)|} \geq \tau \quad \text{and} \quad \frac{|g(R)|}{|g(M)|} \geq \tau,$$

we have

$$\text{for all } x, y \in M, \quad \frac{1}{K} \leq \frac{|g'(x)|}{|g'(y)|} \leq K.$$

Remark 6.3. The conclusion of the real Koebe lemma is summarized by saying that $g|_M$ has bounded distortion.

7. *A priori bounds*

Let f be an infinitely renormalizable $C^{2+|\cdot|}$ unimodal map with quadratic tip at $c \in (0, 1)$. Let $I_0^n = [f^{2^n}(c), f^{2^{n+1}}(c)]$ be the central interval whose first return map corresponds to the nth renormalization. Here, we study the geometry of the cycle consisting of the intervals

$$I_j^n = f^j(I_0^n), \quad j = 0, 1, \ldots, 2^n - 1.$$

Notice that

$$I_j^{n+1}, I_{j+2^n}^{n+1} \subset I_j^n, \quad j = 0, 1, \ldots, 2^n - 1.$$

Let I_l^n and I_r^n be the direct neighbors of I_j^n for $3 \leq j \leq 2^n$.

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LEMMA 7.1. *For each $1 \le i < j$, there exists an interval T that contains I_i^n, such that $f^{j-i} : T \to [I_l^n, I_r^n]$ is monotone and onto.*

Proof. Let $T \subset [0, 1]$ be the maximal interval that contains I_i^n such that $f^{j-i}|_T$ is monotone. Such an interval exists because of monotonicity of $f^{j-i}|_{I_i^n}$. The boundary points of T are $a, b \in [0, 1]$. Suppose that $f^{j-i}(b)$ is to the right of I_j^n. The maximality of T ensures the existence of k, $k < j - i$, such that $f^k(b) = c$. Because $i + k < j \le 2^n$, we have $c \notin I_{i+k}^n$ and so $f^{k+1}(T) \supset I_1^n$. Moreover, $f^{j-i-(k+1)}|_{f^{k+1}(T)}$ is monotone. Hence $f^{j-i-(k+1)}|_{I_1^n}$ is monotone. So $1 + j - i - (k + 1) \le 2^n$. This implies that $f^{j-i}(T)$ contains $I_{1+j-i-(k+1)}^n$. In particular, $f^{j-i}(T)$ contains I_r^n. Similarly we can prove that $f^{j-i}(T)$ contains I_l^n. □

LEMMA 7.2. (Intersection multiplicity) *Let $f^{j-i} : T \to [I_l^n, I_r^n]$ be monotone and onto with $T \supset I_i^n$. Then, for all $x \in [0, 1]$,*

$$\#\{k < j - i \mid f^k(T) \ni x\} \le 7.$$

Proof. Without loss of generality we may restrict ourselves to estimate the intersection multiplicity at a point $x \in U$, where

$$U = [I_l^n, I_r^n] = [u_l, u_r].$$

Let $c_l \in I_l^n$ such that $f^{2^n - l}(c_l) = c$ and

$$C_l = [u_l, c_l] \subset I_l^n.$$

Similarly, define

$$C_r = [c_r, u_r] \subset I_r^n.$$

Let $T_k = f^k(T)$, $k = 0, 1, \ldots, j - i$.

CLAIM. *If $i + k \notin \{l, j, r\}$ and $T_k \cap U \ne \emptyset$, then:*
(i) $I_{i+k}^n \cap U = \emptyset$;
(ii) $U \cap T_k = I_l^n$ *or* C_l *or* I_r^n *or* C_r.

Proof. Let $T \setminus I_i^n = L \cup R$ and then we may assume that $U \cap T_k = U \cap L_k$, where $L_k = f^k(L)$. This holds because $I_{i+k}^n \cap U = \emptyset$. Consider the situation where

$$I_r^n \cap L_k \ne \emptyset.$$

The other possibilities can be treated similarly. Notice that I_r^n cannot be strictly contained in L_k. Otherwise there would be a third 'neighbor' of I_j^n in U. Let $a = \partial L \cap \partial T$. Notice that

$$f^k(a) \in \partial L_k \cap I_r^n.$$

Furthermore, notice that

$$f^{j-k}(f^k(a)) \in \partial U.$$

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This means that $f^{j-k}(f^k(a))$ is a point in the orbit of c. This holds because all boundary points of the interval I_j^n are in the orbit of c. Hence, $f^k(a)$ is a point in the orbit of c or $f^k(a)$ is a preimage of c. The first possibility implies that $f^k(a) \in \partial I_r^n$. This implies that

$$U \cap T_k = U \cap L_k = I_r^n.$$

The second possibility implies that $f^k(a) = c_r$, which means that

$$U \cap T_k = U \cap L_k = C_r.$$

This finishes the proof of the claim. □

This claim gives 7 as a bound for the intersection multiplicity. □

PROPOSITION 7.3. *For $j < 2^n$, $f^{2^n-j} : I_j^n \to I_0^n$ has uniformly bounded distortion.*

Proof. The proof is broken into four steps.

Step 1. Choose $j_0 < 2^n$, such that, for all $j \le 2^n$, we have $|I_{j_0}^n| \le |I_j^n|$. By Lemma 7.1 there exists an interval neighborhood $T_n = L_n^0 \cup I_1^n \cup R_n^0$ such that $f^{j-1} : T_n \to [I_l^n, I_r^n] \supset I_{j_0}^n$ is monotone and onto. Lemma 7.2 together with Theorem 6.1 allow us to apply the Koebe lemma (Lemma 6.2). So, there exists $\tau_0 > 0$ such that

$$|L_n^0|, |R_n^0| \ge \tau_0 |I_1^n|.$$

Let $U_n = I_0^n$, $V_n = f^{-1}(L_n^0 \cup I_1^n \cup R_n^0)$ and let L_n^1, R_n^1 be the components of $V_n \setminus U_n$. From Proposition 5.6 we get $\tau_1 > 0$ such that

$$|L_n^1|, |R_n^1| \ge \tau_1 |U_n|.$$

Step 2. Suppose that $W_n = [I_{l_n}^n, I_{r_n}^n]$, where $I_{l_n}^n$, $I_{r_n}^n$ are the direct neighbors of U_n. We claim that $V_n \subset W_n$. Suppose that it is not. Then, say $I_{r_n}^n \subset \text{int}(V_n)$ implies that $f(I_{r_n}^n) \subset \text{int}(L_n^1)$. So, $f^{j_0-1}|_{f(I_{r_n}^n)}$ is monotone implies that $r_n + j_0 \le 2^n$ and $f^{j_0}(I_{r_n}^n) \subset \text{int}([I_l^n, I_r^n])$. This contradiction concludes that $V_n \subset W_n$.

Step 3. Let L_n, R_n be the components of $W_n \setminus U_n$. Then

$$|L_n|, |R_n| \ge \tau_1 |U_n|.$$

Step 4. For all $j < 2^n$, there exists an interval neighborhood T_j that contains I_j^n such that $f^{2^n-j} : T_j \to W_n$ is monotone and onto. Now Proposition 7.3 follows from Lemma 7.2 together with Theorem 6.1 and the Koebe lemma (Lemma 6.2). □

COROLLARY 7.4. *There exists a constant K such that*

$$|Df^{2^n}|_{I_0^n}| \le K.$$

Proof. Let $x \in I_1^n$. Then from Proposition 7.3 we get $K_1 > 0$ such that, for some $x_0 \in I_1^n$,

$$\begin{aligned} |Df^{2^n-1}(x)| &= \frac{|I_0^n|}{|I_1^n|} \left\{ \frac{Df^{2^n-1}(x)}{Df^{2^n-1}(x_0)} \right\} \\ &\le \frac{|I_0^n|}{|I_1^n|} K_1. \end{aligned}$$

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Proposition 5.6 implies that there exists $K_2 > 0$ such that, for $x \in I_0^n$,

$$|Df(x)| \leq K_2|x - c|$$

and

$$|I_1^n| \geq \frac{1}{K_2}|I_0^n|^2.$$

Now, for $x \in I_0^n$,

$$\begin{aligned} |Df^{2^n}(x)| &\leq K_2|x - c|\frac{|I_0^n|}{|I_1^n|}K_1 \\ &\leq K_2K_1\frac{|I_0^n|^2}{|I_1^n|} \leq K_2^2K_1 = K. \end{aligned}$$

Therefore, we conclude that $|Df^{2^n}|_{I_0^n}| \leq K$. □

Definition 5. (*A priori* bounds) Let f be infinitely renormalizable. We say that f has *a priori* bounds if there exists $\tau > 0$ such that, for all $n \geq 1$ and $j \leq 2^n$, we have

$$\tau < \frac{|I_j^{n+1}|}{|I_j^n|}, \ \frac{|I_{j+2^n}^{n+1}|}{|I_j^n|}, \tag{15}$$

$$\tau < \frac{|I_j^n \setminus (I_j^{n+1} \cup I_{j+2^n}^{n+1})|}{|I_j^n|}, \tag{16}$$

where I_j^{n+1}, $I_{j+2^n}^{n+1}$ are the intervals of the next generation contained in I_j^n.

For a general discussion on real *a priori* bounds, see [**MS**] and the references therein. The proof of the following proposition follows closely the argument in [**MMSS**].

PROPOSITION 7.5. *Every infinitely renormalizable $C^{2+|\cdot|}$ map has* a priori *bounds.*

Proof. The proof consists of five steps.

Step 1. There exists $\tau_1 > 0$ such that $|I_0^{n+1}|/|I_0^n| > \tau_1$. Let $I_0^n = [a_n, a_{n-1}]$ be the central interval, and so $a_n = f^{2^n}(c)$. A similar argument as in the proof of Corollary 7.4 gives $K_1 > 0$ such that

$$|f^{2^n}([a_n, c])| \leq \left(\frac{|a_n - c|}{|I_0^n|}\right)^2 |I_0^n|K_1.$$

Notice that

$$f^{2^n}([a_n, c]) = I_{2^n}^{n+1},$$

thus

$$|I_{2^n}^{n+1}| \leq \frac{|a_n - c|^2}{|I_0^n|}K_1.$$

Note also that

$$f^{2^n}(I_{2^n}^{n+1}) = I_0^{n+1} \supset [a_n, c].$$

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Therefore, by Corollary 7.4,

$$|a_n - c| \le |f^{2^n}(I_{2^n}^{n+1})| \le K|I_{2^n}^{n+1}| \le K\frac{|a_n - c|^2}{|I_0^n|}K_1.$$

This implies that

$$|a_n - c| \ge \frac{1}{K}|I_0^n|,$$

which proves that $|I_0^{n+1}|/|I_0^n| > \tau_1$.

Step 2. There exists $\tau_2 > 0$ such that $|I_{2^n}^{n+1}|/|I_0^n| \ge \tau_2$. From above we get

$$\tau_1|I_0^n| \le |I_0^{n+1}| = |f^{2^n}(I_{2^n}^{n+1})| \le K|I_{2^n}^{n+1}|.$$

This proves that

$$\frac{|I_{2^n}^{n+1}|}{|I_0^n|} \ge \tau_2.$$

Step 3. There exists $\tau_3 > 0$ such that the following holds:

$$\frac{|I_j^{n+1}|}{|I_j^n|}, \ \frac{|I_{j+2^n}^{n+1}|}{|I_j^n|} \ge \tau_3.$$

Because

$$f^{2^n-j}(I_j^{n+1}) = I_0^{n+1}, \quad f^{2^n-j}(I_j^n) = I_0^n,$$

and from Proposition 7.3, we get a $K > 0$ such that

$$\frac{|I_j^{n+1}|}{|I_j^n|} \ge \frac{1}{K}\frac{|I_0^{n+1}|}{|I_0^n|} \ge \frac{\tau_1}{K}.$$

Hence, $|I_j^{n+1}|/|I_j^n| \ge \tau_3$. Similarly we prove that $|I_{j+2^n}^{n+1}|/|I_j^n| \ge \tau_3$, which completes the proof of (15).

Step 4. To complete the proof of the proposition, it remains to show that the gap between the intervals I_0^{n+1}, $I_{2^n}^{n+1}$ and as well as I_j^{n+1}, $I_{j+2^n}^{n+1}$ are not too small. Let

$$G_n = I_0^n \setminus (I_0^{n+1} \cup I_{2^n}^{n+1}).$$

We claim that there exists $\tau_4 > 0$ such that

$$\frac{|G_n|}{|I_0^n|} \ge \tau_4.$$

Let H_n be the image of G_n under f^{2^n}. Then $H_n = f^{2^n}(G_n) \supset I_{3\cdot 2^n}^{n+2}$. The claim follows by using Corollary 7.4 and the bounds we have so far, namely,

$$K|G_n| \ge |H_n| \ge |I_{3\cdot 2^n}^{n+2}| \ge \tau_3|I_{2^n}^{n+1}| \ge \tau_3\tau_2|I_0^n|.$$

This implies that

$$|G_n| \ge \tau_4|I_0^n|.$$

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Step 5. Let $G_j^n = I_j^n \setminus (I_j^{n+1} \cup I_{j+2^n}^{n+1})$, then there exists $\tau_5 > 0$ such that

$$\frac{|G_j^n|}{|I_j^n|} \geq \tau_5.$$

We have $f^{2^n-j}(G_j^n) = G_n$ and $f^{2^n-j}(I_j^n) = I_0^n$. Since f^{2^n-j} has bounded distortion, we immediately get a constant $K > 0$ such that

$$\frac{|G_j^n|}{|I_j^n|} \geq \frac{1}{K}\frac{|G_n|}{|I_0^n|} \geq \frac{\tau_4}{K}.$$

This implies that

$$|G_j^n| \geq \tau_5 |I_j^n|.$$

This completes the proof of (16). □

8. *Approximation of $f|_{I_j^n}$ by a quadratic map*

Let $\phi : [0, 1] \to [0, 1]$ be an orientation-preserving C^2 diffeomorphism with nonlinearity $\eta_\phi : [0, 1] \to \mathbb{R}$. We identify a C^2 diffeomorphism with its nonlinearity, which is a continuous function. Hence, we identify the set of C^2 diffeomorphisms with the vector space of continuous functions equipped with the C^0 norm. In this context

$$|\phi| = |\eta_\phi|_0$$

becomes a norm (nonlinearity norm), see [**M**]. Let $[a, b] \subset [0, 1]$ and $f : [a, b] \to f([a, b])$ be a diffeomorphism. Let

$$1_{[a,b]} : [0, 1] \to [a, b]$$

and

$$1_{f([a,b])} : [0, 1] \to f([a, b])$$

be the affine homeomorphisms with $1_{[a,b]}(0) = a$ and $1_{f([a,b])}(0) = f(a)$. The rescaling $f_{[a,b]} : [0, 1] \to [0, 1]$ is the diffeomorphism

$$f_{[a,b]} = (1_{f([a,b])})^{-1} \circ f \circ 1_{[a,b]}.$$

We say that $0 \in [0, 1]$ corresponds to $a \in [a, b]$.

PROPOSITION 8.1. *Let f be an infinitely renormalizable $C^{2+|\cdot|}$ map with critical point $c \in (0, 1)$. For $n \geq 1$ and $1 \leq j < 2^n$ we have*

$$f_{I_j^n} = \phi_j^n \circ q_j^n,$$

where

$$q_j^n = (q_c)_{I_j^n} : [0, 1] \to [0, 1]$$

such that 0 corresponds to $f^j(c) \in I_j^n$ and $\phi_j^n : [0, 1] \to [0, 1]$ a C^2 diffeomorphism. Moreover

$$\lim_{n\to\infty} \sum_{j=1}^{2^n-1} |\phi_j^n| = 0.$$

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Proof. If $I_j^n \subset [c, 1]$ then use Proposition 5.6 and define

$$\phi_j^n = (\phi_+)_{q_c(I_j^n)} : [0, 1] \to [0, 1]$$

such that $0 \in [0, 1]$ corresponds to $q_c(f^j(c)) \in q_c(I_j^n)$. In the case $I_j^n \in [0, c]$ then let

$$\phi_j^n = (\phi_-)_{q_c(I_j^n)} : [0, 1] \to [0, 1],$$

where again $0 \in [0, 1]$ corresponds to $q_c(f^j(c)) \in q_c(I_j^n)$. Let η_j^n be the nonlinearity of ϕ_j^n. Then the chain rule for nonlinearities [**M**] gives

$$|\eta_j^n(x)| = |q_c(I_j^n)|\, |\eta_{\phi_\pm}(1_j^n(x))|,$$

where $1_j^n : [0, 1] \to q_c(I_j^n)$ is the affine homeomorphism such that $1_j^n(0) = q_c(f^j(c))$. Now use (11) to get

$$\begin{aligned} |\eta_j^n|_0 &\le |q_c(I_j^n)| \frac{(1-c)^2}{2} \frac{1}{\min_{x \in I_j^n}(1 + \bar{\epsilon}(x))} \sup_{x \in I_j^n} \frac{|\delta(x)|}{(x-c)^2} \\ &\le \frac{1}{\min_{x \in [0,1]}(1 + \bar{\epsilon}(x))} |\zeta_j^n - c|\, |I_j^n| \sup_{x \in I_j^n} \frac{|\delta(x)|}{|x-c|^2}, \end{aligned}$$

where

$$|Dq_c(\xi_j^n)| = \frac{|q_c(I_j^n)|}{|I_j^n|}$$

and $\xi_j^n \in I_j^n$. The *a priori* bounds give $K_1 > 0$ such that

$$\operatorname{dist}(c, I_j^n) \ge \frac{1}{K_1} |I_j^n|.$$

This implies that, for some $K > 0$,

$$|\eta_j^n| \le K \sup_{x \in I_j^n} \frac{|\delta(x)|}{|x-c|} |I_j^n|.$$

Therefore,

$$\begin{aligned} \sum_{j=1}^{2^n-1} |\phi_{I_j^n}| &\le K \sum_{j=1}^{2^n-1} \sup_{x \in I_j^n} \frac{|\delta(x)|}{|x-c|} |I_j^n| \\ &= K Z_n. \end{aligned}$$

Let $\Lambda_n = \bigcup_{j=0}^{2^n-1} I_j^n$. The *a priori* bounds imply that there exists $\tau > 0$ such that

$$|\Lambda_n| \le (1 - \tau) |\Lambda_{n-1}|.$$

In particular, $|\Lambda| = 0$, where $\Lambda \cap \Lambda_n$ is the Cantor attractor. Now we go back to our estimate and notice that Z_n is a Riemann sum for

$$\int_{\Lambda_n} \frac{|\delta(x)|}{|x-c|}\, dx.$$

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Suppose that $\limsup Z_n = Z > 0$. Let $n \geq 1$ and $m > n$. Then we can find a Riemann sum $\Sigma_{m,n}$ for

$$\int_{\Lambda_n} \frac{|\delta(x)|}{|x-c|}\, dx$$

by adding positive terms to Z_m. Then

$$\int_{\Lambda_n} \frac{|\delta(x)|}{|x-c|}\, dx = \limsup_{m\to\infty} \Sigma_{m,n} \geq \limsup_{m\to\infty} Z_m \geq Z > 0.$$

Hence,

$$\int_{\Lambda} \frac{|\delta(x)|}{|x-c|}\, dx \geq Z > 0.$$

This is impossible because $|\Lambda| = 0$. Thus we have proved that

$$\sum_{j=1}^{2^n-1} |\phi_{I_j^n}| \longrightarrow 0. \qquad \square$$

9. *Approximation of $R^n f$ by a polynomial map*

The following lemma is a variation on the sandwich lemma from [**M**].

LEMMA 9.1. (Sandwich) *For every $K > 0$ there exists a constant $B > 0$ such that the following holds. Let ψ_1, ψ_2 be the compositions of finitely many $\phi, \phi_j \in \mathrm{Diff}_+^2([0,1])$, $1 \leq j \leq n$:*

$$\psi_1 = \phi_n \circ \cdots \circ \phi_t \circ \ldots \phi_1$$

and

$$\psi_2 = \phi_n \circ \cdots \circ \phi_{t+1} \circ \phi \circ \phi_t \circ \ldots \phi_1.$$

If

$$\sum_j |\phi_j| + |\phi| \leq K,$$

then

$$|\psi_1 - \psi_2|_1 \leq B|\phi|.$$

Proof. Let $x \in [0,1]$. For $1 \leq j \leq n$ let

$$x_j = \phi_{j-1} \circ \cdots \circ \phi_2 \circ \phi_1(x)$$

and

$$D_j = (\phi_{j-1} \circ \cdots \circ \phi_2 \circ \phi_1)'(x).$$

Furthermore, for $t+1 \leq j \leq n$, let

$$x'_j = \phi_{j-1} \circ \cdots \circ \phi_{t+1}(\phi(x_{t+1}))$$

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and

$$D'_j = (\phi_{j-1} \circ \cdots \circ \phi_{t+1})'(x'_{t+1})\phi'(x_{t+1})D_{t+1}.$$

Now we estimate the difference of the derivatives of ψ_1 and ψ_2, namely,

$$\frac{D\psi_2(x)}{D\psi_1(x)} = D\phi(x_{t+1}) \prod_{j \geq t+1} \frac{D\phi_j(x'_j)}{D\phi_j(x_j)}.$$

In the following estimates we will repeatedly apply [**M**, Lemma 10.3], which says that

$$e^{-|\psi|} \leq |D\psi|_0 \leq e^{|\psi|}.$$

This allows us to get an estimate on $|D\psi_1 - D\psi_2|_0$ in terms of $D\psi_2/D\psi_1$. Now

$$D\phi_j(x'_j) = D\phi_j(x_j) + D^2\phi_j(\zeta_j)(x'_j - x_j),$$

and therefore

$$\begin{aligned} \frac{D\phi_j(x'_j)}{D\phi_j(x_j)} &\leq 1 + \frac{|D^2\phi_j|_0}{D\phi_j(x_j)}|x'_j - x_j| \\ &= 1 + O(\phi_j)|x'_j - x_j|. \end{aligned}$$

To continue, we have to estimate $|x'_j - x_j|$. Apply [**M**, Lemma 10.2] to get

$$\begin{aligned} |x'_j - x_j| &= O(|x'_{t+1} - x_{t+1}|) \\ &= O(|\phi|). \end{aligned}$$

Because $\sum |\phi_j| + |\phi| \leq K$, there exists $K_1 > 0$ such that

$$\begin{aligned} \frac{D\psi_2(x)}{D\psi_1(x)} &\leq e^{|\phi|} \prod_{j \geq t+1} (1 + O(|\phi_j|\,|\phi|)) \\ &\leq e^{|\phi|}\, e^{K_1 \sum |\phi_j|\,|\phi|}. \end{aligned}$$

Hence, we have

$$\frac{D\psi_2}{D\psi_1} \leq e^{|\phi|(1+K_1K)}.$$

We get a lower bound in similar way. So there exists $K_2 > 0$ such that

$$e^{-K_2|\phi|} \leq \frac{|D\psi_2|}{|D\psi_1|} \leq e^{K_2|\phi|}.$$

Finally, there exists $B > 0$ such that

$$|D\psi_2(x) - D\psi_1(x)| \leq B|\phi|. \qquad \square$$

Let f be an infinitely renormalizable $C^{2+|\cdot|}$ unimodal map.

LEMMA 9.2. *There exists $K > 0$ such that, for all $n \geq 1$, the following holds:*

$$\sum_{1 \leq j \leq 2^n - 1} |q_j^n| \leq K.$$

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Proof. The nonlinearity norm of q_j^n, $j = 1, \ldots, 2^n - 1$, is

$$|q_j^n| = \frac{|I_j^n|}{\operatorname{dist}(I_j^n, c)}.$$

Let

$$Q_n = \sum_{j=1}^{2^n-1} |q_j^n|.$$

Observe that there exists $\tau > 0$ such that, for $j = 1, 2, \ldots, 2^n - 1$,

$$\begin{aligned} |q_j^{n+1}| + |q_{j+2^n}^{n+1}| &\le \frac{|I_j^{n+1}| + |I_{j+2^n}^{n+1}|}{\operatorname{dist}(I_j^n, c)} \\ &= |q_j^n| \frac{|I_j^{n+1}| + |I_{j+2^n}^{n+1}|}{|I_j^n|} \\ &= |q_j^n| \frac{|I_j^n - G_j^n|}{|I_j^n|} \le |q_j^n|(1 - \tau). \end{aligned}$$

Therefore

$$Q_{n+1} \le (1 - \tau) Q_n + |q_{2^n}^{n+1}|.$$

From the *a priori* bounds we get a constant $K_1 > 0$ such that

$$|q_{2^n}^{n+1}| \le \frac{|I_{2^n}^{n+1}|}{|G_{2^n}^n|} \le K_1.$$

Thus we have

$$Q_{n+1} \le (1 - \tau) Q_n + K_1.$$

This implies the lemma. □

Consider the map $f : I_0^n \to I_1^n$ and rescale affinely range and domain to obtain the unimodal map

$$\hat{f}_n : [0, 1] \to [0, 1].$$

Apply Proposition 5.6 to obtain the following representation of $\hat{f}_n$. There exists $c_n \in (0, 1)$ and diffeomorphisms $\phi_\pm^n : [0, 1] \to [0, 1]$ such that

$$\hat{f}_n(x) = \phi_+^n \circ q_{c_n}(x), \quad x \in [c_n, 1]$$

and

$$\hat{f}_n(x) = \phi_-^n \circ q_{c_n}(x), \quad x \in [0, c_n].$$

Furthermore

$$|\phi_\pm^n| \to 0$$

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when $n \to \infty$. Let $q_0^n = q_{c_n}$. Use Proposition 8.1 to obtain the following representation for the nth renormalization of f:

$$R^n f = (\phi_{2^n-1}^n \circ q_{2^n-1}^n) \circ \cdots \circ (\phi_j^n \circ q_j^n) \circ \cdots \circ (\phi_1^n \circ q_1^n) \circ \phi_\pm^n \circ q_0^n.$$

Inspired by [**AMM**] we introduce the unimodal map

$$f_n = q_{2^n-1}^n \circ \cdots \circ q_j^n \circ \cdots \circ q_1^n \circ q_0^n.$$

PROPOSITION 9.3. *If f is an infinitely renormalizable $C^{2+|\cdot|}$ map, then*

$$\lim_{n\to\infty} |R^n f - f_n|_1 = 0.$$

Proof. Define the diffeomorphisms

$$\psi_j^\pm = q_{2^n-1}^n \circ \cdots \circ q_j^n \circ (\phi_{j-1}^n \circ q_{j-1}^n) \circ \cdots \circ (\phi_1^n \circ q_1^n) \circ \phi_\pm^n$$

with $j = 0, 1, 2, \ldots, 2^n$. Notice that

$$R^n f(x) = \psi_{2^n}^\pm \circ q_0^n(x)$$

and that

$$f_n(x) = \psi_0^\pm \circ q_0^n(x),$$

where we use again the $\pm$ distinction for points $x \in [0, c_n]$ and $x \in [c_n, 1]$. Apply the sandwich lemma (Lemma 9.1) to get a constant $B > 0$ such that

$$|\psi_{j+1}^\pm - \psi_j^\pm|_1 \le B|\phi_j^n|$$

for $j \ge 1$, and also notice that

$$|\psi_1^\pm - \psi_0^\pm|_1 \le B|\phi_\pm^n| \longrightarrow 0.$$

We can now apply Proposition 8.1 to get

$$\lim_{n\to\infty} |\psi_{2^n}^\pm - \psi_0^\pm|_1 \le \lim_{n\to\infty} B \sum_{1\le j\le 2^n-1} |\phi_j^n| + |\phi_\pm^n| = 0,$$

which implies that

$$\lim_{n\to\infty} |R^n f - f_n|_1 = 0. \qquad \square$$

10. *Convergence*

Fix an infinitely renormalizable $C^{2+|\cdot|}$ map f.

LEMMA 10.1. *For every $N_0 \ge 1$, there exists $n_1 \ge 1$ such that f_n is N_0 times renormalizable whenever $n \ge n_1$.*

Proof. The *a priori* bounds from Proposition 7.5 give $d > 0$ such that, for $n \ge 1$,

$$|(R^n f)^i(c) - (R^n f)^j(c)| \ge d$$

for all $i, j \le 2^{N_0+1}$ and $i \ne j$. Now by taking n large enough and using Proposition 9.3 we find

$$|f_n^i(c) - f_n^j(c)| \ge \tfrac{1}{2}d$$

for $i \ne j$ and $i, j \le 2^{N_0+1}$. The *kneading sequence* of f_n (i.e., the sequence of signs of the derivatives of that function) coincides with the kneading sequence of $R^n f$ for at least 2^{N_0+1} positions. We have proved that f_n is N_0 times renormalizable because $R^n f$ is N_0 times renormalizable. $\square$

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The polynomial unimodal maps f_n are in a compact family of quadratic like maps. This follows from Lemma 9.2. The unimodal renormalization theory presented in [**Ly**] gives us the following result.

PROPOSITION 10.2. *There exist $N_0 \geq 1$ and $n_0 \geq 1$ such that f_n is N_0 renormalizable and*

$$\operatorname{dist}_1(R^{N_0} f_n, W^u) \leq \tfrac{1}{3} \operatorname{dist}_1(f_n, W^u).$$

Here, W^u is the unstable manifold of the renormalization fixed point contained in the space of quadratic like maps [**Ly**]. Recall that dist_1 stands for the C^1 distance.

LEMMA 10.3. *There exists $K > 0$ such that, for $n \geq 1$,*

$$\operatorname{dist}_1(R^n f, W^u) \leq K.$$

Proof. This follows from Lemma 9.2 and Proposition 9.3. □

Let $f_*^\omega \in W^u$ be the analytic renormalization fixed point.

THEOREM 10.4. *If f is an infinitely renormalizable $C^{2+|\cdot|}$ unimodal map, then*

$$\lim_{n\to\infty} \operatorname{dist}_0(R^n f, f_*^\omega) = 0.$$

Proof. For every $K > 0$, there exists $A > 0$ such that the following holds. Let f, g be renormalizable unimodal maps with

$$|Df|_0, |Dg|_0 \leq K,$$

then

$$\operatorname{dist}_0(Rf, Rg) \leq A \operatorname{dist}_0(f, g). \tag{17}$$

Let $N_0 \geq 1$ be as in Proposition 10.2. Now

$$\begin{aligned}\operatorname{dist}_0(R^{n+N_0} f, W^u) &\leq \operatorname{dist}_0(R^{N_0}(R^n f), R^{N_0} f_n) + \operatorname{dist}_0(R^{N_0} f_n, W^u)\\ &\leq A^{N_0} \operatorname{dist}_0(R^n f, f_n) + \tfrac{1}{3} \operatorname{dist}_0(f_n, W^u).\end{aligned}$$

Notice that

$$\operatorname{dist}_0(f_n, W^u) \leq \operatorname{dist}_0(f_n, R^n f) + \operatorname{dist}_0(R^n f, W^u).$$

Thus there exists $K > 0$ such that

$$\operatorname{dist}_0(R^{n+N_0} f, W^u) \leq \tfrac{1}{3} \operatorname{dist}_0(R^n f, W^u) + K \operatorname{dist}_0(R^n f, f_n).$$

Let

$$z_n = \operatorname{dist}_0(R^{nN_0} f, W^u)$$

and

$$\delta_n = \operatorname{dist}_0(R^n f, f_n).$$

Then

$$z_{n+1} \leq \tfrac{1}{3} z_n + K \delta_{nN_0}.$$

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This implies that

$$z_n \leq \sum_{j<n} K\delta_{jN_0}\left(\frac{1}{3}\right)^{n-j}.$$

Now we use that $\delta_n \to 0$ (see Proposition 9.3) to get $z_n \to 0$. So we have proved that $R^{nN_0}f$ converges to W^u. Use (17) and $R(W^u) \subset W^u$ to get that $R^n f$ converges to W^u in C^0 sense. Notice that any limit of $R^n f$ is infinitely renormalizable. The only infinitely renormalizable map in W^u is the fixed point f_*^ω. Thus

$$\lim_{n\to\infty} \operatorname{dist}_0(R^n f,\, f_*^\omega) = 0. \qquad \square$$

11. *Slow convergence*

THEOREM 11.1. *Let $d_n > 0$ be any sequence with $d_n \to 0$. There exists an infinitely renormalizable C^2 map f with quadratic tip such that*

$$\operatorname{dist}_0(R^n f,\, f_*^\omega) \geq d_n.$$

The proof needs some preparation. Use the representation

$$f_*^\omega = \phi \circ q_c,$$

where ϕ is an analytic diffeomorphism. The renormalization domains are denoted by I_0^n with

$$c = \bigcap_{n\geq 1} I_0^n.$$

Each I_0^n contains two intervals of the $(n+1)$th generation, namely I_0^{n+1} and $I_{2^n}^{n+1}$. Let

$$G_n = I_0^n \setminus (I_0^{n+1} \cup I_{2^n}^{n+1}),$$
$$\hat{G}_n = q_c(G_n) \subset \hat{I}_0^n = q_c(I_0^n)$$

and $\hat{I}_{2^n}^{n+1} = q_c(I_{2^n}^{n+1})$. The invariant Cantor set of f_*^ω is denoted by Λ. Notice that

$$q_c(\Lambda) \cap \hat{I}_0^n \subset (\hat{I}_0^{n+1} \cup \hat{I}_{2^n}^{n+1}).$$

The gap $\hat{G}_n$ in $\hat{I}_0^n$ does not intersect with Λ. Choose a family of C^2 diffeomorphisms

$$\phi_t : [0,\, 1] \to [0,\, 1]$$

with the following properties:

(i) $D\phi_t(0) = D\phi_t(1) = 1$;

(ii) $D^2\phi_t(0) = D^2\phi(1) = 0$;

(iii) for some $C_1 > 0$, $\operatorname{dist}_0(\phi_t,\, \mathrm{id}) \geq C_1 t$; and

(iv) for some $C_2 > 0$, $|\eta_{\phi_t}|_0 \leq C_2 t$.

Let $m = \min D\phi$ and $t_n = 1/(mC_1|\hat{G}_1|)d_n$. Now we will introduce a perturbation $\tilde{\phi}$ of ϕ. Let

$$1_n : [0,\, 1] \to \hat{G}_n$$

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be the affine orientation-preserving homeomorphism. Define

$$\psi : [0, 1] \to [0, 1]$$

as follows:

$$\psi(x) = \begin{cases} x, & x \notin \bigcup_{n\geq 0} \hat{G}_n, \\ 1_n \circ \phi_{t_n} \circ 1_n^{-1}(x), & x \in \hat{G}_n. \end{cases}$$

Let

$$f = \phi \circ \psi \circ q_c = \tilde{\phi} \circ q_c.$$

Then f is a unimodal map with quadratic tip that is infinitely renormalizable and still has Λ as its invariant Cantor set. This follows from the fact that the perturbation did not affect the critical orbit and it is located in the complement of the Cantor set. In particular, the invariant Cantor set of $R^n f$ is again $\Lambda \subset I_0^1 \cup I_1^1$ and G_1 is the gap of $R^n f$. Notice, by using that f_*^ω is the fixed point of renormalization, that, for $x \in G_1$,

$$R^n f(x) = \phi \circ 1_1 \circ \phi_{t_n} \circ 1_1^{-1} \circ q_c(x).$$

Hence,

$$\begin{aligned} |R^n f - f_*^\omega|_0 &\geq \max_{x\in\hat{G}_1} |R^n f(x) - f_*^\omega(x)| \\ &\geq \max_{x\in\hat{G}_1} m\, |(1_1 \circ \phi_{t_n} \circ 1_1^{-1}) q_c(x) - q_c(x)| \\ &\geq m \max_{x\in\hat{G}_1} |(1_1 \circ \phi_{t_n} \circ 1_1^{-1})(x) - x| \\ &= m|\hat{G}_1|\, |\phi_{t_n} - \mathrm{id}|_0 \\ &\geq m|\hat{G}_1| C_1 t_n = d_n. \end{aligned}$$

It remains to prove that f is C^2. The map f is C^2 on $[0, 1] \setminus \{c\}$ because $f = \tilde{\phi} \circ q_c$ with $\tilde{\phi} = \phi \circ \psi$, where ϕ is an analytic diffeomorphism and ψ is by construction C^2 on $[0, 1)$. Notice that, from (10) we have

$$\begin{aligned} D^2 f(x) = 4\frac{(x-c)^2}{(1-c)^4} D^2\tilde{\phi}(q_c(x)) \\ - 2\frac{1}{(1-c)^2} D\tilde{\phi}(q_c(x)). \end{aligned} \tag{18}$$

We will analyze the above two terms separately. Observe that

$$D\psi(x) = \begin{cases} 1, & x \notin \bigcup_{n\geq 0} \hat{G}_n, \\ |D\phi_{t_n}(1_n^{-1}(x))|, & x \in \hat{G}_n. \end{cases}$$

This implies that, for $x \in G_n$,

$$\begin{aligned} D\tilde{\phi}(q_c(x)) &= D\phi(\psi \circ q_c) D\psi(q_c(x)) \\ &= D\phi(1)(1 + O(\hat{I}_0^n))(1 + O(t_n)). \end{aligned}$$

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For $x \notin \bigcup_{n\geq 1} G_n$ we have

$$D\tilde{\phi}(q_c(x)) = D\phi(q_c(x)),$$

which implies that the term

$$x \longmapsto -2\frac{1}{(1-c)^2} D\tilde{\phi}(q_c(x))$$

extends continuously to the whole domain. The first term in (18) needs more care. Observe that, for $u \in \hat{G}_n$,

$$\begin{aligned} D^2\tilde{\phi}(u) &= D^2\phi(\psi(u))(D\psi(u))^2 + D\phi(\psi(u))D^2\psi(u) \\ &= D^2\phi(1)(1+O(\hat{I}_0^n))(1+O(t_n)) \\ &\quad + D\phi(1)(1+O(\hat{I}_0^n))(1+O(t_n))D^2\psi(u) \\ &= D^2\phi(1)(1+O(\hat{I}_0^n))(1+O(t_n)) \\ &\quad + D\phi(1)(1+O(\hat{I}_0^n))(1+O(t_n))\frac{1}{|\hat{G}_n|}O(t_n). \end{aligned}$$

This implies that

$$4\frac{(x-c)^2}{(1-c)^4} D^2\tilde{\phi}(q_c(x)) = \begin{cases} O((x-c)^2) + O(t_n), & x \in \hat{G}_n, \\ O((x-c)^2), & x \notin \bigcup_{n\geq 0} \hat{G}_n. \end{cases}$$

In particular, the first term of $D^2 f$,

$$x \longmapsto 4\frac{(x-c)^2}{(1-c)^4} D^2\tilde{\phi}(q_c(x)),$$

also extends to a continuous function on $[0, 1]$. Indeed, f is C^2.

Remark 11.2. If the sequence d_n is not summable (and in particular not exponential decaying) then the example constructed above is not $C^{2+|\cdot|}$. This follows from

$$\int_{\hat{G}_n} |\eta_{\tilde{\phi}}(x)|\, dx \asymp t_n.$$

Thus

$$\int |\eta_{\tilde{\phi}}| \asymp \sum d_n = \infty.$$

Now, Proposition 5.6 implies that f is not $C^{2+|\cdot|}$. If the sequence d_n is summable, the previous construction will give an example of a $C^{2+|\cdot|}$ unimodal map whose renormalizations converges only polynomially. Any reasonable metric used on $C^{2+|\cdot|}$ will be stronger than the C^0 distance in which the polynomial convergence occurs. Hence the renormalization fixed point cannot be hyperbolic in any space of $C^{2+|\cdot|}$ unimodal maps.

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