100.09 Extremal distance ratios

Given a line ℓ and two points A and B not on ℓ, we construct the points X of ℓ that make the distance ratio XA/XB extremal. We consider in particular the case of a triangle ABC where ℓ is the bisector of angle C. Our short proofs are simpler than in [1, 2, 3].

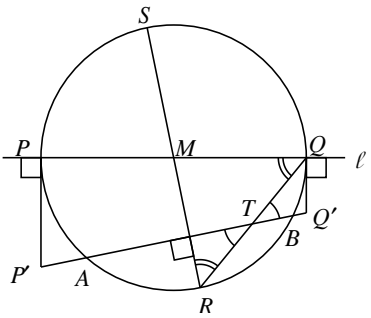


FIGURE 1

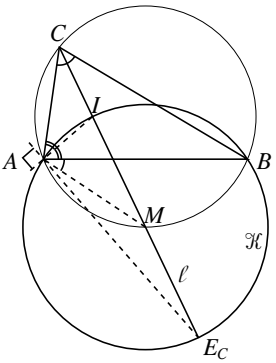


FIGURE 2

We suppose that AB is not perpendicular to ℓ and construct the circle through A and B with diameter PQ on ℓ (Figure 1). We prove that P and Q are the points X we are looking for by considering P' and Q' on the line AB given by $P'P \perp \ell \perp QQ'$. Whether A and B are on one side of ℓ or on opposite sides, the same argument works.

Angles in the same segment of a circle being equal, QR is the angle bisector of $\angle AQB$ as

$$\angle AQB = \angle ASB = 2\angle RSB = 2\angle RQB.$$

Thus Q and T are on the same Apollonius circle $XA/XB = QA/QB$ by the angle bisector theorem. By angle chasing, $TQ' = QQ'$, hence Q' is the centre of the above Apollonius circle $\mathscr{A}$. As $\mathscr{A}$ is tangent to ℓ, Q maximises XA/XB under the constraint $X \in \ell$ for the following reason: XA/XB being a continuous function of X with value ∞ at $X = B$ inside $\mathscr{A}$, an Apollonius circle given by a ratio XA/XB greater than on $\mathscr{A}$ lies around B *inside* $\mathscr{A}$ and has thus no intersection with ℓ.

P minimises XA/XB for $X \in \ell$ as it maximises XB/XA. For $AB \perp \ell$, the solutions are the point at infinity of ℓ and the intersection with the line AB (and every point of ℓ when ℓ is the perpendicular bisector of AB).

There is an even shorter proof. Each circle $XA/XB = \text{constant}$ cuts all circles through A and B at right angles (proof below!). We are looking for circles $XA/XB = \text{constant}$ tangent to ℓ, that is, for circles through A and B that cut ℓ at right angles: we are done!

Choosing $A(-1, 0)$ and $B(1, 0)$ without loss of generality, we prove for the sake of completeness that a true circle $\mathscr{C}$ through A and B always cuts the Apollonius circle $\mathscr{C}'$ given by $XA/AB = \lambda$ at right angles, where $\lambda \neq 1$

https://doi.org/10.1017/mag.2016.18 Published online by Cambridge University Press

is a positive number. (The assertion is obvious when $\mathscr{C}$ is the x-axis or when $\lambda = 1$ and $\mathscr{C}'$ is the y-axis.) $\mathscr{C}$ is centred at $U(0, y_0)$ with squared radius $r^2 = 1 + y_0^2$. $\mathscr{C}'$ has the equation

$$(x + 1)^2 + y^2 = \lambda^2 (x - 1)^2 + \lambda^2 y^2,$$

which can be written as

$$(x - x_0)^2 + y^2 = x_0^2 - 1 \qquad \text{for} \qquad x_0 = \frac{\lambda^2 + 1}{\lambda^2 - 1}.$$

$\mathscr{C}'$ is thus a circle centred at $V(x_0, 0)$ with squared radius $\rho^2 = x_0^2 - 1$. As

$$r^2 + \rho^2 = (1 + y_0^2) + (x_0^2 - 1) = x_0^2 + y_0^2 = UV^2,$$

$\mathscr{C}$ and $\mathscr{C}'$ intersect at right angles.

As inner and outer angle bisectors are perpendicular, the incentre I and the excentre E_C of triangle ABC delimit the diameter of a circle $\mathscr{H}$ through A and B (Figure 2). (The centre of $\mathscr{H}$ is the midpoint M of the circumcircle arc $\overset{\frown}{AB}$ as triangle MAI is isosceles by the exterior angle theorem. This incentre/excentre configuration is well-known: for example, it is Theorem 2.13 in [4].) When the bisector of angle C in triangle ABC is not an altitude, the incentre and C-excentre are thus the points of the angle bisector where the extremal ratios XA/XB occur. Note that ℓ intersects here the segment AB internally. For the same reason, the A- and B-excentres provide the extremal ratios XA/XB on the external angle bisector of C.

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doi:10.1017/mag.2016.18

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https://doi.org/10.1017/mag.2016.18 Published online by Cambridge University Press