Check for updates

Adv. Appl. Probab. **57,** 811–870 (2025)
*doi:*10.1017/apr.2024.63

PROBABILITY THAT N POINTS ARE IN CONVEX POSITION IN A REGULAR κ-GON: ASYMPTOTIC RESULTS

LUDOVIC MORIN,* *Université de Bordeaux, LaBRI*

Abstract

Let $\mathbb{P}_\kappa(n)$ be the probability that n points $z_1, \dots, z_n$ picked uniformly and independently in $\mathfrak{C}_\kappa$, a regular κ-gon with area 1, are in convex position, that is, form the vertex set of a convex polygon. In this paper, we compute $\mathbb{P}_\kappa(n)$ up to asymptotic equivalence, as $n \to +\infty$, for all $\kappa \geq 3$, which improves on a famous result of Bárány (*Ann. Prob.* **27**, 1999). The second purpose of this paper is to establish a limit theorem which describes the fluctuations around the limit shape of an n-tuple of points in convex position when $n \to +\infty$. Finally, we give an asymptotically exact algorithm for the random generation of $z_1, \dots, z_n$, conditioned to be in convex position in $\mathfrak{C}_\kappa$.

Keywords: Random convex chains; random polygon; Sylvester's problem; stochastic geometry

2010 Mathematics Subject Classification: Primary 52A22
Secondary 60D05

1. Introduction

Let $\mathfrak{C}_\kappa$ be the regular κ-gon with area 1 positioned on the x-axis, as represented in Figure 1, let $r_\kappa = \left(4 \tan\left(\frac{\pi}{\kappa}\right)/\kappa\right)^{1/2}$ be its side length, and let $\theta_\kappa = \frac{(\kappa-2)\pi}{\kappa}$ be the interior angle between two consecutive sides.

For any compact convex domain K of area 1 in $\mathbb{R}^2$ with non-empty interior and for any $n \in \mathbb{N}$, we let $\mathbb{U}_K^{(n)}$ denote the law of an n-tuple $\mathbf{z}[n] := (\mathbf{z}_1, \cdots, \mathbf{z}_n)$, where the $\mathbf{z}_i$ are independent and identically distributed (i.i.d.) and uniform in K.

In the special case $K = \mathfrak{C}_\kappa$, we write for short $\mathbb{U}_\kappa^{(n)} := \mathbb{U}_{\mathfrak{C}_\kappa}^{(n)}$.

An n-tuple of points $z[n] \in (\mathbb{R}^2)^n$ is said to be in convex position if $\{z_1, \cdots, z_n\}$ is the vertex set of a convex polygon, which we will refer to as the $z[n]$-gon; the set of such n-tuples $z[n]$ is denoted by $\mathcal{Z}_n$. Hence

$$\mathbb{P}_K(n) := \mathbb{P}\left(\mathbf{z}[n] \in \mathcal{Z}_n\right) = \mathbb{U}_K^{(n)}(\mathcal{Z}_n)$$

is the probability that n i.i.d. random points $\mathbf{z}[n]$ taken uniformly in K are in convex position, and

Received 30 January 2024; accepted 3 October 2024.
* Postal address: Université de Bordeaux, CNRS, Bordeaux INP, LaBRI, UMR 5800, F-33400 Talence, France. Email address: ludovic.morin@labri.fr

© The Author(s), 2025. Published by Cambridge University Press on behalf of Applied Probability Trust. This is an Open Access article, distributed under the terms of the Creative Commons Attribution licence (https://creativecommons.org/licenses/by/4.0/), which permits unrestricted re-use, distribution, and reproduction in any medium, provided the original work is properly cited.

https://doi.org/10.1017/apr.2024.63 Published online by Cambridge University Press

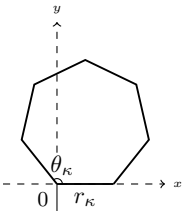


FIGURE 1. $\mathfrak{C}_7$.

$$\mathbb{P}_\kappa(n) := \mathbb{P}_{\mathfrak{C}_\kappa}(n)$$

is the corresponding probability in the regular κ-gon.

The purpose of this paper is threefold. First we give an equivalent of $\mathbb{P}_\kappa(n)$ as $n \to \infty$ (see Theorem 1 below), then we describe the fluctuations of a $z[n]$-gon with distribution $\mathbb{U}_\kappa^{(n)}$ conditioned to be in convex position (Theorem 8), and we conclude by providing an algorithm to sample such an n-tuple $z[n]$ (Section 6). One of the main contributions of this paper is thus the following theorem.

Theorem 1. *Let $\kappa \geq 3$ be an integer. We have*

$$\mathbb{P}_\kappa(n) \underset{n\to+\infty}{\sim} C_\kappa \cdot \frac{e^{2n}}{4^n} \frac{\kappa^{3n} r_\kappa^{2n} \sin(\theta_\kappa)^n}{n^{2n+\kappa/2}},$$

where

$$C_\kappa = \frac{1}{\pi^{\kappa/2}\sqrt{\mathrm{m}_\kappa}} \frac{\sqrt{\kappa}^{\kappa+1}}{4^\kappa (1+\cos(\theta_\kappa))^\kappa},$$

and m_κ is the determinant of a deterministic matrix (see Theorem 7), an explicit formula for which is given by

$$\mathrm{m}_\kappa = \frac{\kappa}{3 \cdot 2^\kappa} \left(2(-1)^{\kappa-1} + (2-\sqrt{3})^\kappa + (2+\sqrt{3})^\kappa \right). \tag{1.1}$$

Theorem 1 actually refines a famous result of Bárány [3] in the case of κ-gons (note, however, that Bárány's result holds under weaker hypotheses).

Theorem 2. (Bárány [3].) *For any compact convex set K of area 1 with non-empty interior,*

$$\lim_{n\to+\infty} n^2 \left(\mathbb{P}_K(n)\right)^{\frac{1}{n}} = \frac{1}{4} e^2 \mathrm{AP}^*(K)^3,$$

where $\mathrm{AP}^*(K)$ is the supremum of the affine perimeters of all convex sets $S \subset K$.

The definition of the affine perimeter will be recalled in Definition 2; we send the interested reader to [2] for additional details. In the κ-gon case, as will be shown in Lemma 17, we have

$$\mathrm{AP}^*(\mathfrak{C}_\kappa) = \kappa \left(r_\kappa^2 \sin(\theta_\kappa) \right)^{1/3},$$

https://doi.org/10.1017/apr.2024.63 Published online by Cambridge University Press

so that one can check that in that particular case, Theorem 1 is compatible with and more precise than Theorem 2.

The quantity $\mathbb{P}_K(n)$ has been widely studied since the 19th century, and for a large variety of convex sets K, not just regular polygons. Sylvester [23] initiated the consideration of this matter, looking at the probability that four points chosen at random in the plane were in convex position. Though Sylvester's question was ill-posed, it matured in its later formulation into the study of $\mathbb{P}_K(4)$, for any convex shape K of area 1 (see Pfiefer [20] for historical notes). In 1917, Blaschke [5] determined the convex domain K that maximizes or minimizes the probability $\mathbb{P}_K(4)$ (on the set of non-flat compact convex domains of $\mathbb{R}^2$) by proving that the lower bound is achieved when $K = \triangle$ is a triangle, and the upper bound when $K = \bigcirc$ is a disk; that is,

$$\frac{2}{3} = \mathbb{P}_{\triangle}(4) \le \mathbb{P}_K(4) \le \mathbb{P}_{\bigcirc}(4) = 1 - \frac{35}{12\pi^2}.$$

In the same direction, Marckert and Rahmani [18] proved in 2021 that

$$\frac{11}{36} = \mathbb{P}_{\triangle}(5) \le \mathbb{P}_K(5) \le \mathbb{P}_{\bigcirc}(5) = 1 - \frac{305}{48\pi^2}.$$

This question can be generalized to different values of n, and other dimensions. To this day, the conjecture in dimension 2, that

$$\mathbb{P}_{\triangle}(n) \le \mathbb{P}_K(n) \le \mathbb{P}_{\bigcirc}(n)$$

for all $n \ge 3$, remains open. Yet the value $\mathbb{P}_{\bigcirc}(n)$ has been known since 2017 to be computable for all $n \ge 3$, thanks to Marckert's algebraic formula [17] in the disk case. Note also that Hilhorst *et al.* [13] managed in 2008 to derive an asymptotic expansion of $\log \mathbb{P}_{\bigcirc}(n)$.

In the case of regular convex polygons, exact formulas are rare, but Valtr proved in 1995 [24] that for K a parallelogram,

$$\mathbb{P}_4(n) = \mathbb{P}_{\square}(n) = \frac{1}{(n!)^2}\binom{2n-2}{n-1}^2 \underset{n\to+\infty}{\sim} \frac{1}{\pi^2 2^5}\frac{4^{2n}e^{2n}}{n^{2n+2}},$$

and in 1996 [25] that when K is a triangle,

$$\mathbb{P}_3(n) = \mathbb{P}_{\triangle}(n) = \frac{2^n(3n-3)!}{(2n)!((n-1)!)^3} \underset{n\to+\infty}{\sim} \frac{\sqrt{3}}{4}\frac{1}{\pi^{3/2}3^3}\frac{3^{3n}e^{2n}}{2^n n^{2n+3/2}}.$$

The equivalents given at the right-hand side are of course consistent with Theorem 1. Note however that our method will allow us to recover Valtr's formulas in Section B (our approach avoids discretization arguments, but it largely relies on Valtr's ideas).

In dimension $d \ge 3$, if $\triangle^d$ and $\bigcirc^d$ denote respectively a simplex and an ellipsoid of volume 1, the following generalization of Sylvester's question—that

$$\mathbb{P}_{\triangle^d}(d+2) \le \mathbb{P}_K(d+2) \le \mathbb{P}_{\bigcirc^d}(d+2)$$

for any convex domain $K \subset \mathbb{R}^d$ of volume 1—is a conjecture that remains to be proven (though the right inequality is known as a generalization of Blaschke's proof in dimension 2). For a comprehensive overview of these matters, we refer to Schneider [21].

Canonical ordering of $z[n]$-gons. An element of $z[n] \in \mathcal{Z}_n$ (in convex position) is said to be in convex canonical order if it satisfies the following conditions (see Figure 2):

https://doi.org/10.1017/apr.2024.63 Published online by Cambridge University Press

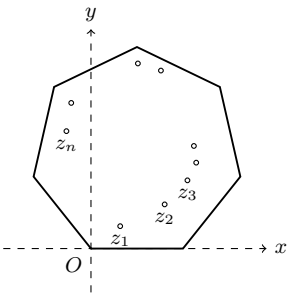


FIGURE 2. Some $z[n]$ in $\mathcal{C}_7(n)$.

- If (x_i, y_i) are the coordinates of z_i in $\mathbb{R}^2$, then $y_1 \le y_i$ for all i (that is, z_1 has the smallest y-component), and among the points having the minimal y-component, z_1 has the smallest x-component.
- The sequence $(\arg(z_{i+1} - z_i),\ 1 \le i \le n-1)$ is non-decreasing in $[0, 2\pi]$.

We denote by $\overset{\curvearrowleft}{\mathcal{Z}}_n$ the subset of $\mathcal{Z}_n$ of n-tuples of points $z[n]$ in convex canonical order. The symmetric group $\mathcal{S}_n$ acts transitively on $\mathcal{Z}_n$ by relabeling the vertex indices; each orbit contains a unique element of $\overset{\curvearrowleft}{\mathcal{Z}}_n$. We put $\mathcal{D}_\kappa(n) = \mathcal{Z}_n \cap (\mathfrak{C}_\kappa)^n$ and $\mathcal{C}_\kappa(n) = \overset{\curvearrowleft}{\mathcal{Z}}_n \cap (\mathfrak{C}_\kappa)^n$.

Since $\mathbf{z}[n]$ is picked according to the uniform distribution on $(\mathfrak{C}_\kappa)^n$, and since this measure is the Lebesgue measure Leb on this set, we have

$$\mathbb{P}_\kappa(n) = \mathsf{Leb}_{2n}(\mathcal{D}_\kappa(n)) = n!\,\mathsf{Leb}_{2n}(\mathcal{C}_\kappa(n)).$$

In what follows, we will abandon $\mathcal{D}_\kappa(n)$ and work mainly in $\mathcal{C}_\kappa(n)$, as the elements of this set are easier to parametrize. The argument we detail for the computation of $\mathsf{Leb}(\mathcal{C}_\kappa(n))$ is mainly deterministic, and we will not really be using random variables in the analysis, even though everything could be rewritten in terms of them (but the proof would then be much more cumbersome).

Notation. From now on, we denote by $\mathbb{Q}_K^{(n)}$ the law of an n-tuple of points with distribution $\mathbb{U}_\kappa^{(n)}$, conditioned to be in $\mathcal{C}_\kappa(n)$, and write for short $\mathbb{Q}_\kappa^{(n)} := \mathbb{Q}_{\mathfrak{C}_\kappa}^{(n)}$; that is,

$$\mathrm{d}\mathbb{Q}_\kappa^{(n)}(z[n]) = \frac{n!}{\mathbb{P}_\kappa(n)} \mathbf{1}_{\{z[n] \in \mathcal{C}_\kappa(n)\}} \mathrm{d}z_1 \dots \mathrm{d}z_n.$$

This formula represents the measure, but it is not amenable to being used in further computations; we will thus need an alternative geometric understanding of $\mathcal{C}_\kappa(n)$, which was inspired by Valtr's papers.

Limit shape. Bárány [2] proved in 1999 that the convex hull of an n-tuple $\mathbf{z}[n]$ with distribution $\mathbb{Q}_\kappa^{(n)}$ converges in probability for the Hausdorff topology to an explicit deterministic domain $\mathsf{Dom}(K)$, which has the important property that

$$\mathrm{AP}^*(K) = \mathrm{AP}(\mathsf{Dom}(K)).$$

https://doi.org/10.1017/apr.2024.63 Published online by Cambridge University Press

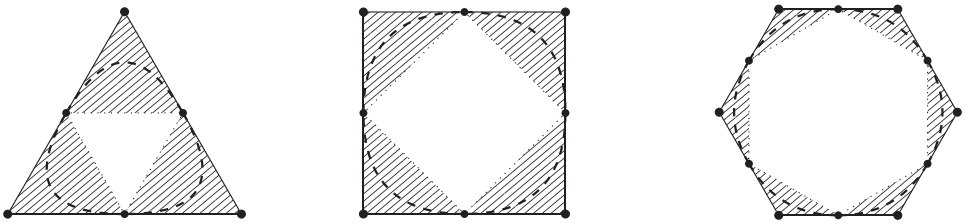


FIGURE 3. For each case $\kappa = 3, 4, 6$, the inner dashed curve delimits a convex domain $\mathsf{Dom}(\mathfrak{C}_\kappa)$ inside $\mathfrak{C}_\kappa$. The dashed curve represents the limit shape of a $\mathbf{z}[n]$-gon taken under $\mathbb{U}_\kappa^{(n)}$, conditioned to be in convex position, as $n \to +\infty$. The curve can be drawn as follows: add the midpoints of the sides of the initial κ-gon, and between two consecutive midpoints, add the arc of the parabola which is tangent to the sides and incident to these inner points. The sum of the hatched areas corresponds to the supremum of affine perimeters (for an explanation see Lemma 17 in the appendix).

In the case of the κ-gon, we represent this domain $\mathsf{Dom}(\mathfrak{C}_\kappa)$ in Figure 3. We will explain in Lemma 16 how $\mathsf{Dom}(\mathfrak{C}_\kappa)$ is determined using the inner symmetries of $\mathfrak{C}_\kappa$.

Denote by d_H the Hausdorff distance on the set of compact subsets of $\mathbb{R}^2$, and for any tuple $z[n] \in (\mathbb{R}^2)^n$, let $\mathsf{conv}(z[n])$ be its convex hull. In the second main contribution of this paper, we detail the fluctuations of the $\mathbf{z}[n]$-gon having distribution $\mathbb{Q}_\kappa^{(n)}$ around its limit $\mathsf{Dom}(\mathfrak{C}_\kappa)$.

Theorem 3. *Let $\kappa \geq 3$ be fixed, and let $\mathbf{z}[n]$ have distribution $\mathbb{Q}_\kappa^{(n)}$. When $n \to +\infty$, we have*

$$n^{1/2} d_H\left(\mathsf{conv}(\mathbf{z}[n]), \mathsf{Dom}(\mathfrak{C}_\kappa)\right) \xrightarrow[n]{(d)} \Delta,$$

where Δ is a non-trivial random variable.

This theorem will turn out to be a consequence of the fluctuations of the $\mathbf{z}[n]$-gon in distribution (at scale $1/\sqrt{n}$ around its limit) in a functional space, as stated in Theorem 8. We refrain from stating the latter theorem at this point, since we would need to introduce too much material to do so; we postpone this work to Section 5.

However, we disclose an element of the proof here: the main idea is to partition each $z[n]$-gon of $\mathcal{C}_\kappa(n)$ into κ suitable convex chains, one per corner of the initial polygon $\mathfrak{C}_\kappa$. Each of the convex chains will be shown to converge separately towards the arc of the parabola associated with the corresponding 'corner' of $\mathfrak{C}_\kappa$, as introduced in Figure 3.

The convergence results stated in Theorem 8 are reminiscent of the limit theorems concerning lattice convex polygons: in this model, an integer n is given, and a convex (lattice) polygon is a convex polygon contained in the square $[-n, n]^2$ and having vertices with integer coordinates (and any number of sides). Vershik asked whether it was possible to determine the number and typical shape of convex lattice polygons contained in $[-n, n]^2$. Three different solutions were brought to light by Bárány [1], Vershik [26], and Sinai [22] in 1994, which we outline below.

A convex lattice polygon can be decomposed naturally into four parts (delimited by the extreme points in the north/east/south/west directions), which determine four 'polygonal convex lines' between them. It is therefore natural to investigate the behavior of these chains, which can be considered, in a first approximation, as convex chains going from $(0, 0)$ to (n, n) in the square $[0, n]^2$ (up to rotations/translations). For these chains, Bárány [1], Vershik [26], and Sinai [22] proved that when $n \to +\infty$,

https://doi.org/10.1017/apr.2024.63 Published online by Cambridge University Press

(1) the number of these convex polygonal lines is $\exp\left(3(\zeta(3)/\zeta(2))^{1/3}n^{2/3}+o(n^{2/3})\right)$, where ζ is the Riemann zeta function,

(2) the random number of vertices in such a chain is concentrated around the quantity $\left(\zeta(3)^2/\zeta(2)\right)^{-1/3}n^{2/3}$, and

(3) the limit shape of such a chain, normalized in both directions by n, is an arc of a parabola.

These results were refined by Bureaux and Enriquez [10] in 2016, and generalized to higher dimensions by Bárány *et al.* [11] in 2018, as well as by Buffière [9] for zonotopes in 2023.

On a related topic, the paper of Bodini *et al.* [7] gives a characterization of digitally convex polyominoes using combinatorics on words.

Random generation of a z[n]-gon with distribution $\mathbb{Q}_\kappa^{(n)}$. The naive way of sampling a $\mathbf{z}[n]$-gon with distribution $\mathbb{Q}_\kappa^{(n)}$ consists in rejection sampling, i.e. sampling points that are $\mathbb{U}_n^{(\kappa)}$-distributed until they are in $\mathcal{C}_\kappa(n)$ (or in $\mathcal{D}_\kappa(n)$). This algorithm works fine for small values of n, but as n grows, computation times become unacceptable (by Theorem 1, the probability of success is less than $\frac{k^n}{n^{2n}}$ for some constant k). In particular, the limit shape theorem proven by Bárány cannot be observed empirically using such a method.

A comprehensive understanding of the distribution $\mathbb{Q}_\kappa^{(n)}$ will allow us to determine another distribution $\mathbb{D}_\kappa^{(n)}$, for which we have an exact sampling algorithm (called κ-sampling and defined in Section 6) that behaves asymptotically like $\mathbb{Q}_\kappa^{(n)}$, meaning that $d_V(\mathbb{Q}_\kappa^{(n)}, \mathbb{D}_\kappa^{(n)}) \underset{n\to+\infty}{\longrightarrow} 0$, where d_V is the total variation distance. The distribution $\mathbb{D}_\kappa^{(n)}$ is defined in Section 3 and can be viewed as $\mathbb{Q}_\kappa^{(n)}$ conditioned to satisfy a property which occurs with probability going to 1. This algorithm is asymptotically exact (in n, for κ fixed), for the $\mathbb{Q}_\kappa^{(n)}$-sampling.

Theorem 4. *The algorithm of κ-sampling samples an n-tuple of points with distribution $\mathbb{D}_\kappa^{(n)}$ with a complexity of $\mathcal{O}\left(n^{\kappa/2+1}\kappa\log(\kappa)\right)$.*

Contents of the paper. In the second section of this paper, we analyze the properties of an n-tuple $z[n]\in\mathcal{C}_\kappa(n)$ in the light of a new geometric description. In Section 3 we derive the distribution of the important variables of this geometric scheme, using which we provide the proof of Theorem 1 in Section 4. Section 5 is dedicated to the proof of Theorem 3 and the understanding of the fluctuations of $z[n]$ around its limit. In Section 6, we provide the aforementioned algorithm of κ-sampling and some alternative (more efficient) versions in the cases $\kappa=3$ and $\kappa=4$. As for the appendices, the first is dedicated to some proofs omitted from the main text, and the second provides a new demonstration of Valtr's formulas in the triangle and the parallelogram.

2. Geometric aspects

Notation. In the sequel, $\kappa\geq 3$ is considered to be fixed. We will work quite a lot with indices j running through the set of integers $\{1,\dots,\kappa\}$. By convention, in the case $j=1$, $j-1$ stands for κ, and when $j=\kappa$, $j+1$ stands for 1 (we do so to avoid tedious notation).

We start by defining the equiangular circumscribed polygon $\mathsf{ECP}(z[n])$ associated to $z[n]\in\mathcal{C}_\kappa(n)$, any n-tuple of points in canonical convex order: as represented (in blue) in Figure 4, this is the polygon equal to the intersection of all equiangular polygons whose sides are parallel, one by one, to those of $\mathfrak{C}_\kappa$, and which contain $z[n]$.

https://doi.org/10.1017/apr.2024.63 Published online by Cambridge University Press

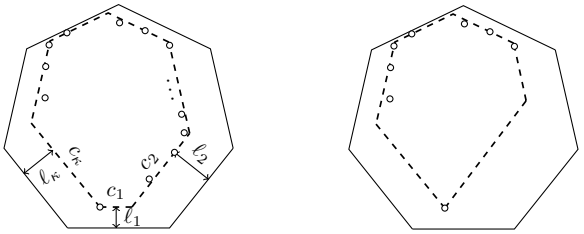


FIGURE 4. On the left, we draw an $\mathsf{ECP}(z[n])$ for an n-tuple taken in $\overset{\curvearrowleft}{\mathcal{C}_7}(n)$, with distances $\ell[7]$ from the sides of $\mathfrak{C}_7$ to those of $\mathsf{ECP}(z[n])$. This latter polygon, whose side lengths are given by the tuple of values $c[7]$, is drawn with a dashed boundary inside $\mathfrak{C}_7$. On the right, a six-sided $\mathsf{ECP}(z[n])$ in $\mathfrak{C}_7$. One of the sides is reduced to a point: this happens when three consecutive values ℓ_{j-1}, ℓ_j, ℓ_{j+1} are defined on the same point z_i in $z[n]$.

We now define some quantities that will allow the description of $z[n]$ in terms of its circumscribed polygon (see also Figure 4).

The distance from the jth side of $\mathfrak{C}_\kappa$ to $z[n]$ is denoted by $\ell_j := \ell_j(z[n])$. The length of the side of $\mathsf{ECP}(z[n])$ parallel to the x-axis is denoted by $c_1 := c_1(z[n])$. Then, the consecutive side lengths of $\mathsf{ECP}(z[n])$, sorted counterclockwise, are denoted by $c_1, c_2, \ldots, c_\kappa$, one or several c_i possibly being zero.

Remark 1. If $\kappa = 3$, the only possible internal polygons within $\mathfrak{C}_3$ are equilateral triangles. If $\kappa = 4$, only rectangles are admitted. In both these cases, an internal polygon within $\mathfrak{C}_\kappa$ has exactly κ sides. This is no longer true for $\kappa \geq 5$, as we can see in the right panel of Figure 4. The number of 'nonzero sides' of $\mathsf{ECP}(z[n])$ is bounded above by κ, and below by 3 (in fact by 4 for the $\kappa = 4$ case; it can technically be 2 if all the points in $z[n]$ are aligned, but we may neglect this case).

Some properties of equiangular circumscribed polygons. A moment's thought allows one to see that $\mathsf{ECP}(z[n])$ is characterized by the κ-tuple of distances $\ell[\kappa] := (\ell_1(z[n]), \cdots, \ell_\kappa(z[n]))$, and that, in turn, $\ell[\kappa]$ determines the side lengths of the ECP, $c[\kappa] := (c_1(z[n]), \cdots, c_\kappa(z[n]))$. In the sequel, since there is no other set of points (except for $z[n]$) for which $\ell[\kappa]$ or $c[\kappa]$ would be defined, we deliberately omit the mention of $z[n]$ when there is no ambiguity.

Proposition 1. *Let $z[n] \in \mathcal{C}_\kappa(n)$, with the corresponding $c[\kappa]$, $\ell[\kappa]$.*

(i) *The vectors $\ell[\kappa]$ and $c[\kappa]$ are related by the κ equations*

$$c_j = r_\kappa - \mathfrak{cl}_j(\ell[\kappa]), \quad \forall j \in \{1, \ldots, \kappa\}, \tag{2.1}$$

where, for all $j \in \{1, \ldots, \kappa\}$, $\mathfrak{cl}_j(\ell[\kappa]) := \big(\ell_{j-1} + \ell_{j+1} + 2\ell_j \cos(\theta_\kappa)\big) / \sin(\theta_\kappa)$ (here $\mathfrak{cl}$ stands for 'linear combination').

(ii) *The set $\mathcal{L}_\kappa = \ell[\kappa]\,(\mathcal{C}_\kappa(n))$ (of all possible vectors $\ell[\kappa]$) is the set of solutions $\ell[\kappa]$ to the system of inequalities*

$$\left\{ \quad \mathfrak{cl}_j(\ell[\kappa]) \leq r_\kappa, \quad \forall j \in \{1, \ldots, \kappa\}, \right. \tag{2.2}$$

together with the conditions $\ell_j \geq 0$, $j \in \{1, \ldots, \kappa\}$.

https://doi.org/10.1017/apr.2024.63 Published online by Cambridge University Press

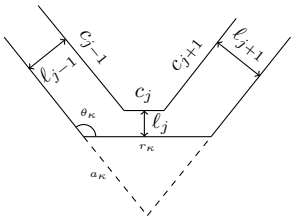


FIGURE 5. Characteristics of an internal polygon.

(iii) *The perimeter of the z[n]-gon satisfies*

$$\sum_{j=1}^{\kappa} c_j + \frac{2(1+\cos(\theta_\kappa))}{\sin(\theta_\kappa)} \sum_{j=1}^{\kappa} \ell_j = \kappa r_\kappa.$$

Proof.

(i) The formulas (2.1) may be deduced from routine computations on the angles and some appropriate applications of Thales's theorem according to Figure 5 below. Indeed, we have

$$\frac{r_\kappa}{\ell_{j-1}/\sin(\theta_\kappa) + c_j + \ell_{j+1}/\sin(\theta_\kappa)} = \frac{a_\kappa}{a_\kappa + \ell_j/\sin(\theta_\kappa)},$$

with (see Figure 5) $a_\kappa = \frac{-r_\kappa}{2\cos(\theta_\kappa)}$.

(ii) It is clear that all elements of $\mathcal{L}_\kappa$ solve the system (2.2). Let $\ell[\kappa]$ be a solution to (2.2). Draw a κ-gon and add a straight line (l_1) at distance ℓ_1 parallel to the first side of $\mathfrak{C}_\kappa$, and another one (l_2) at distance ℓ_2 from $\mathfrak{C}_\kappa$'s second side. The intersection point of these two lines is a vertex b_1 of the ECP. Since $c_2 = r_\kappa - \mathfrak{cl}_2(\ell[\kappa]) \geq 0$, a second vertex b_2 of the ECP is at distance c_2 from b_1 on (l_2). We can draw (l_3) parallel to the third side of $\mathfrak{C}_\kappa$ passing through b_2. With $c_3 = r_\kappa - \mathfrak{cl}_3(\ell[\kappa]) \geq 0$, we can set b_3 as the third vertex of the ECP. Recursively, with all $c_j = r_\kappa - \mathfrak{cl}_j(\ell[\kappa]) \geq 0$, we get all vertices $(\mathsf{b}[\kappa])$ and a full ECP. Hence, each solution of (2.2) is in $\mathcal{L}_\kappa$.

To get (*iii*), just sum all of the equations in (2.1) for all values of j. □

Contact points. For each $z[n]$ in $\mathcal{C}_\kappa(n)$, each side of $\mathsf{ECP}(z[n])$ contains at least one element of $\{z_1, \cdots, z_n\}$. The jth 'contact point' $\mathsf{cp}_j := \mathsf{cp}_j(z[n])$ is the point of $\{z_1, \cdots, z_n\}$ which is on the jth side of $\mathsf{ECP}(z[n])$, and which is the smallest with respect to the lexicographical order among those with this property. Note that we will work with n-tuples $\mathbf{z}[n]$ of random variables, so that when $\mathbf{z}[n]$ is $\mathbb{Q}_\kappa^{(n)}$-distributed, there is a single point of $\{\mathbf{z}_1, \cdots, \mathbf{z}_n\}$ on the jth side of $\mathsf{ECP}(\mathbf{z}[n])$ with probability 1; thus the particular choice of the lexicographical order has no importance. However, $\mathsf{cp}_j = \mathsf{cp}_{j+1}$ is possible and occurs with positive probability for all $n \geq 1$.

Denote by b_j the intersection point between the jth and $(j+1)$th sides of $\mathsf{ECP}(z[n])$ for all $j \in \{1, \ldots, \kappa\}$ (the jth vertex of $\mathsf{ECP}(z[n])$). In the case where the jth side of $\mathsf{ECP}(z[n])$ is reduced to a point, i.e. $c_j = 0$, we have $\mathsf{cp}_{j-1} = \mathsf{b}_{j-1} = \mathsf{cp}_j = \mathsf{b}_j = \mathsf{cp}_{j+1}$.

https://doi.org/10.1017/apr.2024.63 Published online by Cambridge University Press

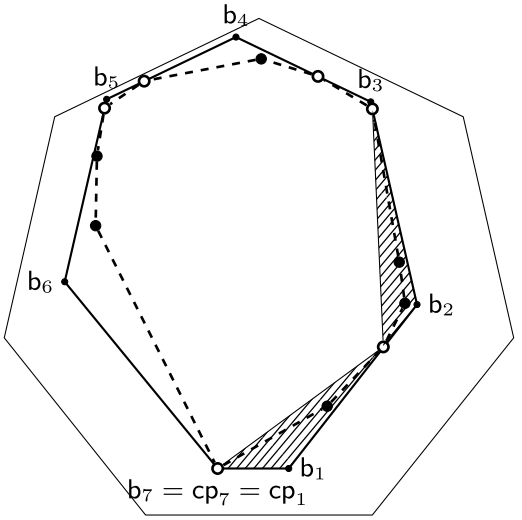


FIGURE 6. In $\mathfrak{C}_7$, an example of a $z[n]$-gon, the $\mathsf{ECP}(z[n])$, and its vertices $\mathsf{b}[7]$, as well as the first and second corners (the hashed areas). Here we have $s[7] = (2, 3, 1, 2, 1, 3, 0)$.

The triangle with vertices cp_j, cp_{j+1}, b_j will be referred to as the jth corner of $\mathsf{ECP}(z[n])$ or $\mathsf{corner}_j(z[n])$ (see Figure 6 below for a summary).

Convex chains between contact points. To get a comprehensive description of $z[n]$ with respect to its circumscribed polygon $\mathsf{ECP}(z[n])$, we need to enrich the decomposition between the contact points. For this purpose, let ABC be the triangle with vertices A,B,C (taken in that order) in the plane. For every integer $m \geq 0$, we denote by $\mathsf{Chain}_m(ABC)$ the set of $(m+1)$-tuples $(A, z'_1, \ldots, z'_{m-1}, B)$ such that $z'_1, \ldots, z'_{m-1}$ are in the triangle ABC, and $(A, z'_1, \ldots, z'_{m-1}, B) \in \overset{\curvearrowleft}{\mathcal{Z}}_{m+1}$. Hence, m is the number of vectors needed to join the points of any convex chain in $\mathsf{Chain}_m(ABC)$. If $A = B$, we define $\mathsf{Chain}_m(ABC)$ only for $m = 0$ as the set reduced to the trivial chain (A, A). We can now decompose the $z[n]$-gon between the contact points as follows.

For all $j \in \{1, \ldots, \kappa\}$, let $k := k(j) \in \{1, \ldots, n\}$ be such that $z_k = \mathsf{cp}_j$, and denote by $s_j := s_j(z[n])$ the integer such that $z_{k+s_j} = \mathsf{cp}_{j+1}$ (eventually $s_j = 0$); the quantity s_j denotes the number of vectors joining the points of the convex chain $(z_k = \mathsf{cp}_j, \ldots, z_{k+s_j} = \mathsf{cp}_{j+1})$. We will refer to the tuple $s[\kappa]$ as the *size vector* (see Figure 6 for an example).

The main technical ingredient of the paper is now tackled in the following structural lemma.

Lemma 1. *For a given $s[\kappa]$ and $j \in \{1, \ldots, \kappa\}$, set $k = \sum_{t<j} s_t$ (so that $\mathsf{cp}_j = z_k$). Given $s[\kappa]$, cp_j, and cp_{j+1}, the set of convex chains $(\mathsf{cp}_j, z_{k+1}, \ldots, z_{k+s_j-1}, \mathsf{cp}_{j+1})$ coincides with the set $\mathsf{Chain}_{s_j}(\mathsf{corner}_j)$.*

Hence, if $\mathbf{z}[n]$ has distribution $\mathbb{Q}_\kappa^{(n)}$, conditional on $(\mathsf{cp}_j, \mathsf{cp}_{j+1}, \mathbf{s}_j = s_j)$, the points in the tuple $(\mathbf{z}_{k+1}, \cdots, \mathbf{z}_{k+s_j-1})$ have the same distribution as that of $s_j - 1$ points $(\mathbf{z}'_1, \ldots, \mathbf{z}'_{s_j-1})$ taken uniformly and independently in the triangle corner_j, conditioned on $(\mathsf{cp}_j, \mathbf{z}'_1, \ldots, \mathbf{z}'_{s_j-1}, \mathsf{cp}_{j+1})$ being in $\overset{\curvearrowleft}{\mathcal{Z}}_{s_j+1}$.

https://doi.org/10.1017/apr.2024.63 Published online by Cambridge University Press

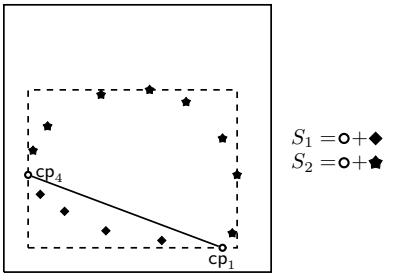


FIGURE 7. An example in the square case, where the ECP is always a rectangle.

Proof. The first statement is equivalent to saying that there are no restrictions on $(z_k, \dots, z_{k+s_j})$ other than those defining $\mathsf{Chain}_{s_j}(\mathsf{corner}_j)$: indeed, it is immediate to check that given two consecutive contact points $z_k = \mathsf{cp}_j$ and $z_{s_j+k} = \mathsf{cp}_{j+1}$, the points of $z[n]$ are in convex position if and only if both subsets S_1 and S_2 of points above and below the straight line joining cp_j and cp_{j+1} (where both S_1 and S_2 contain cp_j and cp_{j+1}) are in convex position. An example is given in Figure 7.

Because of this property, under $\mathbb{Q}_\kappa^{(n)}$, the distribution of $(\mathbf{z}_{k+1}, \cdots, \mathbf{z}_{k+s_j-1})$, conditional on the position of $(\mathsf{cp}_j, \mathsf{cp}_{j+1})$, is the same as that of $(\mathbf{z}_{k+1}, \cdots, \mathbf{z}_{k+s_j-1})$ conditional on the position of all the other points, and it is therefore proportional to the Lebesgue measure on the set of points in convex position in the jth corner (that is, in $\mathsf{Chain}_{s_j}(\mathsf{corner}_j)$), which is equivalent to the second statement of the theorem. □

The law of chain $(A, \mathbf{u}_1, \cdots, \mathbf{u}_k, B)$ conditioned to be in $\mathsf{Chain}_{k+1}(ABC)$ will be called the uniform law in $\mathsf{Chain}_{k+1}(ABC)$.

Denote by $\measuredangle$ the right triangle with vertices (0, 0),(1, 1),(0, 1). For a given non-flat triangle ABC and an integer $m \geq 1$, let Aff_{ABC} be the unique affine map that sends ABC onto $\measuredangle$ (meaning it sends A, B, C to (0, 0),(1, 0),(1, 1), respectively). In the sequel, for $m \geq 0$, we will denote by $\measuredangle\mathsf{CC}_m$ a random variable whose law is uniform in $\mathsf{Chain}_m(\measuredangle)$, and refer to this random variable as a generic $\measuredangle$-normalized convex chain of size m.

From the fundamental property that affine maps preserve convexity, we deduce the following lemma.

Lemma 2.

- *For a triangle ABC (with non-empty interior), and k points* $\mathbf{u}_1, \cdots, \mathbf{u}_k$ *with distribution* $\mathbb{U}_{ABC}^{(k)}$*, the probability that the chain* $(A, \mathbf{u}_1, \cdots, \mathbf{u}_k, B)$ *is in the set* $\mathsf{Chain}_{k+1}(ABC)$ *does not depend on ABC (so that this value is the same as in the right-triangle case).*
- *The map* Aff_{ABC} *sends ABC to* $\measuredangle$*, sends the uniform distribution on ABC to that of* $\measuredangle$ *(as well as* $\mathbb{U}_{ABC}^{(k)}$ *to* $\mathbb{U}_{\measuredangle}^{(k)}$*), and sends the uniform distribution on* $\mathsf{Chain}_m(ABC)$ *to that on* $\mathsf{Chain}_m(\measuredangle)$*.*

The affine map φ. In the following, we will work in the spirit of Lemma 1 by mapping every corner of an ECP to a right triangle. Let $z[n] \in \mathcal{C}_\kappa(n)$, let $c[\kappa]$ be the side lengths of $\mathsf{ECP}(z[n])$, and let $\mathsf{b}[\kappa]$ the vertices of $\mathsf{ECP}(z[n])$. For convenience we impose the condition $s_j > 0$, so as to have $\mathsf{cp}_j \neq \mathsf{cp}_{j+1}$ (the mapping is still definable otherwise).

https://doi.org/10.1017/apr.2024.63 Published online by Cambridge University Press

Let $A'_j = (0, c_j)$, $B'_j = (0, 0)$, $C'_j = (c_{j+1}, 0)$, and define φ_j as the unique affine map that sends b_{j-1}, b_j, b_{j+1} to A'_j, B'_j, C'_j, respectively:

$$A'_j := \varphi_j(\mathsf{b}_{j-1}), \quad B'_j := \varphi_j(\mathsf{b}_j), \quad C'_j := \varphi_j(\mathsf{b}_{j+1}).$$

The map φ_j can be seen as the composition of a rotation of the jth corner so as to place the second side (in the clockwise order) parallel to the x-axis; the straightening of the angle of the triangle thus obtained to produce a right triangle; and a translation (which does not play any role). Therefore, the Jacobian determinant of φ_j is the determinant of the matrix $A_j(\theta_\kappa)$ defined as follows:

$$A_j(\theta_\kappa) := \underbrace{\begin{pmatrix} 1 & \cos(\beta_\kappa) \\ 0 & \sin(\beta_\kappa) \end{pmatrix}^{-1}}_{\text{straightening}} \underbrace{\begin{pmatrix} \cos(-j\beta_\kappa) & \sin(j\beta_\kappa) \\ -\sin(j\beta_\kappa) & \cos(-j\beta_\kappa) \end{pmatrix}}_{\text{rotation}}, \tag{2.3}$$

where $\beta_\kappa = \pi - \theta_\kappa$. The Jacobian determinant of φ_j is thus

$$\mathrm{Jac}\varphi_j = \det\big(A_j(\theta_\kappa)\big) = \frac{1}{\sin(\theta_\kappa)}. \tag{2.4}$$

Encoding convex chains in a triangle by simplex products. For all $\ell \in \mathbb{R}_+$ and $k \in \mathbb{Z}_{>0}$, define the simplex

$$P[\ell, k] = \{(a_1, \ldots, a_k),\ 0 < a_1 < \ldots < a_k < \ell\} \tag{2.5}$$

and the 'reordered' simplex

$$I[\ell, k] = \left\{ (b_1, \ldots, b_k) \text{ where } 0 < b_1 < \ldots < b_k < \ell, \text{ and } \sum_{i=1}^{k} b_i = \ell \right\}. \tag{2.6}$$

An element $(a_1, \ldots, a_k)$ of the set $P[\ell, k]$ encodes k points on the segment $[0, \ell]$, whereas an element $(b_1, \ldots, b_k)$ of $I[\ell, k]$ must be seen as k increasing intervals partitioning the segment $[0, \ell]$.

Nonetheless, the set $I[\ell, k]$ can actually be identified as a subset of $P[\ell, k-1]$ whose increments are increasing. Indeed, if $(a_1, \ldots, a_{k-1})$ is in $P[\ell, k-1]$ and is such that $a_1 < a_2 - a_1 < \ldots < a_{k-1} - a_{k-2} < \ell - a_{k-1}$, then $(a_1, a_2 - a_1, \ldots, a_{k-1} - a_{k-2}, \ell - a_{k-1})$ is in $I[\ell, k]$. We will sometimes make this identification to present some bijections; nevertheless it is important to remember that topologically, $I[\ell, k]$ remains a surface in $\mathbb{R}^k$, a $(k-1)$-dimensional simplex, and that useful bijections in measure theory are those whose Jacobian determinant may be computed.

Note that the Lebesgue measures of these sets are

$$\mathsf{Leb}_k(P[\ell, k]) = \int_{P[\ell,k]} \mathrm{d}a[k] = \frac{\ell^k}{k!},$$

$$\text{and} \quad \mathsf{Leb}_{k-1}(I[\ell, k]) = \int_{I[\ell,k]} \mathrm{d}a[k-1] = \frac{\ell^{k-1}}{k!(k-1)!}, \tag{2.7}$$

where, for any tuple $a[k] = (a_1, \ldots, a_k)$, the notation $\mathrm{d}a[k]$ stands for $\prod_{i=1}^{k} \mathrm{d}a_i$.

https://doi.org/10.1017/apr.2024.63 Published online by Cambridge University Press

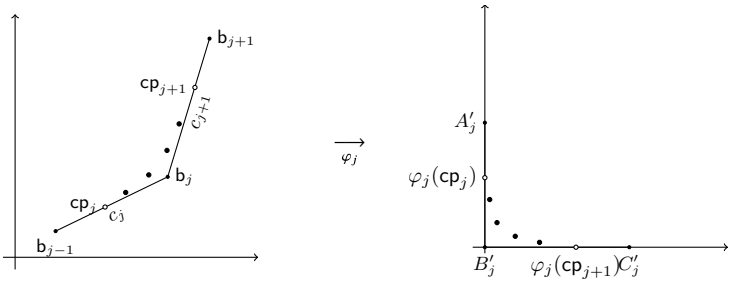


FIGURE 8. The map φ_j.

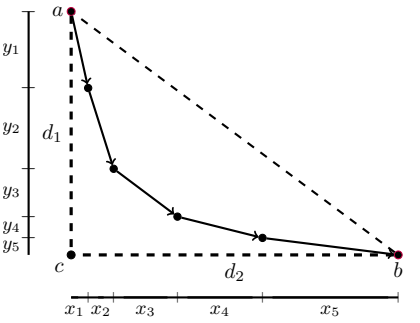


FIGURE 9. A convex chain in a right triangle abc.

An $m!$-to-1 map, piecewise linear, from Chain to a simplex product. Let abc be a right triangle in c of $\mathbb{R}^2$, and let $d_1 = ac$, $d_2 = bc$ denote the distances. For any convex chain $(a, u_1, \cdots, u_{m-1}, b) \in \mathsf{Chain}_m(abc)$, we may consider the vectors $v[m]$ joining the points of the convex chain in their order of appearance. Then let the x- and y-coordinates $x[m]$,$y[m]$ of these vectors be given by $x_i = \pi_1(v_i)$, $y_i = \pi_2(v_i)$ for all $i \in \{1, \ldots, m\}$, and let $(\mathring{x}_1 < \ldots < \mathring{x}_m)$, $(\mathring{y}_1 < \ldots < \mathring{y}_m)$ be the tuples of reordered coordinates (see Figure 9 for an example).

Now consider the following surjective mapping (note that we will be working with vectors that are randomly distributed and such that $\mathbb{P}\left(\exists i \neq j \text{ s.t. } x_i = x_j \text{ or } y_i = y_j\right) = 0$ almost surely (a.s.), which ensures that the map $\mathsf{Order}^{(m)}_{abc}$ is well-defined):

$$\begin{array}{rccc} \mathsf{Order}^{(m)}_{abc}: & \mathsf{Chain}_m(a,b,c) & \longrightarrow & I[d_1,m] \times I[d_2,m] \\ & (a, u_1, \cdots, u_{m-1}, b) & \longmapsto & (\mathring{x}[m], \mathring{y}[m]) \end{array}. \tag{2.8}$$

This map is piecewise linear (see Definition 1 below) and has Jacobian determinant 1 since we are in a right triangle.

Definition 1. *(Piecewise linear map.) A map $g : E \subset \mathbb{R}^n \to \mathbb{R}^n$ is said to be piecewise linear if the following hold:*

- *There exists a collection of polytopes $(P_i)_{i\in\{1,\ldots,m\}}$ such that $\bigcup_{i=1}^m P_i = E$ and the interiors P_i° of the sets $(P_i)_{i\in\{1,\ldots,m\}}$ are pairwise disjoint.*
- *For all $i \in \{1, \ldots, m\}$, $g : P_i^\circ \to \mathbb{R}^n$ is linear.*

https://doi.org/10.1017/apr.2024.63 Published online by Cambridge University Press

Piecewise differentiability may be defined in an analogous way (here the term 'piecewise' must be understood as g being piecewise differentiable on every P_i°).

Of course, 'polytopes' can be replaced by more general Lebesgue-measurable sets, the union of whose interiors would partition E, up to a Lebesgue-negligible set.

Remark 2. Consider the mapping

$$\begin{array}{rccc} g: & \mathbb{R}^3 & \longrightarrow & \mathbb{R}^3 \\ & (x_1, x_2, x_3) & \longmapsto & (x_{(1)}, x_{(2)}, x_{(3)}) \end{array}$$

where $(x_{(1)} \le x_{(2)} \le x_{(3)})$ is the sorted sequence (x_1, x_2, x_3). The map g is clearly not linear; however, for any $(x \neq y \neq z) \in \mathbb{R}^3$, there exists a neighborhood of (x,y,z) on which g is actually linear. At several places in the paper, we use this kind of reordering map, and so we use the term 'piecewise linearity' (and 'piecewise differentiability') in these cases.

Lemma 3. *Let* $(\mathring{x}[m], \mathring{y}[m]) \in I[d_1, m] \times I[d_2, m]$. *We have*

$$\#\left(\mathsf{Order}_{abc}^{(m)}\right)^{(-1)}\left(\mathring{x}[m], \mathring{y}[m]\right) = m!. \tag{2.9}$$

Proof. There are $m!$ distinct ways of pairing every element of $\mathring{x}[m]$ with one of $\mathring{y}[m]$ to form m vectors. There exists a unique order that sorts these vectors by increasing slope. This forms the boundary of a convex chain whose vertices in canonical convex order $(a, u_1, \ldots, u_{m-1}, b)$ are in $\mathsf{Chain}_m(abc)$. □

This lemma allows us to obtain the Lebesgue measure of the set $\mathsf{Chain}_m(a, b, c)$, by carrying the Lebesgue measure of $I[d_1, m] \times I[d_2, m]$ onto $\mathsf{Chain}_m(abc)$. In order to compute the Lebesgue measure of $\mathsf{Chain}_m(abc)$, we need to identify the convex chains with m vectors as a subset of $\mathbb{R}^{2(m-1)}$ (so that its dimension is $2(m-1)$, and appears as such). Therefore we introduce $\mathsf{Chain}'_m(a, b, c) = \{(z_1, \cdots, z_{m-1}) : (a, z_1, \cdots, z_{m-1}, b) \in \mathsf{Chain}_m(abc)\}$. By a change of variables we have

$$\begin{aligned} \mathsf{Leb}_{2(m-1)}\left(\mathsf{Chain}'_m(abc)\right) &= \int_{(\mathbb{R}^2)^{m-1}} \mathbf{1}_{\{\mathsf{Chain}'_m(abc)\}} \mathrm{d}z[m-1] \\ &= m!(m-1)! \cdot \mathsf{Leb}_{m-1}(I[d_1, m]) \cdot \mathsf{Leb}_{m-1}(I[d_2, m]) \\ &= \frac{(d_1 d_2)^{m-1}}{m!(m-1)!}. \end{aligned} \tag{2.10}$$

Note that the term in $(m-1)!$ on the second line accounts for the relabeling of the points $(u_1, \cdots, u_{m-1})$, and $m!$ appears because of Lemma 3.

Intuition. For $\mathbf{z}[n]$ with distribution $\mathbb{Q}_\kappa^{(n)}$, these lemmas reveal that conditional on the position of $\mathsf{ECP}(\mathbf{z}[n])$, $\mathsf{cp}[\kappa](\mathbf{z}[n])$, and $s[\kappa](\mathbf{z}[n])$ (all together), the convex chains in each corner are independent. Thus each corner can be considered separately, and by mapping the jth corner of $\mathsf{ECP}(\mathbf{z}[n])$ to A'_j, B'_j, C'_j with φ_j (see (2.3)), we are brought back to the (simpler) study of a convex chain in a right triangle. However, although this big picture is useful for understanding the limit shape theorem, it is unfortunately not sufficient for computing the full asymptotic expansion of $\mathbb{P}_\kappa(n)$, mainly because of the fact that the joint distribution of

https://doi.org/10.1017/apr.2024.63 Published online by Cambridge University Press

$(\ell[\kappa](\mathbf{z}[n]), s[\kappa](\mathbf{z}[n]), \mathsf{cp}[\kappa](\mathbf{z}[n]))$ is intricate and needs to be understood. Hence we need to introduce some more tools to work with the joint distribution.

Number of sides of $\mathsf{ECP}(z[n])$. For $z[n] \in \mathcal{C}_\kappa(n)$ and the corresponding $c[\kappa]$, define the map NZS as follows:

$$\begin{array}{rccc} \mathrm{NZS}: & \mathcal{C}_\kappa(n) & \longrightarrow & \mathcal{P}(\{1,\dots,\kappa\}) \\ & z[n] & \longmapsto & \{i; c_i \neq 0\} \end{array}. \tag{2.11}$$

This map records the indices corresponding to the nonzero sides of $\mathsf{ECP}(z[n])$. Let us also set

$$\mathbb{N}_\kappa(n) = \{s[\kappa] \in \mathbb{N} \text{ such that } s_1 + \ldots + s_\kappa = n \text{ and } s_{j-1} + s_j \neq 0 \text{ for all } j \in \{1,\dots,\kappa\}\},$$

and define

$$\mathcal{C}_\kappa(\mathbb{N}_\kappa(n)) := \{z[n] \in \mathcal{C}_\kappa(n) \text{ such that } s[\kappa](z[n]) \in \mathbb{N}_\kappa(n)\}.$$

The following proposition states an equivalent condition on $\mathbf{s}^{(n)}[\kappa]$ to ensure a 'full-sided' ECP.

Proposition 2. *Let* $\mathbf{z}[n]$ *have distribution* $\mathbb{Q}_\kappa^{(n)}$. *Then* $\mathrm{NZS}(\mathbf{z}[n]) = \{1,\dots,\kappa\}$ *is equivalent to* $s[\kappa](\mathbf{z}[n]) \in \mathbb{N}_\kappa(n)$.

Proof. Suppose that the $\mathsf{ECP}(z[n])$ has exactly κ nonzero sides, i.e. that if $\mathbf{c}[\kappa] = c[\kappa](\mathbf{z}[n])$, then we have $\mathbf{c}_j > 0$ for all $j \in \{1,\dots,\kappa\}$. Inside the tuple $\mathbf{z}[n]$, consider for all $j \in \{1,\dots,\kappa\}$ the contact points cp_{j-1}, cp_j, and cp_{j+1}. A small picture suffices to show that we cannot have $\mathsf{cp}_{j-1} = \mathsf{cp}_j = \mathsf{cp}_{j+1}$ for this is equivalent to $\mathbf{c}_j = 0$, and thus is also equivalent to the fact that there exists a nonzero vector leading either cp_{j-1} to cp_j (i.e. $\mathbf{s}_{j-1} \geq 1$), or cp_j to cp_{j+1} (i.e. $\mathbf{s}_j \geq 1$). □

Therefore the set $\mathcal{C}_\kappa(\mathbb{N}_\kappa(n))$ admits another equivalent definition:

$$\mathcal{C}_\kappa(\mathbb{N}_\kappa(n)) := \{z[n] \in \mathcal{C}_\kappa(n) \text{ such that } \mathrm{NZS}(z[n]) = \{1,\dots,\kappa\}\}.$$

The following lemma ensures that the overwhelming mass of n-tuples $z[n] \in \mathcal{C}_\kappa(n)$ is actually contained in $\mathcal{C}_\kappa(\mathbb{N}_\kappa(n))$.

Lemma 4. *Let* $\mathbf{z}[n]$ *have distribution* $\mathbb{U}_\kappa^{(n)}$. *Denote by* $\widetilde{\mathbb{P}}_\kappa(n) := n!\ \mathbb{P}(\mathbf{z}[n] \in \mathcal{C}_\kappa(\mathbb{N}_\kappa(n)))$ *the probability that the* $\mathbf{z}[n]$ *are in convex canonical order and additionally that their* ECP *has* κ *nonzero sides. We have*

$$\mathbb{P}_\kappa(n) \underset{n\to+\infty}{\sim} \widetilde{\mathbb{P}}_\kappa(n).$$

The proof of this result requires several arguments related to Bárány's limit shape theorem, so we send the interested reader to Appendix A for a complete overview of the proof.

Remark 3. Lemma 4 is of paramount importance since it allows us to neglect a subset of $\mathcal{C}_\kappa(n)$ whose Lebesgue measure becomes insignificant relative to that of $\mathcal{C}_\kappa(n)$ as $n \to +\infty$. To do so, we will assume that all n-tuples of points $z[n]$ we are working with are in $\mathcal{C}_\kappa(\mathbb{N}_\kappa(n))$, so as to force—by Proposition 2—the number of nonzero sides of $\mathsf{ECP}(z[n])$ to be κ.

Notation. Denote by $\mathbb{D}_\kappa^{(n)}$ the distribution of an n-tuple of random points $\mathbf{z}[n]$ with distribution $\mathbb{U}_\kappa^{(n)}$, conditioned to be in $\mathcal{C}_\kappa(\mathbb{N}_\kappa(n))$.

https://doi.org/10.1017/apr.2024.63 Published online by Cambridge University Press

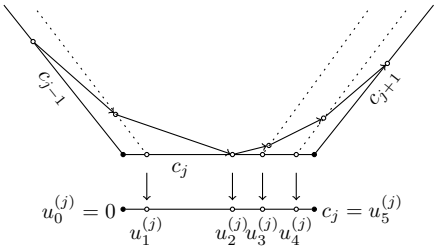


FIGURE 10. The jth side-partition $(0 = u_0^{(j)} < u_1^{(j)} < \ldots < u_{N_j}^{(j)} < u_{N_j+1}^{(j)} = c_j)$ of c_j, with $s_j = 2$, $s_{j+1} = 3$. An alternative way of building the $u^{(j)}[N_j, c_j]$ will be given in Figure 11. Notice here that we see the contact point on c_j, but we do not mark it; we treat it the same as the other points.

3. Distribution of a convex $z[n]$-gon

Notation. From now on, we will work with a fixed size vector $s[\kappa] \in \mathbb{N}_\kappa(n)$. We denote by $\mathcal{C}_\kappa(s[\kappa])$ the subset of all $z[n] \in \mathcal{C}_\kappa(\mathbb{N}_\kappa(n))$ such that $s[\kappa](z[n]) = s[\kappa]$, i.e. the set of n-tuples $z[n] \in \mathcal{C}_\kappa(n)$ with a prescribed size vector $s[\kappa]$. We will write $N_j = s_j + s_{j+1} - 1$ for all $j \in \{1, \ldots, \kappa\}$.

The choice to work with a prescribed size vector is not only a technical tool: as a matter of fact, our analysis relies deeply on the computation of the distribution of the size vector, and then on the description of the chains with a prescribed size vector (a foretaste has been given in Lemma 1, for instance). Later in the paper, we will see that the fluctuations of the $z[n]$-gon in each corner depend also on the fluctuations of the vector $s[\kappa]$, so that considerations of this kind cannot be avoided.

3.1. Encoding $z[n]$-gons into side-partitions of $\mathsf{ECP}(z[n])$

A new geometric description: convex chains between contact points, convex chains in a right triangle, and simplex product. Let us now fix $s[\kappa] \in \mathbb{N}_\kappa(n)$. For all $z[n] \in \mathcal{C}_\kappa(s[\kappa])$, consider the corresponding side lengths $c[\kappa]$ (which are thus all nonzero), and for all $j \in \{1, \ldots, \kappa\}$, define the *side-partition* $u^{(j)}[N_j, c_j] = (u_1^{(j)}, \ldots, u_{N_j}^{(j)})$ of the jth side length c_j of $\mathsf{ECP}(z[n])$, which is defined in Figure 10 below and is an element of $P[c_j, N_j]$. For any side-partition $u^{(j)}[N_j, c_j]$ thus defined, we set $u_0^{(j)} := 0$, $u_{N_j+1}^{(j)} = c_j$, so that we have $u_0^{(j)} < u_1^{(j)} < \ldots < u_{N_j}^{(j)} < u_{N_j+1}^{(j)}$.

Main strategy of the proof. Our main strategy is to consider for all $s[\kappa] \in \mathbb{N}_\kappa(n)$ the extraction mapping, which encodes a convex $z[n]$-gon in terms of its $\mathsf{ECP}(z[n])$ and its side-partitions:

$$\begin{array}{rccl} \chi_{s[\kappa]} : & \mathcal{C}_\kappa(s[\kappa]) & \longrightarrow & \mathsf{NiceSet}(s[\kappa]) \\ & z[n] & \longmapsto & \left(\ell[\kappa], u^{(1)}[N_1, c_1], \ldots, u^{(\kappa)}[N_\kappa, c_\kappa]\right) \end{array}, \tag{3.1}$$

where $\mathsf{NiceSet}(s[\kappa]) := \mathsf{Im}(\chi_{s[\kappa]})$ is a strict subset of $(\mathbb{R}^+)^\kappa \times \prod_{j=1}^\kappa (\mathbb{R}^+)^{N_j}$ that we now discuss. Recall that for all $j \in \{1, \ldots, \kappa\}$ we have set $N_j = s_j + s_{j+1} - 1$.

In what follows, we need to see the map $\chi_{s[\kappa]}$ as a 'nice map' (a piecewise linear map; see Definition 1) with a 'nice inverse' (i.e. with a computable Jacobian determinant), since we will later use this inverse to push forward a measure of $\mathsf{NiceSet}(s[\kappa])$ onto the Lebesgue measure on $\mathcal{C}_\kappa(s[\kappa])$.

https://doi.org/10.1017/apr.2024.63 Published online by Cambridge University Press

Since $\mathcal{C}_\kappa(s[\kappa])$ is a subset of $\mathbb{R}^{2n}$ with non-empty interior, $\mathsf{NiceSet}(s[\kappa])$ will be seen to be identifiable with a subset of a domain with the same dimension. In order to characterize $\mathsf{NiceSet}(s[\kappa])$, it is relevant to notice that since the $s[\kappa]$ are fixed, the $u^{(j)}[N_j, c_j]$ allow us to reconstruct the vectors of the convex chains. Since these vectors have increasing slope (as we progress counterclockwise around the $z[n]$-gon), the $u^{(j)}[N_j, c_j]$ must satisfy a condition that we now detail.

The image set of $\chi_{s[\kappa]}$. Set $\mathcal{L}^*_\kappa := \{\ell[\kappa] \in \mathcal{L}_\kappa \text{ s.t. for all } j \in \{1, \dots, \kappa\},\ c_j > 0\}$. For any $\ell[\kappa] \in \mathcal{L}^*_\kappa$, consider the side lengths $c[\kappa]$ of the ECP induced by $\ell[\kappa]$. For any κ-tuple of side-partitions $\left(u^{(1)}[N_1, c_1], \dots, u^{(\kappa)}[N_\kappa, c_\kappa]\right)$ of $c[\kappa]$, and all $j \in \{1, \dots, \kappa\}$, define the inter-point distances of the side-partition $u^{(j)}[N_j, c_j]$ by $\Delta u^{(j)}_i = u^{(j)}_i - u^{(j)}_{i-1}$ for all $i \in \{1, \dots, N_j + 1\}$. Then define the vectors

$$v^{(j)}_k = \begin{pmatrix} \Delta u^{(j)}_{s_j+k} \\ \Delta u^{(j+1)}_k \end{pmatrix}, \quad \forall k \in \{1, \dots, s_{j+1}\}.$$

In words, summing the vectors $v^{(j)}[s_{j+1}]$ allows one to join the point $(0, 0)$ to $(c_j - u^{(j)}_{s_j}, u^{(j+1)}_{s_{j+1}})$. When reordered by increasing slope, these vectors form the boundary of a convex polygon whose vertices form a convex chain. This condition on the vectors must be encoded in the side-partitions when we decompose a $z[n]$-gon through $\chi_{s[\kappa]}$; this condition allows us to identify the image set $\mathsf{NiceSet}(s[\kappa])$.

We therefore define the following open subset of $\mathbb{R}^\kappa \times \prod_{j=1}^\kappa \mathbb{R}^{N_j}$:

$$\mathcal{S}^{(n)}(s[\kappa]) := \left\{ \left(\ell[\kappa], w^{(1)}, \dots, w^{(\kappa)}\right) \in \mathcal{L}^*_\kappa \times \prod_{j=1}^\kappa \mathbb{R}^{N_j} \right.$$
$$\text{where } w^{(j)} := w^{(j)}[N_j, c_j] \in P[c_j, N_j],$$
$$\left. \text{and } \underbrace{\frac{\Delta w^{(j)}_1}{\Delta w^{(j-1)}_{s_{j-1}+1}} < \dots < \frac{\Delta w^{(j)}_{s_j}}{\Delta w^{(j-1)}_{s_{j-1}+s_j}}}_{\text{condition on order of slopes}} \text{ for all } j \in \{1, \dots, \kappa\} \right\}.$$

Note that we set $\ell[\kappa]$ in $\mathcal{L}^*_\kappa$ so as to force the construction of any ECP possible (except those having a nonzero side) within $\mathfrak{C}_\kappa$. We consider the increments for the side-partitions because they make up the vectors in each corner as described by Figure 11. Recall the family of mappings $(\varphi_j)_{j\in\{1,\dots,\kappa\}}$ introduced in (2.3) together with Figure 8.

A powerful diffeomorphism. It is quite easy to see that, up to a Lebesgue-null set (we want to avoid treating separately the cases in which several points of $z[n]$ are parallel to the lines of $\mathfrak{C}_\kappa$, or more than two z_i are aligned), $\chi_{s[\kappa]}$ is a bijection between $\mathcal{C}_\kappa(s[\kappa])$ and $\mathcal{S}^{(n)}(s[\kappa])$. The following theorem details some even more important properties of the mapping $\chi_{s[\kappa]}$.

Theorem 5. *For all $s[\kappa] \in \mathbb{N}_\kappa(n)$, the mapping*

$$\begin{array}{rccc} \chi_{s[\kappa]}: & \mathcal{C}_\kappa(s[\kappa]) & \longrightarrow & \mathcal{S}^{(n)}(s[\kappa]) \\ & z[n] & \longmapsto & \chi_{s[\kappa]}(z[n]) \end{array} \tag{3.2}$$

is a piecewise diffeomorphism (in the sense of Definition 1) whose Jacobian determinant is constant and equals $1/\sin(\theta_\kappa)^{n-\kappa}$ (hence the Jacobian determinant does not depend on $s[\kappa]$).

https://doi.org/10.1017/apr.2024.63 Published online by Cambridge University Press

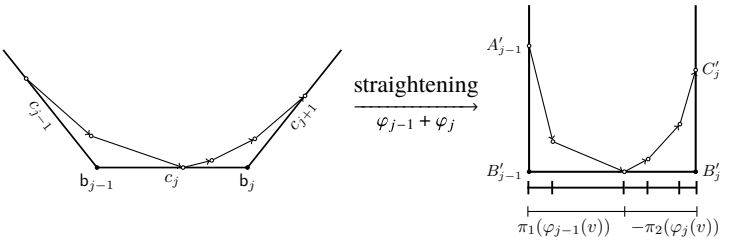


FIGURE 11. The map φ_j (resp. φ_{j-1}), as introduced in Figure 8, sends the triangle corner_j (resp. corner_{j-1}) to the triangle $A'_jB'_jC'_j$ (resp. $A'_{j-1}B'_{j-1}C'_{j-1}$). If we perform one more rotation, which is equivalent to setting $C'_{j-1}=A'_j$ and fixing B'_{j-1}, A'_j, B'_j on the same line, we may interpret the side-partitions just as they appear in the right-hand panel.

In particular, the Lebesgue measure of the set of interest, $\mathcal{C}_\kappa(s[\kappa])$, satisfies

$$\mathsf{Leb}_{2n}\left(\mathcal{C}_\kappa(s[\kappa])\right)=\mathsf{Leb}_{2n}\left(\mathcal{S}^{(n)}(s[\kappa])\right)\sin\left(\theta_\kappa\right)^{n-\kappa}. \tag{3.3}$$

Proof of Theorem 5.We need to detail how the inverse mapping of $\chi_{s[\kappa]}$ is defined to understand its (piecewise) linearity. Pick $\left(\ell[\kappa], u^{(1)}[N_1, c_1], \ldots, u^{(\kappa)}[N_\kappa, c_\kappa]\right) \in \mathcal{S}^{(n)}(s[\kappa])$.

Linearity. Since the tuple $\ell[\kappa]$ is in $\mathcal{L}^*_\kappa$, it defines an equiangular parallel polygon ECP inside $\mathfrak{C}_\kappa$. The map which associates the $\mathsf{b}[\kappa]$ to the $\ell[\kappa]$ is piecewise linear: in the classical Cartesian coordinate system, for any $j\in\{1,\ldots,\kappa\}$, the coordinates of b_j are linear in ℓ_j and ℓ_{j+1}, since

$$\mathsf{b}_j=\left(r_{j-1}+\frac{\ell_j}{\tan\left(\theta_\kappa\right)}-\frac{\ell_{j+1}}{\sin\left(\theta_\kappa\right)},\ell_j\right),$$

up to a rotation.

Then the contact point cp_j is a translation of b_{j-1} by $u^{(j)}_{s_j}$ along the jth side of the ECP. This means that the constructions of the contact points are linear in the $\ell[\kappa]$ and $u^{(j)}_{s_j}, j\in\{1,\ldots,\kappa\}$. To reconstruct the rest of the points, recall the vectors

$$v^{(j)}_k=\begin{pmatrix}\Delta u^{(j)}_{s_j+k}\\ \Delta u^{(j+1)}_k\end{pmatrix},\quad \forall k\in\{1,\ldots,s_{j+1}-1\}.$$

The convexity condition imposed on the slopes in $\mathcal{S}^{(n)}(s[\kappa])$ forces these vectors to appear in order of increasing slope, so that the map φ_j sends these vectors in corner_j to form the boundary of a convex polygon, whose tuple of vertices is thus a convex chain. The construction of the points of this convex chain can hence be rewritten as

$$z^{(j)}_2=\mathsf{cp}_j+A_j(\theta_\kappa)^{-1}v^{(j)}_1,$$

where A_j was introduced in (2.3), and inductively for all $k\in\{2,\ldots,s_{j+1}-1\}$,

$$z^{(j)}_{k+1}=z^{(j)}_k+A_j(\theta_\kappa)^{-1}v^{(j)}_k.$$

https://doi.org/10.1017/apr.2024.63 Published online by Cambridge University Press

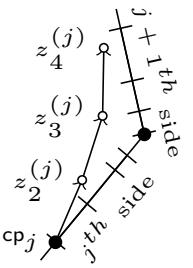


FIGURE 12. Vector-building.

We give an example of this construction in Figure 5. Notice that we have built only $s_{j+1}-1$ vectors, since the s_{j+1}th connects the last point $z_{s_j}^{(j)}$ to cp_{j+1} and is thus determined.

We obtain n points $(z_1, \dots, z_n) = (\underbrace{\mathsf{cp}_1, z_2^{(1)}, \dots, z_{s_1}^{(1)}}_{s_1 \text{ points}}, \underbrace{\mathsf{cp}_2, z_2^{(2)}, \dots, z_{s_2}^{(2)}}_{s_2 \text{ points}}, \dots, \underbrace{\mathsf{cp}_\kappa, z_2^{(\kappa)}, \dots, z_{s_\kappa}^{(\kappa)}}_{s_\kappa \text{ points}})$. In the end, the whole construction includes only maps that are piecewise linear and piecewise differentiable (Definition 1) in the data $\big(\ell[\kappa], u^{(1)}[N_1, c_1], \dots, u^{(\kappa)}[N_\kappa, c_\kappa]\big)$, and thus $\chi_{s[\kappa]}$ also has these properties.

Jacobian. Let us compute the Jacobian determinant of the inverse mapping $(\chi_{s[\kappa]})^{-1}$. This requires first the Jacobian determinant of the construction of the contact points $\mathsf{cp}[\kappa]$. To build a contact point, we build the vertices $\mathsf{b}[\kappa]$: we fix the y-coordinate of b_κ and b_1 as ℓ_1. Now, rotate the figure by $\pi/2 - \theta_\kappa$: in this new system of coordinates, the y-coordinate of b_1 and b_2 is ℓ_2. This determines the coordinates of b_1, and from one rotation to the other, those of b_j for all $j \in \{1, \dots, \kappa\}$. The Jacobian determinant of the whole construction of the $\mathsf{b}[\kappa]$ is the determinant of a product of rotation matrices, and is thus 1.

Then, as said before, the contact point cp_j is built as a translation of $u_{s_j}^{(j)}$ from b_{j-1} on the jth side of ECP. This operation has Jacobian determinant 1 as well.

For $j \in \{1, \dots, \kappa\}$, the building of $z_k^{(j)}$, $k \in \{2, \dots, s_j\}$, is a translation from $z_{k-1}^{(j)}$ with the product of the matrix $A_j(\theta_\kappa)^{-1}$ with the vector $v_{k-1}^{(j)}$, for all $j \in \{1, \dots, \kappa\}$. So we have

$$\begin{aligned} \operatorname{Jac}\left((\chi_{s[\kappa]})^{-1}\right) &= \left|\prod_{j=1}^{\kappa} \det\left(A_j(\theta_\kappa)^{-1}\right)^{s_j-1}\right| \\ &= \sin(\theta_\kappa)^{n-\kappa}. \end{aligned} \tag{3.4}$$

□

3.2. Working at fixed $\ell[\kappa]$

Above, we performed a first 'conditioning' based on the size vector $s[\kappa]$ of the vectors forming the boundary of any $z[n]$-gon. From this point, the map $\chi_{s[\kappa]}$ encodes $z[n]$ in two parts: the 'coordinates' $\ell[\kappa]$ of the $\mathsf{ECP}(z[n])$ (in the sense that their data is equivalent) and the side-partitions $\big(u^{(1)}[N_1, c_1], \dots, u^{(\kappa)}[N_\kappa, c_\kappa]\big)$. We may now perform a second conditioning on the coordinates $\ell[\kappa]$, by introducing the set

https://doi.org/10.1017/apr.2024.63 Published online by Cambridge University Press

$$\mathcal{S}^{(n)}(\ell[\kappa], s[\kappa]) = \left\{ \left(w^{(1)}, \dots, w^{(\kappa)}\right) \text{ such that } \left(\ell[\kappa], w^{(1)}, \dots, w^{(\kappa)}\right) \in \mathcal{S}^{(n)}(s[\kappa]) \right\}. \quad (3.5)$$

This conditioning actually reveals the mass of $z[n]$-gons contained in an ECP of coordinates $\ell[\kappa]$ with a repartition $s[\kappa]$. Indeed, we have the following lemma.

Lemma 5. *For all $\ell[\kappa] \in \mathcal{L}_\kappa$, $s[\kappa] \in \mathbb{N}_\kappa(n)$,*

$$\mathsf{Leb}_{2n}(\mathcal{C}_\kappa(s[\kappa])) = \sin(\theta_\kappa)^{n-\kappa} \int_{\mathbb{R}^\kappa} \mathbf{1}_{\{\ell[\kappa] \in \mathcal{L}_\kappa\}} \mathsf{Leb}_{2n-\kappa}\left(\mathcal{S}^{(n)}(\ell[\kappa], s[\kappa])\right) \mathrm{d}\ell[\kappa]. \quad (3.6)$$

Proof. We have

$$\mathsf{Leb}_{2n}(\mathcal{S}^{(n)}(s[\kappa])) = \int_{\mathbb{R}^\kappa} \mathbf{1}_{\{\ell[\kappa] \in \mathcal{L}_\kappa\}} \underbrace{\int_{\mathbb{R}^{2n-\kappa}} \mathbf{1}_{\{(u^{(1)}[N_1, c_1], \dots, u^{(\kappa)}[N_\kappa, c_\kappa]) \in \mathcal{S}^{(n)}(\ell[\kappa], s[\kappa])\}} \mathrm{d}u[2n-\kappa]}_{\mathsf{Leb}_{2n-\kappa}\left(\mathcal{S}^{(n)}(\ell[\kappa], s[\kappa])\right)} \mathrm{d}\ell[\kappa], \quad (3.7)$$

where we let $\mathrm{d}u[2n-\kappa] = \prod_{j=1}^{\kappa} \mathrm{d}u^{(j)}[N_j, c_j]$ to lighten the notation. Hence, (3.3) allows us to conclude. □

This lemma encodes an n-tuple $z[n]$ in convex position in terms of a new geometric description embodied in the coordinates $\left(\ell[\kappa], u^{(1)}[N_1, c_1], \dots, u^{(\kappa)}[N_\kappa, c_\kappa]\right)$. This change of variables comes at the price of the Jacobian computed in Theorem 5. The next step, as suggested by Lemma 5, is to compute, for fixed $(\ell[\kappa], s[\kappa])$, the Lebesgue measure of the set $\mathcal{S}^{(n)}(\ell[\kappa], s[\kappa])$.

The Lebesgue measure of $\mathcal{S}^{(n)}(\ell[\kappa], s[\kappa])$. Pick $\ell[\kappa] \in \mathcal{L}_\kappa$ and $\left(u^{(1)}, \dots, u^{(\kappa)}\right) \in \mathcal{S}^{(n)}(\ell[\kappa], s[\kappa])$. This tuple of side-partitions $\left(u^{(1)}, \dots, u^{(\kappa)}\right)$ can be seen as an element of the set $\prod_{j=1}^{\kappa} P[c_j, N_j]$. Indeed, a side-partition $u^{(j)} := u^{(j)}[N_j, c_j]$ marks N_j points on the segment $[0, c_j]$. Nonetheless, just as we did after (2.5) and (2.6), we may instead consider the tuples of distances between points, and reorder each $u^{(j)}$ into increasing increments so as to form $(\Delta\widetilde{u}^{(1)}[N_1+1], \dots, \Delta\widetilde{u}^{(\kappa)}[N_\kappa+1])$, which is thus an element of $\prod_{j=1}^{\kappa} I[c_j, N_j+1]$. Considering the elements of $\prod_{j=1}^{\kappa} I[c_j, N_j+1]$ rather than those of $\prod_{j=1}^{\kappa} P[c_j, N_j]$ prevents us from forming the same convex chain twice. Next we define

$$\begin{array}{rccc} \mathsf{Order}_{\ell[\kappa], s[\kappa]}: & \mathcal{S}^{(n)}(\ell[\kappa], s[\kappa]) & \longrightarrow & \prod_{j=1}^{\kappa} I[c_j, N_j+1] \\ & \left(u^{(1)}, \dots, u^{(\kappa)}\right) & \longmapsto & (\Delta\widetilde{u}^{(1)}[N_1+1], \dots, \Delta\widetilde{u}^{(\kappa)}[N_\kappa+1])' \end{array}$$

a piecewise linear mapping. Given $(\Delta\widetilde{u}^{(1)}[N_1+1], \dots, \Delta\widetilde{u}^{(\kappa)}[N_\kappa+1]) \in \prod_{j=1}^{\kappa} I[c_j, N_j+1]$, how many distinct n-tuples $\left(u^{(1)}, \dots, u^{(\kappa)}\right) \in \mathcal{S}^{(n)}(\ell[\kappa], s[\kappa])$ can we build out of this object? We answer this question in the following lemma.

Lemma 6. *Let $s[\kappa] \in \mathbb{N}_\kappa(n)$, $\ell[\kappa] \in \mathcal{L}_\kappa$, with the corresponding $c[\kappa]$. Consider a tuple $(\Delta\widetilde{u}^{(1)}[N_1+1], \dots, \Delta\widetilde{u}^{(\kappa)}[N_\kappa+1]) \in \prod_{j=1}^{\kappa} I[c_j, N_j+1]$. Then*

$$\#\mathsf{Order}^{-1}_{\ell[\kappa], s[\kappa]}\left(\Delta\widetilde{u}^{(1)}[N_1+1], \dots, \Delta\widetilde{u}^{(\kappa)}[N_\kappa+1]\right) = \prod_{j=1}^{\kappa} \binom{s_j + s_{j+1}}{s_j} s_j!. \quad (3.8)$$

https://doi.org/10.1017/apr.2024.63 Published online by Cambridge University Press

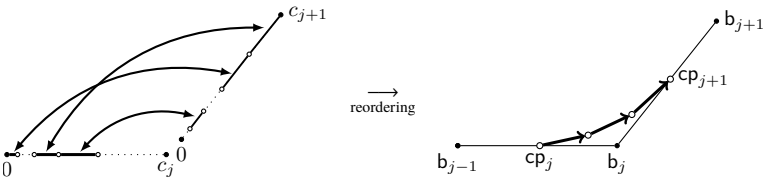


FIGURE 13. In the first drawing, given a partition in $I[c_j, s_{j-1}+s_j]$ and a partition in $I[c_{j+1}, s_j+s_{j+1}]$, we randomly pair s_j pieces of c_j with s_j pieces of c_{j+1} to form the vectors in the jth corner. Note that an affine transformation is hiding in the construction of these vectors. In the second drawing, vectors have been reordered by increasing slope. The points cp_j, cp_{j+1} naturally appear as the edges of the convex chain formed by those vectors. In these particular drawings, we took $s_j = 3$, $s_{j-1} = 2$, $s_{j+1} = 2$.

Proof. We need to build κ sets of vectors, the jth being devoted to the construction of the convex chain in the jth corner of the ECP. To form the s_j vectors in the jth corner, we select s_j pieces in $\Delta\widetilde{u}^{(j)}[N_j+1]$ that will account for the x-contributions of the vectors, and we select s_j pieces (or, complementarily, s_{j+1} pieces) in $\Delta\widetilde{u}^{(j+1)}[N_{j+1}+1]$ that will account for the y-contributions. There are $\prod_{j=1}^{\kappa}\binom{s_j+s_{j+1}}{s_j}$ ways of choosing these pieces, and $\prod_{j=1}^{\kappa} s_j!$ ways to pair these elements to form the s_j vectors in each corner (see Figure 13 for an example of the construction).

There exists a unique order that sorts these vectors into convex order in each corner, so that, put together, these pieces form a convex polygon whose set of vertices is a 'distinct' n-tuple $z[n] \in \mathcal{C}_\kappa(s[\kappa])$ with $\ell[\kappa](z[n]) = \ell[\kappa]$. Now, consider $\chi_{s[\kappa]}(z[n]) = \big(\ell[\kappa], u^{(1)}[N_1, c_1], \dots, u^{(\kappa)}[N_\kappa, c_\kappa]\big)$: the last entries $\big(u^{(1)}, \dots, u^{(\kappa)}\big) := \big(u^{(1)}[N_1, c_1], \dots, u^{(\kappa)}[N_\kappa, c_\kappa]\big)$ of this tuple form a new distinct element (since $z[n]$ is one as well) of $\mathcal{S}^{(n)}(\ell[\kappa], s[\kappa])$. □

This allows us to compute the Lebesgue measure of $\mathcal{S}^{(n)}(\ell[\kappa], s[\kappa])$. Indeed, the map $\mathsf{Order}_{\ell[\kappa],s[\kappa]}$ carries the Lebesgue measure of $\mathcal{S}^{(n)}(\ell[\kappa], s[\kappa])$ onto that of $\prod_{j=1}^{\kappa} I[c_j, N_j+1]$.

Corollary 1. *For $s[\kappa] \in \mathbb{N}_\kappa(n)$, and $\ell[\kappa] \in \mathcal{L}_\kappa$ fixed, we have*

$$\mathsf{Leb}_{2n-\kappa}\left(\mathcal{S}^{(n)}(\ell[\kappa], s[\kappa])\right) = \prod_{j=1}^{\kappa} \frac{c_j^{s_j+s_{j+1}-1}}{s_j!(s_j+s_{j+1}-1)!}. \tag{3.9}$$

Proof. Indeed, by the previous lemmas, we obtain

$$\mathsf{Leb}_{2n-\kappa}\left(\mathcal{S}^{(n)}(\ell[\kappa], s[\kappa])\right) = \prod_{j=1}^{\kappa}\binom{s_j+s_{j+1}}{s_j} s_j! \cdot \mathsf{Leb}_{N_j}\left(I[c_j, N_j+1]\right), \tag{3.10}$$

and we conclude by (2.7). □

3.3. The joint distribution of the pair $(\boldsymbol{\ell}^{(n)}[\kappa], \mathbf{s}^{(n)}[\kappa])$

Theorem 5 concretizes our understanding of this new equivalent geometric description of the set $\mathcal{C}_\kappa(s[\kappa])$ in terms of the ECP. Let $\mathbf{z}[n]$ have distribution $\mathbb{D}_\kappa^{(n)}$, and set $\boldsymbol{\ell}^{(n)}[\kappa] = \ell[\kappa](\mathbf{z}[n])$, $\mathbf{s}^{(n)}[\kappa] = s[\kappa](\mathbf{z}[n])$. By computing the Lebesgue measure of the set $\mathcal{S}^{(n)}(\ell[\kappa], s[\kappa])$,

https://doi.org/10.1017/apr.2024.63 Published online by Cambridge University Press

we managed to understand the weight of all $z[n]$-gons contained in any $(\ell[\kappa], s[\kappa])$-fibration, which is the key to the computation of the joint distribution of the pair $(\boldsymbol{\ell}^{(n)}[\kappa], \mathbf{s}^{(n)}[\kappa])$.

Theorem 6. *Let* $\mathbf{z}[n]$ *have distribution* $\mathbb{D}_\kappa^{(n)}$*, and consider the random variables* $\boldsymbol{\ell}^{(n)}[\kappa] = \ell[\kappa](\mathbf{z}[n])$, $\mathbf{s}^{(n)}[\kappa] = s[\kappa](\mathbf{z}[n])$*. Then for a given* $s[\kappa] \in \mathbb{N}^\kappa$*, the pair* $(\boldsymbol{\ell}^{(n)}[\kappa], \mathbf{s}^{(n)}[\kappa])$ *has the joint distribution*

$$\mathbb{P}\left(\boldsymbol{\ell}^{(n)}[\kappa] \in \mathrm{d}\ell[\kappa], \mathbf{s}^{(n)}[\kappa] = s[\kappa]\right) = f_\kappa^{(n)}\left(\ell[\kappa], s[\kappa]\right) \mathrm{d}\ell[\kappa], \tag{3.11}$$

where

$$f_\kappa^{(n)}\left(\ell[\kappa], s[\kappa]\right) = \frac{n! \sin(\theta_\kappa)^{n-\kappa}}{\widetilde{\mathbb{P}}_\kappa(n)} \mathbf{1}_{\{s[\kappa] \in \mathbb{N}_\kappa(n)\}} \mathbf{1}_{\{\ell[\kappa] \in \mathcal{L}_\kappa\}} \prod_{j=1}^{\kappa} \frac{c_j^{s_{j-1}+s_j-1}}{s_j!(s_{j-1}+s_j-1)!}. \tag{3.12}$$

Proof. Write

$$\mathrm{d}\mathbb{D}_\kappa^{(n)}(z[n]) = \frac{n!}{\widetilde{\mathbb{P}}_\kappa(n)} \mathbf{1}_{\{z[n] \in \mathcal{C}_\kappa(n)\}} \mathbf{1}_{\{s[\kappa](z[n]) \in \mathbb{N}_\kappa(n)\}} \mathrm{d}z[n] \tag{3.13}$$

$$= \frac{n!}{\widetilde{\mathbb{P}}_\kappa(n)} \sum_{s[\kappa] \in \mathbb{N}_\kappa(n)} \mathbf{1}_{\{z[n] \in \mathcal{C}_\kappa(s[\kappa])\}} \mathrm{d}z[n]. \tag{3.14}$$

For any continuous bounded test function $\eta : \mathbb{R}^\kappa \times \mathbb{N}^\kappa \to \mathbb{R}$, we have

$$\mathbb{E}\left[\eta\left(\boldsymbol{\ell}^{(n)}[\kappa], \mathbf{s}^{(n)}[\kappa]\right)\right] = \int_{(\mathbb{R}^2)^n} \eta\left(\left(\ell[\kappa](z[n]), s[\kappa](z[n])\right)\right) \mathrm{d}\mathbb{D}_\kappa^{(n)}(z[n]),$$

which, after the change of variables $\chi_{s[\kappa]}(z[n]) = \left(\ell[\kappa], u^{(1)}[N_1, c_1], \ldots, u^{(\kappa)}[N_\kappa, c_\kappa]\right)$, performed at fixed $(\ell[\kappa], s[\kappa])$, gives

$$\mathbb{E}\left[\eta\left(\boldsymbol{\ell}^{(n)}[\kappa], \mathbf{s}^{(n)}[\kappa]\right)\right] = \frac{n!}{\widetilde{\mathbb{P}}_\kappa(n)} \sum_{s[\kappa] \in \mathbb{N}_\kappa(n)} \mathrm{Jac}\left(\left(\chi_{s[\kappa]}\right)^{-1}\right) \int_{\mathbb{R}^\kappa} \eta(\ell[\kappa], s[\kappa])$$
$$\times \left[\int_{\mathbb{R}^{2n-\kappa}} \mathbf{1}_{\{(\ell[\kappa], u^{(1)}[N_1, c_1], \ldots, u^{(\kappa)}[N_\kappa, c_\kappa]) \in \mathcal{S}^{(n)}(s[\kappa])\}} \mathrm{d}u[2n-\kappa]\right] \mathrm{d}\ell[\kappa]. \tag{3.15}$$

Now, $\mathrm{Jac}\left(\left(\chi_{s[\kappa]}\right)^{-1}\right) = \sin(\theta_\kappa)^{n-\kappa}$ for all $s[\kappa] \in \mathbb{N}_\kappa(n)$ by (3.4), and the last bracket in (3.15) is nothing but the Lebesgue measure of $\mathcal{S}^{(n)}(\ell[\kappa], s[\kappa])$, which we computed in Corollary 1! Hence substituting (3.9) in (3.15) gives Theorem 6. □

In the next section, we are going to exploit the asymptotic stochastic behavior of the pair $(\boldsymbol{\ell}^{(n)}[\kappa], \mathbf{s}^{(n)}[\kappa])$ to deduce an equivalent of $\widetilde{\mathbb{P}}_\kappa(n)$. However, in the particular cases $\kappa \in \{3, 4\}$, we have $\mathbb{D}_\kappa^{(n)} = \mathbb{Q}_\kappa^{(n)}$, and the set $\mathcal{L}_\kappa$ is easily computable. Hence we can immediately compute the exact value of $\mathbb{P}_\kappa(n)$ from $\mathbb{Q}_\kappa^{(n)}$. In Appendix B, we take a look at these computations to recover Valtr's famous results for the triangle and the parallelogram.

4. An asymptotic result for convex regular polygons

Let $\mathbf{z}[n]$ have distribution $\mathbb{D}_\kappa^{(n)}$, and consider $\boldsymbol{\ell}^{(n)}[\kappa] = \ell[\kappa](\mathbf{z}[n])$, $\mathbf{s}^{(n)}[\kappa] = s[\kappa](\mathbf{z}[n])$. By symmetry, we have $\mathbf{s}_j^{(n)} \overset{(d)}{=} \mathbf{s}_1^{(n)}$ for all $j \in \{1, \ldots, \kappa\}$, and since $\sum_{j=1}^{\kappa} \mathbf{s}_j^{(n)} = n$, the expectation

https://doi.org/10.1017/apr.2024.63 Published online by Cambridge University Press

of $\mathbf{s}_j^{(n)}$ is given by $\mathbb{E}[\mathbf{s}_j^{(n)}] = n/\kappa$. In the sequel we will set $\mathbf{s}_\kappa^{(n)} = n - \sum_{j=1}^{\kappa-1} \mathbf{s}_j^{(n)}$, and we will describe $\mathbf{s}^{(n)}[\kappa-1]$ since the last value is determined by the other ones. What we are interested in here are the fluctuations of $\mathbf{s}^{(n)}[\kappa-1]$ around its expectation, and the asymptotic behavior of the variables $\boldsymbol{\ell}^{(n)}[\kappa]$ as n grows. This is all contained in the following theorem.

Theorem 7. *Let* $\mathbf{z}[n]$ *have distribution* $\mathbb{D}_\kappa^{(n)}$, *and consider* $\boldsymbol{\ell}^{(n)}[\kappa] = \ell[\kappa](\mathbf{z}[n])$, $\mathbf{s}^{(n)}[\kappa] = s[\kappa](\mathbf{z}[n])$. *We introduce the random variables* $\overline{\boldsymbol{\ell}}^{(n)}[\kappa] = n\boldsymbol{\ell}^{(n)}[\kappa]$ *and* $\mathbf{x}_j^{(n)} = \frac{\mathbf{s}_j^{(n)} - n/\kappa}{\sqrt{n/\kappa}}$, *for all* $j \in \{1, \dots, \kappa\}$. *The following convergence in distribution holds in* $\mathbb{R}^{2\kappa-1}$:

$$\left(\overline{\boldsymbol{\ell}}_1^{(n)}, \dots, \overline{\boldsymbol{\ell}}_\kappa^{(n)}, \mathbf{x}_1^{(n)}, \dots, \mathbf{x}_{\kappa-1}^{(n)}\right) \xrightarrow[n]{(d)} \left(\overline{\boldsymbol{\ell}}_1, \dots, \overline{\boldsymbol{\ell}}_\kappa, \mathbf{x}_1, \dots, \mathbf{x}_{\kappa-1}\right),$$

where the variables $\overline{\boldsymbol{\ell}}[\kappa]$ *are independent from the* $\mathbf{x}[\kappa-1]$; *the* $\overline{\boldsymbol{\ell}}[\kappa]$ *are* κ *random variables that are exponentially distributed with rate* $\frac{2w_\kappa}{\kappa r_\kappa}$, *where* $w_\kappa = \frac{1+\cos(\theta_\kappa)}{\sin(\theta_\kappa)}$; *and* $\mathbf{x} = \mathbf{x}[\kappa-1]$ *is a centered Gaussian random vector whose inverse covariance matrix* Σ_κ^{-1} *of size* $(\kappa-1)\times(\kappa-1)$ *is given by*

$$\Sigma_3^{-1} = \frac{1}{2}\begin{pmatrix} 6 & 3 \\ 3 & 6 \end{pmatrix}, \quad \Sigma_4^{-1} = \frac{1}{2}\begin{pmatrix} 6 & 4 & 2 \\ 4 & 8 & 4 \\ 2 & 4 & 6 \end{pmatrix}, \quad \Sigma_5^{-1} = \frac{1}{2}\begin{pmatrix} 6 & 4 & 3 & 2 \\ 4 & 8 & 5 & 3 \\ 3 & 5 & 8 & 4 \\ 2 & 3 & 4 & 6 \end{pmatrix},$$

and more generally

$$\Sigma_\kappa^{-1} = \frac{1}{2}\begin{pmatrix} 6 & 4 & 3 & \cdots & \cdots & 3 & 2 \\ 4 & 8 & 5 & 4 & \cdots & 4 & 3 \\ 3 & 5 & \ddots & \ddots & \ddots & \vdots & \vdots \\ \vdots & 4 & \ddots & \ddots & \ddots & 4 & \vdots \\ \vdots & \vdots & \ddots & \ddots & \ddots & 5 & 3 \\ 3 & 4 & \cdots & 4 & 5 & 8 & 4 \\ 2 & 3 & \cdots & \cdots & 3 & 4 & 6 \end{pmatrix}. \text{ for } \kappa \geq 6,$$

The determinant $\mathrm{m}_\kappa = \det\left(\Sigma_\kappa^{-1}\right)$ *of the latter matrix has already been mentioned in Theorem 1. The value of this determinant, i.e.*

$$\mathrm{m}_\kappa = \frac{\kappa}{3\cdot 2^\kappa}\left(2(-1)^{\kappa-1} + (2-\sqrt{3})^\kappa + (2+\sqrt{3})^\kappa\right),$$

stated in (1.1), is computed in Appendix C.

We first state two important intermediate lemmas which will allow us to prove Theorem 7.

Lemma 7. *(Local limit theorem for Poisson variables.) Let* κ *be a positive integer and* Y_n *a Poisson variable of mean* $\frac{n}{\kappa}$. *We have*

$$\sup_y \left|\sqrt{\frac{n}{\kappa}}\mathbb{P}\left(Y_n = \left\lfloor \frac{n}{\kappa} + y\sqrt{\frac{n}{\kappa}} \right\rfloor\right) - \frac{e^{-y^2/2}}{\sqrt{2\pi}}\right| \xrightarrow[n\to\infty]{} 0.$$

https://doi.org/10.1017/apr.2024.63 Published online by Cambridge University Press

Proof. Pick n i.i.d. random Poisson variables $X_1, \dots, X_n$ of mean $1/\kappa$ and apply the local limit theorem [19, Theorem VII.1.1] to $\widetilde{X}_i = \sqrt{\kappa}(X_i - \frac{1}{\kappa})$. The support of $\widetilde{X}_1$ is included in $\sqrt{\kappa}\mathbb{Z} - 1/\sqrt{\kappa}$, and $X_1 + \cdots + X_n$ is a Poisson variable of mean n/κ. □

Lemma 8. *Let $(g_n)_{n\in\mathbb{N}}$ be a sequence of nonnegative measurable functions on $\mathbb{R}^d$. Assume that for every $\varepsilon > 0$ there exists a compact set K_ε such that for all n large enough, $\int_{K_\varepsilon^c} g_n < \varepsilon$ (where K_ε^c is the complement of K_ε in $\mathbb{R}^d$), and that g_n uniformly converges on all compact subsets of $\mathbb{R}^d$ towards a density g (with respect to the Lebesgue measure on $\mathbb{R}^d$). Then there exists a sequence $(\alpha_n)_{n\in\mathbb{N}}$ such that for n large enough (for small values of n, g_n could be zero), $\frac{1}{\alpha_n} g_n$ is a density and $\alpha_n \underset{n\to+\infty}{\longrightarrow} 1$.*

Proof. Take $\varepsilon > 0$, and choose K such that for n large enough, $\int_{K^c} g_n < \varepsilon$. Since g is a density, there exists a compact set H such that $\int_H g \geq 1 - \varepsilon$. Let $S = K \cup H$. By the uniform convergence, there exists $m \in \mathbb{N}$ such that for all $n \geq m$, we have $\int_S |g_n - g| \leq \varepsilon$. Then the triangle inequality gives

$$\int_S g_n \geq \int_S g - \int_S |g_n - g| \geq 1 - 2\varepsilon$$

$$\text{and} \qquad \int_{\mathbb{R}^d} g_n \leq \int_S g + \int_S |g_n - g| + \int_{S^c} g_n \leq 1 + 2\varepsilon. \quad (4.1)$$

This shows $\alpha_n = \int_{\mathbb{R}^d} g_n$ is finite, well-defined, and nonzero for n large enough, and $\frac{1}{\alpha_n} g_n$ is a density on $\mathbb{R}^d$. From (4.1), we have $\alpha_n \underset{n\to+\infty}{\longrightarrow} 1$, and this concludes the proof. □

Notation. Recall that by Proposition 1, if $z[n] \in \mathcal{C}_\kappa(n)$, we can express the side lengths $c[\kappa]$ as a function of the boundary distances $\ell[\kappa]$ of the $\mathsf{ECP}(z[n])$:

$$c_j = r_\kappa - \mathfrak{cl}_j(\ell[\kappa]), \text{ for all } j \in \{1, \dots, \kappa\},$$

with $\mathfrak{cl}_j(\ell[\kappa]) = (\ell_{j-1} + \ell_{j+1} + 2\ell_j \cos(\theta_\kappa))/\sin(\theta_\kappa)$.

Proof of Theorem 7. Note that the joint density of $\left(\left(\overline{\boldsymbol{\ell}}_1, \dots, \overline{\boldsymbol{\ell}}_\kappa\right), (\mathbf{x}_1, \dots, \mathbf{x}_{\kappa-1})\right)$ on $\mathbb{R}_+^\kappa \times \mathbb{R}^{\kappa-1}$ is given by

$$g_\kappa(\overline{\ell}[\kappa], x[\kappa-1]) = \left(\sqrt{\frac{\mathrm{m}_\kappa}{(2\pi)^{\kappa-1}}} \exp\left(-\frac{1}{2} x^t \Sigma_\kappa^{-1} x\right)\right) \left(\frac{2w_\kappa}{\kappa r_\kappa}\right)^\kappa \exp\left(-\frac{2w_\kappa}{\kappa r_\kappa} \sum_{j=1}^{\kappa} \overline{\ell}_j\right).$$

The proof of Theorem 7 is carried out in two steps:

(i) We show the uniform convergence on compact sets of the 'density' of the pair

$$\left(\overline{\boldsymbol{\ell}}^{(n)}[\kappa], \mathbf{x}^{(n)}[\kappa-1]\right).$$

More precisely, we show the uniform convergence on compact sets of a density $g_\kappa^{(n)}$, introduced in (4.4), that is associated to these random variables.

(i) (2) We give an argument of uniform integrability for this limit, which allows us to apply Lemma 8 and conclude.

https://doi.org/10.1017/apr.2024.63 Published online by Cambridge University Press

Step 1: Let $\psi : \mathbb{R}_+^\kappa \times \mathbb{R}^{\kappa-1} \to \mathbb{R}$ a bounded continuous function, and let us pass to the limit in the expectation

$$\mathbb{E}\left[\psi\left(\overline{\boldsymbol{\ell}}^{(n)}[\kappa], \mathbf{x}^{(n)}[\kappa-1]\right)\right] = \sum_{s[\kappa]\in\mathbb{N}_\kappa(n)} \int_{\mathbb{R}^\kappa} \psi\left(n\ell[\kappa], \frac{s[\kappa-1]-n/\kappa}{\sqrt{n/\kappa}}\right) f_\kappa^{(n)}\left(\ell[\kappa], s[\kappa]\right) \mathrm{d}\ell[\kappa], \quad (4.2)$$

where the joint distribution $f_\kappa^{(n)}$ of the pair $(\boldsymbol{\ell}^{(n)}[\kappa], \mathbf{s}^{(n)}[\kappa])$ is given in Theorem 6.

We perform both substitutions $\overline{\ell}_j = n\ell_j$ and $x_j = (s_j - n/\kappa)/\sqrt{n/\kappa}$ in the right-hand side of (4.2). We turn our sum over $s[\kappa] \in \mathbb{N}_\kappa(n)$ into an integral, in the following way:

$$\frac{n!\sin(\theta_\kappa)^{n-\kappa}}{\widetilde{\mathbb{P}}_\kappa(n)} \int_{\overline{\mathbb{N}_{\kappa-1}}(n)} \int_{\mathcal{L}_\kappa} \psi\left(n\ell[\kappa], \frac{\lfloor q[\kappa-1]\rfloor - n/\kappa}{\sqrt{n/\kappa}}\right) \times \prod_{j=1}^{\kappa} \frac{\left(r_\kappa - \mathfrak{cl}_j(\ell[\kappa])\right)^{\lfloor q_j\rfloor+\lfloor q_{j+1}\rfloor-1}}{\lfloor q_j\rfloor!(\lfloor q_j\rfloor+\lfloor q_{j+1}\rfloor-1)!} \cdot \mathrm{d}\ell[\kappa]\mathrm{d}q[\kappa-1],$$

where $\lfloor q_\kappa \rfloor$ is set to satisfy $\lfloor q_\kappa\rfloor = n - \sum_{j=1}^{\kappa-1} \lfloor q_j \rfloor$ (notice that there is no integration with respect to q_κ) and the integration is now done on the region

$$\overline{\mathbb{N}_{\kappa-1}}(n) := \left\{ q[\kappa-1], \text{ with } q_j > 0 \text{ and } \sum_{j=1}^{\kappa-1} q_j \le n \right\}.$$

Let us consider the term

$$\frac{n!\sin(\theta_\kappa)^{n-\kappa}}{\widetilde{\mathbb{P}}_\kappa(n)} \int_{\overline{\mathbb{N}_{\kappa-1}}(n)} \int_{\mathcal{L}_\kappa} \psi\left(n\ell[\kappa], \frac{q[\kappa-1]-n/\kappa}{\sqrt{n/\kappa}}\right) \times \prod_{j=1}^{\kappa} \frac{\left(r_\kappa - \mathfrak{cl}_j(\ell[\kappa])\right)^{\lfloor q_j\rfloor+\lfloor q_{j+1}\rfloor-1}}{\lfloor q_j\rfloor!(\lfloor q_j\rfloor+\lfloor q_{j+1}\rfloor-1)!} \cdot \mathrm{d}\ell[\kappa]\mathrm{d}q[\kappa-1]$$

(we have removed the floor function in ψ). This quantity turns out to be the expectation

$$\mathbb{E}\left[\psi\left(\overline{\boldsymbol{\ell}}^{(n)}[\kappa], \overline{\mathbf{x}}^{(n)}[\kappa-1]\right)\right],$$

where for all $j \in \{1, \dots, \kappa-1\}$ we set $\overline{\mathbf{x}}_j^{(n)} := \mathbf{x}_j^{(n)} + \mathbf{U}_j/\sqrt{n/\kappa}$, with $\mathbf{U}_j$ a random variable uniformly distributed in $[0, 1]$.

We have replaced a sum by an integral, which amounts to representing a discrete random variable by a continuous one; i.e. if X has a discrete law, $\mathbb{P}(X = k) = p_k$, $k \in \mathbb{Z}$, then $X \overset{(d)}{=} \lfloor X + U \rfloor$, where U is uniform in $[0, 1]$. Then

$$\sum_{k\in\mathbb{Z}} p_k f(k) = \int_{\mathbb{R}} f(\lfloor x\rfloor) p_{\lfloor x\rfloor} \mathrm{d}x.$$

https://doi.org/10.1017/apr.2024.63 Published online by Cambridge University Press

We are going to prove first that $\mathbb{E}\left[\psi\left(\overline{\boldsymbol{\ell}}^{(n)}[\kappa], \overline{\mathbf{x}}^{(n)}[\kappa-1]\right)\right]$ converges to deduce that its counterpart $\mathbb{E}\left[\psi\left(\overline{\boldsymbol{\ell}}^{(n)}[\kappa], \mathbf{x}^{(n)}[\kappa-1]\right)\right]$ converges as well, to the same limit.

After substitution, we obtain

$$\mathbb{E}\left[\psi\left(\overline{\boldsymbol{\ell}}^{(n)}[\kappa], \overline{\mathbf{x}}^{(n)}[\kappa-1]\right)\right] = \int_{\mathbb{R}^{\kappa-1}} \int_{\mathbb{R}_+^{\kappa}} \psi(\overline{\ell}[\kappa], x[\kappa-1]) g_{\kappa}^{(n)}(\overline{\ell}[\kappa], x[\kappa-1]) \mathrm{d}\overline{\ell}[\kappa] \mathrm{d}x[\kappa-1], \quad (4.3)$$

where, for all $n \geq 3$, $g_{\kappa}^{(n)}$ stands for the joint distribution of a pair $\left(\overline{\boldsymbol{\ell}}^{(n)}[\kappa], \overline{\mathbf{x}}^{(n)}[\kappa-1]\right)$. With the convention $x_{\kappa} = -\sum_{i=1}^{\kappa-1} x_i$, the function $g_{\kappa}^{(n)}$ can be decomposed as follows:

$$g_{\kappa}^{(n)}\left(\overline{\ell}[\kappa], x[\kappa-1]\right) := \omega(n,\kappa)\, h_n^{(1)}(\overline{\ell}[\kappa], x[\kappa-1])\, h_n^{(2)}(x[\kappa-1]), \quad (4.4)$$

with

$$\omega(n,\kappa) = \frac{n! \sin(\theta_{\kappa})^{n-\kappa}}{\widetilde{\mathbb{P}}_{\kappa}(n)} \left[\frac{1}{\sqrt{2\pi^{\kappa+1}\mathrm{m}_{\kappa}}} \frac{\kappa^{3n} e^{3n}}{4^n n^{3n}}\right] \left[\left(\frac{\kappa r_{\kappa}}{2w_{\kappa}}\right)^{\kappa} r_{\kappa}^{2n-\kappa}\right] \frac{1}{n^{\kappa}} \sqrt{\frac{n}{\kappa}}^{\kappa-1}, \quad (4.5)$$

and

$$h_n^{(1)}\left(\overline{\ell}[\kappa], x[\kappa-1]\right) = \frac{1}{r_{\kappa}^{2n-\kappa}} \left(\frac{2w_{\kappa}}{\kappa r_{\kappa}}\right)^{\kappa} \mathbf{1}_{\{\overline{\ell}[\kappa] \in n\mathcal{L}_{\kappa}\}} \prod_{j=1}^{\kappa} \left(r_{\kappa} - \frac{1}{n}\mathfrak{l}_j(\overline{\ell}[\kappa])\right)^{d_j^{(2)}(x[\kappa-1])}, \quad (4.6)$$

$$h_n^{(2)}\left(x[\kappa-1]\right) = \sqrt{2\pi^{\kappa+1}\mathrm{m}_{\kappa}} \frac{4^n n^{3n}}{\kappa^{3n} e^{3n}} \prod_{j=1}^{\kappa} \frac{1}{d_j^{(1)}(x[\kappa-1])! \cdot d_j^{(2)}(x[\kappa-1])!}, \quad (4.7)$$

where

$$d_j^{(1)}\left(x[\kappa-1]\right) = \lfloor n/\kappa + \sqrt{n/\kappa} x_j \rfloor, \text{ for all } j \in \{1, \ldots, \kappa\}, \quad (4.8)$$

$$d_j^{(2)}\left(x[\kappa-1]\right) = \lfloor 2n/\kappa - 1 + \sqrt{n/\kappa}(x_j + x_{j+1}) \rfloor, \text{ for all } j \in \{1, \ldots, \kappa\}. \quad (4.9)$$

We have arranged the factors so that, as we will see, $h_n^{(1)}$ and $h_n^{(2)}$ converge to some probability densities.

Note first that there exists $\eta > 0$ such that $[0,\eta]^{\kappa} \subset \mathcal{L}_{\kappa}$, and thus we have $n\mathcal{L}_{\kappa} \underset{n\to\infty}{\longrightarrow} \mathbb{R}_+^{\kappa}$. Then, for every compact $K \subset \mathbb{R}_+^{\kappa}$ and every $\varepsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$, $K \subset n\mathcal{L}_{\kappa}$, i.e. $||\mathbf{1}_{\{\overline{\ell}[\kappa] \in n\mathcal{L}_{\kappa}-1\}}|| = 0 < \varepsilon$, so that the map $\overline{\ell}[\kappa] \mapsto \mathbf{1}_{\{\overline{\ell}[\kappa] \in n\mathcal{L}_{\kappa}\}}$ converges uniformly to the constant function 1 on every compact set of $\mathbb{R}_+^{\kappa}$. Now by the standard approximation $\left(1 - \frac{a}{n}\right)^{nb} \underset{n\to+\infty}{\longrightarrow} e^{-ab}$ uniformly for (a,b) on every compact set, we get that $h_n^{(1)}$ converges uniformly on every compact set of $\mathbb{R}_+^{\kappa} \times \mathbb{R}^{\kappa-1}$ towards $h^{(1)}$ with

$$h^{(1)}\left(\overline{\ell}[\kappa]\right) = \left(\frac{2w_{\kappa}}{\kappa r_{\kappa}}\right)^{\kappa} \prod_{j=1}^{\kappa} \exp\left(-\frac{2w_{\kappa}}{\kappa r_{\kappa}} \overline{\ell}_j\right). \quad (4.10)$$

https://doi.org/10.1017/apr.2024.63 Published online by Cambridge University Press

Now, thanks to Corollary 7, for $x[\kappa-1]$ fixed in $\mathbb{R}^{\kappa-1}$ we have

$$h_n^{(2)}(x[\kappa-1]) \underset{n\to+\infty}{\sim} \sqrt{2\pi^{\kappa+1}\mathrm{m}_\kappa}\,\frac{4^n n^{3n}}{\kappa^{3n}e^{3n}}\left(\frac{\kappa}{n}\right)^n\left(\frac{\kappa}{2n-\kappa}\right)^{2n-\kappa} e^{3n-\kappa} \times\prod_{j=1}^{\kappa}\frac{\sqrt{\kappa}e^{-x_j^2/2}}{\sqrt{2\pi n}}\,\frac{\sqrt{\kappa}e^{-(x_j+x_{j+1})^2/4}}{\sqrt{2\pi(2n-\kappa)}}.$$

After simplifications, this actually can be rewritten as the convergence on every compact set of $\mathbb{R}^{\kappa-1}$ of $h_n^{(2)}$ towards $h^{(2)}$ where

$$h^{(2)}\,(x[\kappa-1]) = \sqrt{\frac{\mathrm{m}_\kappa}{(2\pi)^{\kappa-1}}}\exp\left(-\frac{1}{2}x^t\Sigma_\kappa^{-1}x\right). \tag{4.11}$$

We have established the following uniform convergence on every compact set of $\mathbb{R}_+^\kappa\times\mathbb{R}^{\kappa-1}$:

$$\frac{1}{\omega(n,\kappa)}g_\kappa^{(n)}(\bar{\ell}[\kappa],x[\kappa-1]) \underset{n\to+\infty}{\longrightarrow} g_\kappa(\bar{\ell}[\kappa],x[\kappa-1]). \tag{4.12}$$

This concludes Step 1 of our proof.

Step 2: We will apply Lemma 8 to the sequence of functions $g_n=\frac{1}{\omega(n,\kappa)}g_\kappa^{(n)}$, and to $g=g_\kappa$, which is already known to be a density. We therefore need to check that we control the mass of $\frac{1}{\omega(n,\kappa)}g_\kappa^{(n)}$ outside of a certain compact set.

For any compact $K'\subset\mathbb{R}_+^\kappa$, we have

$$\mathbf{1}_{\{\bar{\ell}[\kappa]\in n\mathcal{L}_\kappa\cap(K')^c\}}\le\mathbf{1}_{\{\bar{\ell}[\kappa]\in n\mathcal{L}_\kappa\}},$$

and with $n\mathcal{L}_\kappa$ being a compact set of $\mathbb{R}_+^\kappa$, there exists some $N\in\mathbb{N}$ such that when $n\ge N$, we have the following for all $(\bar{\ell}[\kappa],x[\kappa-1])\in(K')^c\times\mathbb{R}^{\kappa-1}$:

$$h_n^{(1)}(\bar{\ell}[\kappa],x[\kappa-1])\le 2h^{(1)}(\bar{\ell}[\kappa]). \tag{4.13}$$

Now let $\varepsilon>0$ be fixed for the rest of this proof, and let us build the compact set K_ε outside of which we control the mass of $h_n^{(2)}$. We can reinterpret the map $h_n^{(2)}$ as follows: if M_1,M_2 are two independent multinomial variables, with $M_1\sim\mathcal{M}(n;\frac{1}{\kappa},\dots,\frac{1}{\kappa})$ and $M_2\sim\mathcal{M}(2n-\kappa;\frac{1}{\kappa},\dots,\frac{1}{\kappa})$, for

$$\mathbf{P}\,(x[\kappa-1]) = \mathbb{P}\left(M_1=d^{(1)}(x[\kappa-1])[\kappa],M_2=d^{(2)}(x[\kappa-1])[\kappa]\right),$$

we have

$$\begin{aligned}h_n^{(2)}(x[\kappa-1]) &= \frac{\sqrt{2\pi^{\kappa+1}\mathrm{m}_\kappa}}{\kappa^\kappa}\frac{4^n n^{3n}}{e^{3n}n!(2n-\kappa)!}\mathbf{P}\,(x[\kappa-1])\\ &\le \frac{2^\kappa\sqrt{\pi}^{\kappa-1}\sqrt{\mathrm{m}_\kappa}}{B\kappa^\kappa}n^{\kappa-1}\mathbf{P}\,(x[\kappa-1])\,.\end{aligned} \tag{4.14}$$

For n large enough we have both $n!\ge 2^{-1/2}\sqrt{\pi}n^{n+1/2}e^{-n}$ and the existence of a constant α such that for all n, $(2n-\kappa)^{2n-\kappa+1/2}\ge\alpha(2n)^{2n-\kappa+1/2}$, so, setting $B:=2\pi e^\kappa$, we have

https://doi.org/10.1017/apr.2024.63 Published online by Cambridge University Press

$$n!(2n-\kappa)! \geq B\frac{4^n n^{3n}}{e^{3n}n^{\kappa-1}}, \text{ for all } n \geq 1.$$

Let $M_k(i)$, for $k \in \{1, 2\}$ and $i \in \{1, \ldots, \kappa\}$, be the ith entry of the multinomial random variable M_k. Recall that the entry $M_1(i)$ is a binomial random variable $\mathcal{B}(n, \frac{1}{\kappa})$, and that, for $i \neq j$, the law of $M_1(i)$ conditioned on $M_1(j) = k_j$ is a binomial distribution $\mathcal{B}(n - k_j, \frac{1}{\kappa-1})$. Analogous results hold for $M_2(i)$. Now, since these marginals are binomial random variables, they are concentrated around their mean. We will design a compact K_ε such that K_ε^c contains the elements $x[\kappa-1]$ whose ith entry (for at least one i) is far from its expected value (which will give us exponential small bounds).

Let us rewrite by presenting the multinomial random variable as Markov chains of a sort. We have

$$\mathbf{P}(x[\kappa-1]) = \prod_{i=1}^{\kappa-1} \mathbb{P}\left(M_1(i) = \left\lfloor \frac{n}{\kappa} + x_i\sqrt{\frac{n}{\kappa}} \right\rfloor, M_2(i) = \left\lfloor \frac{2n-\kappa}{\kappa} + (x_i + x_{i-1})\sqrt{\frac{n}{\kappa}} \right\rfloor \middle| G_1(x,i), G_2(x,i)\right) \tag{4.15}$$

where

$$G_1(x,i) := \bigcap_{j=1}^{i-1} \left\{ M_1(j) = \left\lfloor \frac{n}{\kappa} + x_j\sqrt{\frac{n}{\kappa}} \right\rfloor \right\} \text{ and}$$

$$G_2(x,i) := \bigcap_{j=1}^{i-1} \left\{ M_2(j) = \left\lfloor \frac{2n-\kappa}{\kappa} + (x_j + x_{j-1})\sqrt{\frac{n}{\kappa}} \right\rfloor \right\}.$$

For all $x[\kappa-1] \in \mathbb{R}^{\kappa-1}$ and $i \in \{1, \ldots, \kappa-1\}$, let $Y_i(x)$ be a binomial random variable with the same law as $M_1(i)|G_1(x,i)$, i.e.

$$Y_i(x) \sim \mathcal{B}\left(\left\lfloor \frac{n}{\kappa}(\kappa - i + 1) - \sqrt{\frac{n}{\kappa}}(x_1 + \ldots + x_{i-1}) \right\rfloor, \frac{1}{\kappa - i + 1}\right).$$

A standard inequality for binomial distributions $\mathbf{x} \sim \mathcal{B}(m, q)$ with $q \in [a, b]$, $0 < a < b < 1$, [19, III.5.2], gives the existence of a constant $C > 0$ such that for all x,

$$\mathbb{P}(\mathbf{x} = x) \leq \frac{1}{C\sqrt{m}}. \tag{4.16}$$

For n large enough, this implies the existence of a constant $C_\kappa > 0$ such that

$$\mathbf{P}(x[\kappa-1]) \leq \frac{\prod_{i=1}^{\kappa-1} \mathbb{P}\left(M_1(i) = \left\lfloor \frac{n}{\kappa} + x_i\sqrt{\frac{n}{\kappa}} \right\rfloor \mid G_1(x,i)\right)}{C_\kappa \sqrt{n}^{\kappa-1}}. \tag{4.17}$$

Controlling the map $\mathbf{P}$ allows one to control the map $h_n^{(2)}$ (recall (4.14)), and thus g_n (recall (4.4)). Define the sequence $e[\kappa]$ as follows: $e_1 = \sqrt{\kappa}$, and for all $j \in \{2, \ldots, \kappa-1\}$,

$$e_j = \sqrt{\kappa} + \frac{e_1 + \ldots + e_{j-1}}{\kappa - j + 1}. \tag{4.18}$$

https://doi.org/10.1017/apr.2024.63 Published online by Cambridge University Press

We will use $e[\kappa]$ to define an event of $\mathbf{P}$ that has small probability. Let $M > 0$ and $t \in \{1, \dots, \kappa - 1\}$. We define the set

$$\widehat{B}_M(t) = \left\{ w[\kappa - 1] \in \mathbb{R}^{\kappa - 1} \text{ such that } |w_t| > Me_t \text{ and } \left|w_j\right| \leq Me_j,\ 1 \leq j \leq t - 1 \right\}. \tag{4.19}$$

Intuitively, forcing the multinomial variable M_k, $k \in \{1, 2\}$, to be in $\widehat{B}_M(t)$ is a huge condition for M large, since this requires M_k to have a coordinate far from its mean. Since for all $x[\kappa - 1] \in \widehat{B}_M(t)$, by (4.18), we have

$$\sqrt{\frac{n}{\kappa}} \left| x_t + \frac{x_1 + \cdots + x_{t-1}}{\kappa - t + 1} \right| > \sqrt{\frac{n}{\kappa}} M\sqrt{\kappa} = M\sqrt{n},$$

we may write (making the small change of variables $t_i = \frac{n}{\kappa} + x_i\sqrt{\frac{n}{\kappa}}$ and then taking the supremum on $\widehat{B}_M(t)$), by (4.17),

$$\begin{aligned}
&\int_{\widehat{B}_M(t)} \prod_{i=1}^{\kappa-1} \mathbb{P}\left(M_1(i) = \left\lfloor \frac{n}{\kappa} + x_i\sqrt{\frac{n}{\kappa}} \right\rfloor \,\middle|\, G_1(x, i) \right) \mathrm{d}x \\
&\qquad \leq \sup_{x \in \widehat{B}_M(t)} \left[\prod_{i=1}^{t-1} \mathbb{P}\left(|Y_i(x) - \mathbb{E}[Y_i(x)]| \leq M\sqrt{n} \right) \right] \\
&\qquad \times \mathbb{P}\left(|Y_t(x) - \mathbb{E}[Y_t(x)]| > M\sqrt{n} \right) \left[\prod_{i=t+1}^{\kappa-1} \mathbb{P}\left(|Y_i(x) - \mathbb{E}[Y_i(x)]| \in \mathbb{R} \right) \right] \frac{1}{\sqrt{n}^{\kappa-1}}.
\end{aligned} \tag{4.20}$$

We bound the terms different from t in the product by 1, and we handle the term in t by Hoeffding's inequality [19, III.5.8], i.e.

$$\mathbb{P}\left(|Y_t(x) - \mathbb{E}[Y_t(x)]| > M\sqrt{n} \right) \leq 2\exp\left(-2M^2\right), \text{ for all } x[\kappa - 1] \in \widehat{B}_M(t).$$

Let us check that there exists $M := M(\varepsilon)$ large enough so that the integral of the map $\mathbf{P}$ outside of $K_\varepsilon := [-M\sqrt{\kappa}, M\sqrt{\kappa}]^{\kappa-1}$ is controlled.

With the decomposition $A_M := \mathbb{R}^{\kappa-1} \backslash [-M\sqrt{\kappa}, M\sqrt{\kappa}]^{\kappa-1} = \bigsqcup_{t=1}^{\kappa-1} \widehat{B}_M(t)$, we may write

$$\int_{A_M} \mathbf{P}(x[\kappa - 1])\mathrm{d}x = \sum_{t=1}^{\kappa-1} \int_{\widehat{B}_M(t)} \mathbf{P}(x[\kappa - 1])\mathrm{d}x \leq \frac{2(\kappa - 1)}{C_\kappa n^{\kappa-1}} \exp\left(-2M^2\right), \tag{4.21}$$

where we set $\mathrm{d}x = \prod_{i=1}^{\kappa-1} \mathrm{d}x_i$. With ε being fixed, we may now choose $M := M(\varepsilon) > 0$ sufficiently large so that

$$\frac{2^\kappa \sqrt{\pi}^{\kappa-1} \sqrt{\mathrm{m}_\kappa}}{B\kappa^\kappa} \frac{2(\kappa - 1)}{C_\kappa} \exp\left(-2M^2\right) < \frac{1}{2}\varepsilon. \tag{4.22}$$

For such an M, we put $K_\varepsilon = [-M\sqrt{\kappa}, M\sqrt{\kappa}]^{\kappa-1}$. In this case, we have indeed

$$\begin{aligned}
\int_{K_\varepsilon^c} h_n^{(2)} &\leq \frac{2^\kappa \sqrt{\pi}^{\kappa-1} \sqrt{\mathrm{m}_\kappa}}{B\kappa^\kappa} n^{\kappa-1} \int_{A_M} \mathbf{P}(x[\kappa - 1])\mathrm{d}x \\
&\leq \frac{2^\kappa \sqrt{\pi}^{\kappa-1} \sqrt{\mathrm{m}_\kappa}}{B\kappa^\kappa} \frac{2(\kappa - 1)}{C_\kappa} \exp\left(-2M^2\right) < \frac{1}{2}\varepsilon.
\end{aligned} \tag{4.23}$$

https://doi.org/10.1017/apr.2024.63 Published online by Cambridge University Press

This completes the proof of Step 2. Indeed, if we sum up, we have uniform convergence of $\frac{1}{\omega(n,\kappa)} g_\kappa^{(n)}$ to g_κ on every compact set, and we have built two compact sets $K' \subset \mathbb{R}_+^\kappa$ and $K_\varepsilon \subset \mathbb{R}^{\kappa-1}$ such that for all $n \geq N$,

$$\int_{(K')^c \times K_\varepsilon^c} \frac{1}{\omega(n,\kappa)} g_\kappa^{(n)} = \frac{1}{\omega(n,\kappa)} \int_{(K')^c \times K_\varepsilon^c} h_n^{(1)}\, h_n^{(2)} \tag{4.24}$$

$$\leq 2 \underbrace{\int_{(K')^c} h^{(1)}}_{\leq 1} \int_{K_\varepsilon^c} h_n^{(2)} < \varepsilon, \tag{4.25}$$

where the first line comes from (4.4) and the second line comes from the bounds we gave in (4.13) and (4.23).

We may now conclude: we apply Lemma 8, so that there exists a (unique!) sequence normalizing $\frac{1}{\omega(n,\kappa)} g_\kappa^{(n)}$ into a density. Since $g_\kappa^{(n)}$ is already a density, this sequence is nothing but $\omega(n,\kappa)$; hence we have

$$\omega(n,\kappa) \underset{n\to\infty}{\longrightarrow} 1. \tag{4.26}$$

This also proves that $g_\kappa^{(n)}$ converges pointwise to g_κ, or, by definition, that

$$(\bar{\boldsymbol{\ell}}^{(n)}[\kappa], \bar{\mathbf{x}}^{(n)}[\kappa-1]) \overset{(d)}{\underset{n}{\longrightarrow}} (\bar{\boldsymbol{\ell}}[\kappa], \mathbf{x}[\kappa-1]).$$

Now, since

$$\left|\bar{\mathbf{x}}^{(n)}[\kappa-1] - \mathbf{x}^{(n)}[\kappa-1]\right| \overset{(\mathbb{P})}{\underset{n}{\longrightarrow}} 0,$$

by Slutsky's lemma we have

$$(\bar{\boldsymbol{\ell}}^{(n)}[\kappa], \mathbf{x}^{(n)}[\kappa-1]) \overset{(d)}{\underset{n}{\longrightarrow}} (\bar{\boldsymbol{\ell}}[\kappa], \mathbf{x}[\kappa-1]).$$

This ends the proof. □

Theorem 1 actually turns out to be a nice corollary of Theorem 7.

Proof of Theorem 1. We saw in (4.26) that the sequence $\omega(n,\kappa)$ introduced in (4.5) satisfies $\omega(n,\kappa) \underset{n\to+\infty}{\longrightarrow} 1$. This allows us to determine $\widetilde{\mathbb{P}}_\kappa(n)$. Indeed, Stirling's formula yields

$$\widetilde{\mathbb{P}}_\kappa(n) \underset{n\to+\infty}{\sim} \frac{1}{\pi^{\kappa/2}\sqrt{\mathrm{m}_\kappa}} \frac{\sqrt{\kappa}^{\kappa+1}}{4^\kappa (1+\cos(\theta_\kappa))^\kappa} \frac{e^{2n} \kappa^{3n} r_\kappa^{2n} \sin(\theta_\kappa)^n}{4^n n^{2n+\kappa/2}}.$$

Having $\widetilde{\mathbb{P}}_\kappa(n) \underset{n\to+\infty}{\sim} \mathbb{P}_\kappa(n)$ by Lemma 4, we obtain the expected result. □

5. Fluctuations around the limit shape

5.1. Basic brick fluctuations

Notation. We say that a point z is the α-barycenter of (a, b) if $z = \alpha a + (1-\alpha)b$; in this case, α is called the barycenter parameter.

Basic building bricks of a $z[n]$-gon. For any $\mathbf{z}[n]$ with distribution $\mathbb{D}_\kappa^{(n)}$, we have provided an alternative geometric description of the $\mathbf{z}[n]$-gon in terms of its $\mathsf{ECP}(\mathbf{z}[n])$, and more specifically in terms of the following:

https://doi.org/10.1017/apr.2024.63 Published online by Cambridge University Press

- the boundary distances $\boldsymbol{\ell}^{(n)}[\kappa]$ of this ECP,
- the tuple $\mathbf{s}^{(n)}[\kappa]$ counting the vectors in the corners of the ECP, and
- the fragmentation of the sides $\mathbf{c}^{(n)}[\kappa]$ into side-partitions $\mathbf{u}^{(1)}[\mathbf{N}_1^{(n)}], \dots, \mathbf{u}^{(\kappa)}[\mathbf{N}_\kappa^{(n)}]$, where $\mathbf{N}_j^{(n)} = \mathbf{s}_j^{(n)} + \mathbf{s}_{j+1}^{(n)} - 1$ for all $j \in \{1, \dots, \kappa\}$.

Notice that the contact points $\mathsf{cp}^{(n)}[\kappa]$ and the vertices $\mathsf{b}^{(n)}[\kappa]$ of the $\mathsf{ECP}(\mathbf{z}[n])$ can be recovered using these three data. Indeed, $\boldsymbol{\ell}^{(n)}[\kappa]$ determines the ECP and thus its vertices $\mathsf{b}^{(n)}[\kappa]$.

In (3.15), we see that, conditional on the jth side length $\mathbf{c}_j^{(n)} = c_j$ and on the size vector $\mathbf{s}^{(n)}[\kappa]$, the side-partition $\left(u_1^{(j)} < \ldots < u_{\mathbf{N}_j}^{(j)}\right)$ has the law of a reordered $\mathbf{N}_j$-tuple of i.i.d. uniform random variables drawn in $[0, c_j]$. The contact point $\mathsf{cp}_j^{(n)}$ is placed on the jth side of the ECP (on the segment $[\mathsf{b}_{j-1}^{(n)}, \mathsf{b}_j^{(n)}]$), at the coordinate $\mathbf{u}_{\mathbf{s}_j^{(n)}}^{(j)}$ for all $j \in \{1, \dots, \kappa\}$. This means that conditional on $(\boldsymbol{\ell}^{(n)}[\kappa], \mathbf{s}^{(n)}[\kappa])$, the tuple $\mathsf{cp}^{(n)}[\kappa]$ has independent entries, and $\mathsf{cp}_j^{(n)}$ is a $\boldsymbol{\beta}_j^{(n)}$-barycenter of $\left(\mathsf{b}_{j-1}^{(n)}, \mathsf{b}_j^{(n)}\right)$, where $\boldsymbol{\beta}_j^{(n)}$ is β-distributed with parameters $\left(\mathbf{s}_j^{(n)}, \mathbf{s}_{j+1}^{(n)}\right)$.

For $j \in \{1, \dots, \kappa\}$, the random variable

$$\boldsymbol{\delta}_j^{(n)} := \sqrt{\frac{n}{\kappa}} \left(\boldsymbol{\beta}_j^{(n)} - 1/2\right), \text{ for all } j \in \{1, \dots, \kappa\}, \tag{5.1}$$

provides the fluctuations of the barycenter parameter and thus encodes the fluctuations of the contact point $\mathsf{cp}_j^{(n)}$ on the jth side $\mathbf{c}_j^{(n)}$ of $\mathsf{ECP}(\mathbf{z}[n])$.

Fluctuations of basic bricks. We have proved that the boundary distances $\boldsymbol{\ell}^{(n)}[\kappa]$ behaved as $\frac{1}{n}$ times an exponential distribution, and we will prove in the sequel that the contact points $\mathsf{cp}^{(n)}[\kappa]$ are typically at distance $\frac{1}{\sqrt{n}}$ around their limit (this is visible in (5.1)). So in order to describe the fluctuations of the $\mathbf{z}[n]$-gon around its limit, we will state a theorem describing the joint distribution of all basic bricks taken together, rather than providing the fluctuations in $\frac{1}{\sqrt{n}}$ of a complicated object for the whole process, which would crush the behavior of some of the bricks. A comprehensive picture of the fluctuations would rely on the concatenation of all the corners' fluctuations, adjusted to take into account the contact points; we believe that presenting such a picture would not bring any new insight, so we leave it to the interested reader as an exercise.

The $\measuredangle$-convex chains. Recall Lemma 1 and its notation. Consider for all $j \in \{1, \dots, \kappa\}$ the convex chain $\mathsf{CC}_j^{(n)} = \mathsf{CC}_j^{(n)}(\mathbf{z}[n])$ lying in the jth corner of $\mathsf{ECP}(\mathbf{z}[n])$, defined as an element of $\mathsf{Chain}_{\mathbf{s}_j^{(n)}}(\mathsf{corner}_j(\mathbf{z}[n]))$ (this accounts for the decomposition of $\mathbf{z}[n]$ into convex chains). In order to understand the fluctuations of the jth corner, we will use the normalized version of each of these convex chains:

$$\measuredangle\mathsf{CC}_j^{(n)} = \mathsf{Aff}_j(\mathsf{CC}_j^{(n)}) \text{ where } \mathsf{Aff}_j = \mathsf{Aff}_{\mathsf{corner}_j(\mathbf{z}[n])}, \tag{5.2}$$

where, for a given non-flat triangle ABC, the mapping Aff_{ABC} is the unique map that sends A, B, C to (0, 0),(1, 0),(1, 1), as introduced in Lemma 2. The law of $\measuredangle\mathsf{CC}_j^{(n)}$ depends only on $\mathbf{s}_j^{(n)}$,

https://doi.org/10.1017/apr.2024.63 Published online by Cambridge University Press

but the affine mapping Aff_j depends on the coordinates of $\mathsf{ECP}(\mathbf{z}[n])$. However, determining the fluctuations of $\mathsf{CC}_j^{(n)}$ amounts to looking at those of $\measuredangle\mathsf{CC}_j^{(n)}$.

Recall that a generic $\measuredangle$-normalized convex chain $\measuredangle\mathsf{CC}_m$ of size m is a random variable whose law is that of a convex chain taken uniformly in $\mathsf{Chain}_m(\measuredangle)$. Therefore, a consequence of Lemma 1 and Lemma 2 is the following lemma.

Lemma 9. *Conditional on* $(\boldsymbol{\ell}^{(n)}[\kappa], \mathbf{s}^{(n)}[\kappa], \boldsymbol{\delta}^{(n)}[\kappa])$*, the convex chains* $\measuredangle\mathsf{CC}^{(n)}[\kappa]$ *are independent. Furthermore, for all* $j \in \{1, \dots, \kappa\}$*, the distribution of* $\measuredangle\mathsf{CC}_j^{(n)}$ *is that of a generic* $\measuredangle$*-normalized convex chain of size* $\mathbf{s}_j^{(n)}$*, i.e.* $\measuredangle\mathsf{CC}_j^{(n)} \overset{(d)}{=} \measuredangle\mathsf{CC}_{\mathbf{s}_j^{(n)}}$.

Thanks to this lemma, it should be clear that we can work on each convex chain separately when $\mathbf{s}^{(n)}[\kappa]$ is fixed. We introduce some processes in order to describe these fluctuations.

5.2. A parametrization of the normalized convex chains

Let $m \geq 0$, and let $\measuredangle\mathsf{CC}_m = ((0,0), \mathbf{z}_1, \dots, \mathbf{z}_{m-1}, (1,1))$ be a generic $\measuredangle$-normalized convex chain of size m. Rather than considering the tuple of points $((0,0), \mathbf{z}_1, \dots, \mathbf{z}_{m-1}, (1,1))$, we consider the m vectors composing the convex chain. Recall that these vectors are obtained by forming the tuples $\mathbf{u}[m]$, $\mathbf{v}[m]$ of increments of two elements taken uniformly (and independently) in the simplex $P[1, m-1]$. Then the vectors $(\mathbf{u}_i, \mathbf{v}_i)$, $i \in \{1, \dots, m\}$, are reordered by increasing slope to form this chain.

In order to use the toolbox of stochastic processes, it is convenient for us to see $\measuredangle\mathsf{CC}_m$ as a linear process. For this we will need a suitable parametrization (several choices are possible). For technical reasons, we choose the local slope of $\measuredangle\mathsf{CC}_m$ as the time parameter, and apply the arctan function in order to remain in a compact set.

Consider the point $(x_m(u), y_m(u))$ for $u \in [0,1]$, corresponding to the contributions of the previous vectors whose slope are smaller than $\tan\left(\frac{\pi}{2}u\right)$, and the process C_m defined as $\mathsf{C}_m(u) := (x_m(u), y_m(u))$ for $u \in [0,1]$. Hence, we have

$$x_m(u) = \sum_{i=1}^{m} \mathbf{u}_i \mathbf{1}_{\left\{\frac{\mathbf{v}_i}{\mathbf{u}_i} \leq \tan\left(\frac{\pi}{2}u\right)\right\}} \quad \text{and} \quad y_m(u) = \sum_{i=1}^{m} \mathbf{v}_i \mathbf{1}_{\left\{\frac{\mathbf{v}_i}{\mathbf{u}_i} \leq \tan\left(\frac{\pi}{2}u\right)\right\}}, \tag{5.3}$$

where we set $\tan\left(\frac{\pi}{2} \cdot 1\right) = +\infty$ so that $x_m(1) = y_m(1) = 1$. Note that the tuple $\measuredangle\mathsf{CC}_m$ (seen as a set) coincides with the (finite) set $\{\mathsf{C}_m(u), u \in [0,1]\}$.

Define further the curve $\mathsf{C}_\infty(u) := (x(u), y(u))$ for all $u \in [0,1]$, where

$$x(u) := 1 - \frac{1}{\left(1 + \tan\left(\frac{\pi}{2}u\right)\right)^2} \quad \text{and} \quad y(u) := \frac{\tan\left(\frac{\pi}{2}u\right)^2}{\left(1 + \tan\left(\frac{\pi}{2}u\right)\right)^2}. \tag{5.4}$$

We have once more $x(1) = y(1) = 1$. This curve is actually the parametrization of the parabolic arc lying in $\measuredangle$, tangent at $(0,0)$ and $(1,1)$. It is not surprising to encounter C_∞ here, since the convergence in distribution of C_m to C_∞ that will be stated in Theorem 9 can be seen as a consequence of Bárány's work on affine perimeters (not a direct consequence, however).

For all $j \in \{1, \dots, \kappa\}$, we let $\mathsf{C}_j^{(n)}$ be the parametrization in terms of the slope of the jth $\measuredangle$-normalized convex chain $\measuredangle\mathsf{CC}_j^{(n)}$. So now by Lemma 9, we have

$$\mathcal{L}\left(\mathsf{C}_j^{(n)}, 1 \leq j \leq \kappa \mid \mathbf{s}^{(n)}[\kappa]\right) = \mathcal{L}\left(\mathsf{C}_{\mathbf{s}_j^{(n)}}, 1 \leq j \leq \kappa\right),$$

https://doi.org/10.1017/apr.2024.63 Published online by Cambridge University Press

and conditional on $\mathbf{s}^{(n)}[\kappa]$, the $\mathsf{C}_j^{(n)}$, $j \in \{1, \dots, \kappa\}$, are independent.

Theorem 9 will also include the convergence in distribution of the fluctuation process $\mathsf{Y}_m := \sqrt{m}(\mathsf{C}_m - \mathsf{C}_\infty)$ to a Gaussian process. This convergence is actually the key that will help us understand the fluctuations of the convex chain $\measuredangle\mathsf{CC}_j^{(n)}$ around its limit $\measuredangle\mathsf{CC}_\infty$. Indeed, conditional on $(\boldsymbol{\ell}^{(n)}[\kappa], \mathbf{s}^{(n)}[\kappa])$, the processes

$$\mathsf{Y}_j^{(n)} := \sqrt{\frac{n}{\kappa}} \left(\mathsf{C}_j^{(n)} - \mathsf{C}_\infty \right), \quad 1 \le j \le \kappa, \tag{5.5}$$

describe the successive fluctuations of the $\measuredangle$-convex chain in their corners. We are now able to state the most important result of this section (in which we borrow the notation of Theorem 7).

Theorem 8. *(0) Conditional on $(\boldsymbol{\ell}^{(n)}[\kappa], \mathbf{s}^{(n)}[\kappa])$, the processes $\mathsf{C}^{(n)}[\kappa]$ are independent and have the same distribution. For all $j \in \{1, \dots, \kappa\}$, the process $\mathsf{C}_j^{(n)}$ converges in $D([0,1], \mathbb{R})$, which we endow with the Skorokhod topology for the rest of this paper, to the deterministic process C_∞ introduced in (5.4).*

Furthermore, the 'fluctuation' tuple $\left(\overline{\boldsymbol{\ell}}^{(n)}[\kappa], \mathbf{x}^{(n)}[\kappa - 1], \boldsymbol{\delta}^{(n)}[\kappa], \mathsf{Y}^{(n)}[\kappa]\right)$ converges in distribution in $\mathbb{R}^\kappa \times \mathbb{R}^{\kappa-1} \times \mathbb{R}^\kappa \times D([0,1], \mathbb{R})^\kappa$ (equipped with the corresponding product topology) to a tuple $\left(\overline{\boldsymbol{\ell}}[\kappa], \mathbf{x}[\kappa - 1], \boldsymbol{\delta}[\kappa], \mathsf{Y}[\kappa]\right)$ with the following properties:

(1) *The variables $\left(\overline{\boldsymbol{\ell}}[\kappa], \mathbf{x}[\kappa - 1]\right)$ and their fluctuations are as already described in Theorem 7.*

(2) *Conditional on $\left(\overline{\boldsymbol{\ell}}[\kappa], \mathbf{x}[\kappa - 1]\right)$, the variables $\boldsymbol{\delta}_1, \dots, \boldsymbol{\delta}_\kappa$ are independent, and for all $j \in \{1, \dots, \kappa\}$, $\boldsymbol{\delta}_j$ is a normal random variable with mean $(\mathbf{x}_j - \mathbf{x}_{j+1})/4$ and variance $1/8$.*

(3) *The tuple $\mathsf{Y}^{(n)}[\kappa]$ converges in distribution in $D([0,1])$ to the tuple $\mathsf{Y}[\kappa]$, where the processes $\mathsf{Y}_1, \dots, \mathsf{Y}_\kappa$ are i.i.d., and their common distribution is that of Y_∞, a Gaussian process whose law will be detailed in Theorem 9.*

This convergence theorem contains the fluctuations of every basic brick of $\mathsf{ECP}(\mathbf{z}[n])$.

Proof. We start with the proof of Part 2. Recall (5.1). Conditional on $(\boldsymbol{\ell}^{(n)}[\kappa], \mathbf{s}^{(n)}[\kappa])$, the $\mathsf{cp}^{(n)}[\kappa]$ are independent, which provides the independence of the $\boldsymbol{\delta}^{(n)}[\kappa]$ (given $(\boldsymbol{\ell}^{(n)}[\kappa], \mathbf{s}^{(n)}[\kappa])$). It suffices then, to prove the convergence of the marginals; by symmetry, we will prove only the convergence of $\boldsymbol{\delta}_j^{(n)} = \sqrt{\frac{n}{\kappa}} \left(\boldsymbol{\beta}_j(n) - \frac{1}{2} \right)$.

According to the Skorokhod representation theorem, up to a change of probability space, we may assume that

$$\frac{\mathbf{s}_j^{(n)} - \frac{n}{\kappa}}{\sqrt{n/\kappa}} \xrightarrow[n]{(a.s.)} \mathbf{x}_j$$

for all $j \in \{1, \dots, \kappa\}$. We may then assume that $\mathbf{s}_j^{(n)} = \frac{n}{\kappa} + \mathbf{x}_j \sqrt{\frac{n}{\kappa}} + o(\sqrt{n})$. In the sequel we drop the $o(\sqrt{n})$ since it provides only negligible contributions.

Since $\beta_j^{(n)}$ is β-distributed with parameters $(\mathbf{s}_j^{(n)}, \mathbf{s}_{j+1}^{(n)})$, we have $\beta_j^{(n)} \overset{(d)}{=} \frac{\mathbf{T}_j}{\mathbf{T}_j + \mathbf{T}_{j+1}}$ where $\mathbf{T}_j$ (resp. $\mathbf{T}_{j+1}$) is a random variable that is gamma-distributed with parameter $\mathbf{s}_j^{(n)}$ (resp. $\mathbf{s}_{j+1}^{(n)}$), these variables being independent.

https://doi.org/10.1017/apr.2024.63 Published online by Cambridge University Press

By the central limit theorem, we have

$$\left(\frac{\mathbf{T}_j - \frac{n}{\kappa}}{\sqrt{n/\kappa}}, \frac{\mathbf{T}_{j+1} - \frac{n}{\kappa}}{\sqrt{n/\kappa}}\right) \xrightarrow[n]{(d)} \left(\mathbf{x}_j + \mathbf{q}_1, \mathbf{x}_{j+1} + \mathbf{q}_2\right),$$

for $\mathbf{q}_1, \mathbf{q}_2$ two i.i.d. standard Gaussian random variables $\mathcal{N}(0, 1)$. Hence

$$\sqrt{\frac{n}{\kappa}}\left(\boldsymbol{\beta}_j^{(n)} - \frac{1}{2}\right) \overset{(d)}{=} \sqrt{\frac{n}{\kappa}}\left(\frac{\mathbf{T}_j}{\mathbf{T}_j + \mathbf{T}_{j+1}} - \frac{1}{2}\right) \tag{5.6}$$

$$\overset{(d)}{=} \frac{n/\kappa}{2(\mathbf{T}_j + \mathbf{T}_{j+1})}\left(\frac{\mathbf{T}_j - \frac{n}{\kappa}}{\sqrt{n/\kappa}} - \frac{\mathbf{T}_{j+1} - \frac{n}{\kappa}}{\sqrt{n/\kappa}}\right) \tag{5.7}$$

$$\xrightarrow[n]{(d)} \frac{1}{4}\left(\mathbf{x}_j - \mathbf{x}_{j+1} + \mathbf{q}_1 - \mathbf{q}_2\right). \tag{5.8}$$

This gives the expected result.

Notice now that Part 3 implies Part 0. Indeed, the weak convergence of the processes $\mathsf{C}_1^{(n)}, \ldots, \mathsf{C}_\kappa^{(n)}$ to C_∞ is a consequence of that of $\mathsf{Y}_1^{(n)}, \ldots, \mathsf{Y}_\kappa^{(n)}$ to Y_∞. This latter convergence will be the main object of the rest of this section. This is by far the most complicated proof, and we will need to introduce quite a lot of tools to achieve it. We will come back to this proof later. □

5.3. Convergence of the normalized convex chain

In order to prove the convergence of $\mathsf{Y}^{(n)}[\kappa]$, a new parametrization is required.

Notation. Define the maps $g : t \in [0, 1] \mapsto \tan\left(\frac{\pi}{2}t\right)$ and $h : t \in [0, 1] \mapsto \dfrac{1}{1 + g(t)}$. Let us introduce the following mappings for all $(s, t) \in [0, 1]^2$:

$$f(s, t) = h(s) - h(t), \tag{5.9}$$

$$e_1(s, t) = h(s)^2 - h(t)^2, \tag{5.10}$$

$$e_2(s, t) = (g(t)h(t))^2 - (g(s)h(s))^2, \tag{5.11}$$

$$v_1(s, t) = 2\left(h(s)^3 - h(t)^3\right) - e_1(s, t)^2, \tag{5.12}$$

$$v_2(s, t) = 2\left((g(t)h(t))^3 - (g(s)h(s))^3\right) - e_2(s, t)^2. \tag{5.13}$$

In the following, we will denote by $\left(\mathsf{Z}^{(1)}, \mathsf{Z}^{(2)}\right)$ the coordinates of any process Z taking values in $\mathbb{R}^2$.

Recall the definition of the process Y_m given in (5.5).

Theorem 9. *The convergence in distribution*

$$\mathsf{C}_m \xrightarrow[n]{(d)} \mathsf{C}_\infty \tag{5.14}$$

holds in $D([0, 1])^2$.

https://doi.org/10.1017/apr.2024.63 Published online by Cambridge University Press

The sequence of processes (Y_m) *converges in distribution to a centered Gaussian process* Y_∞ *in* $D([0,1])^2$*, whose coordinates can be represented as*

$$\mathsf{Y}_\infty^{(1)}(t) = \overline{\mathsf{Y}}^{(1)}(t) + \mathbf{g}_1 \cdot x(t) + (\mathbf{g}_1 - \mathbf{g}_2) \cdot \frac{2}{\pi} \frac{\tan\left(\frac{\pi}{2}t\right)}{1+\tan\left(\frac{\pi}{2}t\right)^2} \cdot x'(t), \tag{5.15}$$

$$\mathsf{Y}_\infty^{(2)}(t) = \overline{\mathsf{Y}}^{(2)}(t) + \mathbf{g}_1 \cdot y(t) + (\mathbf{g}_1 - \mathbf{g}_2) \cdot \frac{2}{\pi} \frac{\tan\left(\frac{\pi}{2}t\right)}{1+\tan\left(\frac{\pi}{2}t\right)^2} \cdot y'(t), \tag{5.16}$$

and where the following hold:

(i) *The mappings x',y' are the derivatives of* $t \mapsto x(t) = \mathsf{C}_\infty^{(1)}(t)$, $t \mapsto y(t) = \mathsf{C}_\infty^{(2)}(t)$ *(they are deterministic processes).*

(ii) *The random variables* $\mathbf{g}_1, \mathbf{g}_2$ *are independent standard Gaussian variables* $\mathcal{N}(0,1)$.

(iii) *The process* $\overline{\mathsf{Y}}$ *is a centered Gaussian process with variance function*

$$\mathbb{V}\left[\overline{\mathsf{Y}}^{(p)}(t)\right] = f(0,t)v_p(0,t) + f(0,t)(1-f(0,t))e_p(0,t)^2 \quad \forall p \in \{1,2\} \tag{5.17}$$

and with covariance function determined, for $(s<t) \in [0,1]^2$*, by*

$$\operatorname{cov}\left(\overline{\mathsf{Y}}^{(p)}(s), \overline{\mathsf{Y}}^{(q)}(t) - \overline{\mathsf{Y}}^{(q)}(s)\right) = -f(0,s)f(s,t)e_p(0,s)e_q(s,t) \quad \forall (p,q) \in \{1,2\}^2. \tag{5.18}$$

(iv) *The processes* $\overline{\mathsf{Y}}^{(1)}, \overline{\mathsf{Y}}^{(2)}$ *are independent from* $\mathbf{g}_1$ *and* $\mathbf{g}_2$.

The reason why the functions f, e_p, v_p appear will be revealed in Proposition 3. Notice that

$$\lim_{t\to 1} \frac{2}{\pi} \frac{\tan\left(\frac{\pi}{2}t\right)}{1+\tan\left(\frac{\pi}{2}t\right)^2} \cdot x'(t) \quad \text{and} \quad \lim_{t\to 1} \frac{2}{\pi} \frac{\tan\left(\frac{\pi}{2}t\right)}{1+\tan\left(\frac{\pi}{2}t\right)^2} \cdot y'(t)$$

are finite, so the process Y_∞ is also well-defined at $t=1$.

Once more, the first assertion (5.14) is a consequence of the second; we will prove only the second one.

Remark 4. (*Back to Theorem 8.*) The proof of Part 4 of Theorem 8 is an immediate consequence of Theorem 9! Indeed, for all $j \in \{1,\dots,\kappa\}$ we have

$$\mathsf{Y}_j^{(n)} = \sqrt{\frac{n/\kappa}{\mathbf{s}_j^{(n)}}} \sqrt{\mathbf{s}_j^{(n)}} \left(\mathsf{C}_{\mathbf{s}_j^{(n)}} - \mathsf{C}_\infty\right). \tag{5.19}$$

By Theorem 9, the term $\sqrt{\mathbf{s}_j^{(n)}}\left(\mathsf{C}_{\mathbf{s}_j^{(n)}} - \mathsf{C}_\infty\right)$ converges in distribution to Y_∞, and we know furthermore that $\frac{\mathbf{s}_j^{(n)}}{n/\kappa} \xrightarrow[n]{(d)} 1$. Slutsky's lemma allows us to conclude.

Remark 5. An immediate consequence of Theorem 9 is that the curve $\mathcal{C}_m = \{(t, \mathsf{C}_m(t)), t \in [0,1]\}$, seen as a compact set, converges in distribution to $\mathcal{C}_\infty = \{(t, \mathsf{C}_\infty(t)), t \in [0,1]\}$, for the Hausdorff distance, as $m \to +\infty$. Furthermore, the term $\sqrt{m} d_H(\mathcal{C}_m, \mathcal{C}_\infty)$ converges in distribution to a non-trivial random value.

https://doi.org/10.1017/apr.2024.63 Published online by Cambridge University Press

5.4. Proof of Theorem 9

The parametrization of C_m in terms of the variables $(\mathbf{u}[m], \mathbf{v}[m])$ is tricky, since these variables are interconnected (they sum to 1), paired, and then sorted by increasing slope (even if the parametrization in terms of the indicator of the slope allows one to get rid of this difficulty).

Exponential model. Let $\boldsymbol{\zeta}[m], \boldsymbol{\xi}[m]$ be two m-tuples of random variables exponentially distributed with mean 1, all these variables being independent. Set $\boldsymbol{\alpha}_m^{(1)} = \sum_{i=1}^m \boldsymbol{\zeta}_i$, $\boldsymbol{\alpha}_m^{(2)} = \sum_{i=1}^m \boldsymbol{\xi}_i$. The following equalities in distribution are classically used to represent order statistics by exponential random variables (a more general result can be found in [14, Theorem 3]):

$$\mathbf{u}[m] \overset{(d)}{=} \frac{1}{\boldsymbol{\alpha}_m^{(1)}} \boldsymbol{\zeta}[m] \quad \text{and} \quad \mathbf{v}[m] \overset{(d)}{=} \frac{1}{\boldsymbol{\alpha}_m^{(2)}} \boldsymbol{\xi}[m]. \tag{5.20}$$

Lemma 10. *Let $\mathbf{g}_1, \mathbf{g}_2$ be two independent standard Gaussian variables. The following convergence in distribution holds in $\mathbb{R}^3$:*

$$\sqrt{m}\left[\left(\frac{m}{\boldsymbol{\alpha}_m^{(1)}} - 1\right), \left(\frac{m}{\boldsymbol{\alpha}_m^{(2)}} - 1\right), \left(\frac{\boldsymbol{\alpha}_m^{(2)}}{\boldsymbol{\alpha}_m^{(1)}} - 1\right)\right] \xrightarrow[m]{(d)} \left[\mathbf{g}_1, \mathbf{g}_2, \mathbf{g}_1 - \mathbf{g}_2\right]. \tag{5.21}$$

Proof. Let us write

$$\sqrt{m}\left(\frac{m}{\boldsymbol{\alpha}_m^{(j)}} - 1\right) = \frac{m}{\boldsymbol{\alpha}_m^{(j)}} \times \frac{m - \boldsymbol{\alpha}_m^{(j)}}{\sqrt{m}} \quad \text{for all } j \in \{1, 2\}.$$

As for the third marginal in the left-hand side of (5.21), write

$$\sqrt{m}\left(\frac{\boldsymbol{\alpha}_m^{(2)}}{\boldsymbol{\alpha}_m^{(1)}} - 1\right) = \frac{m}{\boldsymbol{\alpha}_m^{(1)}} \times \left[\frac{m - \boldsymbol{\alpha}_m^{(1)}}{\sqrt{m}} - \frac{m - \boldsymbol{\alpha}_m^{(2)}}{\sqrt{m}}\right].$$

Now, in all these cases, Slutsky's lemma together with the central limit theorem gives the expected convergence. □

We now build an object $\overline{\mathsf{C}}_m$ close to C_m whose convergence is easier to prove. We pair the tuples $\boldsymbol{\zeta}[m], \boldsymbol{\xi}[m]$ to form m vectors $\mathbf{w}_i = (\boldsymbol{\zeta}_i, \boldsymbol{\xi}_i)$ for all $i \in \{1, \dots, m\}$. When ordered by increasing slope and summed one by one, these vectors form the boundary of a convex polygon, whose vertices form a convex chain in the triangle $\mathsf{Tri}(m)$ of vertices $(0, 0)$, $(\boldsymbol{\alpha}_m^{(1)}, 0)$, $(\boldsymbol{\alpha}_m^{(1)}, \boldsymbol{\alpha}_m^{(2)})$. If we renormalize the x-coordinates of these vectors by $\boldsymbol{\alpha}_m^{(1)}$ and the y-coordinates by $\boldsymbol{\alpha}_m^{(2)}$, we obtain a convex chain in $\measuredangle$ whose law is that of a generic $\measuredangle$-normalized convex chain. However, we want to study the convex chain before renormalization. Hence, to obtain the analogous process before normalization, we consider the contribution $\overline{\mathsf{C}}_m(u)$ of the vectors $(\boldsymbol{\zeta}_i, \boldsymbol{\xi}_i)$ with slope smaller or equal than $u \in [0, 1]$,

$$\overline{\mathsf{C}}_m(u) := \frac{1}{m} \sum_{i=1}^m (\boldsymbol{\zeta}_i, \boldsymbol{\xi}_i) \mathbf{1}_{\left\{\frac{\boldsymbol{\xi}_i}{\boldsymbol{\zeta}_i} \le \tan\left(\frac{\pi}{2}u\right)\right\}}. \tag{5.22}$$

We will eventually send $\overline{\mathsf{C}}_m$ to C_m by sending $\mathsf{Tri}(m)$ to $\measuredangle$; in order to control the induced slope modification, we introduce the function $\boldsymbol{\alpha}_m$ defined in $[0, 1]$ by

$$\boldsymbol{\alpha}_m(u) = \frac{2}{\pi} \arctan\left(\frac{\boldsymbol{\alpha}_m^{(2)}}{\boldsymbol{\alpha}_m^{(1)}} \tan\left(\frac{\pi}{2}u\right)\right), \tag{5.23}$$

https://doi.org/10.1017/apr.2024.63 Published online by Cambridge University Press

and such that $\tan\left(\frac{\pi}{2}\boldsymbol{\alpha}_m(u)\right) = \frac{\boldsymbol{\alpha}_m^{(2)}}{\boldsymbol{\alpha}_m^{(1)}}\tan\left(\frac{\pi}{2}u\right)$.

By what we just explained, the link between $\overline{\mathsf{C}}_m$ and C_m is the following:

$$\mathsf{C}_m \overset{(d)}{=} \left(\frac{m}{\boldsymbol{\alpha}_m^{(1)}}\overline{\mathsf{C}}_m^{(1)}\left(\boldsymbol{\alpha}_m(u)\right), \frac{m}{\boldsymbol{\alpha}_m^{(2)}}\overline{\mathsf{C}}_m^{(2)}\left(\boldsymbol{\alpha}_m(u)\right), u \in [0,1]\right). \tag{5.24}$$

Indeed, for the example of the first coordinate,

$$\frac{m}{\boldsymbol{\alpha}_m^{(1)}}\overline{\mathsf{C}}_m^{(1)}\left(\boldsymbol{\alpha}_m(u)\right) = \frac{1}{\boldsymbol{\alpha}_m^{(1)}}\sum_{i=1}^{m}\boldsymbol{\zeta}_i \mathbf{1}_{\left\{\frac{\boldsymbol{\alpha}_m^{(1)}\boldsymbol{\xi}_i}{\boldsymbol{\alpha}_m^{(2)}\boldsymbol{\zeta}_i} \le \tan\left(\frac{\pi}{2}u\right)\right\}},$$

which takes into account the dilatation of the vectors composing the boundary of the convex chain, as well as the normalization of the slope that is effected by an affine dilatation of $\measuredangle$.

Decomposition of Y_m according to the exponential model. Let us now decompose the process Y_m into several processes that are easier to manipulate. Let $u \in [0,1]$; for the pth coordinate, $p \in \{1,2\}$, we have

$$\mathsf{Y}_m^{(p)}(u) = \sqrt{m}\left[\frac{m}{\boldsymbol{\alpha}_m^{(1)}}\left(\overline{\mathsf{C}}_m^{(p)}\left(\boldsymbol{\alpha}_m(u)\right) - \mathsf{C}_\infty^{(p)}\left(\boldsymbol{\alpha}_m(u)\right)\right) + \frac{m}{\boldsymbol{\alpha}_m^{(1)}}\left(\mathsf{C}_\infty^{(p)}\left(\boldsymbol{\alpha}_m(u)\right) - \mathsf{C}_\infty^{(p)}(u)\right) + \left(\frac{m}{\boldsymbol{\alpha}_m^{(1)}} - 1\right)\mathsf{C}_\infty^{(p)}(u)\right]. \tag{5.25}$$

Consider $\mathbf{g}_1, \mathbf{g}_2$, the two independent standard Gaussian variables of Lemma 10, and let us handle the last two terms of (5.25). Notice first that

$$\frac{\mathsf{C}_\infty^{(1)}\left(\frac{\boldsymbol{\alpha}_m^{(2)}}{\boldsymbol{\alpha}_m^{(1)}}u\right) - \mathsf{C}_\infty^{(1)}(u)}{\left(\frac{\boldsymbol{\alpha}_m^{(2)}}{\boldsymbol{\alpha}_m^{(1)}} - 1\right)} \overset{(d)}{\underset{m}{\longrightarrow}} \frac{2}{\pi}\frac{\tan\left(\frac{\pi}{2}u\right)}{1+\tan\left(\frac{\pi}{2}u\right)^2}\cdot x'(u),$$

where we have used $\frac{\boldsymbol{\alpha}_m^{(2)}}{\boldsymbol{\alpha}_m^{(1)}} \overset{(d)}{\underset{n}{\longrightarrow}} 1$ by (5.21). This means that with Lemma 10 we have

$$\sqrt{m}\left[\frac{m}{\boldsymbol{\alpha}_m^{(1)}}\left(\mathsf{C}_\infty^{(1)}\left(\frac{\boldsymbol{\alpha}_m^{(2)}}{\boldsymbol{\alpha}_m^{(1)}}u\right) - \mathsf{C}_\infty^{(1)}(u)\right) + \left(\frac{m}{\boldsymbol{\alpha}_m^{(1)}} - 1\right)\mathsf{C}_\infty^{(1)}(u)\right] \overset{(d)}{\underset{m}{\longrightarrow}} (\mathbf{g}_1 - \mathbf{g}_2)\cdot\frac{2}{\pi}\frac{\tan\left(\frac{\pi}{2}u\right)}{1+\tan\left(\frac{\pi}{2}u\right)^2}\cdot x'(u) + \mathbf{g}_1\cdot x(u). \tag{5.26}$$

For the second coordinate, we get a similar convergence of the two last terms:

$$\sqrt{m}\left[\frac{m}{\boldsymbol{\alpha}_m^{(1)}}\left(\mathsf{C}_\infty^{(2)}\left(\frac{\boldsymbol{\alpha}_m^{(2)}}{\boldsymbol{\alpha}_m^{(1)}}u\right) - \mathsf{C}_\infty^{(2)}(u)\right) + \left(\frac{m}{\boldsymbol{\alpha}_m^{(1)}} - 1\right)\mathsf{C}_\infty^{(2)}(u)\right] \overset{(d)}{\underset{m}{\longrightarrow}} (\mathbf{g}_1 - \mathbf{g}_2)\cdot\frac{2}{\pi}\frac{\tan\left(\frac{\pi}{2}u\right)}{1+\tan\left(\frac{\pi}{2}u\right)^2}\cdot y'(u) + \mathbf{g}_1\cdot y(u), \tag{5.27}$$

https://doi.org/10.1017/apr.2024.63 Published online by Cambridge University Press

where the convergence of (5.26) and (5.27) has to be thought of as a joint convergence including the same Gaussian standard random variables $\mathbf{g}_1, \mathbf{g}_2$. This allows us to recover the last two processes mentioned in Theorem 8. For the first one, corresponding to the first term of the decomposition of (5.25), we first need to prove an intermediary lemma.

Lemma 11. *Suppose that we have the following convergence in distribution:*

$$\sqrt{m}\left[\overline{\mathsf{C}}_m - \mathsf{C}_\infty\right] \xrightarrow[m]{(d)} \overline{\mathsf{Y}}, \tag{5.28}$$

in $D([0,1])^2$*, for some process* $\overline{\mathsf{Y}}$*. Then, for all* $u \in [0,1]$*, we have*

$$\sqrt{m}\left[\overline{\mathsf{C}}_m(u) - \mathsf{C}_\infty(u),\ \frac{m}{\boldsymbol{\alpha}_m^{(1)}}\left(\overline{\mathsf{C}}_m\big(\boldsymbol{\alpha}_m(u)\big) - \mathsf{C}_\infty\big(\boldsymbol{\alpha}_m(u)\big)\right)\right] \xrightarrow[m]{(d)} \left[\overline{\mathsf{Y}}(u), \overline{\mathsf{Y}}(u)\right]. \tag{5.29}$$

In words, the limiting processes of the two processes on the left-hand side are equal.

Proof. By the strong law of large numbers we have $\frac{m}{\boldsymbol{\alpha}_m^{(1)}} \xrightarrow[n]{(a.s.)} 1$, as well as $\frac{\boldsymbol{\alpha}_m^{(2)}}{\boldsymbol{\alpha}_m^{(1)}}$ and thus $\boldsymbol{\alpha}_m(u) \xrightarrow[n]{(a.s.)} u$ for all $u \in [0,1]$. By (5.28), the sequence of processes $\left(\sqrt{m}\left(\overline{\mathsf{C}}_m - \mathsf{C}_\infty\right)\right)$ is tight in $D([0,1]^2)$. The map $F{:}u \mapsto \tan\left(\frac{\pi}{2}u\right)$ is a continuous non-decreasing surjective map from $[0,1]$ to $[0,+\infty]$ (where $[0,+\infty]$ is seen as a compact set). Let (X_n) be a sequence of processes (with values in $\mathbb{R}$) that converges in distribution to X in $D([0,+\infty])$ (where, for all $n \in \mathbb{N}$, $\lim_{t\to+\infty} \mathsf{X}_n(t)$ is finite, as is $\lim_{t\to+\infty} \mathsf{X}(t)$). This implies that $\mathsf{X}_n \circ F$ converges in distribution to $\mathsf{X} \circ F$ in $D([0,1])$. We claim that if $(a_n, b_n) \xrightarrow[n]{(a.s.)} 1$, then

$$\big(a_n \mathsf{X}_n(b_n F(u)),\ u \in [0,1]\big) \xrightarrow[n]{(d)} \big(\mathsf{X}(F(u)),\ u \in [0,1]\big)$$

for the same topology. A proof runs as follows: by the Skorokhod representation theorem, there exists a probability space in which are simultaneously defined some copies $(\overline{a}_n, \overline{b}_n, \overline{\mathsf{X}}_n)$ of (a_n, b_n, X_n) (and $\bar{\mathsf{X}}$ a copy of X) such that $(\overline{a}_n, \overline{b}_n, \overline{\mathsf{X}}_n \circ F) \xrightarrow[n]{(a.s.)} (a, b, \overline{\mathsf{X}} \circ F)$.

Set $\lambda_n : u \mapsto \frac{2}{\pi} \arctan\left(\tan\left(\frac{\pi}{2}u\right)/b_n\right)$, which is a sequence of strictly increasing continuous functions mapping $[0,1]$ to itself, with $\lambda_n(0) = 0$ and $\lambda_n(1) = 1$. In particular, $\overline{\mathsf{X}}_n(b_n F(\lambda_n(u))) = \overline{\mathsf{X}}_n(F(u))$. Hence

$$\sup_u \left|\overline{\mathsf{X}}_n(b_n F(\lambda_n(u))) - \overline{\mathsf{X}}(F(u))\right| \to 0,$$

since $(\overline{\mathsf{X}}_n \circ F)$ converges to $(\overline{\mathsf{X}} \circ F)$ in $D([0,1])$ (and this holds ω by ω). Furthermore, $\lambda_n(u) \to u$ uniformly in $[0,1]$, and then, according to Billingsley [4, p. 124], we may conclude that the claim holds true. □

Notation. In order to prove Theorem 9 (and thus Theorem 8), it remains to prove (5.28). We set $\overline{\mathsf{Y}}_m := \sqrt{m}\left(\overline{\mathsf{C}}_m - \mathsf{C}_\infty\right)$.

Lemma 12. *The process* $\overline{\mathsf{Y}}_m$ *converges in distribution to* $\overline{\mathsf{Y}}$ *in* $D([0,1], \mathbb{R}^2)$*, where the process* $\overline{\mathsf{Y}}$ *is the process introduced in Theorem 9.*

https://doi.org/10.1017/apr.2024.63 Published online by Cambridge University Press

The proof of this lemma is in two steps, including first the convergence of the finite-dimensional distributions (FDDs) and the tightness of the process. Therefore, we need a suitable parametrization to complete this proof.

Parametrization in the exponential model. Fix some $k \geq 1$ and $(0 = u_0 < u_1 < \ldots < u_{k-1} < u_k = 1)$. We will prove the convergence of $\mathsf{Y}_m(u_i)$ for all $i \in \{1, \ldots, k\}$ (recall that $\mathsf{Y}_m(u_0) = 0$ a.s.). Fix, for the moment, $i \in \{1, \ldots, k\}$. The random variable

$$\mathbf{n}(u_i) = \mathbf{n}_i := \left| \left\{ j; \tan\left(\frac{\pi}{2} u_{i-1}\right) < \frac{\boldsymbol{\xi}_j}{\boldsymbol{\zeta}_j} \leq \tan\left(\frac{\pi}{2} u_i\right) \right\} \right|$$

counts the number of vectors whose slope is in the interval $\left(\tan\left(\frac{\pi}{2}u_{i-1}\right), \tan\left(\frac{\pi}{2}u_i\right)\right]$. We denote by $\left(\mathbf{w}_1^{(i)}, \ldots, \mathbf{w}_{\mathbf{n}_i}^{(i)}\right)$ the sequence of these vectors taken in their initial order. We have

$$\overline{\mathsf{C}}_m(u_i) = \frac{1}{m} \sum_{s=1}^{i} \sum_{j=1}^{\mathbf{n}_s} \mathbf{w}_j^{(s)}, \tag{5.30}$$

since $\overline{\mathsf{C}}_m(u_i)$ is obtained by taking the sum of vectors with slopes smaller than or equal to $\tan\left(\frac{\pi}{2}u_i\right)$. For this i, the variables $\mathbf{w}_j^{(i)}$ are independent and distributed as the law of a pair $(\boldsymbol{\zeta}, \boldsymbol{\xi})$ conditioned on $\left\{\tan\left(\frac{\pi}{2}u_{i-1}\right) < \frac{\boldsymbol{\xi}}{\boldsymbol{\zeta}} \leq \tan\left(\frac{\pi}{2}u_i\right)\right\}$, where $\boldsymbol{\zeta}, \boldsymbol{\xi}$ are independent exponential variables with mean 1. Denote by $\mathbf{w}^{(i)} = (\boldsymbol{\zeta}^{(i)}, \boldsymbol{\xi}^{(i)})$ a generic random value with this conditional law. Note that the vectors of the sequence $\left(\mathbf{w}_1^{(i)}, \ldots, \mathbf{w}_{\mathbf{n}_i}^{(i)}\right)$ have the same distribution as $\mathbf{w}^{(i)}$.

By setting the non-decreasing mapping

$$q : u \in [0, 1] \mapsto \mathbb{P}\left(\frac{\boldsymbol{\xi}}{\boldsymbol{\zeta}} \leq \tan\left(\frac{\pi}{2}u\right)\right) = 1 - \frac{1}{1 + \tan\left(\frac{\pi}{2}u\right)}, \tag{5.31}$$

we may then set

$$p_i := q(u_i) - q(u_{i-1}),$$

so that the tuple $\mathbf{n}[k]$ has a multinomial distribution $\mathcal{M}(m, p[k])$.

Proposition 3. *Recall the mappings introduced from (5.9) to (5.13). The following properties of $\boldsymbol{\zeta}^{(i)}$ and $\boldsymbol{\xi}^{(i)}$ hold:*

1. $p_i = f(u_{i-1}, u_i)$;
2. $\mathbb{E}\left[\boldsymbol{\zeta}^{(i)}\right] = e_1(u_{i-1}, u_i)/p_i$, and $\mathbb{E}\left[\boldsymbol{\xi}^{(i)}\right] = e_2(u_{i-1}, u_i)/p_i$;
3. $\mathbb{V}\left[\boldsymbol{\zeta}^{(i)}\right] = v_1(u_{i-1}, u_i)/p_i$, and $\mathbb{V}\left[\boldsymbol{\xi}^{(i)}\right] = v_2(u_{i-1}, u_i)/p_i$;
4. *for all* $s \in \mathbb{N}$, $\mathbb{E}\left[\left|\boldsymbol{\zeta}^{(i)}\right|^s\right] \leq s!$ and $\mathbb{E}\left[\left|\boldsymbol{\xi}^{(i)}\right|^s\right] \leq s!$.

Proof. The first three statements come from standard integral computations. As an illustrative example, let us compute $\mathbb{E}\left[\boldsymbol{\xi}^{(i)}\right]$:

https://doi.org/10.1017/apr.2024.63 Published online by Cambridge University Press

$$\begin{aligned}\mathbb{E}\left[\boldsymbol{\xi}^{(i)}\right] &= \frac{1}{p_i}\int_{(\mathbb{R}_+)^2} ye^{-x-y}\mathbf{1}_{\{\tan(\frac{\pi}{2}u_{i-1})<\frac{y}{x}\le\tan(\frac{\pi}{2}u_i)\}}\mathrm{d}x\mathrm{d}y\\ &= \frac{1}{p_i}\int_0^{+\infty} ye^{-y}\int_{\frac{y}{\tan(\frac{\pi}{2}u_i)}}^{\frac{y}{\tan(\frac{\pi}{2}u_{i-1})}} e^{-x}\mathrm{d}x\mathrm{d}y\\ &= \frac{1}{p_i}\int_0^{+\infty} ye^{-(1+\frac{1}{\tan(\frac{\pi}{2}u_i)})y}\mathrm{d}y - \int_0^{+\infty} ye^{-(1+\frac{1}{\tan(\frac{\pi}{2}u_{i-1})})y}\mathrm{d}y\\ &= \frac{1}{p_i}e_2(u_{i-1},u_i). \end{aligned} \tag{5.32}$$

The fourth statement comes from the fact the sth moment of an exponential random variable of mean 1 equals $s!$. □

The following proposition establishes the link between our parametrization and the limit parabola.

Proposition 4. *For all $u\in[0,1]$, we have, for all $m\ge 1$,*

$$\mathbb{E}\left[\overline{\mathsf{C}}_m(u)\right] = \mathsf{C}_\infty(u), \tag{5.33}$$

where $\mathsf{C}_\infty(u)$ is given in (5.4).

Proof. We have

$$\overline{\mathsf{C}}_m(u) = \frac{1}{m}\sum_{j=1}^{m}(\boldsymbol{\zeta}_j,\boldsymbol{\xi}_j)\,\mathbf{1}_{\left\{\frac{\xi_j}{\zeta_j}\le\tan(\frac{\pi}{2}u)\right\}}.$$

By linearity of the expectation, it suffices to prove that

$$\mathsf{C}_\infty(u) = \mathbb{E}\left[(\boldsymbol{\zeta},\boldsymbol{\xi})\,\mathbf{1}_{\left\{\frac{\xi}{\zeta}\le\tan(\frac{\pi}{2}u)\right\}}\right].$$

Standard integral computations very similar to (5.32) allow us to compute

$$\mathbb{E}\left[\boldsymbol{\zeta}_j\mathbf{1}_{\left\{\frac{\xi_j}{\zeta_j}\le\tan(\frac{\pi}{2}u)\right\}}\right] = x(u) \quad\text{and}\quad \mathbb{E}\left[\boldsymbol{\xi}_j\mathbf{1}_{\left\{\frac{\xi_j}{\zeta_j}\le\tan(\frac{\pi}{2}u)\right\}}\right] = y(u)$$

for all $j\in\{1,\dots,m\}$, which is (5.4). It also follows that

$$\mathsf{C}_\infty(u_i) = \sum_{s=1}^{i}\mathbb{E}\left[(\boldsymbol{\zeta},\boldsymbol{\xi})\,\mathbf{1}_{\left\{\tan(\frac{\pi}{2}u_{s-1})<\frac{\xi}{\zeta}\le\tan(\frac{\pi}{2}u_s)\right\}}\right] = \sum_{s=1}^{i}p_s\mathbb{E}\left[\mathbf{w}^{(s)}\right]. \tag{5.34}$$

□

We may now come back to the proof of Lemma 12.

Proof of Lemma 12. We start by proving the convergence of the FDDs of $\overline{\mathsf{Y}}_m$ to those of $\overline{\mathsf{Y}}$.

FDDs. Let us split $\overline{\mathsf{Y}}_m(u_i)$ in order to make more visible the convergence result we want to prove. For $i\in\{1,\dots,k\}$, we have

$$\overline{\mathsf{Y}}_m(u_i) = \frac{1}{\sqrt{m}}\left(\sum_{s=1}^{i}\sum_{j=1}^{\mathbf{n}_s}\left(\mathbf{w}_j^{(s)} - \mathbb{E}\left[\mathbf{w}_j^{(s)}\right]\right)\right) + \left(\frac{1}{\sqrt{m}}\sum_{s=1}^{i}\mathbf{n}_s\mathbb{E}\left[\mathbf{w}^{(s)}\right] - \sqrt{m}\mathsf{C}_\infty(u_i)\right). \tag{5.35}$$

https://doi.org/10.1017/apr.2024.63 Published online by Cambridge University Press

We decompose the process as suggested by (5.35):

$$\overline{\mathsf{Y}}_m(u_i) = \mathsf{A}_m(u_i) + \mathsf{B}_m(u_i), \tag{5.36}$$

where $\mathsf{A}_m(u_i)$ is the first contribution of the right-hand side and the second one is rewritten as

$$\mathsf{B}_m(u_i) = \sum_{s=1}^{i} \left(\frac{\mathbf{n}_s - mp_s}{\sqrt{m}} \right) \mathbb{E}\left[\mathbf{w}^{(s)} \right]. \tag{5.37}$$

From the fact that $\mathbf{n}[k] \sim \mathcal{M}(m, p[k])$ and (5.34), a standard consequence of the central limit theorem is that

$$\left(\frac{\mathbf{n}_s - mp_s}{\sqrt{m}}, s \in \{1, \dots, k\} \right) \xrightarrow[m\to\infty]{(d)} (\mathbf{b}_s, s \in \{1, \dots, k\}), \tag{5.38}$$

where $(\mathbf{b}_s, s \in \{1, \dots, k\})$ is a centered Gaussian vector with covariance function

$$\operatorname{cov}(\mathbf{b}_k, \mathbf{b}_\ell) = -p_k p_\ell + p_k \mathbf{1}_{\{k=\ell\}}.$$

Using a concentration result for $\mathbf{n}_s$ around mp_s (for example the Hoeffding inequalities), by the central limit theorem,

$$\left(\sum_{j=1}^{\mathbf{n}_s} \frac{\mathbf{w}_j^{(s)} - \mathbb{E}\left[\mathbf{w}_j^{(s)} \right]}{\sqrt{m}}, s \in \{1, \dots, k\} \right) \xrightarrow[m\to\infty]{(d)} \left(\sqrt{p_s} \mathbf{G}_s, s \in \{1, \dots, k\} \right), \tag{5.39}$$

where, for all $s \in \{1, \dots, k\}$, the random variable $\mathbf{G}_s := (\mathbf{G}_s^{(1)}, \mathbf{G}_s^{(2)})$ is such that the random variables $\mathbf{G}_s^{(1)}$ and $\mathbf{G}_s^{(2)}$ are independent, and $\mathbf{G}_s^{(1)}$ (resp. $\mathbf{G}_s^{(2)}$) is a centered Gaussian random variable with variance $\mathbb{V}\left[\boldsymbol{\zeta}^{(s)} \right]$ (resp. $\mathbb{V}\left[\boldsymbol{\xi}^{(s)} \right]$), these variables being independent of $\mathbf{b}_s$. These considerations also allow us to prove that the families of random variables $(\mathbf{G}_s^{(p)}, s \in \{1, \dots, k\})$ and $(\mathbf{b}_s, s \in \{1, \dots, k\})$ are independent for all $p \in \{1, 2\}$.

This proves that the FDDs of $\overline{\mathsf{Y}}_m$ converge to those of $\overline{\mathsf{Y}}$, where

$$\overline{\mathsf{Y}}(u_i) := \sum_{s=1}^{i} \left(\sqrt{p_s} \mathbf{G}_s + \mathbf{b}_s \mathbb{E}\left[\left(\boldsymbol{\xi}^{(s)}, \boldsymbol{\zeta}^{(s)} \right) \right] \right), \qquad i \in \{1, \dots, k\}. \tag{5.40}$$

Tightness. Proving the tightness of the sequence $(\overline{\mathsf{Y}}_m)_{m\geq 0}$ in $D([0, 1])$ is the tough part of this section. The key point is the following lemma.

Lemma 13. *Let $(\mathsf{X}_m)_{m\geq 0}$ be a sequence of processes taking their values in $D([0, 1])$. Assume that for any m, $\mathsf{X}_m = \mathsf{X}_m^{(1)} + \mathsf{X}_m^{(2)}$ where $\mathsf{X}_m^{(1)}$ is a continuous process and $\mathsf{X}_m^{(2)}$ is a càdlàg process. If $\mathsf{X}_m^{(1)}$ converges in distribution to $\mathsf{X}^{(1)}$ in $C([0, 1])$, and if $\sup \left| \mathsf{X}_m^{(2)} \right| \xrightarrow[m]{(d)} 0$, then (X_m) converges in distribution to $\mathsf{X}^{(2)}$ in $D([0, 1])$.*

From this point, the proof is broken into three steps:

1. We consider the decomposition $\overline{\mathsf{Y}}_m = \overline{\mathcal{Y}}_m + \left(\overline{\mathsf{Y}}_m - \overline{\mathcal{Y}}_m \right)$, where $\overline{\mathcal{Y}}_m$ is a continuous process and $\overline{\mathsf{Y}}_m - \overline{\mathcal{Y}}_m$ is a càdlàg process (on $[0, 1]$). We want to apply Lemma 13 to $\mathsf{X}_m^{(1)} = \overline{\mathcal{Y}}_m$ and $\mathsf{X}_m^{(2)} = \overline{\mathsf{Y}}_m - \overline{\mathcal{Y}}_m$.

https://doi.org/10.1017/apr.2024.63 Published online by Cambridge University Press

2. We prove the convergence in distribution of the sequence $(\overline{\mathcal{Y}}_m)$ to $\overline{\mathsf{Y}}$ in $C([0,1])$.
3. We prove that $\sup_{t\in[0,1]}\left|\overline{\mathsf{Y}}_m-\overline{\mathcal{Y}}_m\right|\xrightarrow[m]{(d)}0$ to conclude that $(\overline{\mathsf{Y}}_m)$ converges in distribution to the same limit $\overline{\mathsf{Y}}$ as does $(\overline{\mathcal{Y}}_m)$.

Step 1: Let us define the process $\overline{\mathcal{Y}}_m$ properly. Recall the mapping q introduced in (5.31), and for each $j\in\{0,\dots,m\}$ define the 'jth m-tile'

$$v_j(m)=\inf\left\{u;q(u)\geq\frac{j}{m}\right\},\tag{5.41}$$

as well as the interval

$$I_j(m)=[v_j(m),v_{j+1}(m)).\tag{5.42}$$

Recall the definition of $\mathsf{A}_m,\mathsf{B}_m$ in (5.37), which satisfies $\overline{\mathsf{Y}}_m(u_i)=\mathsf{A}_m(u_i)+\mathsf{B}_m(u_i)$ for any generic point $u_i\in[0,1]$. We define $\overline{\mathcal{Y}}_m$ as

$$\overline{\mathcal{Y}}_m=\mathcal{A}_m+\mathcal{B}_m,\tag{5.43}$$

where for all $j\in\{0,\dots,m\}$,

$$\mathcal{A}_m(v_j(m))=\mathsf{A}_m(v_j(m)),\tag{5.44}$$

$$\mathcal{B}_m(v_j(m))=\mathsf{B}_m(v_j(m)),\tag{5.45}$$

and $\mathcal{A}_m,\mathcal{B}_m$ are interpolated between the points $v_j(m)$, in order to embed them in the space of continuous processes $C([0,1])$, and thus, so is $\overline{\mathcal{Y}}_m$. We have replace the generic points u_i by the m-tiles $v_j(m)$, which are more suitable for proving the tightness of $\mathcal{A}_m,\mathcal{B}_m$ (and thus that of $\overline{\mathcal{Y}}_m$).

Step 2: By construction, the FDDs of $\overline{\mathcal{Y}}_m$ are the same as those of $\overline{\mathsf{Y}}_m$ on the m-tiles $v_j(m)$, $j\in\{1,\dots,m\}$, so that it remains to prove only the tightness of $(\overline{\mathcal{Y}}_m)$ to prove its convergence in distribution towards $\overline{\mathsf{Y}}$. By Lemma 14 above (see [16, Lemma 8]), it suffices to prove the tightness of $\mathcal{A}_m$ and that of $\mathcal{B}_m$ separately. In fact this establishes the convergence of the FDDs in the sense that $\left(\overline{\mathcal{Y}}_m\left(\frac{\lfloor mu_i\rfloor}{m}\right),1\leq i\leq k\right)$ converges, that is, at the discretization points. The tightness allows us to see that

$$\sup\left|\overline{\mathcal{Y}}_m\left(\frac{\lfloor mu_i\rfloor}{m}\right)-\overline{\mathcal{Y}}_m\left(u_i\right)\right|\xrightarrow[m]{(\mathbb{P})}0$$

(the argument is direct from (5.54)).

Lemma 14. *Let* $\left(\mathsf{Z}_m^{(1)},\mathsf{Z}_m^{(2)}\right)_{m\geq0}$ *be a sequence of pairs of processes in* $C([0,1])^2$. *The tightnesses of both families* $\left(\mathsf{Z}_m^{(1)}\right)_{m\geq0}$ *and* $\left(\mathsf{Z}_m^{(2)}\right)_{m\geq0}$ *in* $C([0,1])$ *imply that of* $\left(\mathsf{Z}_m^{(1)},\mathsf{Z}_m^{(2)}\right)_{m\geq0}$ *in* $C([0,1])^2$.

A criterion for tightness in $C([0,1])$ is the following [4, Theorem II.12.3].

Lemma 15. *Let* $(\mathsf{X}_m)_{m\geq0}$ *be a sequence of stochastic processes in* $C([0,1],\mathbb{R})$. *If there exist some positive numbers* $\alpha>1$, $\beta\geq0$ *and a non-decreasing continuous function* F *on [0, 1]*

https://doi.org/10.1017/apr.2024.63 Published online by Cambridge University Press

such that, for all $m \in \mathbb{N}$ *and* $(s, t) \in [0, 1]^2$ *with* $0 \le s \le t \le 1$,

$$\mathbb{P}\left[|\mathsf{X}_m(t) - \mathsf{X}_m(s)| \ge \lambda\right] \le \frac{1}{\lambda^\beta} |F(t) - F(r)|^\alpha \tag{5.46}$$

for all $\lambda > 0$, *then* $(\mathsf{X}_m)_{m\ge 0}$ *is tight in* $C([0, 1])$.

Note that by Markov's inequality, it suffices to prove that $\mathbb{E}\left[|\mathsf{X}_m(t) - \mathsf{X}_m(s)|^\beta\right] \le |F(t) - F(r)|^\alpha$.

We are working in $\mathbb{R}^2$, so we will apply this criterion twice, one on every coordinate $\overline{\mathcal{Y}}_m^{(p)}$, $p \in \{1, 2\}$. We shall prove that for both processes, $\beta = 4$, $\alpha = 2$ does the job, together with the continuous non-decreasing map $F = q$ introduced in (5.31), which we extend by setting $F(1) = q(1) = 1$.

Let $(s, t) \in [0, 1]^2$ be such that $0 \le s < t \le 1$. There exist $j_1 < j_2 \in \{0, \dots, m\}$ such that $s \in I_{j_1}(m)$ and $t \in I_{j_2}(m)$, and we assume that $j_1 \ne j_2$ (the case $j_1 = j_2$ will be treated afterwards). We may write, for all $p \in \{1, 2\}$,

$$\begin{aligned}\mathbb{E}\left[\left|\mathcal{B}_m^{(p)}(t) - \mathcal{B}_m^{(p)}(s)\right|^4\right] \le \mathrm{Cste}\Bigg(&\mathbb{E}\left[\left|\mathcal{B}_m^{(p)}(t) - \mathcal{B}_m^{(p)}(v_{j_2}(m))\right|^4\right] \\ &+ \mathbb{E}\left[\left|\mathcal{B}_m^{(p)}(v_{j_2}(m)) - \mathcal{B}_m^{(p)}(v_{j_1+1}(m))\right|^4\right] + \mathbb{E}\left[\left|\mathcal{B}_m^{(p)}(v_{j_1+1}(m)) - \mathcal{B}_m^{(p)}(s)\right|^4\right]\Bigg).\end{aligned} \tag{5.47}$$

Notation. For the sequel, we introduce the random variable $\mathsf{N}_{[s,t]}$ for $(s \le t) \in [0, 1]^2$, which is binomial-$(m, q(t) - q(s))$-distributed. If $j \in \{0, \dots, m\}$ and $s = v_j(m)$ (or t), we will abuse notation and write $\mathsf{N}_{[j,t]} = \mathsf{N}_{[v_j(m),t]}$. Notice that $q(v_j(m)) = \frac{j}{m}$, so that $\mathsf{N}_{[j,j+1]}$ is binomial-$(m, \frac{1}{m})$-distributed.

Recall (5.37). We have the following equalities in distribution:

$$\mathcal{B}_m^{(1)}(v_{j_2}(m)) - \mathcal{B}_m^{(1)}(v_{j_1+1}(m)) \overset{(d)}{=} \frac{\mathsf{N}_{[j_1+1,j_2]} - (j_2 - j_1 - 1)}{\sqrt{m}} \mathbb{E}[\boldsymbol{\zeta}^{(j_1,j_2)}], \tag{5.48}$$

$$\mathcal{B}_m^{(2)}(v_{j_2}(m)) - \mathcal{B}_m^{(2)}(v_{j_1+1}(m)) \overset{(d)}{=} \frac{\mathsf{N}_{[j_1+1,j_2]} - (j_2 - j_1 - 1)}{\sqrt{m}} \mathbb{E}[\boldsymbol{\xi}^{(j_1,j_2)}]. \tag{5.49}$$

where $\mathbf{w}^{(j_1,j_2)} = (\boldsymbol{\zeta}^{(j_1,j_2)}, \boldsymbol{\xi}^{(j_1,j_2)})$ is a random variable distributed as the law of a pair $(\boldsymbol{\zeta}, \boldsymbol{\xi})$ conditioned on

$$\left\{\tan\left(\frac{\pi}{2} v_{j_1+1}(m)\right) < \frac{\boldsymbol{\xi}}{\boldsymbol{\zeta}} \le \tan\left(\frac{\pi}{2} v_{j_2}(m)\right)\right\},$$

where $\boldsymbol{\zeta}, \boldsymbol{\xi}$ are independent exponential variables with mean 1.

Note that for any random variable $\mathbf{x}$ that is binomial-(m, p)-distributed,

$$\mathbb{E}\left[|\mathbf{x} - mp|^4\right] = mp(1 - p) + (3m - 6)mp^2(1 - p)^2 \le mp + 3m^2p^2. \tag{5.50}$$

Since $mp \le m^2p^2$, for $p \ge 1/m$ (and here $q(v_{j_2}(m)) - q(v_{j_1+1}(m)) \ge 1/m$), we obtain

$$\mathbb{E}\left[\left|\mathcal{B}_m^{(p)}(v_{j_2}(m)) - \mathcal{B}_m^{(p)}(v_{j_1+1}(m))\right|^4\right] = \mathrm{Cste}\left|q(v_{j_2}(m)) - q(v_{j_1+1}(m))\right|^2, \tag{5.51}$$

https://doi.org/10.1017/apr.2024.63 Published online by Cambridge University Press

where we have used Proposition 3 to bound $\mathbb{E}[\left(\boldsymbol{\xi}^{(j_1,j_2)}\right)^4] \leq 4!$.

Now, by the definition of $\mathcal{B}_m$,

$$\mathcal{B}_m(t) - \mathcal{B}_m(v_{j_2}(m)) = \left[\mathcal{B}_m(v_{j_2+1}(m)) - \mathcal{B}_m(v_{j_2}(m))\right] \cdot \frac{t - v_{j_2}(m)}{v_{j_2+1}(m) - v_{j_2}(m)}, \tag{5.52}$$

and since

$$\frac{t - v_{j_2}(m)}{v_{j_2+1}(m) - v_{j_2}(m)} \leq \frac{q(t) - q(v_{j_2}(m))}{q(v_{j_2+1}(m)) - q(v_{j_2}(m))}$$

because $t \mapsto q(t)$ is convex on $[0, 1]$, using also (5.51), we reach

$$\mathbb{E}\left[\left|\mathcal{B}_m^{(p)}(t) - \mathcal{B}_m^{(p)}(v_{j_2}(m))\right|^4\right] \leq \text{Cste}\left|q(t) - q(v_{j_2}(m))\right|^2. \tag{5.53}$$

In the end, using (5.51) and (5.53) twice in (5.47), we obtain

$$\mathbb{E}\left[\left|\mathcal{B}_m^{(p)}(t) - \mathcal{B}_m^{(p)}(s)\right|^4\right] \leq \text{Cste}\,|q(t) - q(s)|^2. \tag{5.54}$$

We may now work on the sequence $(\mathcal{A}_m)_{m\geq 0}$, so again fix $(s, t) \in [0, 1]^2$ such that $0 \leq s < t \leq 1$. Recall (5.37) and let $\mathbf{w}_1^{(j_1,j_2)}, \ldots, \mathbf{w}_{\mathsf{N}_{[j_1+1,j_2]}}^{(j_1,j_2)}$ be a sequence of i.i.d. random variables having the same distribution as $\mathbf{w}^{(j_1,j_2)}$. We have the equality in distribution

$$\mathcal{A}_m(v_{j_2}(m)) - \mathcal{A}_m(v_{j_1+1}(m)) \overset{(d)}{=} \sum_{i=1}^{\mathsf{N}_{[j_1+1,j_2]}} \frac{\mathbf{w}_1^{(j_1,j_2)} - \mathbb{E}[\mathbf{w}^{(j_1,j_2)}]}{\sqrt{m}}, \tag{5.55}$$

so that

$$\mathbb{E}\left[\left|\mathcal{A}_m(v_{j_2}(m)) - \mathcal{A}_m(v_{j_1+1}(m))\right|^4\right] = \sum_{r=0}^{m} \frac{1}{m^2}\mathbb{E}\left[\left|\sum_{j=1}^{r} \mathbf{w}_1^{(j_1,j_2)} - \mathbb{E}[\mathbf{w}^{(j_1,j_2)}]\right|^4\right] \mathbb{P}\left(\mathsf{N}_{[j_1+1,j_2]} = r\right). \tag{5.56}$$

If $\mathbf{H}^{(j_1,j_2)}$ is a random variable having the law of $\mathbf{w}_1^{(j_1,j_2)} - \mathbb{E}[\mathbf{w}^{(j_1,j_2)}]$, then for all $r \in \{1, \ldots, m\}$,

$$\frac{1}{m^2}\mathbb{E}\left[\left|\sum_{j=1}^{r} \mathbf{w}_1^{(j_1,j_2)} - \mathbb{E}[\mathbf{w}^{(j_1,j_2)}]\right|^4\right] \leq \frac{r\mathbb{E}\left[(\mathbf{H}^{(j_1,j_2)})^4\right]}{m^2} + \frac{r^2\mathbb{E}\left[(\mathbf{H}^{(j_1,j_2)})^2\right]}{2m^2}.$$

Therefore,

$$\mathbb{E}\left[\left|\mathcal{A}_m(v_{j_2}(m)) - \mathcal{A}_m(v_{j_1+1}(m))\right|^4\right] \leq \frac{\mathbb{E}\left[\mathsf{N}_{[j_1+1,j_2]}\right]\mathbb{E}\left[(\mathbf{H}^{(j_1,j_2)})^4\right]}{m^2} + \frac{\mathbb{E}\left[\mathsf{N}_{[j_1+1,j_2]}^2\right]\mathbb{E}\left[(\mathbf{H}^{(j_1,j_2)})^2\right]}{2m^2}. \tag{5.57}$$

https://doi.org/10.1017/apr.2024.63 Published online by Cambridge University Press

In the same spirit as in Part 4 of Proposition 3, we may see that for all $s \in \mathbb{N}$, we have $\mathbb{E}\left[(\mathbf{H}^{(j_1,j_2)})^s\right] \leq s!$. For any random variable $\mathbf{x}$ that is binomial-(m,p)-distributed, we have

$$\mathbb{E}[\mathbf{x}] \leq m^2p^2 \quad \text{and} \quad \mathbb{E}\left[\mathbf{x}^2\right] \leq 2m^2p^2$$

as soon as $p \geq 1/m$ (and here $q(v_{j_2}(m)) - q(v_{j_1+1}(m)) \geq 1/m$); hence

$$\mathbb{E}\left[\left|\mathcal{A}_m^p(v_{j_2}(m)) - \mathcal{A}_m^{(p)}(v_{j_1+1}(m))\right|^4\right] \leq \text{Cste}\left|q(v_{j_2}(m)) - q(v_{j_1+1}(m))\right|^2. \tag{5.58}$$

We can treat the other terms $\mathcal{A}_m^{(p)}(t) - \mathcal{A}_m^{(p)}(v_{j_2}(m))$ and $\mathcal{A}_m^{(p)}(v_{j_1+1}(m)) - \mathcal{A}_m^{(p)}(s)$ exactly as we did for the process $\mathcal{B}_m$ in (5.53) and finally find that

$$\mathbb{E}\left[\left|\mathcal{A}_m^{(p)}(t) - \mathcal{A}_m^{(p)}(s)\right|^4\right] \leq \text{Cste}\,|q(t) - q(s)|^2. \tag{5.59}$$

In the case $j_1 = j_2$, we may immediately write, in both cases $\mathcal{A}_m$, $\mathcal{B}_m$ (we take $\mathcal{B}_m$ here as an example), just as in (5.52),

$$\mathcal{B}_m(t) - \mathcal{B}_m(s) = \left[\mathcal{B}_m(v_{j_2+1}(m)) - \mathcal{B}_m(v_{j_2}(m))\right] \cdot \frac{t-s}{v_{j_2+1}(m) - v_{j_2}(m)}, \tag{5.60}$$

so that

$$\mathbb{E}\left[\left|\mathcal{B}_m^{(p)}(t) - \mathcal{B}_m^{(p)}(s)\right|^4\right] \leq \text{Cste}\,|q(t) - q(s)|^2 \tag{5.61}$$

for every $p \in \{1, 2\}$, by the same arguments as before. This proves the tightnesses of the sequences of processes $(\mathcal{A}_m)$, $(\mathcal{B}_m)$.

Step 3: In the next step, we prove that

$$\sup_{t\in[0,1]} \left|\overline{\Upsilon}_m^{(p)}(t) - \overline{\mathcal{Y}}_m^{(p)}(t)\right| \xrightarrow[m]{(\mathbb{P})} 0$$

for every $p \in \{1, 2\}$. Since

$$\sup_{t\in[0,1]} \left|\overline{\Upsilon}_m^{(p)}(t) - \overline{\mathcal{Y}}_m^{(p)}(t)\right| = \sup_{j\in\{0,\ldots,m-1\}} \sup_{t\in I_j(m)} \left|\overline{\Upsilon}_m^{(p)}(t) - \overline{\mathcal{Y}}_m^{(p)}(t)\right|, \tag{5.62}$$

and for all $j \in \{0, \ldots, m-1\}$,

$$\begin{aligned}\sup_{t\in I_j(m)} \left|\overline{\Upsilon}_m^{(p)}(t) - \overline{\mathcal{Y}}_m^{(p)}(t)\right| \leq \sup_{t\in I_j(m)} &\left|\overline{\Upsilon}_m^{(p)}(t) - \overline{\Upsilon}_m^{(p)}(v_j(m))\right| \\ &+ \sup_{t\in I_j(m)} \left|\overline{\mathcal{Y}}_m^{(p)}(v_j(m)) - \overline{\mathcal{Y}}_m^{(p)}(t)\right|, \end{aligned} \tag{5.63}$$

the proof of this part consists in showing that both processes $(\overline{\Upsilon}_m^{(p)})$, $(\overline{\mathcal{Y}}_m^{(p)})$ admit fluctuations larger than $\varepsilon > 0$ on an interval $I_j(m)$ with probability $o(1/m)$. Fix $j \in \{0, \ldots, m-1\}$ for the rest of the proof. Since $(\overline{\mathcal{Y}}_m^{(p)})$ is linear on every interval $I_j(m)$, we have, first,

$$\sup_{t\in I_j(m)} \left|\overline{\mathcal{Y}}_m^{(p)}(v_j(m)) - \overline{\mathcal{Y}}_m^{(p)}(t)\right| \leq \left|\overline{\mathcal{Y}}_m^{(p)}(v_j(m)) - \overline{\mathcal{Y}}_m^{(p)}(v_{j+1}(m))\right|, \tag{5.64}$$

https://doi.org/10.1017/apr.2024.63 Published online by Cambridge University Press

and second, for all $\varepsilon > 0$,

$$\mathbb{P}\left(\left|\overline{\mathcal{Y}}_m^{(p)}(v_j(m)) - \overline{\mathcal{Y}}_m^{(p)}(v_{j+1}(m))\right| \geq \varepsilon\right) \leq \frac{\mathbb{E}\left[\left|\overline{\mathcal{Y}}_m^{(p)}(v_j(m)) - \overline{\mathcal{Y}}_m^{(p)}(v_{j+1}(m))\right|^4\right]}{\varepsilon^4} \leq \frac{\text{Cste}}{\varepsilon^4 m^2},$$

where the last inequality comes from the combination of (5.53) and (5.59). By the union bound, we obtain

$$\sup_{j\in\{0,\dots,m-1\}} \sup_{t\in I_j(m)} \left|\overline{\mathcal{Y}}_m^{(p)}(v_j(m)) - \overline{\mathcal{Y}}_m^{(p)}(t)\right| \xrightarrow[m]{(\mathbb{P})} 0.$$

Let us now handle the process $(\overline{\mathsf{Y}}_m^{(p)})$. Recall its definition (on page 37): $\overline{\mathsf{Y}}_m^{(p)} = \sqrt{m}\left(\mathsf{C}_m^{(p)} - \mathsf{C}_\infty^{(p)}\right)$. We can write

$$\sup_{t\in I_j(m)} \left|\overline{\mathsf{Y}}_m^{(p)}(t) - \overline{\mathsf{Y}}_m^{(p)}(v_j(m))\right| \leq L_1^{(p)}(j,m) + L_2^{(p)}(j,m), \tag{5.65}$$

where

$$L_1^{(p)}(j,m) := \sup_{t\in I_j(m)} \sqrt{m}\left|\mathsf{C}_\infty^{(p)}(t) - \mathsf{C}_\infty^{(p)}(v_j(m))\right|$$

is deterministic since $\mathsf{C}_\infty(t) = (x(t), y(t))$ (see (5.4)). This term is easily handled:

$$\sup_{j\in\{0,\dots,m-1\}} \sup_{t\in I_j(m)} L_1^{(p)}(j,m) \leq \frac{\text{Cste}}{\sqrt{m}} \xrightarrow[m]{} 0.$$

So it suffices to prove that

$$L_2^{(p)}(j,m) := \sup_{t\in I_j(m)} \sqrt{m}\left|\overline{\mathsf{C}}_m^{(p)}(t) - \overline{\mathsf{C}}_m^{(p)}(v_j(m))\right| \xrightarrow[m]{(\mathbb{P})} 0. \tag{5.66}$$

We return to the definition of $\overline{\mathsf{C}}_m$ in (5.22). The term $\left|\overline{\mathsf{C}}_m(t) - \overline{\mathsf{C}}_m(v_j(m))\right|$ preserves the contribution of the vectors $\mathbf{w}[m]$ whose slopes are in the interval $\left[\tan\left(\frac{\pi}{2}v_j(m)\right), \tan\left(\frac{\pi}{2}t\right)\right)$, i.e.

$$\sqrt{m}(\overline{\mathsf{C}}_m(t) - \overline{\mathsf{C}}_m(v_j(m))) = \frac{1}{\sqrt{m}}\sum_{i=1}^{m} (\boldsymbol{\zeta}_i, \boldsymbol{\xi}_i)\mathbf{1}_{\left\{\tan\left(\frac{\pi}{2}v_j(m)\right) \leq \frac{\xi_i}{\zeta_i} < \tan\left(\frac{\pi}{2}t\right)\right\}}. \tag{5.67}$$

In the right-hand side of (5.67), the sum is a sum of non-negative terms; it is thus immediate that

$$\sup_{j\in\{0,\dots,m-1\}} L_2^{(1)}(j,m) \leq \frac{1}{\sqrt{m}} \max\{\boldsymbol{\zeta}_i,\ 1 \leq i \leq m\} \max_{j\in\{0,\dots,m-1\}} \mathsf{N}_{[j,j+1]} \tag{5.68}$$

(and a similar property holds for $\sup_j L_2^{(2)}(j,m)$). By the union bound, $\mathbb{P}(\max\{\boldsymbol{\zeta}_i,\ 1 \leq i \leq m\} \geq 3\log m) \leq 1/m^2$. By Markov, the binomial random variable $\mathsf{N}_{[j,j+1]}$ satisfies

$$\mathbb{P}\left(\mathsf{N}_{[j,j+1]} \geq \lfloor 2\log(m)\rfloor\right) \leq \mathbb{E}\left[e^{\mathsf{N}_{[j,j+1]}}\right] e^{-\lfloor 2\log(m)\rfloor} \leq e^{e-1}/m^2.$$

These two bounds, together with (5.68), yield (5.66).

This completes Step 3 of the proof, as well as the proof as a whole. Indeed, by Lemma 13 and what we proved in Steps 2 and 3, the sequence of processes $\left(\overline{\mathsf{Y}}_m\right)$ converges in distribution to the same limit as the sequence $\left(\overline{\mathcal{Y}}_m\right)$. This limit has been proven to be $\overline{\mathsf{Y}}$.

https://doi.org/10.1017/apr.2024.63 Published online by Cambridge University Press

6. Random generation

In this section, given $\kappa \geq 3$, we provide an algorithm to sample an n-tuple of points $z[n]$ with distribution $\mathbb{D}_\kappa^{(n)}$. In Figure 14, we plotted two drawings of κ-sampling in the cases $\kappa = 5$ and $\kappa = 7$. In the special cases $\kappa = 3, 4$ where we have $\mathbb{D}_\kappa^{(n)} = \mathbb{Q}_\kappa^{(n)}$ (recall the discussion on pages 4 and 14), we will provide two alternative sharpened algorithms.

6.1. Algorithm of κ-sampling

The algorithm starts with the computation of the ECP. We will use the following notation to express the side lengths with respect to $\ell[\kappa]$: put $P_\kappa = \kappa r_\kappa / w_\kappa$ with $w_\kappa := \dfrac{1+\cos(\theta_\kappa)}{\sin(\theta_\kappa)}$, and recall that by Proposition 1, for $\ell[\kappa] \in \mathcal{L}_\kappa$ and $\widetilde{c}[\kappa] = \frac{1}{w_\kappa}(r_\kappa - \mathfrak{cl}_j(\ell[\kappa]))_{j\in\{1,\dots,\kappa\}}$ with $\widetilde{c} = \sum_{j=1}^{\kappa} \widetilde{c}_j$, we have

$$\widetilde{c} + \sum_{j=1}^{\kappa} \ell_j = P_\kappa.$$

Let us prove that this algorithm returns an n-tuple $z[n]$ that is $\mathbb{D}_\kappa^{(n)}$-distributed, and that this is done within a reasonable amount of time. The notation $\propto$ means that two quantities are proportional.

Proof of Theorem 4. Denote by $\mathbb{P}_{\mathrm{Alg}}$ the probability of an event in our κ-sampling. The distribution induced in the first step of the algorithm (which is nothing but a rejection algorithm) satisfies

$$\mathbb{P}_{\mathrm{Alg}}\left(\boldsymbol{\ell}^{(n)}[\kappa] \in \prod_{i=1}^{\kappa} \mathrm{d}\ell_i\right) \propto \mathbf{1}_{\{\ell[\kappa]\in\mathcal{L}_\kappa\}} \left(P_\kappa - \sum_{j=1}^{\kappa} \ell_j\right)^{2n-\kappa} \prod_{j=1}^{\kappa} \mathrm{d}\ell_j = \mathbf{1}_{\{\ell[\kappa]\in\mathcal{L}_\kappa\}} \widetilde{c}^{2n-\kappa} \prod_{j=1}^{\kappa} \mathrm{d}\ell_j. \tag{6.1}$$

The second step (which is also a rejection algorithm working at $\ell[\kappa]$ fixed) induces the distribution

$$\begin{aligned}\mathbb{P}_{\mathrm{Alg}}\left(\mathbf{s}^{(n)}[\kappa] = s[\kappa] \,\middle|\, \boldsymbol{\ell}^{(n)}[\kappa] \in \prod_{i=1}^{\kappa} \mathrm{d}\ell_i\right) &\propto \mathbf{1}_{\{s[\kappa]\in\mathbb{N}_\kappa(n)\}} \prod_{j=1}^{\kappa} \frac{(1/\kappa)^{s_j}}{s_j!} \frac{(\widetilde{c}_j/\widetilde{c})^{N_j}}{N_j!} \\ &\propto \mathbf{1}_{\{s[\kappa]\in\mathbb{N}_\kappa(n)\}} \prod_{j=1}^{\kappa} \frac{(\widetilde{c}_j/\widetilde{c})^{s_j+s_{j+1}-1}}{s_j!(s_j+s_{j+1}-1)!}.\end{aligned} \tag{6.2}$$

This gives the appropriate joint distribution for $(\boldsymbol{\ell}^{(n)}[\kappa], \mathbf{s}^{(n)}[\kappa])$ as computed in Theorem 6. Now, given $\ell[\kappa]$, $c[\kappa]$, $s[\kappa]$, and since, conditionally on the $c[\kappa]$, the projections of the vectors $v[n]$ are uniform in the $c[\kappa]$, the way we are constructing our vectors is valid and so is the building of $z[n]$.

We saw that the $\ell[\kappa]$ behaves in c/n in Theorem 7, so that $\mathbf{1}_{\{\ell[\kappa]\in n\mathcal{L}_\kappa\}} \to 1$ pointwise and thus the condition $\ell[\kappa] \in \mathcal{L}_\kappa$ of **Step 1** is satisfied with a probability going to 1 with n. This step costs only the drawing of $2n$ uniform variates requiring $\mathcal{O}(n \log(n))$ operations. Notice that to perfect the algorithm, one could reuse the unused uniform variates of **Step 1** for the next steps.

https://doi.org/10.1017/apr.2024.63 Published online by Cambridge University Press

Algorithm 1 κ-sampling

Step 1: Sample $\ell[\kappa]$; Construct $\widetilde{c}[\kappa]$;

do{

Draw a beta random variable $S \sim P_\kappa \cdot \beta(k+1, n)$ and draw k i.i.d. uniform random variables $(u_1, \ldots, u_k)$ in [0,S];

Sort them in increasing order into $(u_{(1)}, \ldots, u_{(k)})$;

$\ell[\kappa] \leftarrow (u_{(1)}, u_{(2)} - u_{(1)}, \ldots, u_{(\kappa)} - u_{(\kappa-1)})$;}

While$\{\ell[\kappa] \notin \mathcal{L}_\kappa;\}$

• Compute $\widetilde{c}[\kappa] = \frac{1}{w_\kappa}(r_\kappa - \mathfrak{cl}_j(\ell[\kappa]))_{j\in\{1,\ldots,\kappa\}}$;

Step 2: Sample $s[\kappa]$;

do{

$s[\kappa] \leftarrow \mathcal{M}(n, \frac{1}{\kappa}, \ldots, \frac{1}{\kappa})$# *sampling of multinomial random variable*;

$N[\kappa] \leftarrow \mathcal{M}(2n-\kappa, \frac{\widetilde{c}_1}{\widetilde{c}}, \ldots, \frac{\widetilde{c}_\kappa}{\widetilde{c}})$;}

while$\{\exists j \in \{1, \ldots, \kappa\}, N_j \neq s_j + s_{j+1} - 1$ and $s[\kappa] \notin \mathbb{N}_n(\kappa);\}$

Step 3: Compute z[n];

for$j \in \{1, \ldots, \kappa\}$**{**

Draw N_j random variables i.i.d. uniform $(u_1^{(j)}, \ldots, u_{N_j}^{(j)})$ in the segment $[0, c_j]$;

Sort them into $(u_{(1)}^{(j)}, \ldots, u_{(N_j)}^{(j)})$;

Build $\Delta u^{(j)}[N_j+1] = (u_{(1)}^{(j)}, u_{(2)}^{(j)} - u_{(1)}^{(j)} \ldots, u_{(N_j)}^{(j)} - u_{(N_j-1)}^{(j)}, c_j - u_{(N_j)}^{(j)})$;}

end for

for$j \in \{1, \ldots, \kappa\}$**{**

Build the vectors $v_k^{(j)} = A_j(\theta_\kappa)^{-1} \begin{pmatrix} \Delta u_{s_{j-1}+k}^{(j-1)} \\ \Delta u_k^{(j)} \end{pmatrix}$;

Sort them into $(v_{(1)}^{(j)}, \ldots, v_{(s_j)}^{(j)})$ by increasing slope;}

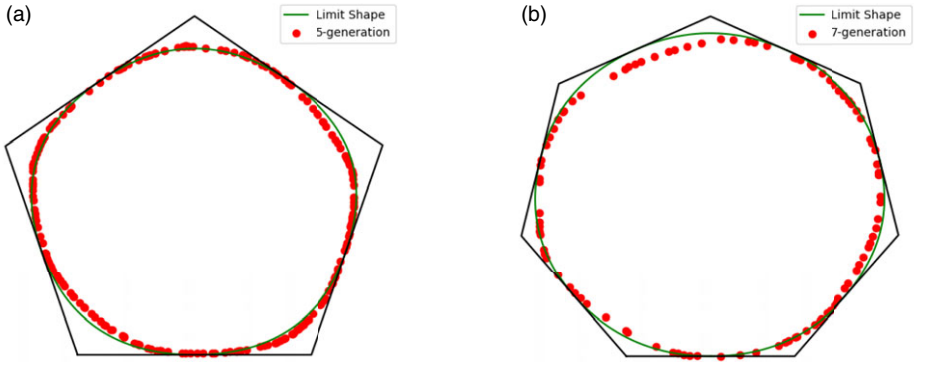
end for

• Gather all vectors into $v[n] = (v_{(1)}^{(1)}, \ldots, v_{(s_1)}^{(1)}, \ldots, v_{(1)}^{(\kappa)}, \ldots, v_{(s_\kappa)}^{(\kappa)})$;

• Build the ECP at distance $\ell[\kappa]$ from the sides of $\mathfrak{C}_\kappa$. The vectors $v[n]$ form the boundary of a unique convex polygon circumscribed in this ECP, whose set of vertices is an n-tuple $z[n]$ in $\mathcal{C}_\kappa(n)$;

As for the second step, the probability that the two multinomial samples coincide behaves in $n^{-\kappa/2}$. A standard efficient algorithm to simulate a multinomial distribution is the alias method presented by Walker [27], the theoretical basis of which was provided by Kronmal and Peterson [15]. In our case the complexity of the alias method is $\mathcal{O}(n\kappa \log(\kappa))$. (There exists other efficient procedures, such as the two-stage method of Brown and Bromberg [8]. For a discussion of the most suitable method of multinomial sampling, we refer to [12].)

https://doi.org/10.1017/apr.2024.63 Published online by Cambridge University Press



5-sampling, $n = 200$. 7-sampling, $n = 100$.

FIGURE 14. Two examples of κ-sampling. The set of points is the set of vertices of a convex $z[n]$-gon, whose boundary is very close to the limit shape drawn inside the κ-gon.

The last step includes several iterations of sampling and sorting $2n$ variates, which also has a complexity in $\mathcal{O}(n \log(n))$.

The second step is obviously the most costly, and it implies a global complexity of $\mathcal{O}\left(n^{\kappa/2+1}\kappa \log(\kappa)\right)$. Despite considerable effort, we were not able to make any significant progress in finding an efficient algorithm to reduce the cost of this step. □

6.2. Exact and fast algorithm of Δ-sampling

In the triangular case, the algorithm of △-sampling avoids the rejection-sampling steps 1 and 2 included in the algorithm of κ-sampling, which makes the sampling direct and immediately implies a reasonable computation time. Indeed, it happens to be that in the case $\kappa = 3$, the joint distribution of $\boldsymbol{\ell}^{(n)}[3]$, $\mathbf{s}^{(n)}[3]$ comes with simplifications (we will show this in Theorem 11) and becomes

$$\mathbb{P}\left(\boldsymbol{\ell}^{(n)}[3] \in \prod_{j=1}^{3} \mathrm{d}\ell_j, \mathbf{s}^{(n)}[3] = (i,j,k)\right) = \frac{n!\sin(\theta_\kappa)^{n-3}}{\mathbb{P}_{\triangle}(n)((n-1)!)^3}\mathbf{1}_{\left\{\ell_1+\ell_2+\ell_3 \le \frac{\sqrt{3}}{2}r_3\right\}}$$
$$\times\left(r_3 - \frac{2}{\sqrt{3}}(\ell_1+\ell_2+\ell_3)\right)^{2n-3}\binom{n-1}{i}\binom{n-1}{j}\binom{n-1}{k}\mathbf{1}_{\{i+j+k=n\}}\prod_{j=1}^{3}\mathrm{d}\ell_j. \quad (6.3)$$

In Figure 15, we plotted an example of △-sampling. To draw (s_1, s_2, s_3) according to three binomial distributions $\mathcal{B}(n-1, \frac{1}{2})$ and conditioned on $s_1 + s_2 + s_3 = n$, we may do the following:

- Draw $3n-3$ Bernoulli($\frac{1}{3}$) random variables $\mathbf{x}[3n-3]$.
- Set $(s_1, s_2, s_3) = \left(\sum_{i=1}^{n-1}\mathbf{x}_i, \sum_{i=n}^{2n-2}\mathbf{x}_i, \sum_{i=2n-1}^{3n-3}\mathbf{x}_i\right)$.
- Correct $\mathbf{x}[3n-3]$ to have $\sum_{i=1}^{3n-3}\mathbf{x}_i = n$.

https://doi.org/10.1017/apr.2024.63 Published online by Cambridge University Press

Algorithm 2 △-sampling

Step 1: Sample s[3];

• Draw (s_1, s_2, s_3) as three binomial random variables $\mathcal{B}(n-1, \frac{1}{2})$ conditioned on $s_1 + s_2 + s_3 = n$.
• $N[3] \leftarrow (s_1 + s_2 - 1, s_2 + s_3 - 1, s_3 + s_1 - 1)$;

Step 2: Sample ℓ[3];

• Draw $2n$ uniform $(u_1, \ldots, u_{2n})$ in the segment $[0, r_3]$;
• Sort them into $(u_{(1)}, \ldots, u_{(2n)})$;
• $\ell[3] \leftarrow (u_{(1)}, u_{(2)} - u_{(1)}, u_{(3)} - u_{(2)})$; # *only the first* 3u_(i) *are used*
• Compute $c = r_3 - (\ell_1 + \ell_2 + \ell_3)$;

Step 3: Compute z[n];

for $1 \le j \le 3$ {

Pick uniformly N_j points among the $u_{(j)}, j \in \{4, \ldots, 2n\}$, to form a partition of [0,c]: $(u^{(j)}_{(1)}, \ldots, u^{(j)}_{(N_j)})$;

Build $\Delta u^{(j)}[N_j + 1] = (u^{(j)}_{(1)}, u^{(j)}_{(2)} - u^{(j)}_{(1)} \ldots, u^{(j)}_{(N_j)} - u^{(j)}_{(N_j-1)}, c - u^{(j)}_{(N_j)})$;}

end for

for $1 \le j \le 3$ {

Build the vectors $v^{(j)}_k = A_j(\frac{\pi}{3})^{-1} \begin{pmatrix} \Delta u^{(j-1)}_{s_{j-1}+k} \\ \Delta u^{(j)}_k \end{pmatrix}$ for $1 \le k \le s_j$;

Sort them into $(v^{(j)}_{(1)}, \ldots, v^{(j)}_{(s_j)})$ by increasing slope;}

• Gather vectors into $v[n] = (v^{(1)}_{(1)}, \ldots, v^{(1)}_{(s_1)}, \ldots, v^{(3)}_{(1)}, \ldots, v^{(3)}_{(s_3)})$;
• Build the equilateral triangle T of side length c at distance $\ell[3]$ from the sides of $\mathfrak{C}_3$;
• The vectors $(v_{(1)}, \ldots, v_{(n)})$, sorted by increasing slope, form the boundary of a convex polygon circumscribed in T. The set of its vertices is an n-tuple $z[n]$ in convex position.

end for

The binomial correction works as follows: if $\sum_{i=1}^{3n-3} \mathbf{x}_i = n$, then we have the right distribution. Otherwise, if $\sum_{i=1}^{3n-3} \mathbf{x}_i < n$, pick uniformly some $j \in \{i; \mathbf{x}_i = 0\}$ and put $\mathbf{x}_j = 1$ until $\sum_{i=1}^{3n-3} \mathbf{x}_i = n$. If $\sum_{i=1}^{3n-3} \mathbf{x}_i > n$, pick uniformly some $j \in \{i; \mathbf{x}_i = 1\}$ and put $\mathbf{x}_j = 0$ until $\sum_{i=1}^{3n-3} \mathbf{x}_i = n$.

At the end we get that

$$\mathbb{P}_{\mathrm{Alg}}\left(\mathbf{s}^{(n)}[3] = (i, j, k)\right) \propto \binom{n-1}{i}\left(\frac{1}{2}\right)^{i}\left(\frac{1}{2}\right)^{n-1-i}\binom{n-1}{j}\left(\frac{1}{2}\right)^{n-1} \times \binom{n-1}{k}\left(\frac{1}{2}\right)^{n-1}\mathbf{1}_{\{i+j+k=n\}},$$

https://doi.org/10.1017/apr.2024.63 Published online by Cambridge University Press

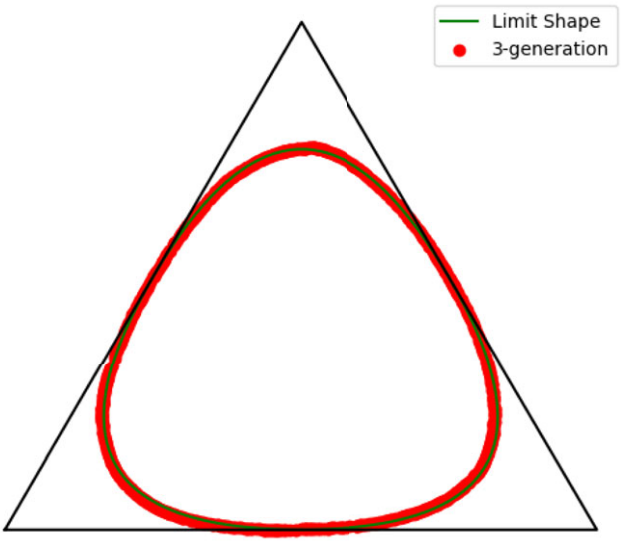


FIGURE 15. A △-sampling, some $z[n]$-gon for $n = 1000$, close to the limit shape.

so that

$$\mathbb{P}_{\mathrm{Alg}}\left(\mathbf{s}^{(n)}[3] = (i,j,k)\right) \propto \binom{n-1}{i}\binom{n-1}{j}\binom{n-1}{k}\mathbf{1}_{\{i+j+k=n\}}. \tag{6.4}$$

The analysis of the sampling of $\boldsymbol{\ell}_n[3]$ is the same as in the general κ-sampling. In the end, the algorithm of △-generation admits a global complexity of $\mathcal{O}(n\log(n))$ (in expectation).

6.3. Exact and fast algorithm of □-sampling

In the square case, a fast □-sampling can be proposed, which is slightly different from the △-sampling and from the κ-sampling. In Figure 16, we plotted an example of this □-sampling.

Indeed, in this case, no affine mapping intervenes in the proof since all corners are already right triangles. We may thus consider the law of the positive x-components $\mathbf{i}_n = \mathbf{s}_4^{(n)} + \mathbf{s}_1^{(n)}$ and y-components $\mathbf{j}_n = \mathbf{s}_1^{(n)} + \mathbf{s}_2^{(n)}$ of the vectors forming the boundary of a $z[n]$-gon. A quick calculus allows us to obtain

$$\mathbb{P}\left(\boldsymbol{\ell}_n[4] \in \prod_{j=1}^{4} \mathrm{d}\ell_j, \mathbf{i}_n = i, \mathbf{j}_n = j\right) = \frac{(n!)^2}{\mathbb{P}_{\square}(n)((n-1)!)^4}\mathbf{1}_{\{\ell_1+\ell_3\le 1\}}\mathbf{1}_{\{\ell_2+\ell_4\le 1\}}$$

$$\times (1-(\ell_1+\ell_3))^{n-2}(1-(\ell_2+\ell_4))^{n-2}\binom{n-1}{i}\binom{n-1}{i-1}\binom{n-1}{j}\binom{n-1}{j-1}\prod_{j=1}^{4}\mathrm{d}\ell_j. \tag{6.5}$$

https://doi.org/10.1017/apr.2024.63 Published online by Cambridge University Press

Algorithm 3. □-sampling

Step 1: Sampling of **i**, **j**;

- For both **i** and **j**, draw two binomial random variables $\mathcal{B}(n-1, \frac{1}{2})$ until their sum reaches n and set **i** (or **j**) as the result of the first.

Step 2: Sampling of $\ell[4]$;

- Draw n i.i.d. uniform random variables $(u_1, \dots, u_n)$ in the segment $[0, 1]$ and sort them into $(u_{(1)}, \dots, u_{(n)})$;
- $(\ell_1, \ell_3) \leftarrow (u_{(1)}, u_{(2)} - u_{(1)})$ *# use the two smallest*
- Draw n i.i.d. uniform random variables $(\hat{u}_1, \dots, \hat{u}_n)$ in the segment $[0, 1]$ and sort them into $(\hat{u}_{(1)}, \dots, \hat{u}_{(n)})$;
- $(\ell_2, \ell_4) \leftarrow (\hat{u}_{(1)}, \hat{u}_{(2)} - \hat{u}_{(1)})$;
- Compute $c_1 = 1 - (\ell_1 + \ell_3)$ and $c_2 = 1 - (\ell_2 + \ell_4)$;

Step 3: Compute z[n];

- Pick uniformly **i** points $(x_1 < \dots < x_{\mathbf{i}})$ among the $u_{(k)}$, $k \in \{3, \dots, n\}$, and form $h^+[\mathbf{i}+1] = (x_1, x_2 - x_1, \dots, c_1 - x_{\mathbf{i}})$ (positive increments). The $n-2-\mathbf{i}$ points $(\tilde{x}_1 < \dots < \tilde{x}_{n-2-\mathbf{i}})$ remaining in $u_{(k)}$, $k \in \{3, \dots, n\}$ are used to form $h^-[n-1-\mathbf{i}] = (-\tilde{x}_1, -\tilde{x}_2 + \tilde{x}_1, \dots, -c_1 + \tilde{x}_{n-2-\mathbf{i}})$ (negative increments);
- Pick uniformly **j** points $(y_1 < \dots < y_{\mathbf{j}})$ among the $\hat{u}_{(k)}$, $k \in \{3, \dots, n\}$, and form $v^+[\mathbf{j}+1] = (y_1, y_2 - y_1, \dots, c_1 - y_{\mathbf{j}})$ (positive increments). The $n-2-\mathbf{j}$ points $(\tilde{y}_1 < \dots < \tilde{y}_{n-2-\mathbf{j}})$ remaining in $\hat{u}_{(k)}$, $k \in \{3, \dots, n\}$ are used to form $v^-[n-1-\mathbf{j}] = (-\tilde{y}_1, -\tilde{y}_2 + \tilde{y}_1, \dots, -c_1 + \tilde{y}_{n-2-\mathbf{j}})$ (negative increments);
- $h[n] = \mathsf{Merge}(h^+, h^-)$ and $v[n] = \mathsf{Merge}(v^+, v^-)$;
- Pick uniformly in $\mathcal{S}_n$ a permutation σ and build for $1 \le i \le n$ the vector $w_i = (h_i, v_{\sigma(i)})$;
- Sort them into $(w_{(1)}, \dots, w_{(n)})$ by increasing slope;
- Build the rectangle R of side lengths c_1 (horizontally) and c_2 (vertically) at distance $\ell[4]$ from the sides of $\mathfrak{C}_4$;
- The vectors $(w_{(1)}, \dots, w_{(n)})$, sorted by increasing slope, form the boundary of a convex polygon inscribed in R. The set of its vertices is an n-tuple $z[n]$ in convex position.

In particular, this proves that $\mathbf{i}_n, \mathbf{j}_n$ are independent, and their law is explicit. It is easier to draw according to this distribution than to consider $\mathbf{s}^{(n)}[4]$. This approach is, once more, inspired by Valtr's paper [24].

The law of a random variable **k** described in the first step satisfies

$$\mathbb{P}(\mathbf{k} = k) \propto \binom{n-1}{k}\binom{n-1}{d}\mathbf{1}_{\{k+d=n\}} = \binom{n-1}{k}\binom{n-1}{k-1}.$$

The probability that two binomial samples are equal is typically $\frac{1}{\sqrt{n}}$. A binomial sampling requires $\mathcal{O}(n)$ operations, and since this step is the most costly in the □-generation, the whole algorithm has a global complexity of $\mathcal{O}(n^{3/2})$.

https://doi.org/10.1017/apr.2024.63 Published online by Cambridge University Press

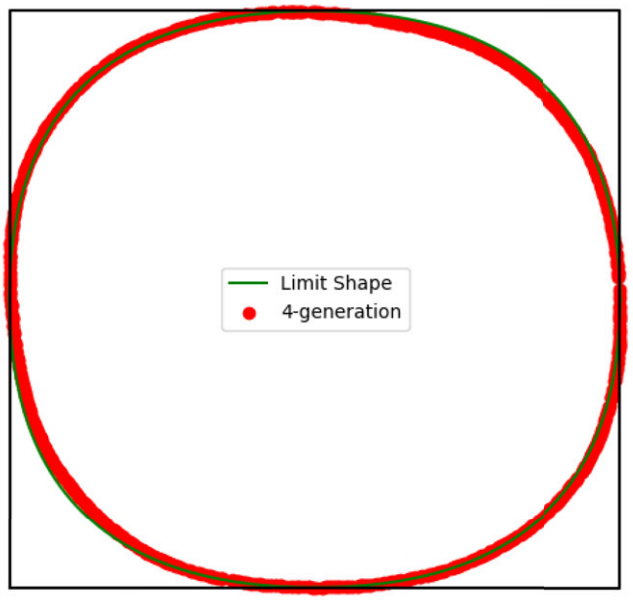


FIGURE 16. A □-sampling, some $z[n]$-gon for $n = 1000$, close to the limit shape.

Appendix A. Proof of Lemma 4

We decompose $\mathbb{P}_\kappa(n)$ according to the number of sides of the ECP we are considering. Let $\mathbf{z}[n]$ have distribution $\mathbb{U}_\kappa^{(n)}$, and write

$$\mathbb{P}_\kappa(n) = n! \sum_{\substack{\mathcal{J}\subset\{1,\dots,\kappa\}\\ |\mathcal{J}|\geq 3}} \mathbb{P}\left(\mathbf{z}[n] \in \mathcal{C}_\kappa(n) \cap \mathrm{NZS}(\mathbf{z}[n]) = \mathcal{J}\right).$$

We now borrow some considerations from Bárány ([2, 3]).

Definition 2. *Given S a convex compact set (non-flat), let $x_1, \dots, x_m, x_{m+1} = x_1$ be a subdivision of the boundary ∂S and let d_i be the line supporting S at x_i for all $i \in \{1, \dots, m\}$. Write y_i for the intersection of d_i and d_{i+1} (if $d_i = d_{i+1}$ then y_i can be any point between x_i and x_{i+1}). Let T_i denote the triangle with vertices x_i, y_i, y_{i+1} and also its area. We define the affine perimeter of the convex set S as*

$$\mathrm{AP}(S) = 2 \lim \sum_{i=1}^{m} \sqrt[3]{T_i},$$

where the limit is taken over all sequences of subdivisions $x[m]$ with $\max_{1,\dots,m} |x_i - x_{i+1}| \to 0$.

Theorem 10. (Limit shape theorem, Bárány [2].) *Let K be a compact convex domain of $\mathbb{R}^2$ with nonempty interior.*

(1) *There exists a convex domain $\mathsf{Dom}(K) \subset K$ such that $\mathrm{AP}(\mathsf{Dom}(K)) > \mathrm{AP}(S)$ for all convex sets $S \subset K$ different from $\mathsf{Dom}(K)$.*

https://doi.org/10.1017/apr.2024.63 Published online by Cambridge University Press

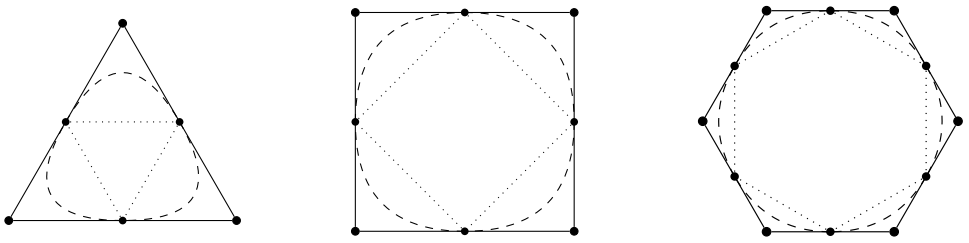


FIGURE 17. For each of the cases $\kappa = 3, 4, 6$, the inner dashed curve is the boundary of the domain $\mathsf{Dom}(\mathfrak{C}_\kappa)$. By the limit shape theorem, it also represents the boundary of a $\mathbf{z}[n]$-gon where $\mathbf{z}[n]$ is taken under $\mathbb{Q}_\kappa^{(n)}$, when $n \to +\infty$.

(2) *Let $n \geq 3$, and let $\mathbf{z}[n]$ have distribution $\mathbb{Q}_K^{(n)}$. Then for all $\varepsilon > 0$,*

$$\lim_{n\to+\infty} \mathbb{P}\left(d_H(\mathsf{conv}(\mathbf{z}[n]), \mathsf{Dom}(K)) < \varepsilon\right) = 1.$$

Definition 3. *Define* $\mathrm{AP}^*(K) = \max\{\mathrm{AP}(S), S \text{ convex sets included in } K\}$.

In the case of the regular κ-gons, the following result comes as a corollary of the properties of the affine perimeter.

Lemma 16. *Let $p_1, \ldots, p_\kappa$ be the midpoints of the consecutives sides of $\mathfrak{C}_\kappa$, and $y_1, \ldots, y_\kappa$ the vertices of $\mathfrak{C}_\kappa$, so that p_i is the middle of the segment $[y_i, y_{i+1}]$ (modulo κ). Let $\mathcal{C}_i$ be the unique parabola tangent to $p_i y_{i+1}$ at p_i and tangent to $y_{i+1}p_{i+1}$ at p_{i+1}. The convex domain $\mathsf{Dom}(\mathfrak{C}_\kappa)$ is the subset of $\mathfrak{C}_\kappa$ whose boundary is formed by the parabolas $(\mathcal{C}_i)_{1\leq i\leq\kappa}$. The set $\mathsf{Dom}(\mathfrak{C}_\kappa)$ is thus tangent to $\mathfrak{C}_\kappa$ in the κ points $p_1, \ldots, p_\kappa$.*

In Figure 17, we represented 3 instances of the convex set $\mathsf{Dom}(\mathfrak{C}_\kappa)$ in the cases $\kappa = 3, 4, 6$, which boundary is the dashed curve contained in $\mathfrak{C}_\kappa$.

Proof. Theorem 10 indicates that $\mathsf{Dom}(\mathfrak{C}_\kappa)$ is the convex domain contained in $\mathfrak{C}_\kappa$ which maximizes the affine perimeter. By the definition of the affine perimeter, we have $\mathrm{AP}(\mathfrak{C}_\kappa) = 0$, so that $\mathsf{Dom}(\mathfrak{C}_\kappa)$ lies within the interior of $\mathfrak{C}_\kappa$. In this case (by Bárány [2]), the boundary of $\mathsf{Dom}(\mathfrak{C}_\kappa)$ is composed of finitely many arcs of parabolas. In order to maximize the affine perimeter, $\mathsf{Dom}(\mathfrak{C}_\kappa)$ has to be tangent to at least three sides of $\mathfrak{C}_\kappa$, and the symmetry of $\mathfrak{C}_\kappa$ forces these tangency points to be the κ midpoints of the sides of $\mathfrak{C}_\kappa$. Hence between two consecutive midpoints lies an arc of a parabola. □

Lemma 17. *For all $\kappa \geq 3$, the supremum of affine perimeters $\mathrm{AP}^*(\mathfrak{C}_\kappa)$ is*

$$\mathrm{AP}^*(\mathfrak{C}_\kappa) = \mathrm{AP}(\mathsf{Dom}(\mathfrak{C}_\kappa)) = \kappa\left(r_\kappa^2 \sin(\theta_\kappa)\right)^{\frac{1}{3}}. \tag{A.1}$$

Proof. Using the notation of Lemma 16, we have of course, by symmetry,

$$\mathrm{AP}(\mathsf{Dom}(\mathfrak{C}_\kappa)) = \sum_{i=1}^{\kappa} \mathrm{AP}(\mathcal{C}_i) = \kappa\,\mathrm{AP}(\mathcal{C}_1).$$

Let $\mathcal{T}_1$ be the area of the triangle with vertices p_1, y_2, p_1. We claim that

$$\mathrm{AP}(\mathcal{C}_1) = \mathsf{Area}\,(\mathcal{T}_1)^{1/3}. \tag{A.2}$$

https://doi.org/10.1017/apr.2024.63 Published online by Cambridge University Press

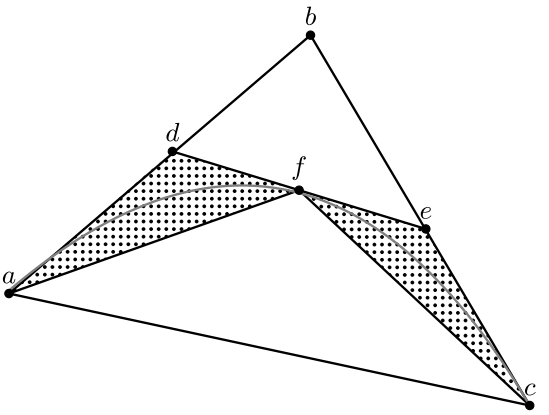


FIGURE 18. Blaschke's property for arcs of parabolas.

This property comes from the following fact, due to Blaschke [6, p. 38]. Consider a triangle T with vertices a, b, c and with subtriangles (both with dotted areas) $T^{(1)}$ (with vertices a, d, f) and $T^{(2)}$ (with vertices f, e, c) defined so that $(d, e) \in [a, b] \times [b, c]$ and the segment $[d, e]$ is tangent to the arc of the parabola $\mathcal{C}$ at f, just as in Figure 18.

In this case, we have

$$\mathsf{Area}\,(T)^{1/3} = \mathsf{Area}\left(T^{(1)}\right)^{1/3} + \mathsf{Area}\left(T^{(2)}\right)^{1/3}.$$

Therefore, for any integer m, any tuple of points $x[m] \in \mathcal{C}_1$, and triangles T_i, $i \in \{1, \dots, m\}$ (both defined as in Definition 2), the quantity $\lim_{x[m]} \sum_{i=1}^{m} \mathsf{Area}\,(T_i)^{1/3}$ is constant; hence

$$\mathrm{AP}(\mathcal{C}_1) := \mathsf{Area}\,(\mathcal{T}_1)^{1/3}.$$

Now, from easy computations, one gets

$$\mathsf{Area}\,(\mathcal{T}_1)^{1/3} = \frac{1}{2}\left(r_\kappa^2 \sin(\theta_\kappa)\right)^{1/3},$$

which is (A.1). □

Proof of Lemma 4. For an n-tuple $\mathbf{z}[n]$ that is $\mathbb{U}_\kappa^{(n)}$-distributed, Bárány's Theorem 10 states that for all $\varepsilon > 0$,

$$\mathbb{P}\big(d_H(\mathsf{conv}(\mathbf{z}[n]), \mathsf{Dom}(\mathfrak{C}_\kappa)) > \varepsilon \mid \mathbf{z}[n] \in \mathcal{C}_\kappa(n)\big) \underset{n\to+\infty}{\longrightarrow} 0,$$

which implies immediately that

$$\mathbb{P}\big(\mathrm{NZS}(\mathbf{z}[n]) = \{1, \dots, \kappa\} \mid \mathbf{z}[n] \in \mathcal{C}_\kappa(n)\big) = \frac{\widetilde{\mathbb{P}}_\kappa(n)}{\mathbb{P}_\kappa(n)} \underset{n\to+\infty}{\longrightarrow} 1.$$

Appendix B. Valtr's results

The surprising simplicity of Valtr's formulas in the cases of the parallelogram and the triangle can be seen as a consequence of the fact that the sets $\mathcal{L}_4$ and $\mathcal{L}_3$ are easily computable. Let us recover these results with Theorem 6.

https://doi.org/10.1017/apr.2024.63 Published online by Cambridge University Press

B.1. The triangle

Theorem 11. *(Valtr [25].) For all $n \geq 3$, we have*

$$\mathbb{P}_{\triangle}(n) = \frac{2^n(3n-3)!}{(2n)!((n-1)!)^3}.$$

We propose a new proof of Valtr's result.

Proof. In the case $\kappa = 3$, the side length of $\mathfrak{C}_3$ is $r_3 = 2/3^{1/4}$. Pick $c_1, c_2, c_3, \ell_1, \ell_2, \ell_3$ satisfying the equations $(\mathcal{C}_j)_{1\leq j\leq 3}$. Since the only equiangular polygon with three sides is the equilateral triangle, we have $c_1 = c_2 = c_3 = c$. This forces

$$(\mathcal{C}_1) = (\mathcal{C}_2) = (\mathcal{C}_3) : c + \frac{2}{\sqrt{3}}(\ell_1 + \ell_2 + \ell_3) = r_3.$$

We thus understand that

$$\mathcal{L}_3 = \left\{ (\ell_1, \ell_2, \ell_3) \in \left[0, \frac{\sqrt{3}}{2} r_3\right]^3 \text{ with } \ell_1 + \ell_2 + \ell_3 \leq \frac{\sqrt{3}}{2} r_3 \right\}.$$

We also have $\mathbb{N}_3(n) = \big\{(i, j, k) \in \{0, \dots, n-1\}^3, i + j + k = n\big\}$, so that we have $\mathbb{D}_n^{(3)} = \mathbb{Q}_n^{(3)}$, and combining this with

$$\sum_{s[3]\in\mathbb{N}_3(n)} \int_{\mathbb{R}^3} f_n^{(3)}(s[3], \ell[3])\, \mathrm{d}\ell_1 \mathrm{d}\ell_2 \mathrm{d}\ell_3 = 1,$$

we obtain

$$\mathbb{P}_{\triangle}(n) = n! \sin\left(\frac{\pi}{3}\right)^{n-3} \sum_{s[3]\in\mathbb{N}_3(n)} \int_{\ell[3]\in\mathcal{L}_3} \prod_{j=1}^{3} \frac{c^{s_{j-1}+s_j-1}}{s_j!(s_{j-1}+s_j-1)!} \mathrm{d}\ell_1 \mathrm{d}\ell_2 \mathrm{d}\ell_3.$$

Put $(i, j, k) = s[3]$ and perform the substitution $\ell = \frac{2}{\sqrt{3}}(\ell_1 + \ell_2 + \ell_3)$ to get

$$\begin{aligned}
\mathbb{P}_{\triangle}(n) &= n! \sin\left(\frac{\pi}{3}\right)^{n-3} \sum_{(i,j,k)\in\mathbb{N}_3(n)} \int_0^{r_3} \frac{1}{2}\ell^2 \frac{(r_3-\ell)^{i+j-1}}{i!(i+j-1)!} \frac{(r_3-\ell)^{j+k-1}}{j!(j+k-1)!} \frac{(r_3-\ell)^{k+i-1}}{k!(k+i-1)!} \left(\frac{\sqrt{3}}{2}\right)^3 \mathrm{d}\ell \\
&= n! \left[\frac{1}{((n-1)!)^3} \sum_{i+j+k=n} \binom{n-1}{i}\binom{n-1}{j}\binom{n-1}{k}\right] \left(\frac{\sqrt{3}}{2}\right)^n \int_0^{r_3} \frac{1}{2}\ell^2 (r_3-\ell)^{2n-3} \mathrm{d}\ell \\
&= \frac{n!}{((n-1)!)^3} \binom{3n-3}{n} \frac{(2n-3)!}{(2n)!} \left(\frac{\sqrt{3}}{2}\right)^n r_3^{2n} \\
&= \frac{2^n(3n-3)!}{(2n)!((n-1)!)^3}.
\end{aligned}$$

□

https://doi.org/10.1017/apr.2024.63 Published online by Cambridge University Press

B.2. The square

Theorem 12. *(Valtr [24].) For all $n \geq 3$, we have*

$$\mathbb{P}_{\square}(n) = \frac{1}{(n!)^2} \binom{2n-2}{n-1}^2.$$

Again, we propose a new proof of Valtr's result.

Proof. Consider a square (i.e. the case $\kappa = 4$) of side length $r_4 = 1$. Pick c_1, c_2, c_3, c_4 and $\ell_1, \ell_2, \ell_3, \ell_4$ satisfying the equations $(\mathcal{C}_j)_{1 \leq j \leq 4}$. Since the only equiangular polygons with four sides are rectangles, we have $c_1 = c_3$ and $c_2 = c_4$. This implies

$$(\mathcal{C}_1 = \mathcal{C}_3) : c_1 + \ell_1 + \ell_3 = 1,$$

$$(\mathcal{C}_2 = \mathcal{C}_4) : c_2 + \ell_2 + \ell_4 = 1.$$

This means in particular that

$$\mathcal{L}_4 = \big\{(\ell_1, \ell_2, \ell_3, \ell_4) \in [0,1]^4 \text{ with } \ell_1 + \ell_3 \leq 1 \text{ and } \ell_2 + \ell_4 \leq 1\big\}.$$

Just as before, we have $\mathbb{D}_n^{(4)} = \mathbb{Q}_n^{(4)}$, so that

$$\sum_{s[4] \in \mathbb{N}_4(n)} \int_{\mathbb{R}^4} f_n^{(4)}(s[4], \ell[4])\, \mathrm{d}\ell_1 \mathrm{d}\ell_2 \mathrm{d}\ell_3 \mathrm{d}\ell_4 = 1,$$

from which we deduce

$$\mathbb{P}_{\square}(n) = n! \sum_{s[4] \in \mathbb{N}_4(n)} \int_{\ell[4] \in \mathcal{L}_4} \prod_{j=1}^{4} \frac{c_j^{s_{j-1}+s_j}}{s_j!(s_{j-1}+s_j)!} \mathrm{d}\ell_1 \mathrm{d}\ell_2 \mathrm{d}\ell_3 \mathrm{d}\ell_4,$$

where $\mathbb{N}_4(n) = \big\{(s_1, s_2, s_3, s_4) \in \mathbb{N}^4 \text{ such that } s_1 + s_2 + s_3 + s_4 = n \text{ and } s_j + s_{j+1} \geq 1\big\}$. Put $(h, i, j, k) = s[4]$ and perform the substitutions $c_1 = 1 - (\ell_1 + \ell_3)$, $c_2 = 1 - (\ell_2 + \ell_4)$ to get

$$\mathbb{P}_{\square}(n) = n! \sum_{(h,i,j,k) \in \mathbb{N}_4(n)} \int_0^1 \int_0^1 \frac{(1-c_1)c_1^{h+i-1}}{h!(h+i-1)!} \frac{(1-c_2)c_2^{i+j-1}}{i!(i+j-1)!} \times \frac{c_1^{j+k-1}}{j!(j+k-1)!} \frac{c_2^{h+k-1}}{k!(h+k-1)!} \mathrm{d}c_1 \mathrm{d}c_2.$$

This leads to

$$\mathbb{P}_{\square}(n) = n! \left(\int_0^1 (1-c)c^{n-2} \mathrm{d}c \right)^2 \times \sum_{(h,i,j,k) \in \mathbb{N}_4(n)} \frac{1}{h!(h+i-1)!} \frac{1}{i!(i+j-1)!} \frac{1}{j!(j+k-1)!} \frac{1}{k!(h+k-1)!}. \quad \text{(B.1)}$$

The integral is a standard beta-integral, so that

$$n! \left(\int_0^1 (1-c)c^{n-2} \mathrm{d}c \right)^2 = n! \left(\frac{(n-2)!}{n!} \right)^2.$$

https://doi.org/10.1017/apr.2024.63 Published online by Cambridge University Press

It remains to compute the big sum S that appears separately in (B.1):

$$\begin{aligned} S &= \frac{1}{(n-2)!((n-1)!)^2} \sum_{(h,i,j,k)\in\mathbb{N}_4(n)} \binom{n-1}{h+i}\binom{n-1}{j+k}\binom{h+i}{h}\binom{j+k}{k}\binom{n-2}{i+j-1} \\ &= \frac{1}{(n-2)!((n-1)!)^2} \sum_{r=1}^{n-1} \binom{n-1}{r}\binom{n-1}{n-r}\binom{2n-2}{n-1} \\ &= \frac{1}{n!((n-2)!)^2} \binom{2n-2}{n-1}^2, \end{aligned}$$

which is Valtr's formula. □

Appendix C. Computation of m_κ

In this section, we aim to prove that the determinant m_κ of the matrix Σ_κ^{-1} of size $(\kappa-1)\times(\kappa-1)$, defined in Theorem 7 as

$$\Sigma_\kappa^{-1} := \frac{1}{2}\begin{pmatrix} 6 & 4 & 3 & \cdots & \cdots & 3 & 2 \\ 4 & 8 & 5 & 4 & \cdots & 4 & 3 \\ 3 & 5 & \ddots & \ddots & \ddots & \vdots & \vdots \\ \vdots & 4 & \ddots & \ddots & \ddots & 4 & \vdots \\ \vdots & \vdots & \ddots & \ddots & \ddots & 5 & 3 \\ 3 & 4 & \cdots & 4 & 5 & 8 & 4 \\ 2 & 3 & \cdots & \cdots & 3 & 4 & 6 \end{pmatrix} \text{ for } \kappa \text{ large enough,}$$

is indeed

$$\mathrm{m}_\kappa = \frac{\kappa}{3\cdot 2^\kappa}\left(2(-1)^{\kappa-1} + (2-\sqrt{3})^\kappa + (2+\sqrt{3})^\kappa\right),$$

as given in Theorem 1.

Proof. We define the matrix D_κ as

$$D_\kappa := 2\Sigma_\kappa^{-1}\begin{pmatrix} 4/3 & 0 & \cdots & & 0 \\ 0 & 1 & & & \\ & \ddots & \ddots & \ddots & \vdots \\ \vdots & & & 1 & 0 \\ 0 & \cdots & & 0 & 4/3 \end{pmatrix} = \begin{pmatrix} 8 & 5 & 3 & \cdots & \cdots & 3 & 8/3 \\ 16/3 & 8 & 5 & 4 & \cdots & 4 & 4 \\ 4 & 5 & \ddots & \ddots & \ddots & \vdots & \vdots \\ \vdots & 4 & \ddots & \ddots & \ddots & 4 & \vdots \\ \vdots & \vdots & \ddots & \ddots & \ddots & 5 & 4 \\ 4 & 4 & \cdots & 4 & 5 & 8 & 16/3 \\ 8/3 & 3 & \cdots & \cdots & 3 & 5 & 8 \end{pmatrix},$$

where, the factor 2 aside, we have just multiplied by $\frac{4}{3}$ the first and last columns of Σ_κ^{-1}. This means of course that

$$\det(D_\kappa) = 2^{\kappa-1}(4/3)^2\mathrm{m}_\kappa.$$

https://doi.org/10.1017/apr.2024.63 Published online by Cambridge University Press

We now decompose D_κ as $D_\kappa := Q_\kappa + E_\kappa$ with

$$Q_\kappa := \begin{pmatrix} 3 & \cdots & \cdots & 3 \\ 4 & \cdots & \cdots & 4 \\ \vdots & & & \vdots \\ 4 & \cdots & \cdots & 4 \\ 3 & \cdots & \cdots & 3 \end{pmatrix} \quad \text{and} \quad E_\kappa := \begin{pmatrix} 5 & 1 & 0 & \cdots & \cdots & 0 & -1/3 \\ 4/3 & 4 & \ddots & 0 & \cdots & 0 & 0 \\ 0 & 1 & \ddots & \ddots & \ddots & \vdots & \vdots \\ \vdots & 0 & \ddots & \ddots & \ddots & 0 & \vdots \\ \vdots & \vdots & \ddots & \ddots & \ddots & 1 & 0 \\ 0 & 0 & \cdots & 0 & \ddots & 4 & 4/3 \\ -1/3 & 0 & \cdots & \cdots & 0 & 1 & 5 \end{pmatrix}.$$

We first claim that

$$\det(E_\kappa) = \frac{4}{9}\left(2(-1)^{\kappa-1} + (2-\sqrt{3})^\kappa + (2+\sqrt{3})^\kappa\right). \tag{C.1}$$

To prove this, notice first that taking two Laplace expansions of the determinant of the $(m \times m)$ matrix

$$L_m := \begin{pmatrix} 4 & 1 & 0 & \cdots & 0 \\ 1 & \ddots & \ddots & \ddots & \vdots \\ 0 & \ddots & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & 1 \\ 0 & \cdots & 0 & 1 & 4 \end{pmatrix}$$

gives a constant-recursive sequence of order 2 for its determinant:

$$\det(L_m) = 4\det(L_{m-1}) - \det(L_{m-2}),$$

which can be solved immediately to get

$$\det(L_m) = \frac{1}{2\sqrt{3}}\left[(2+\sqrt{3})^{m+1} - (2-\sqrt{3})^{m+1}\right], \qquad m \geq 1.$$

Taking several Laplace expansions of $\det(E_\kappa)$ along the first column allows one to either deal with diagonal matrices (leading to the term $\frac{8}{9}(-1)^{\kappa-1}$)), or with $L_{\kappa-2}$ and $L_{\kappa-3}$ to ultimately get (C.1).

How do we compute $\det(D_\kappa) = \det(Q_\kappa + E_\kappa)$? In general, since the determinant is a multilinear alternating map of the columns of the matrix, for two $(m \times m)$ matrices $A = (A_i)_{1\leq i\leq m}$ and $B = (B_i)_{1\leq i\leq m}$, (where A_i is the ith column of A), we can write

$$\begin{aligned} \det(A+B) &= \det(A_1 + B_1, \ldots, A_m + B_m) \\ &= \sum_{I \sqcup J = \{1,\ldots,m\}} \det\left((A_i \mathbf{1}_{\{i\in I\}} + B_i \mathbf{1}_{\{i\in J\}})_{i\in\{1,\ldots,m\}}\right), \end{aligned}$$

where $I \sqcup J = \{1, \ldots, m\}$ means that I,J forms a partition of $\{1, \ldots, m\}$.

https://doi.org/10.1017/apr.2024.63 Published online by Cambridge University Press

In the case where all columns of B are the same, the sum above only keeps the partitions (I,J) of $\{1,\dots,m\}$ where either $|J|=0$ (hence we retrieve $\det(A)$), or $|J|=1$. We therefore introduce the matrix $E_\kappa^{(i)}$, for all $i\in\{1,\dots,\kappa-1\}$, which is the matrix E_κ with its ith column replaced by $(3\ 4\cdots 4\ 3)^t$. By the previous argument we have

$$\det(D_\kappa)=\det(E_\kappa)+\sum_{i=1}^{\kappa-1}\det(E_\kappa^{(i)}). \tag{C.2}$$

We now make the following claim. □

Lemma 18 *For all κ large enough, we have the following:*

1. $\det(E_\kappa^{(i)})=\frac{2}{3}\det(E_\kappa)$ *for all* $i\in\{2,\dots,\kappa-2\}$,
2. $\det(E_\kappa^{(1)})=\det(E_\kappa^{(\kappa-1)})=\frac{1}{2}\det(E_\kappa)$.

This lemma allows us to conclude, since by (C.2), we now have

$$\det(D_\kappa)=\det(E_\kappa)\left(1+\frac{2}{3}(\kappa-3)+1\right)=\frac{2}{3}\kappa\det(E_\kappa).$$

Proof of Lemma 18. The proof relies on some determinant-preserving column manipulations on $E_\kappa^{(i)}$, which provide matrices equal to E_κ up to a constant factor.

Pick $i\in\{2,\dots,\kappa-2\}$, and consider the matrix $E_\kappa^{(i)}$ where the ith column is multiplied by $3/2$. Then subtract all other columns from the ith. Then add $-1/4$ times the first and last columns to the ith column, to obtain E_κ. This gives the first part.

If $i=1$ or $\kappa-1$ the same reasoning works: multiply the ith column by 2, and then there exists a linear combination of the columns other than i that can be added to the ith column to retrieve E_κ. □

Acknowledgements

I would like to express my deepest gratitude to Jean-François Marckert for his valuable guidance and advice throughout the long research and writing process of this paper. I also thank Zoé Varin for her precious help in my struggles with TikZ. Some of the figures in this paper were entirely created by her expert hand. Many thanks to the referees for their wise comments and suggestions, which considerably increased the readability of this paper.

Funding information

There are no funding bodies to thank in relation to the creation of this article.

Competing interests

There were no competing interests to declare which arose during the preparation or publication process of this article.

References

[1] BÁRÁNY, I. (1995). The limit shape of convex lattice polygons. *Discrete Comput. Geom.* **13**, 279–295.
[2] BÁRÁNY, I. (1997). Affine perimeter and limit shape. *J. reine angew. Math.* **484**, 71–84.
[3] BÁRÁNY, I. (1999). Sylvester's question: the probability that n points are in convex position. *Ann. Prob.* **27**, 2020–2034.

https://doi.org/10.1017/apr.2024.63 Published online by Cambridge University Press

[4] BILLINGSLEY, P. (1999). *Convergence of Probability Measures*. John Wiley, New York.
[5] BLASCHKE, W. (1917). Über affine Geometrie XI: Lösung des "Vierpunktproblems" von Sylvester aus der Theorie der geometrischen Wahrscheinlichkeiten. *Leipzig Ber.* **69**, 436–453.
[6] BLASCHKE, W. (1923). *Vorlesungen über Differentialgeometrie. II. Affine Differentialgeometrie*. Springer, Berlin.
[7] BODINI, O., JACQUOT, A., DUCHON, P. AND MUTAFCHIEV, L. R. (2013). Asymptotic analysis and random sampling of digitally convex polyominoes. In *Discrete Geometry for Computer Imagery (DGCI 2013)*, eds R. Gonzalez-Diaz, M.-J. Jimenez, and B. Medrano, Springer, Berlin, Heidelberg, pp. 95–106.
[8] BROWN, M. B. AND BROMBERG, J. (1984). An efficient two-stage procedure for generating random variates from the multinomial distribution. *Amer. Statistician* **38**, 216–219.
[9] BUFFIÈRE, T. (2023). *Théorèmes combinatoires et probabilistes sur certaines familles de polytopes*. Doctoral Thesis, Université Paris-Nord—Paris XIII.
[10] BUREAUX, J. AND ENRIQUEZ, N. (2016). On the number of lattice convex chains. *Discrete Anal.* **19**, 15 pp.
[11] BÁRÁNY, I., BUREAUX, J. AND LUND, B. (2018). Convex cones, integral zonotopes, limit shape. *Adv. Math.* **331**, 143–169.
[12] DAVIS, C. S. (1993). The computer generation of multinomial random variates. *Comput. Statist. Data Anal.* **16**, 205–217.
[13] HILHORST, H. J., CALKA, P. AND SCHEHR, G. (2008). Sylvester's question and the Random Acceleration Process. *J. Statist. Mech.* 2008, article no. P10010.
[14] JAMBUNATHAN, M. V. (1954). Some properties of beta and gamma distributions. *Ann. Math. Statist.* **25**, 401–405.
[15] KRONMAL, R. A. AND PETERSON, A. V., Jr. (1979). On the alias method for generating random variables from a discrete distribution. *Amer. Statistician* **33**, 214–218.
[16] MARCKERT, J.-F. (2008). One more approach to the convergence of the empirical process to the Brownian bridge. *Electron. J. Statist.* **2**, 118–126.
[17] MARCKERT, J.-F. (2017). The probability that n random points in a disk are in convex position. *Brazilian J. Prob. Statist.* **31**, 320–337.
[18] MARCKERT, J.-F. AND RAHMANI, S. (2021). Around Sylvester's question in the plane. *Mathematika* **67**, 860–884.
[19] PETROV, V. (1975). *Sums of Independent Random Variables*. Springer, Berlin, Heidelberg.
[20] PFIEFER, R. E. (1989). The historical development of J. J. Sylvester's four point problem. *Math. Magazine* 62, 309–317.
[21] SCHNEIDER, R. (2017). Discrete aspects of stochastic geometry. In *Handbook of Discrete and Computational Geometry*, 3rd edn, Chapman and Hall/CRC, Boca Raton, pp. 299–329.
[22] SINAI, Y. (1994). Probabilistic approach to the analysis of statistics for convex polygonal lines. *Funct. Anal. Appl.* **28**, 108–113.
[23] SYLVESTER, J. J. (1864). Problem 1491. Educational Times.
[24] VALTR, P. (1995). Probability that n random points are in convex position. *Discrete Comput. Geom.* **13**, 637–643.
[25] VALTR, P. (1995). The probability that n random points in a triangle are in convex position. *Combinatorica* **16**, 567–573.
[26] VERSHIK, A. (1994). The limit shape of convex lattice polygons and related topics. *Funct. Anal. Appl.* **28**, 13–20.
[27] WALKER, A. J. (1977). An efficient method for generating discrete random variables with general distributions. *ACM Trans. Math. Software* **3**, 253–256.

https://doi.org/10.1017/apr.2024.63 Published online by Cambridge University Press