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NONCOMMUTATIVE COTLAR IDENTITIES FOR GROUPS ACTING ON TREE-LIKE STRUCTURES

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Abstract Let T_m be a noncommutative Fourier multiplier. In recent work, Mei and Ricard introduced a noncommutative analogue of Cotlar's identity to prove that certain multipliers are bounded on the noncommutative L_p-spaces of a free group. Here, we study Cotlar-type identities in full generality, giving a closed characterization for them in terms of m:

$$\big(m(gh)-m(g)\big)\big(m(g^{-1})-m(h)\big)=0,\ \forall g\in \mathrm{G}\setminus\{e\},h\in \mathrm{G}.$$

Using a geometric argument, we prove that if X is a tree —or more generally an $\mathbb{R}$-tree— on which G acts and m lifts to a function $\widetilde{m}: X\to\mathbb{C}$ that is constant on the connected subsets of $X\setminus\{x_0\}$, then m satisfies Cotlar's identity and thus T_m is bounded in L_p for $1<p<\infty$.

This result establishes a new connection between group actions on $\mathbb{R}$-trees and Fourier multipliers. This machinery allows us to simultaneously generalize the free group transforms of Mei and Ricard and the theory of Hilbert transforms in left-orderable groups, which follows from Arveson's subdiagonal algebras. Using Bass-Serre theory, we construct new examples of Fourier multipliers on groups. These include new families such as Baumslag-Solitar groups. We also show that a natural Hilbert transform in $\mathrm{PSL}_2(\mathbb{C})$ satisfies Cotlar's identity when restricted to the Bianchi group $\mathrm{PSL}_2(\mathbb{Z}[\sqrt{-1}])$.

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1. Introduction

The Hilbert transform was introduced by Hilbert in 1912 as part of his investigation of the Riemann problem in the realm of complex analysis [38]. Indeed, it may be regarded as the operator describing the boundary behaviour of the harmonic conjugate in the upper half plane. Explicitly, it is given by

$$Hf(x) = \frac{1}{\pi} \lim_{\varepsilon \to 0^+} \int_{|x-y|>\varepsilon} \frac{f(y)}{x-y}\, dy. \tag{HT}$$

Equivalently, it is the Fourier multiplier $(Hf)^\wedge(\xi) = i\,\mathrm{sgn}(\xi)\,\widehat{f}(\xi)$ [25]. In 1924, M. Riesz proved its L_p-boundedness for all $p \in 2\mathbb{Z}_+$ using an *ad hoc* complex analysis argument [75, 76]. By duality and Marcinkiewicz's interpolation this yields L_p-boundedness for every $1 < p < \infty$. Afterwards, Kolmogorov and Calderón-Zygmund found proofs giving the weak type (1,1) [50, 81, 10].

In 1955, Cotlar proved the L_p-boundedness of the Hilbert transform through an extremely simple argument [17]. He showed that H is L_p-bounded for $p = 2^k$ recursively – from the trivial case $p = 2$ – using his elegant *Cotlar identity*

$$\big(Hf\big)^2 = 2H\big(f \cdot Hf\big) + f^2. \tag{Classical Cotlar}$$

Note that for the Hilbert transform, we have $H^2 = -\mathrm{id}$ whenever it is well-defined. Therefore, the Cotlar identity above can be rewritten as

$$\big(Hf\big)^2 = 2H\big(f \cdot Hf\big) - H\big(H(f^2)\big).$$

A key point in our work is to notice that Cotlar's identity and its generalizations have a nicer expression at the frequency side of the Fourier transform. As an illustration, notice that a Euclidean Fourier multiplier $(T_m f)^\wedge(\xi) = m(\xi)\,\widehat{f}(\xi)$ satisfies Cotlar's identity precisely when

$$\big(m(\xi+\eta) - m(\xi)\big)\big(m(-\xi) - m(\eta)\big) = 0, \quad \text{a.e. } \xi,\eta \in \mathbb{R}. \tag{Classical $\widehat{\text{Cotlar}}$}$$

In fact, any bounded function satisfying the classical Cotlar identity above will be bounded on $L_p(\mathbb{R})$ for every $1 < p < \infty$. In this article, we will investigate similar identities on more general locally compact groups and their von Neumann algebras. A pioneering work in this direction was due to Mei and Ricard [55], where the authors deployed a noncommutative analogue of the Cotlar identity, which holds in the context of amalgamated free product von Neumann algebras. The main goal of this article is to further Mei and Ricard's technique beyond free groups by illuminating the hidden connection between noncommutative Cotlar identities and group actions on trees and other tree-like structures – like $\mathbb{R}$-trees and uniquely arcwise connected spaces.

Noncommutative Fourier multipliers. Here we will deal with operators analogous to the Hilbert transform over group algebras. Let G be a locally compact group. Its (left regular) group von Neumann algebra is defined as

$$\mathcal{L}\mathrm{G} := \overline{\mathrm{span}}^{\mathrm{w}^*}\big\{\lambda_g : g \in \mathrm{G}\big\} \subseteq \mathcal{B}(L_2\mathrm{G}).$$

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When G is Abelian, $\mathcal{L}\mathrm{G}$ is isomorphic to $L_\infty(\widehat{\mathrm{G}})$, the L_∞-space over the Pontryagin dual of G. Thus, $\mathcal{L}\mathrm{G}$ behaves as a generalization of the Pontryagin dual for noncommutative groups. When G is unimodular, the algebra $\mathcal{L}\mathrm{G}$ admits a normal, semifinite, and faithful tracial weight called the *Plancherel trace* [67, Chapter 8], with respect to which the noncommutative L_p-spaces $L_p(\mathcal{L}\mathrm{G})$ are defined [84, 72]. These L_p-spaces straightforwardly generalize the spaces of L_p-integrable elements over the dual of G. As such, many classical problems of Fourier analysis over L_p-spaces find an analogue in the noncommutative setting. A prominent example is the study of the L_p-boundedness of *Fourier multipliers*. Indeed, given $m : \mathrm{G} \to \mathbb{C}$, the *Fourier multiplier of symbol* m will be the – potentially unbounded – linear operator $T_m : D \subseteq \mathcal{L}\mathrm{G} \to \mathcal{L}\mathrm{G}$ given by linear extension of

$$T_m(\lambda_g) = m(g)\,\lambda_g.$$

The study of bounded Fourier multipliers on L_p-spaces of group von Neumann algebras has gained significant attention due to its deep connections with harmonic analysis and group theory. For $p = \infty$, many classical approximation properties of groups can be reformulated in terms of sequences of Fourier multipliers; see [36, 18, 21]. This viewpoint enabled the use of harmonic analysis techniques in group theory and led, for example, to Haagerup's proof that higher-rank Lie groups are not weakly amenable [37]. For $1 < p < \infty$, the analogous approximation property on L_p was studied by Lafforgue, de la Salle and de Laat [51, 23], who showed that lattices in higher-rank Lie groups do not admit uniformly bounded Fourier multiplier approximate units on L_p, for a range of p that increases with the rank. Determining the exact range of exponents and whether the critical threshold p_0 encodes the rank remains a major open question [65, 15, 22, 63].

Fourier multipliers are also interesting for purely analytical reasons. Many long-standing problems in harmonic analysis on $\mathbb{R}^n$ can be formulated in terms of L_p-bounded multipliers; see, for example, [82] and [83, Lecture 5]. Consequently, a complete characterization of bounded Fourier multipliers is not expected beyond the cases $p = 1, 2, \infty$. Finding sufficient conditions for the boundedness of Fourier multipliers in the group setting presents additional challenges that do not arise in the abelian case. First, the definition of the L_p-spaces requires working within von Neumann algebras, where many of the necessary functional-analytic tools, such as operator space theory [26, 70, 71], mixed-norm L_p-spaces [69, 42, 46], and maximal inequalities [19, 52, 48, 9], were fully developed only in the past two decades. A second major difficulty is the lack, in the von Neumann algebra setting, of many singular integral techniques that are fundamental in the analysis on $\mathbb{R}^n$. Although there has been progress toward a noncommutative Calderón-Zygmund theory [62, 33, 45, 8], a comprehensive theory capable of producing weak type (1,1) bounds remains unavailable outside semicommutative or nilpotent contexts. These technical obstacles have significantly delayed the systematic study of Fourier multipliers on noncommutative L_p-spaces until relatively recently.

As hinted before, one possible way of overcoming the lack of a Calderón-Zygmund theory was found in Mei and Ricard's article [55]. Their noncommutative analogue of the Cotlar identity allowed them to prove that functions $m : \mathbb{F}_2 \to \mathbb{C}$ over the free group

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whose value $m(\omega)$ depends only on the starting letter $\{a,a^{-1},b,b^{-1}\}$ of the reduced word $\omega \in \mathbb{F}_2$ give rise to bounded Fourier multipliers on L_p, i.e.,

$$\big\|T_m : L_p(\mathcal{L}\mathbb{F}_2) \to L_p(\mathcal{L}\mathbb{F}_2)\big\|_{\mathrm{cb}} < \infty, \quad \text{for every } 1<p<\infty. \tag{MR}$$

In this article, we will study a new geometric way to define L_p-bounded Fourier multipliers on groups admitting actions on tree-like structures, and we will see in Section 5 that this recovers (MR) as a particular example.

Noncommutative Cotlar identities. Let $\mathrm{G}_0 \subseteq \mathrm{G}$ be an open subgroup of the locally compact and unimodular group G. It is trivial to see that G_0 is also unimodular and that $\mathcal{L}\mathrm{G}_0 \subseteq \mathcal{L}\mathrm{G}$ is a complemented inclusion, that is, an inclusion admitting a normal conditional expectation $\mathbb{E} : \mathcal{L}\mathrm{G} \to \mathcal{L}\mathrm{G}_0$. We will say that a (potentially unbounded) multiplier T_m satisfies a *noncommutative Cotlar identity* with respect to the von Neumann subalgebra $\mathcal{L}\mathrm{G}_0$ iff

$$\mathbb{E}^{\perp}\big[T_m(f)\,T_m(f)^*\big] = \mathbb{E}^{\perp}\Big[T_m\big(f\,T_m(f)^*\big) + T_m\big(f\,T_m(f)^*\big)^* - T_m\big(T_m(ff^*)^*\big)\Big], \tag{Cotlar}$$

where $\mathbb{E}^{\perp} = (\mathrm{id} - \mathbb{E})$.

We have distilled an easily verifiable closed formula ($\widehat{\text{Cotlar}}$) for m that is equivalent to (Cotlar) above; see Theorem 2.4. Since with an additional assumption on the symbol, the Cotlar formula implies L_p-boundedness, we obtain the following theorem.

Theorem A. *Let* G *be a locally compact unimodular group,* $\mathrm{G}_0 \subseteq \mathrm{G}$ *a closed subgroup and* $m : \mathrm{G} \to \mathbb{C}$ *a left* G_0*-invariant bounded and measurable function. If* m *satisfies that*

$$\big(m(g^{-1}) - m(h)\big)\big(m(gh) - m(g)\big) = 0, \quad \textit{for almost every } \ g \in \mathrm{G}\setminus\mathrm{G}_0, h \in \mathrm{G}, \tag{$\widehat{\text{Cotlar}}$}$$

then T_m *is* L_p*-bounded for* $1<p<\infty$ *and furthermore*

$$\big\|T_m : L_p(\mathcal{L}\mathrm{G}) \to L_p(\mathcal{L}\mathrm{G})\big\| \lesssim \left(\frac{p^2}{p-1}\right)^{\beta} \|m\|_{\infty}, \quad \textit{with } \beta = \log_2(1+\sqrt{2}) \approx 1.272\ldots. \tag{1.1}$$

The theorem above decouples into two different statements depending on whether $\mathrm{G}_0 \subseteq \mathrm{G}$ is open or has empty interior. In the case of an open subgroup, it holds that $0<\mu(\mathrm{G}_0)$ and thus condition ($\widehat{\text{Cotlar}}$) has to be verified in a non-total set of pairs g,h. In this case, the associated Fourier multiplier T_m satisfies (Cotlar) relative to the conditional expectation from $\mathcal{L}\mathrm{G}$ to $\mathcal{L}\mathrm{G}_0$. In contrast, in the case of a subgroup G_0 of empty interior we have that $\mu(\mathrm{G}_0) = 0$ and thus ($\widehat{\text{Cotlar}}$) has to be verified almost everywhere, in which case its associated Fourier multiplier satisfies the non-relative version of the Cotlar identity ($\text{Cotlar}_{\mathrm{nr}}$); see below. Furthermore, in the case of G_0 of empty interior, the condition of $m : \mathrm{G} \to \mathbb{C}$ being left G_0-invariant can be dropped, see Remark 2.5. Although our most novel examples would be discrete groups, the case of $\mu(\mathrm{G}_0) = 0$ is still useful as it recovers the classical case of the Hilbert transform on $\mathbb{R}$.

The first advantage of the result above is that it makes the verification of the Cotlar identity for previously known cases almost trivial. Indeed, restricting ourselves to the discrete case for clarity, we can easily show that it holds in the following situations.

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(1) **Classical case.** In the classical case of $\mathrm{G} = \mathbb{Z}$ and $\mathrm{G}_0 = \{0\}$ with $m(x) = \mathrm{sgn}(x)$ we only have to verify that either $m(x+y) = m(x)$ or $m(-x) = m(y)$. But that is trivial since either x and y have different signs or $x+y$ and x share the same sign.

(2) **Free product case.** If $\mathrm{G} = \mathrm{G}_1 * \mathrm{G}_1$ and $\mathrm{G}_0 = \{e\}$ with both G_2 and G_2 discrete, then any function $m(\omega)$ such that its value depends on the first letter of the reduced word of g satisfies (Cotlar). Indeed, we need to prove that either $m(gh) = m(g)$ or that $m(g^{-1}) = m(h)$. Assume the first equality fails, then the first letter of gh and that of g are different, but that can only happen if the reduced word of h begins with the reduced word of g^{-1}. If that is the case g^{-1} and h have the same starting letter and thus $m(g^{-1}) = m(h)$. This family of examples was explored by Mei and Ricard [55].

A natural question that we answer affirmatively is whether there are examples of groups that go beyond those two categories. In order to explore that question it seems natural to search for bounded functions $m : \mathrm{G} \to \mathbb{C}$ satisfying (Cotlar) with G having Serre's property (FA) [79]. A group G is said to have Serre's property (FA) iff every orientation-preserving action of G on a tree has a global fixed point (a vertex in the tree which is fixed by the action of any $g \in \mathrm{G}$). More deeply, Serre proved that a discrete group G has property (FA) iff it is finitely generated, it does not possess a quotient isomorphic to $\mathbb{Z}$ and it cannot be expressed as a nontrivial amalgamated free product $\mathrm{G} = \mathrm{G}_1 *_A \mathrm{G}_2$. Therefore a group with property (FA) is excluded from cases 1 and 2. Although there are interesting examples with property (FA), we also give applications to groups like Baumslag-Solitar groups $\mathrm{BS}(1,m)$ and the Bianchi group $\mathrm{PSL}_2(\mathcal{O}_{-1})$, which, despite failing property (FA), admit bounded functions satisfying Cotlar's identity for reasons unrelated to them having $\mathbb{Z}$-quotients or being free products.

Groups acting on uniquely arcwise connected spaces. The closed formula in (Cotlar) highlights a surprising connection between Cotlar's identity and geometric group theory. A topological space X is arcwise connected iff given two points $x, y \in X$ there exists an injective continuous path $\gamma : [0,1] \to X$ joining x and y. An arcwise connected space will be said to be a *uniquely arcwise connected space* or *UAC space* iff the path joining x and y is unique [5]. Let $\mathrm{G} \curvearrowright X$ be a topological action on a UAC space and fix a root $x_0 \in X$ with G_0 being the stabilizer St_{x_0} of x_0. Observe that $X \setminus \{x_0\}$ is given by

$$X \setminus \{x_0\} = \bigsqcup_{\beta} X_\beta, \tag{1.2}$$

where each X_β is arcwise connected. The decomposition into disjoint sets in identity (1.2) above is, a priori, purely set-theoretic, and we do not assume the sets X_β to be closed. Observe that the action of G_0 restricted to $X \setminus \{x_0\}$ permutes the sets X_β. The following theorem gives a machinery to get multipliers satisfying Cotlar's identity from actions on UAC spaces.

Theorem B. *Let $\mathrm{G} \curvearrowright X$ be a topological action on a UAC space. Fix $x_0 \in X$, G_0 being the stabilizer of x_0, i.e., $\mathrm{G}_0 = \mathrm{St}_{x_0}$, and let $\widetilde{m} : X \to \mathbb{C}$ be a bounded function satisfying that*

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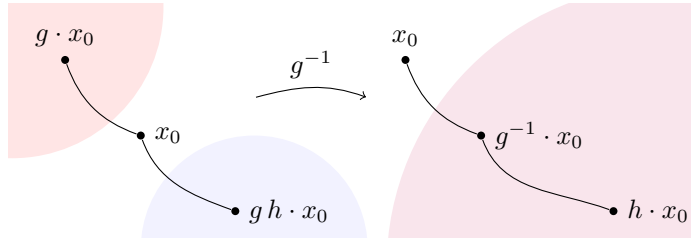


Figure 1. Action over paths.

(i) $\widetilde{m}$ *is constant over each* X_β *of (1.2).*

(ii) $\widetilde{m}$ *is constant over* G_0 *orbits, i.e.,* $\widetilde{m}|_{X_\beta} = \widetilde{m}|_{X_\alpha}$ *if there is an element* $h \in \mathrm{G}_0$ *such that* $X_\beta = h \cdot X_\alpha$.

Then the function $m : \mathrm{G} \to \mathbb{C}$ *given by* $m(g) = \widetilde{m}(g \cdot x_0)$ *satisfies (*$\widehat{\text{Cotlar}}$*) and therefore Theorem A applies.*

The proof of Theorem B is so neat that it can be tightly summarized here.

First, notice that condition (ii) ensures that m is left-G_0-invariant. Condition (i), on the other hand, implies that ($\widehat{\text{Cotlar}}$) holds. To see this, assume that $m(gh) \neq m(g)$. Since the two values are different, the arcwise connected subsets of $X \setminus \{x_0\}$ containing $gh \cdot x_0$ and $g \cdot x_0$ must be different, see Figure 1. Thus, there is a unique arc joining these two points that passes through x_0. Applying g^{-1} to this arc, since $\mathcal{G}$ acts by homeomorphisms, we get an arc starting at x_0 that passes through $g^{-1} \cdot x_0$ and then $h \cdot x_0$. It follows that $g^{-1} \cdot x_0$ and $h \cdot x_0$ lie in the same arcwise connected subset of $X \setminus \{x_0\}$ and thus $m(h) = m(g^{-1})$. By Remarks 2.5 and 3.2, condition (ii) can be dropped when the interior of St_{x_0} is empty.

Left-orderable groups. A family of examples that fits right into the model of Theorem A is that of *left-orderable groups*. Those are groups admitting a *total* or *linear* order $(\mathrm{G}, \preceq)$ that is invariant under left translations, i.e., $g \preceq h \iff ag \preceq ah$ for every $a, g, h \in \mathrm{G}$. We will write $g \prec h$ when $g \preceq h$ but $g \neq h$. For left-orderable groups the following result holds for their sign function.

Theorem C. *Let* G *be a left-orderable group and* $\mathrm{sgn} : \mathrm{G} \to \mathbb{C}$ *be the function*

$$\mathrm{sgn}(g) = \begin{cases} 1 & \text{when } e \prec g \\ 0 & \text{when } g = e \\ -1 & \text{when } g \prec e. \end{cases}$$

Then $H = T_{\mathrm{sgn}} : L_p(\mathcal{L}\mathrm{G}) \to L_p(\mathcal{L}\mathrm{G})$ *satisfies that*

$$\big\|H : L_p(\mathcal{L}\mathrm{G}) \to L_p(\mathcal{L}\mathrm{G})\big\| \lesssim \frac{p^2}{p-1}, \qquad \text{for every } 1 < p < \infty.$$

The boundedness of H in Theorem C can be obtained by showing directly that ($\widehat{\text{Cotlar}}$) holds. Alternatively, it is known that a countable group is left-orderable iff it acts faithfully

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on $\mathbb{R}$ by order-preserving homeomorphisms, see for instance [60, Proposition 2.1]. The observation that $\mathbb{R}$ is a UAC space allows us to prove the result as a consequence of Theorem B. In principle, proving Cotlar's identity gives just that the L_p-norm grows like $\mathrm{O}(p^\beta)$ with $\beta = \log_2(1+\sqrt{2})$, as $p \to \infty$. Nevertheless, a more careful argument allows us to show that the constant can be lowered down to the optimal order $\mathrm{O}(p)$, as long as $m(g)\overline{m(g^{-1})} = -1$ for $g \in \mathrm{G} \setminus \{e\}$, see Corollary 2.8. It is also worth noticing that the above transforms for left-orderable group algebras can be seen as a particular example of the Hilbert transforms associated with Arveson's subdiagonal algebras [2], for which the weak type (1,1) was proved by Randrianantoanina [74], see also [72, Theorem 8.4]. Therefore our geometric model in Theorem B generalizes simultaneously Mei and Ricard's free Hilbert transforms and subdiagonal Hilbert transforms, recovering the best known constants in both cases. Left-orderable groups include:

- Torsion-free Abelian groups;
- Torsion-free nilpotent groups;
- Free groups $\mathbb{F}_r$;
- Braid groups B_n;
- Right-angled Artin groups;
- Baumslag-Solitar groups $\mathrm{BS}(1,n)$ for $n \geq 2$;
- Surface groups;
- The Thompson group F.

It is possible to obtain explicit examples of L_p-bounded Fourier multipliers satisfying Cotlar's identity in each of these families of groups. Furthermore, there are known examples of left-orderable groups that have Serre's property (FA). Some of these examples arise from groups of isometries of the hyperbolic plane. For instance, let $D(2,3,7)$ denote the (2,3,7) von Dyck group (sometimes called the 'ordinary' triangle group). This is the group of orientation-preserving isometries of a tiling of the hyperbolic plane by geodesic triangles with angles $\frac{\pi}{2}$, $\frac{\pi}{3}$ and $\frac{\pi}{7}$. $D(2,3,7)$ admits a presentation as follows.

$$D(2,3,7) = \left\langle x, y \,\middle|\, x^2 = y^3 = (xy)^7 = 1 \right\rangle.$$

Let $\widetilde{\mathrm{PSL}}_2(\mathbb{R}) \twoheadrightarrow \mathrm{PSL}_2(\mathbb{R})$ be the universal cover of $\mathrm{PSL}_2(\mathbb{R})$. The lifting Γ of $D(2,3,7)$ to $\widetilde{\mathrm{PSL}}_2(\mathbb{R})$ is an example of a group with Serre's property (FA) that is also left-orderable [4, 16, 49, 79]. See the proof of Proposition 5.5.

Graphs of groups and Bass-Serre theory. A wealth of examples of multipliers satisfying ($\widehat{\text{Cotlar}}$) can be obtained from Bass-Serre theory – which allows us to classify groups acting on trees without edge inversions – see [79]. Indeed, given a group acting on a tree $\mathrm{G} \curvearrowright T$, it is possible to build a graph by taking the quotient with respect to the action $X = T/\mathrm{G}$ and associating to each vertex and to each edge its corresponding stabilizer. Due to the lack of edge inversions, the stabilizer of an edge embeds into the stabilizers of its extremes. This structure – a graph with groups on its edges and vertices and such that the groups at the edges embed into the extremes of said edge – is called a *graph of groups*. Let us denote it by $\mathbb{X}$. Like in the case of graphs, it is possible to define a sort of universal cover of $\mathbb{X}$ such that its underlying graph is a tree $\widetilde{X}$ and its fundamental group $\pi_1(\mathbb{X})$ acts as deck transformations of the covering. The main point of the theory is that $\pi_1(\mathbb{X}) \cong \mathrm{G}$ and the action of $\pi_1(\mathbb{X}) \curvearrowright \widetilde{X}$ recovers $\mathrm{G} \curvearrowright T$.

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Our main theorem in Section 5 would be that the multiplier results for symbols depending on the starting letter, like inequality (MR) from [55], extend from free products to general graphs of groups. Let us fix some notation. First, when working with a graph of groups, we will consider that our graph is oriented, that both directions of the edge occur and that there can be both loops and multiple edges between two given vertices. As it is customary, given an edge y, we will denote by $\bar{y}$ the *reverse edge* and by $\mathrm{o}(y) \in \mathrm{Vert}(X)$ and $\mathrm{t}(y) \in \mathrm{Vert}(X)$ the *origin* and *target* vertices of the edge. An orientation would be a subset $\mathrm{Edge}_+ \subseteq \mathrm{Edge}$ that contains either y or $\bar{y}$ for each edge. We will also denote by G_x the groups associated to vertices and by H_y the groups associated to edges. By construction, we have that $\mathrm{H}_y = \mathrm{H}_{\bar{y}}$ and that there are injective homomorphisms $\alpha_y : \mathrm{H}_y \to \mathrm{G}_{\mathrm{t}(y)}$ and $\alpha_{\bar{y}} : \mathrm{H}_y \to \mathrm{G}_{\mathrm{o}(y)}$. We will denote the image group $\alpha_y[\mathrm{H}_y] \subseteq \mathrm{G}_{\mathrm{t}(y)}$ by H_y^y. Similarly $\mathrm{H}_{\bar{y}}^{\bar{y}}$ will denote the image of $\alpha_{\bar{y}}$. We can now recall the construction of $\pi_1(\mathbb{X})$. Define the auxiliary group $F(\mathbb{X})$ as the free product of all the groups G_x – where x runs over the vertex set $\mathrm{Vert}(X)$ – together with $\mathbb{F}_{\mathrm{Edge}(X)}$, the free group generated by all the edges, with the following extra relations imposed

$$y^{-1} = \bar{y} \quad \text{and} \quad \alpha_y(h)\, y = y\, \alpha_{\bar{y}}(h),\ \forall y \in \mathrm{Edge}(X), h \in \mathrm{H}_y. \tag{1.3}$$

To construct the fundamental group, assume X is connected and choose a base point $x_0 \in \mathrm{Vert}(X)$. A closed path c that starts and ends at x_0 will be a sequence of edges $c = y_1\, y_2 \cdots y_m$, with $\mathrm{t}(y_i) = \mathrm{o}(y_{i+1})$ and $\mathrm{o}(y_1) = x_0 = \mathrm{t}(y_m)$. The group $\pi_1(\mathbb{X}, x_0)$ is given by the subset of $F(\mathbb{X})$ of elements of the form:

$$g = |c, r| = r_0\, y_1\, r_1\, y_2\, r_2 \cdots y_m\, r_m, \tag{1.4}$$

where $r_j \in \mathrm{G}_{\mathrm{o}(y_{j+1})}$ for $j \leq m-1$ and $r_m \in \mathrm{G}_{x_0}$. Here r is just notation for the tuple $r = (r_0, r_1, \cdots r_m)$. We will call the pair (c, r) a *word of type* c. Similarly, a word $e \neq g = |c, r| \in \pi$ of type c would be said to be in *normal form* if it cannot be shortened by applying the relations (1.3).

We would like to define a multiplier $m(g)$ depending only on the starting segment $r_0\, y_1$ of its associated word. But, given $h \in \mathrm{H}_{y_1}^{y_1}$, we have, in $F(\mathbb{X})$, that

$$r_0 y_1 = r_0\, \alpha_{y_1}(h^{-1})\, \alpha_{y_1}(h)\, y_1 = r_0\, \alpha_{y_1}(h^{-1})\, y_1 \alpha_{\bar{y}_1}(h),$$

as such, two words in normal form representing the same group element g may have different starting segments. Nevertheless, r_0 can only change by an element in $\mathrm{H}_{\bar{y}_1}^{y_1}$ acting on the right. This motivates the definition of the *space of starting segments* on $x_0 \in \mathrm{Vert}(X)$ as

$$W_{x_0} = \bigsqcup_{y \,|\, \mathrm{o}(y) = x_0} \mathrm{G}_{x_0} / \mathrm{H}_{\bar{y}}^{\bar{y}} \tag{1.5}$$

Given a symbol $m : \pi = \pi_1(\mathbb{X}, x_0) \to \mathbb{C}$ we will say that

- *m depends on the starting segment* if there exists a function $\widetilde{m} : W_{x_0} \to \mathbb{C}$ such that

$$m(g) = \begin{cases} \widetilde{m}\big(r_0 \cdot \mathrm{H}_{\bar{y}_1}^{\bar{y}_1}\big), & \text{when } g \notin \mathrm{G}_{x_0} \\ 0 & \text{when } g \in \mathrm{G}_{x_0}, \end{cases} \tag{1.6}$$

  where g is as in (1.4).

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- *m depends on the starting edge* if it not only depends on the starting segment, but it is also constant in each of the components $\mathrm{G}_{x_0}/\mathrm{H}_{\bar{y}}^{\bar{y}}$, for each y starting in x_0, in the disjoint union that forms W_{x_0}.

Theorem D. *Let $\pi = \pi_1(\mathbb{X}, x_0)$ be the fundamental group of a graph of groups and let $m : \pi \to \mathbb{C}$ be a function depending only on the starting segment. It holds that*

(i) *m satisfies Cotlar's identity* ($\mathrm{Cotlar}_{\mathbb{E}\perp}$) *relative to* G_{x_0}.

(ii) *Furthermore, if m only depends on the starting edge, then it is left G_{x_0}-invariant. Thus, Theorem A applies.*

The hypothesis that m depends only on the starting edge is indeed quite restrictive. For instance, if there is only one edge, up to inversions, starting at x_0, then a symbol depending on the starting edge is constant for $g \notin \mathrm{G}_{x_0}$. Luckily, in many interesting examples the group G_{x_0} will be Abelian. In that case, decomposing $\widetilde{m}|_{\mathrm{G}_{x_0}/\mathrm{H}_y^y}$ into characters allows us to work with functions depending on the starting segment, see Remark 2.6 and Theorem 5.2. We will provide a direct proof of Theorem D. Nevertheless, it also follows from noticing that, if m depends on the starting segment, then it lifts to a function $\widetilde{m} : \widetilde{X} \to \mathbb{C}$ on the Bass-Serre tree of $\pi_1(\mathbb{X}, x_0)$ that is constant in connected components of $\widetilde{X} \setminus \{\widetilde{x}_0\}$, see Proposition 5.1 for the details. Therefore, symbols depending on the starting segment lie within the template of Theorem B. We will illustrate Theorem D in the case of free products and Higman-Neumann-Neumann (HNN) extensions. HNN extensions include the Baumslag-Solitar groups $\mathrm{BS}(n,m)$ for which a slightly different model of multipliers satisfying Cotlar's identity would also be given, see Proposition 5.3.

$\mathrm{PSL}_2(\mathbf{K})$, its lattices and open questions. Natural models of Hilbert transforms on a group G often appear via the following straightforward idea. Let $\mathcal{X}$ be a geometric object on which G acts and assume $\mathcal{X}$ contains a *barrier* $\mathcal{F} \subseteq \mathcal{X}$ such that $\mathcal{X} \setminus \mathcal{F}$ is divided into two separated halves $\mathcal{X} \setminus \mathcal{F} = \mathcal{X}_+ \sqcup \mathcal{X}_-$. Then, given $x_0 \in \mathcal{F}$, a symbol m can be defined as

$$m(g) = \mathbf{1}_{\mathcal{X}_+}(g \cdot x_0) - \mathbf{1}_{\mathcal{X}_-}(g \cdot x_0).$$

When $\mathcal{X} = \mathbb{R}$ and $\mathcal{F} = \{0\}$, the group multiplier induced by $\mathbb{R} \curvearrowright \mathcal{X}$ coincides with the sign function. Thus, these symbols are natural generalizations of the Hilbert transform and we will refer to them as such. Important instances of this include:

(1) **Hilbert space model.** Let $\mathcal{X} = \mathcal{H}$ be a (real) Hilbert space in which G acts by affine isometries $\pi(g)$. These isometries are given by $\xi \mapsto \alpha(g)\xi + \beta(g)$, where $\alpha(g)$ is an orthogonal transformation. Let $\mathcal{F} = \langle v \rangle^\perp$ be the codimension 1 subspace of vectors perpendicular to $v \in \mathcal{H} \setminus \{0\}$. Choosing $x_0 = 0$ gives the symbol $m(g) = \mathrm{sgn}(\langle \beta(g), v \rangle)$. These symbols have been studied for finite-dimensional $\mathcal{H}$ in [12, Appendix A] and [66].

(2) **Manifold model.** Choose $\mathcal{X} = M$ as an n-dimensional Riemannian manifold in which G acts by isometries $\alpha : \mathrm{G} \to \mathrm{Iso}(M)$ and let $\mathcal{F} \subseteq \mathcal{X}$ be a $(n-1)$-dimensional geodesic submanifold such that $\mathcal{X} \setminus \mathcal{F}$ has two connected components.

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(3) **Tree model.** $\mathcal{X} = T$ being a tree on which G acts. Choose $x_0 \in T$ to be a vertex, that we will henceforth call the *root*. Then, $\mathcal{X} \setminus \{x_0\}$ is made up of r connected components, with r being the valence of x_0, that we can arrange into two families $\mathcal{X}_+$ and $\mathcal{X}_-$. This is an instance of the model described in Theorem B above.

It is very interesting to point out that, in many examples, the same idempotent Fourier multiplier on a group can be obtained from more than one of the three different models above. Here we will illustrate that phenomenon with the continuous groups $\mathrm{PSL}_2(\mathbb{R})$ and $\mathrm{PSL}_2(\mathbb{C})$, which will explain part of our original motivation. Let $\mathrm{SL}_2(\mathbb{K})$ be the group of 2×2 matrices of determinant 1 with entries over a field $\mathbb{K}$ that in our examples will be $\mathbb{R}$ or $\mathbb{C}$. Then, $\mathrm{PSL}_2(\mathbb{K})$ will denote the quotient of $\mathrm{SL}_2(\mathbb{K})$ by scalar matrices $\{\pm \mathrm{id}\}$. Both groups, $\mathrm{PSL}_2(\mathbb{R})$ and $\mathrm{PSL}_2(\mathbb{C})$, act faithfully and transitively by isometries on the real hyperbolic spaces of dimension 2 and 3, $\mathrm{PSL}_2(\mathbb{R}) \curvearrowright \mathbb{H}^2$ and $\mathrm{PSL}_2(\mathbb{C}) \curvearrowright \mathbb{H}^3$ – which we will identify with their upper half plane and upper half space models. Let us denote the coordinates of $\mathbb{H}^2$ by (x,y) and the coordinates of $\mathbb{H}^3$ by (x,y,z). We can take the geodesic $\{x=0\} \subseteq \mathbb{H}^2$ as separating space in the first example. A calculation yields that the Hilbert transform in the sense of the manifold model is

$$m\underbrace{\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}}_{g} = \mathrm{sgn}\big(\mathrm{Re}\{g\cdot i\}\big) = \mathrm{sgn}\big(a_{11}a_{21} + a_{12}a_{22}\big), \tag{1.7}$$

where g is the class $\pm[a_{i,j}]_{i,j}$. This multiplier can be related to the other two models. Indeed, for the Hilbert space model, it is possible to construct a metrically proper 1-cocycle $\beta : \mathrm{PSL}_2(\mathbb{K}) \to \mathcal{H}$ into an infinite-dimensional Hilbert space $\mathcal{H}$ and choose a unit vector $u \in \mathcal{H}$ such that $m(g) = \mathrm{sgn}\langle \beta(g), u\rangle$, see [27, 13]. While the group $\mathrm{PSL}_2(\mathbb{R})$ is continuous, and thus it is unable to act on trees in an interesting way, the tree model interpretation is indeed available for the restriction of (1.7) to $\mathrm{PSL}_2(\mathbb{Z})$. The key observation is that

$$\mathrm{PSL}_2(\mathbb{Z}) \cong \mathbb{Z}_2 * \mathbb{Z}_3.$$

This free-product decomposition yields an action of $\mathrm{PSL}_2(\mathbb{Z})$ on its Bass-Serre tree with respect to which the multiplier $m|_{\mathrm{PSL}_2(\mathbb{Z})}$ can be recovered, see Figure 10.

For the complex case, the separating subspace is given by the 2-dimensional geodesic submanifold $\{x=0\} \subseteq \mathbb{H}^2$, which readily gives that

$$m\underbrace{\begin{pmatrix} z_{11} & z_{12} \\ z_{21} & z_{22} \end{pmatrix}}_{g} = \mathrm{sgn}\big(\mathrm{Re}\{p(g\cdot x_0)\}\big) = \mathrm{sgn}\big(\mathrm{Re}\{z_{11}\overline{z}_{21} + z_{12}\overline{z}_{22}\}\big), \tag{1.8}$$

where $x_0 = (0,1) \in \mathbb{C}\times\mathbb{R}_+$, $p(z,r) = z$ and $g = \pm[z_{i,j}]_{i,j}$. In this case, the relationship with the other two models is more involved. Nevertheless, it is still possible to describe the multiplier (1.8) above in terms of proper infinite-dimensional 1-cocycles with respect to a natural direction u. For the tree model the situation is a lot more contentious. Indeed, let $\mathcal{O}_{-d}$ be the ring of integers of the algebraic field $\mathbb{Q}(\sqrt{-d})$, where d is a square-free integer. The lattices $\mathrm{PSL}_2(\mathcal{O}_{-d}) \subseteq \mathrm{PSL}_2(\mathbb{C})$ are the *Bianchi groups*. It is known that all

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of them except for $d=3$ admit nontrivial actions on trees, see [29]. For instance, when $d=1$, $\mathcal{O}_{-1}$ is the ring of Gaussian integers $\mathbb{Z}[i]$. In this case, the following rather intricate free product decomposition is known:

$$\mathrm{PSL}_2(\mathcal{O}_{-1}) \cong \big(S_3 *_{\mathbb{Z}_3} A_4\big) *_{\mathrm{PSL}_2(\mathbb{Z})} \big(S_3 *_{\mathbb{Z}_2} V\big),$$

where S_n are the permutation groups, A_n are the alternating groups and V is the Klein 4 group, see [29, Theorem 2.1. (i)]. It is possible that m, when restricted to $\mathrm{PSL}_2(\mathcal{O}_{-d})$, may have an expression in terms of a nontrivial action on a tree. Nevertheless, the complexity of the amalgamated free product decompositions obtained makes it a difficult approach to work with. On the other hand, the strength of our characterization in Theorem A allows us to prove the boundedness of $m|_{\mathrm{PSL}_2(\mathcal{O}_{-1})}$ directly.

Theorem E. *Let* $\mathrm{G} = \mathrm{PSL}_2(\mathcal{O}_{-1}) \subseteq \mathrm{PSL}_2(\mathbb{C})$ *and* $m : \mathrm{PSL}_2(\mathcal{O}_{-1}) \to \mathbb{C}$ *be the function given by*

$$m\begin{pmatrix} a_{11}+ib_{11} & a_{12}+ib_{12} \\ a_{21}+ib_{21} & a_{22}+ib_{22} \end{pmatrix} = \mathrm{sgn}\big(a_{11}a_{21}+b_{11}b_{21}+a_{12}a_{22}+b_{12}b_{22}\big).$$

Then, $T_m : L_p(\mathcal{L}\mathrm{G}) \to L_p(\mathcal{L}\mathrm{G})$ *is bounded for every* $1 < p < \infty$ *with the bound in Theorem A.*

Foreword: In the first released version of this text we left two natural problems open. The first was whether $m|_\Gamma$ is L_p-bounded for lattices Γ other than $\mathrm{PSL}_2(\mathcal{O}_{-1})$. In the years following its release, significant progress toward a positive answer has been made by Jorge Pérez García [73]. The second problem, originally highlighted as Problem A, asked if the multipliers (1.7) and (1.8) are L_p-bounded over the whole Lie group. This has been answered in the negative by the second-named author, together with de la Salle and Tablate [64], via an extension of Fefferman's ball multiplier construction in the context of Schur multipliers.

2. Cotlar identities and multipliers

Noncommutative integration. Throughout this text we will use liberally *noncommutative integration theory* and the theory of noncommutative L_p-spaces. Let $\mathcal{M} \subseteq \mathcal{B}(\mathcal{H})$ be a *von Neumann algebra* admitting a *normal semifinite and faithful tracial weight* $\tau : \mathcal{M}_+ \to [0,\infty]$ that we will henceforth just refer to as a *n.s.f. trace*. It is possible to construct the noncommutative L_p-spaces associated to $(\mathcal{M},\tau)$ as the subset of τ-measurable operators $L_p(\mathcal{M},\tau) \subseteq L_0(\mathcal{M},\tau)$ satisfying that

$$f \in L_p(\mathcal{M},\tau) \quad \Longleftrightarrow \quad \|f\|_p := \tau\big(|f|^p\big)^{\frac{1}{p}} < \infty.$$

This theory, which goes back all the way to Dixmier and Segal [24, 78], is already well understood and the interested reader can consult [84, 72, 32].

Let G be a locally compact group that we will throughout the text assume to be second countable, and let $L_2(\mathrm{G})$ be its L_2-space with respect to the *left Haar measure* μ [28]. As usual, we will denote by $\lambda : \mathrm{G} \to \mathcal{U}(L_2(\mathrm{G}))$ the *left regular representation*, which is the unitary representation $g \mapsto \lambda_g$ that acts by sending $\xi(h)$ to $\xi(g^{-1}h)$. The *left regular von*

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Neumann algebra $\mathcal{L}\mathrm{G} \subseteq \mathcal{B}(L_2(\mathrm{G}))$ of G is given by

$$\overline{\lambda[L_1(\mathrm{G})]}^{\mathrm{w}^*} = \overline{\left\{\lambda(\varphi) := \int_{\mathrm{G}} \varphi(g)\,\lambda_g\,d\mu(g) : \varphi \in L_1(\mathrm{G})\right\}}^{\mathrm{w}^*} \subseteq \mathcal{B}(L_2(\mathrm{G})).$$

This von Neumann algebra admits a normal, semifinite and faithful weight $\tau : \mathcal{L}\mathrm{G}_+ \to [0,\infty]$ that satisfies the Plancherel identity, meaning that $\varphi \mapsto \lambda(\varphi)$ extends to a unitary isometry from $L_2(\mathrm{G})$ to the Gelfand-Neumark-Segal space $L_2(\mathcal{L}\mathrm{G};\tau)$. This weight is usually referred to as the *Plancherel weight* [67, Chapter 7]. The weight τ is a n.s.f. trace precisely when G is unimodular. Thus, we will work in the natural setting of unimodular groups and refer to τ as the *Plancherel trace*. In this context, the Plancherel trace is given by

$$\tau\left(\int_{\mathrm{G}} \varphi(g)\,\lambda_g\,d\mu(g)\right) = \varphi(e), \qquad \text{for every } \varphi \in C_c(\mathrm{G}) * C_c(\mathrm{G}).$$

We will denote the noncommutative L_p-spaces associated to τ simply by $L_p(\mathcal{L}\mathrm{G})$. By analogy with the classical Fourier transform, we will denote by $\widehat{f}(g)$ the value $\tau(\lambda_g^* f)$, which is well-defined whenever $f \in L_1(\mathcal{L}\mathrm{G})$. It is also worth noticing that, by Plancherel's theorem, the map $f \mapsto \widehat{f}$ is well-defined for any $f \in L_2(\mathcal{L}\mathrm{G})$. In fact, we have that $f \in L_2(\mathcal{L}\mathrm{G})$ can be expressed as

$$f = \lambda(\widehat{f}) := \int_{\mathrm{G}} \widehat{f}(g)\,\lambda_g\,d\mu(g), \tag{2.1}$$

where the integral on the right-hand side converges in the L_2 norm.

Conditional expectations. Let $\mathcal{N} \subseteq \mathcal{M}$ be a von Neumann subalgebra of $\mathcal{M}$, that is a $*$-subalgebra that is also ultraweakly closed. If τ is a n.s.f. trace over $\mathcal{M}$ and $\tau|_{\mathcal{N}}$ is still semifinite, then it is easy to see that the inclusion $\iota : L_1(\mathcal{N}) \hookrightarrow L_1(\mathcal{M})$ is isometric and its dual map is a trace preserving and normal (i.e., ultraweakly continuous) *conditional expectation* $\mathbb{E} : \mathcal{M} \to \mathcal{N} \subseteq \mathcal{M}$. By conditional expectation we mean a *unital and completely positive* map such that $\mathbb{E}|_{\mathcal{N}} = \mathrm{id}_{\mathcal{N}}$. Recall that conditional expectations satisfy that $\mathbb{E} \circ \mathbb{E} = \mathbb{E}$ and that, by Tomiyama's Theorem, see [7, Theorem 1.5.10], $\mathbb{E}$ is automatically $\mathcal{N}$-bimodular.

Let $\mathrm{G}_0 \subseteq \mathrm{G}$ be two groups such that G_0 is open inside G. Then G_0 is unimodular if G is. Furthermore, the Plancherel trace τ_{G_0} of $\mathcal{L}\mathrm{G}_0$ coincides with the Plancherel trace of $\mathcal{L}\mathrm{G}$ restricted to $\mathcal{L}\mathrm{G}_0$. Therefore there is a normal and trace-preserving conditional expectation $\mathbb{E} : \mathcal{L}\mathrm{G} \to \mathcal{L}\mathrm{G}_0 \subseteq \mathcal{L}\mathrm{G}$ that is given by the Fourier multiplier associated to $\mathbf{1}_{\mathrm{G}_0}$, i.e.,

$$\mathbb{E}(f) = \mathbb{E}\left(\int_{\mathrm{G}} \widehat{f}(g)\,\lambda_g\,d\mu(g)\right) = \int_{\mathrm{G}} \mathbf{1}_{\mathrm{G}_0}(g)\,\widehat{f}(g)\,\lambda_g\,d\mu(g).$$

The fact that $\mathbb{E}$ is trace preserving allows us to extend $\mathbb{E}$ as a contraction to all the L_p-spaces $1 \le p \le \infty$, $\mathbb{E} : L_p(\mathcal{L}\mathrm{G}) \to L_p(\mathcal{L}\mathrm{G}_0) \subseteq L_p(\mathcal{L}\mathrm{G})$.

Noncommutative Cotlar identities. Most of this section up until the closed-formula characterization of the Cotlar identity in the proof of Theorem A follows closely the results obtained by Mei and Ricard and it is included here for the sake of completeness.

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Definition 2.1 [55, from Proposition 3.2(iv)]. Let $\mathcal{M}$, $\mathcal{N}$ be as above, $\mathbb{E}:\mathcal{M}\to\mathcal{N}$ be the conditional expectation and let $H:L_2(\mathcal{M})\to L_2(\mathcal{M})$ be a bounded operator

(i) H satisfies the (non-relative) Cotlar identity if

$$H(f)H(f)^* = H\big(fH(f)^*\big)+H\big(fH(f)^*\big)^* - H\big(H(ff^*)\big)^*, \tag{Cotlar$_{\mathrm{nr}}$}$$

for every $f\in\mathcal{M}\cap L_2(\mathcal{M})$.

(ii) H is said to satisfy the Cotlar identity (relative to $\mathcal{N}$) if

$$\mathbb{E}^\perp\big[H(f)H(f)^*\big] = \mathbb{E}^\perp\Big[H\big(fH(f)^*\big)+H\big(fH(f)^*\big)^* - H\big(H(ff^*)^*\big)\Big], \tag{Cotlar$_{\mathbb{E}\perp}$}$$

for every $f\in L_2(\mathcal{M})\cap\mathcal{M}$, where $\mathbb{E}^\perp=(\mathrm{id}-\mathbb{E})$.

We are assuming that H is bounded in $L_2(\mathcal{M})$ *a priori*; this is not a restrictive imposition since it holds trivially in the case of Fourier multipliers. Observe as well that, although both Cotlar identities require applying H to products of functions that therefore may not be in $L_2(\mathcal{M})$, the fact that $f\in L_2(\mathcal{M})\cap\mathcal{M}$ and that $L_2(\mathcal{M})$ is stable by multiplications by $\mathcal{M}$ makes both equations meaningful.

We will need the following lemma. Notice that if $H:L_2(\mathcal{M})\to L_2(\mathcal{M})$ is left $\mathcal{N}$-modular, then its restriction to $L_2(\mathcal{N})$ composed with the expectation $\mathbb{E}:L_2(\mathcal{M})\to L_2(\mathcal{N})$ gives a map

$$\mathbb{E}H|_{L_2(\mathcal{N})}:L_2(\mathcal{N})\to L_2(\mathcal{N})$$

that is both bounded with norm equal to that of H and and left $\mathcal{N}$-modular. But all left $\mathcal{N}$-modular maps in $L_2(\mathcal{N})$ are right multiplicators by an element in $\mathcal{N}$ of norm equal to the operator norm. As such, $\mathbb{E}H|_{L_2(\mathcal{N})\cap L_p(N)}$ extends to a bounded operator in $L_p(\mathcal{N})$ and

$$\big\|\mathbb{E}H|_{L_p(\mathcal{N})}:L_p(\mathcal{N})\to L_p(\mathcal{N})\big\|\le\big\|H:L_2(\mathcal{M})\to L_2(\mathcal{M})\big\|. \tag{2.2}$$

With this observation in hand, we can proceed to prove the following lemma, which is analogous to [55, Proposition 3.4], but in the setting of general operators.

Lemma 2.2. *Let $\mathcal{M}$, $\mathcal{N}$ and $\mathbb{E}$ be as above and let $H:L_2(\mathcal{M})\to L_2(\mathcal{M})$ be a bounded and left $\mathcal{N}$-modular map. For every $f\in L_2(\mathcal{M})\cap\mathcal{M}$, it holds that*

$$\mathbb{E}\big[H(f)H(f)^*\big]\le\|H:L_2(\mathcal{M})\to L_2(\mathcal{M})\|^2\,\mathbb{E}\big[ff^*\big]. \tag{2.3}$$

Furthermore, if $\mathbb{E}H=H\mathbb{E}$, we have that for every $1\le p\le\infty$

$$\big\|\mathbb{E}\big[H\big(fH(f)^*\big)\big]\big\|_p\le\big\|H:L_2(\mathcal{M})\to L_2(\mathcal{M})\big\|^2\,\big\|\mathbb{E}\big[ff^*\big]\big\|_p \tag{2.4}$$

$$\big\|\mathbb{E}\big[H\big(H(ff^*)^*\big)\big]\big\|_p\le\big\|H:L_2(\mathcal{M})\to L_2(\mathcal{M})\big\|^2\,\big\|\mathbb{E}\big[ff^*\big]\big\|_p \tag{2.5}$$

Proof. All of the points are elementary. For (2.3) first notice that every state of $\mathcal{N}$ is of the form $f\mapsto\tau(\delta f)$, where δ is a positive element of norm 1 in the space $L_1(\mathcal{N})$.

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Decomposing it as $\delta = \delta^{\frac{1}{2}}\delta^{\frac{1}{2}}$ gives

$$\begin{aligned}
\tau\left\{\delta\mathbb{E}\big[H(f)H(f)^*\big]\right\} &= \tau\left\{\delta^{\frac{1}{2}}\mathbb{E}\big[H(f)H(f)^*\big]\delta^{\frac{1}{2}}\right\} \\
&= \tau\left\{\mathbb{E}\big[\delta^{\frac{1}{2}}H(f)H(f)^*\delta^{\frac{1}{2}}\big]\right\} \\
&= \tau\left\{\mathbb{E}\big[H(\delta^{\frac{1}{2}}f)H(\delta^{\frac{1}{2}}f)^*\big]\right\} \\
&= \tau\left\{H(\delta^{\frac{1}{2}}f)H(\delta^{\frac{1}{2}}f)^*\right\} \\
&\le \big\|H : L_2(\mathcal{M}) \to L_2(\mathcal{M})\big\|^2 \tau\left\{(\delta^{\frac{1}{2}}f)(\delta^{\frac{1}{2}}f)^*\right\} \\
&= \big\|H : L_2(\mathcal{M}) \to L_2(\mathcal{M})\big\|^2 (\tau\circ\mathbb{E})\left\{(\delta^{\frac{1}{2}}f)(\delta^{\frac{1}{2}}f)^*\right\} \\
&= \big\|H : L_2(\mathcal{M}) \to L_2(\mathcal{M})\big\|^2 \tau\left\{\delta\mathbb{E}\big[ff^*\big]\right\}.
\end{aligned}$$

Since this is true for every state, the operator inequality (2.3) holds.

For (2.4) we use that $\mathbb{E}H = H\mathbb{E}$ to rewrite $\mathbb{E}\big[H\big(f\,H(f)^*\big)\big]$ as $(\mathbb{E}H\mathbb{E})\circ\mathbb{E}\big[\big(f\,H(f)^*\big)\big]$. The operator norm on $\mathbb{E}H\mathbb{E} : L_p(\mathcal{M}) \to L_p(\mathcal{M})$ is bounded by that of $\mathbb{E}H|_{L_p(\mathcal{N})} : L_p(\mathcal{N}) \to L_p(\mathcal{N})$, which is bounded by the norm of H by (2.2). To estimate the term $\mathbb{E}\big[\big(f\,H(f)^*\big)\big]$ we will use the following version of Hölder's inequality [41, Inequality (2.1)]

$$\big\|\mathbb{E}\big[f\,g^*\big]\big\|_p \le \big\|\mathbb{E}\big[f\,f^*\big]^{\frac{1}{2}}\big\|_r \big\|\mathbb{E}\big[g\,g^*\big]^{\frac{1}{2}}\big\|_s \quad \text{when} \quad \frac{1}{p} = \frac{1}{r} + \frac{1}{s},$$

with $r = s = 2p$ and $g = H(f)$ to obtain that

$$\big\|\mathbb{E}\big[f\,H(f)^*\big]\big\|_p \le \big\|\mathbb{E}\big[f\,f^*\big]\big\|_p^{\frac{1}{2}} \big\|\mathbb{E}\big[H(f)\,H(f)^*\big]\big\|_p^{\frac{1}{2}}.$$

Applying the inequality in (2.3) gives the result. Identity (2.5) follows immediately after using two times the fact that H and $\mathbb{E}$ commute and that $\mathbb{E}H\mathbb{E}$ has a norm in L_p bounded by the norm in L_2 of H by (2.2). □

We can now prove the following extrapolation result, which generalizes [55, Theorem 3.5] to the setting of general operators.

Proposition 2.3. *Let $\mathcal{N} \subseteq \mathcal{M}$ be as before and let $H : L_2(\mathcal{M}) \to L_2(\mathcal{M})$ be a left $\mathcal{N}$-modular operator commuting with $\mathbb{E} : \mathcal{M} \to \mathcal{N}$. If H satisfies (Cotlar$_{\mathbb{E}\perp}$) then $\forall\, 2 \le p < \infty$*

$$\big\|H : L_p(\mathcal{M}) \to L_p(\mathcal{M})\big\| \lesssim p^{\beta}\big\|H : L_2(\mathcal{M}) \to L_2(\mathcal{M})\big\|, \ \text{where } \beta = \log_2(1+\sqrt{2}). \quad (2.6)$$

Similarly, if $H : L_2(\mathcal{M}) \to L_2(\mathcal{M})$ is a general bounded linear map that satisfies the non-relative Cotlar identity (Cotlar$_{nr}$), then the same extrapolation inequality (2.6) holds.

Proof. First, let us denote the operator norm on L_p of H by $c_p := \big\|H : L_p(\mathcal{M}) \to L_p(\mathcal{M})\big\|$. We are going to proceed by induction, assuming that $c_p < \infty$ to prove that $c_{2p} < \infty$.

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Choose $f \in \mathcal{M} \cap L_2(\mathcal{M})$ with $\|f\|_{2p} \leq 1$ and notice that

$$\begin{aligned}
\|H(f)\|_{2p}^2 &= \|H(f)H(f)^*\|_p \\
&\leq \big\|\mathbb{E}\big[H(f)H(f)^*\big]\big\|_p + \big\|\mathbb{E}^\perp\big[H(f)H(f)^*\big]\big\|_p \\
&\leq c_2^2\,\big\|\mathbb{E}\big[ff^*\big]\big\|_p + \big\|\mathbb{E}^\perp\big[H\big(f\,H(f)^*\big) + H\big(f\,H(f)^*\big)^* - H\big(H(ff^*)^*\big)\big]\big\|_p \qquad (2.7)\\
&\leq c_2^2\,\|f\|_{2p}^2 + \big\|H\big(f\,H(f)^*\big)\big\|_p + \big\|H\big(f\,H(f)^*\big)^*\big\|_p + \big\|H\big(H(ff^*)^*\big)\big\|_p \\
&\quad + \big\|\mathbb{E}\big[H\big(f\,H(f)^*\big)\big]\big\|_p + \big\|\mathbb{E}\big[H\big(f\,H(f)^*\big)^*\big]\big\|_p + \big\|\mathbb{E}\big[H\big(H(ff^*)^*\big)\big]\big\|_p \qquad (2.8)\\
&\leq 2\,c_p\,\|H(f)\|_{2p}\|f\|_{2p} + c_p^2\|f\|_{2p} + 4\,c_2^2\|f\|_{2p}^2 \\
&\leq 2\,c_p\,\|H(f)\|_{2p} + c_p^2 + 4\,c_2^2 \qquad (2.9)
\end{aligned}$$

We have used ($\mathrm{Cotlar}_{\mathbb{E}\perp}$) in the second term of the sum of (2.7) and estimate (2.3) of Lemma 2.2 in the first. To pass from (2.8) to (2.9) we have used the other two identities of Lemma 2.2. Notice that (2.9) is a quadratic inequality of the form $0 \leq -t^2 + 2c_p\,t + c_p^2 + 4\,c_2^2$, where $t = \|H(f)\|_{2p}$. Since the leading term is negative, this implies a bound of $\|H(f)\|_{2p}$ in terms of c_p and c_2 only. Taking supremum over f and using the norm density of $\mathcal{M} \cap L_2(\mathcal{M})$ in $L_{2p}(\mathcal{M})$ implies that c_{2p} is finite. More explicitly, setting $a_p = c_p/c_2$ gives the recursive inequality

$$a_{2p}^2 \leq 2\,a_p\,a_{2p} + a_p^2 + 4. \qquad (2.10)$$

Adding a_{2p}^2 to both sides in order to complete squares gives

$$2a_{2p}^2 \leq a_{2p}^2 + 2\,a_p\,a_{2p} + a_p^2 + 4 = (a_{2p} + a_p)^2 + 4 \leq (a_{2p} + a_p + 2)^2.$$

After taking square roots and recursively applying the inequality above, the following is obtained

$$a_{2^k} \leq C\,(1+\sqrt{2})^{k-1}, \quad \text{with} \quad C \leq 3 + \sqrt{2}.$$

This, together with Marcinkiewicz interpolation for intermediate values of p, gives the desired inequality. The non-relative case works similarly, but the 4 extra terms that give $4\,c_2^2$ in the right-hand side of inequality (2.9) do not appear, which only affects the absolute constant C in the final inequality. □

Our source of examples for the extrapolation theorem above is taken when $\mathcal{M} = \mathcal{L}\mathrm{G}$ and $H = T_m$ is a Fourier multiplier. We will work with Cotlar identities relative to subalgebras induced by subgroups $\mathrm{G}_0 \subseteq \mathrm{G}$. In that setting it holds that there is a normal and trace-preserving conditional expectation $\mathbb{E} : \mathcal{L}\mathrm{G} \to \mathcal{L}\mathrm{G}_0$ if and only if $\mathrm{G}_0 \subseteq \mathrm{G}$ is open. Thus, our relative Cotlar identities would be taken only with respect to open subgroups. In general, given a group G, when speaking about the Cotlar identity for a multiplier T_m – without specifying any subgroup – we will mean, that it satisfies the non-relative Cotlar identity ($\mathrm{Cotlar}_{\mathrm{nr}}$) when the group G is continuous i.e. $\mu(\{e\}) = 0$, while in the discrete case $\mu(\{e\}) \neq 0$, we will instead mean that it satisfies ($\mathrm{Cotlar}_{\mathbb{E}\perp}$) with respect to the subgroup $\{e\}$.

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We are now going to prove the equivalence between Cotlar's identity ($\text{Cotlar}_{\mathbb{E}^\perp}$) and the closed formula in Theorem A.

Theorem 2.4. *Let $\mathrm{G}_0 \subseteq \mathrm{G}$ be an open subgroup of G and $m : \mathrm{G} \to \mathbb{C}$ be a bounded function. The following properties are equivalent:*

(i) *T_m satisfies ($\text{Cotlar}_{\mathbb{E}^\perp}$).*

(ii) *The function m satisfies ($\widehat{\text{Cotlar}}$), i.e., $\big(m(g^{-1}) - m(h)\big)\big(m(gh) - m(g)\big) = 0$, for almost every $g \in \mathrm{G} \setminus \mathrm{G}_0, h \in \mathrm{G}$.*

Proof. Expanding ($\text{Cotlar}_{\mathbb{E}^\perp}$) for T_m gives

$$0 = \underbrace{\mathbb{E}^\perp\Big[T_m(f)\,T_m(f)^*\Big]}_{\text{(I)}} - \underbrace{\mathbb{E}^\perp\Big[T_m\big(f\,T_m(f)^*\big)\Big]}_{\text{(II)}} - \underbrace{\mathbb{E}^\perp\Big[T_m\big(f\,T_m(f)^*\big)^*\Big]}_{\text{(III)}} + \underbrace{\mathbb{E}^\perp\Big[T_m\big(T_m(ff^*)^*\big)\Big]}_{\text{(IV)}}.$$

Now, elementary computations yield that

$$\begin{aligned}
\text{(I)} &= \int_{\mathrm{G}\setminus\mathrm{G}_0}\int_{\mathrm{G}} \widehat{f}(gh)\,\overline{\widehat{f}(h)}\,m(gh)\,\overline{m(h)}\,\lambda_g\,d\mu(h)\,d\mu(g)\\
\text{(II)} &= \int_{\mathrm{G}\setminus\mathrm{G}_0}\int_{\mathrm{G}} \widehat{f}(gh)\,\overline{\widehat{f}(h)}\,m(g)\,\overline{m(h)}\,\lambda_g\,d\mu(h)\,d\mu(g)\\
\text{(III)} &= \int_{\mathrm{G}\setminus\mathrm{G}_0}\int_{\mathrm{G}} \widehat{f}(gh)\,\overline{\widehat{f}(h)}\,m(gh)\,\overline{m(g^{-1})}\,\lambda_g\,d\mu(h)\,d\mu(g)\\
\text{(IV)} &= \int_{\mathrm{G}\setminus\mathrm{G}_0}\int_{\mathrm{G}} \widehat{f}(gh)\,\overline{\widehat{f}(h)}\,m(g)\,\overline{m(g^{-1})}\,\lambda_g\,d\mu(h)\,d\mu(g),
\end{aligned}$$

where $\widehat{f}$ is given as in (2.1). This in turn implies, using the Plancherel theorem, that

$$0 = \int_{\mathrm{G}} \widehat{f}(gh)\,\overline{\widehat{f}(h)}\,\big(m(gh) - m(g)\big)\,\overline{\big(m(h) - m(g^{-1})\big)}\,d\mu(h) \qquad \text{for almost every } g \in \mathrm{G}\setminus\mathrm{G}_0.$$

Obviously, if the factor $G_g(h) = (m(gh) - m(g))\,\overline{(m(h) - m(g^{-1}))}$ is equal to 0 so is the above integral and therefore ($\text{Cotlar}_{\mathbb{E}^\perp}$) holds. The reciprocal is immediate in the case of discrete groups. Indeed, choose any $h_0 \in \mathrm{G}$ and assume that $g \in \mathrm{G}\setminus\mathrm{G}_0$ is fixed. Pick $\widehat{f} = \delta_{h_0} + \delta_{gh_0}$. In order to evaluate the integral, notice that

$$\widehat{f}(gh)\,\overline{\widehat{f}(h)} = \delta_{\{h=h_0\}} + \delta_{\{g^2=e\}}\cdot\delta_{\{h=gh_0\}}.$$

The term $\delta_{h=h_0}$ in the above sum gives $G_g(h_0)$ in the integral. The term in which $g^2 = e$ and $h = gh_0$ gives $\overline{G_g(h_0)}$. Therefore, $\mathrm{Re}\{G_g(h)\} = 0$ for any $h \in \mathrm{G}$. The imaginary part is similarly shown to be 0. In the case of a continuous group G it is necessary to replace δ_{h_0} with an approximation of the unit. □

Remark 2.5. Notice that Theorem 2.4 works similarly in the non-relative case. In that case it holds that T_m has the ($\text{Cotlar}_{\text{nr}}$) if and only if m satisfies the identity $(m(g^{-1}) - m(h))\,(m(gh) - m(g)) = 0$ for almost every $g, h \in \mathrm{G}$.

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With all that at hand we are ready to prove Theorem A.

Proof of Theorem A. Since G_0 is closed, we have two situations, either it is open or of empty interior. In the first case, we have that $\mu(\mathrm{G}_0) > 0$ and we have that the formula in (ii) is equivalent to ($\mathrm{Cotlar}_{\mathbb{E}\perp}$) by Theorem 2.4. Observe that, if m is left-G_0 invariant, then T_m is left $\mathcal{L}\mathrm{G}_0$-modular. Thus, we can apply Proposition 2.3 to obtain the bound (1.1) for $p \geq 2$. Using that, by Plancherel theorem, $\|T_m : L_2(\mathcal{L}\mathrm{G}) \to L_2(\mathcal{L}\mathrm{G})\| = \|m\|_\infty$ we get the result for $2 \geq p$. For $1 < p < 2$ the result follows by standard duality arguments. In the case of G_0 of empty interior the proof follows similarly by using Remark 2.5 and the last assertion of Proposition 2.3. □

Remark 2.6. Let $\alpha : \mathcal{N} \to \mathcal{N}$ be a normal and trace-preserving $*$-homomorphism. It is immediate that both Proposition 2.3 and Lemma 2.2 hold if we change the condition of H being left $\mathcal{N}$-modular by that of being left $\mathcal{N}$-modular relative to α, i.e.,

$$H(f\,g) = \alpha(f)\,H(g), \quad \text{for } f \in \mathcal{N}, g \in L_2(\mathcal{M}) \cap \mathcal{M}.$$

In the case of multipliers this easy observation has deep consequences. For instance, let $\chi : \mathrm{G}_0 \to \mathbb{T}$ be a (multiplicative) character. It is a straightforward consequence of Fell's absorption principle that the map $\lambda_g \mapsto \chi(g)\,\lambda_g$ induces a normal and trace-preserving $*$-automorphism $\alpha_\chi : \mathcal{L}\mathrm{G}_0 \to \mathcal{L}\mathrm{G}_0$. Let $H = T_m$ be a Fourier multiplier on $\mathcal{L}\mathrm{G}$. We have that it is left $\mathcal{L}\mathrm{G}_0$-modular with respect to α_χ, i.e., $H(f\,g) = \alpha_\chi(f)\,H(g)$, for every $f \in \mathcal{N}$ and $g \in L_2(\mathcal{M}) \cap \mathcal{M}$ iff

$$m(k\,g) = \chi(k)\,m(g), \quad \text{for every } k \in \mathrm{G}_0, g \in \mathrm{G}. \tag{2.11}$$

This is specially useful when G_0 is Abelian since, in that case, every function on G_0 can be expressed as a limit of linear combinations of characters by the Fourier transform. This will be exploited in Theorem 5.2. It would also be used in a forthcoming paper of the third-named author.

Tightening the constant. It is known that, on the real line $\mathrm{G} = \mathbb{R}$ the operator L_p-norm of the classical Hilbert transform (HT) is given by

$$\big\|H : L_p(\mathbb{R}) \to L_p(\mathbb{R})\big\| = \max\Big\{\tan\Big(\frac{\pi}{2p}\Big),\ \cot\Big(\frac{\pi}{2p}\Big)\Big\}, \quad \text{for } 1 < p < \infty$$

see [68] or [34] for a simplified proof. These constants grow asymptotically like p as $p \to \infty$ and like $1/(p-1)$ as $p \to 1^+$, and those are the growth orders that we conjecture optimal in the noncommutative case as well. An interesting observation, originally made by Gokhberg and Krupnik in the classical case [31] is that Cotlar's identity on the real line gives the optimal order of growth for the constant in terms of p. Indeed, in the classical case, the fact that $H^2 = -\mathrm{id}$ yields a recurrence relation of the form

$$c_{2p}^2 \leq 2c_p\,c_{2p} + 1 \tag{2.12}$$

instead of (2.10). The lack of a term depending on c_p^2 gives a decisively smaller bound. Solving the quadratic inequality in (2.12) gives

$$c_{2p} \leq c_p + \sqrt{c_p^2 + 1}$$

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and that results, after applying duality and interpolation, in the optimal growth order for the constant.

In the noncommutative case, the same type of argument holds for operators $H : L_2(\mathcal{M}) \to L_2(\mathcal{M})$ satisfying that $H H_{\rm op} = -{\rm id}$, where $H_{\rm op}(f) = H(f^*)^*$. We have the following improvement over Proposition 2.3.

Proposition 2.7. *Let $\mathcal{N} \subseteq \mathcal{M}$ and $H : L_2(\mathcal{M}) \to L_2(\mathcal{M})$ be as before and assume H is left $\mathcal{N}$-modular, commutes with $\mathbb{E} : \mathcal{M} \to \mathcal{N}$ and satisfies that $\mathbb{E}^\perp H H_{\rm op} = -\mathbb{E}^\perp$. If H satisfies (Cotlar$_{\mathbb{E}^\perp}$), then*

$$\big\|H : L_p(\mathcal{M}) \to L_p(\mathcal{M})\big\| \lesssim p \big\|H : L_2(\mathcal{M}) \to L_2(\mathcal{M})\big\| \quad \text{for every } p \geq 2.$$

The same inequality holds in the non-relative case if $H H_{\rm op} = -{\rm id}$.

Proof. The proof is immediate once it is noticed that the property $\mathbb{E}^\perp H H_{\rm op} = -\mathbb{E}^\perp$ implies that (Cotlar$_{\mathbb{E}^\perp}$) can be rewritten as

$$\mathbb{E}^\perp\big[H(f)H(f)^*\big] = \mathbb{E}^\perp\Big[H\big(fH(f)^*\big) + H\big(fH(f)^*\big)^* + ff^*\Big].$$

Applying the same proof as that of Proposition 2.3 gives the recurrence $c_{2p}^2 \leq 2c_{2p}c_p + 1 + 3c_2^2$. After solving the quadratic inequality, we obtain $c_{2p} \leq c_p + \sqrt{c_p^2 + \kappa}$, where $\kappa = 1 + 3c_2^2$. Iterating and applying Marcinkiewicz interpolation gives the bound. The same proof works in the non-relative case. □

Observe that if $H = T_m$ is a Fourier multiplier, then $(T_m)_{\rm op} = T_{\widetilde{m}}$, for $\widetilde{m}(g) = \overline{m(g^{-1})}$. Thus, we are asking that $m(g)\overline{m(g^{-1})} = -1$ for every $g \in {\rm G} \setminus {\rm G}_0$. Similarly, since in the case of multipliers $\|\mathbb{E}H(\mathbf{1})\|_\infty = \|m\,\mathbf{1}_{{\rm G}_0}\|_\infty \leq \|m\|_\infty = c_2$ we can simplify the recurrence above assuming $c_2 = 1$. In particular, we obtain the following corollary.

Corollary 2.8. *Let ${\rm G}$ be a group and let $m : {\rm G} \to \mathbb{C}$ be a function satisfying (Cotlar) relative to a closed subgroup ${\rm G}_0 \subseteq {\rm G}$, and such that m is left ${\rm G}_0$-invariant and $m(g)\overline{m(g^{-1})} = -1$, for every $g \in {\rm G} \setminus {\rm G}_0$. Then*

$$\big\|T_m : L_p(\mathcal{L}{\rm G}) \to L_p(\mathcal{L}{\rm G})\big\|_{\rm cb} \lesssim \Big(\frac{p^2}{p-1}\Big)\|m\|_\infty.$$

Both Proposition 2.7 and Corollary 2.8 are still true if one changes the value of $m(g)\overline{m(g^{-1})}$ from -1 to any other constant independent of g.

Non-Fourier multiplier examples. All of the machinery developed or reviewed in this Section is formulated in a way that works beyond the case of Fourier multipliers. As an illustration, another family of examples comes from Schur multipliers. Indeed, let $\mathcal{D} \subset \mathcal{B}(\ell_2\mathbb{Z})$ be the finite span of the matrix units e_{jk}. A Schur multiplier is a linear map $S_m : \mathcal{D} \subseteq \mathcal{B}(\ell_2\mathbb{Z}) \to \mathcal{B}(\ell_2\mathbb{Z})$ defined by sending the matrix unit e_{jk} to $m(j,k)\,e_{jk}$ for some function $m : \mathbb{Z} \times \mathbb{Z} \to \mathbb{C}$ – called the symbol of S_m. Schur multipliers are $\ell_\infty(\mathbb{Z})$-bimodular, with $\ell_\infty(\mathbb{Z})$ sitting inside $\mathcal{B}(\ell_2\mathbb{Z})$ as the diagonal subalgebra. As such, if we take in Theorem 2.3 $\mathcal{M}$ to be $\mathcal{B}(\ell_2\mathbb{Z})$ and $\mathcal{N}$ to be $\ell_\infty(\mathbb{Z})$, we have that any bounded symbol m satisfying (Cotlar$_{\mathbb{E}^\perp}$) extends boundedly to all Schatten classes S_p with $1 <$

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$p<\infty$. The same computations of Theorem 2.4 can be performed in this setting, yielding that

$$S_m \text{ satisfies } (Cotlar_{\mathbb{E}^\perp}) \text{ relative to } \ell_\infty \iff \big(m(j,k)-m(j,\ell)\big)\big(m(\ell,j)-m(\ell,k)\big)\mathbf{1}_{\{j\neq\ell\}}=0,\ \forall j,k,\ell\in\mathbb{Z}. \tag{2.13}$$

This makes verifying the Cotlar identity for Schur multipliers straightforward. For instance, the triangular truncation $H=S_m$ is the Schur multiplier with symbol $m(j,k)=\operatorname{sgn}(j-k)$. It is trivial to verify that it satisfies identity (2.13). Indeed, if $m(j,k)\neq m(j,\ell)$ it is because $j<\ell$ and $j\geq k$ or vice–versa. If the first case holds, then $k\leq j<\ell$, which implies that $m(\ell,j)=m(\ell,k)$. The other case is checked similarly. Since $H\,H^{\mathrm{op}}=-\mathrm{id}$, we recover the sharp bound of the operator-norm on S_p as $p\to 1^+,\infty$. This is not a novel result, since the triangular truncation is a very well understood object whose S_p-boundedness can be obtained from an array of techniques ranging from subdiagonal algebras [74] to Fourier-Schur transference for amenable groups [61, 11] and smooth Schur multiplier techniques [14]. Nevertheless, it signals the possibility of exploiting Cotlar identities in settings beyond Fourier multipliers.

The convex hull of Cotlar-type multipliers. A natural question is which class of multipliers m can be shown to be bounded in $L_p(\mathcal{L}\mathrm{G})$ by being represented as a convex combination of multipliers satisfying ($\mathrm{Cotlar}_{\mathbb{E}^\perp}$) or natural modifications of them. To that end, notice that if m is an L_p-bounded multiplier, then so is $g\mapsto m(h^{-1}\theta(g)r)$, where $h,r\in\mathrm{G}$, $\theta\in\mathrm{Aut}(\mathrm{G})$ and their norms coincide. Let us denote the group of transformations of G given by $g\mapsto h^{-1}\theta(g)r$ by the *affine transformations* $\mathrm{Aff}(\mathrm{G})$ *of* G. Observe that, if we define $\mathrm{G}^{\Delta}=(\mathrm{G}\oplus\mathrm{G})/\Delta$, where Δ is the subgroup of diagonal central elements, $\Delta=\{(a,a):a\in\mathcal{Z}(\mathrm{G})\}\subseteq\mathrm{G}\oplus\mathrm{G}$, then there is a faithful representation that sends (h,r) to $g\mapsto h^{-1}gr$. A trivial computation gives that

$$\mathrm{Aff}(\mathrm{G})\cong\mathrm{G}^{\Delta}\rtimes\mathrm{Aut}(\mathrm{G}),$$

with the natural action. It is clear that, if $\mu\in M(\mathrm{Aut}(\mathrm{G}))$ is a finite signed measure and $m:\mathrm{Aff}(\mathrm{G})\times\mathrm{G}\to\mathbb{C}$ is a bounded map such that $g\mapsto m(\alpha,g)$ satisfies ($\mathrm{Cotlar}_{\mathbb{E}^\perp}$) for every α, then

$$m(g)=\int_{\mathrm{Aff}(\mathrm{G})} m(\alpha,\alpha(g))\,d\mu(\alpha),$$

is clearly bounded in $L_p(\mathcal{L}\mathrm{G})$ for every $1<p<\infty$. We could add more flexibility to this technique by allowing the map $g\mapsto m(\alpha,g)$ to be the product of k terms satisfying ($\mathrm{Cotlar}_{\mathbb{E}^\perp}$). Let us call this class $\mathrm{coCot}^k(\mathrm{G})$. We leave mostly unexplored the following natural problem:

Problem 2.9. Let $\mathrm{G}_0\subseteq\mathrm{G}$ and $m:\mathrm{G}\to\mathbb{C}$ be as above. Are there sufficient conditions, for example in terms of smoothness, implying that $m\in\mathrm{coCot}^k(\mathrm{G})$?

This remains as an underexplored approach to prove the boundedness of Fourier multipliers over groups without recourse to noncommutative analogues of singular integral theory. In fact, observe that in the classical case of $\mathbb{R}$ any function of bounded variation

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lies in the convex hull of (translations of) the classical Hilbert transform and therefore $m \in \mathrm{BV}(\mathbb{R}) \implies m \in \mathrm{coCot}^1(\mathbb{R})$, see [25, Corollary 3.8]. In higher dimensions the behaviour is even richer. For instance, let $m : \mathbb{R}^2 \to \mathbb{C}$ be a homogeneous function, i.e., a function satisfying that $m(\lambda\xi) = m(\xi)$ for every $\lambda > 0$. Clearly, m depends only on its angular component $m|_{\mathbb{T}}$. We have that

$$m|_{\mathbb{T}} \in \mathrm{BV}(\mathbb{T}) \quad \implies \quad m \in \mathrm{coCot}^2(\mathbb{R}^2),$$

where $\mathrm{BV}(\mathbb{T})$ is the space of functions of bounded variation on the torus. To see that, just notice that the indicator function $\mathbf{1}_{\Sigma_\theta}$ of the sector $\Sigma_\theta \subseteq \mathbb{R}^2$ of all vectors whose polar angle ω lies in $[0,\theta)$ can be expressed as the product of the characteristic functions of two half-planes, each of which satisfies a Cotlar identity. Homogeneous functions of angular bounded variation can be expressed as convex combinations of different sector indicators.

Generalizations. It worth noticing that the identity ($\mathrm{Cotlar}_{\mathbb{E}\perp}$) can be generalized in a natural way by changing the equality by an operator inequality

$$\mathbb{E}^\perp\big[H(f)\,H(f)^*\big] \le \mathbb{E}^\perp\Big[H\big(f\,H(f)^*\big) + H\big(f\,H(f)^*\big)^* - H\big(H(ff^*)^*\big)\Big] + \Lambda\,\mathbb{E}[f\,f^*], \tag{$\mathrm{Cotlar}^{\le}_{\mathbb{E}\perp}$}$$

for every $f \in \mathcal{M} \cap L_2(\mathcal{M})$ and some constant $\Lambda \ge 0$. It is clear that this inequality implies the same bound (2.6) of Proposition 2.3 for left $\mathcal{N}$-modular operators, just with a constant growing like $\mathrm{O}(1+\sqrt{\Lambda})$. Similarly, there exists a closed-formula characterization of Fourier multipliers T_m with left G_0-invariant symbol m satisfying ($\mathrm{Cotlar}^{\le}_{\mathbb{E}\perp}$). To formulate such characterization let us define

$$\Omega_m(g,h) = \big(m(g) - m(gh^{-1})\big)\,\overline{\big(m(h) - m(hg^{-1})\big)}\,\mathbf{1}_{\mathrm{G}\setminus\mathrm{G}_0}(gh^{-1}) + \Lambda\,\mathbf{1}_{\mathrm{G}_0}(gh^{-1})$$

and notice that ($\mathrm{Cotlar}^{\le}_{\mathbb{E}\perp}$) is actually equivalent to

$$0 \le \int_{\mathrm{G}}\int_{\mathrm{G}} \widehat{f}(gh)\,\overline{\widehat{f}(h)}\,\Omega_m(gh,h)\,\lambda_g\,d\mu(h)\,d\mu(g).$$

The inequality above is equivalent to the (integral) positivity of the operator-valued kernel $\Omega_m(g,h)\,\lambda_{gh^{-1}}$. Since multiplication by $\lambda_{g^{-1}} \cdot \lambda_h$ preserves positivity, this is equivalent to requiring that Ω_m be an integrally positive kernel, i.e.,

$$0 \le \int_{\mathrm{G}}\int_{\mathrm{G}} \widehat{f}(g)\,\overline{\widehat{f}(h)}\,\Omega_m(g,h)\,d\mu(h)\,d\mu(g), \quad \text{for every } \widehat{f} \in L_1(\mathrm{G}) \cap L_\infty(\mathrm{G}). \tag{2.14}$$

When G is discrete, this condition implies that Ω_m factors as

$$\Omega_m(g,h) = \big\langle \xi(h), \xi(g) \big\rangle,$$

where $\xi : \mathrm{G} \to \mathcal{H}$ is a bounded function with values in a Hilbert space $\mathcal{H}$, see [3, Theorem C.1.4]. The reason why we need to add the term $\Lambda\,\mathbb{E}[f\,f^*]$ to the Cotlar identity when G_0 is open is that, without it, we will be forcing the kernel $\Omega_m(g,h)$ to be positive definite while also vanishing on the diagonal $g = h$, which would imply that it is identically zero. In the non-relative case, this extra term can be omitted. This more general Cotlar identity leads to the following problem, which we leave unexplored.

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Problem 2.10. Let $G_0 \subseteq G$ and $m : G \to \mathbb{C}$ be as above. Is there a geometric model of $G \curvearrowright \mathcal{X}$, possibly generalizing that of actions on UAC spaces in Theorem B, such that if $m(g)$ lifts to $\mathcal{X}$ via a function $m(g) = \widetilde{m}(g \cdot x_0)$, then its associated Fourier multiplier satisfies (Cotlar$^{\leq}_{\mathbb{E}^\perp}$)?

3. Groups acting on $\mathbb{R}$-trees

Let X be a Hausdorff topological space. An arc γ on X is a subset of X that is the image of an injective continuous function mapping [0,1] onto γ. The space X is said to be *uniquely arcwise connected*, or UAC, if any two points in X are joined by a unique arc. We say that a group G acts on a UAC space X if G acts by homeomorphisms on X.

If, in addition, the UAC space X is metrisable and there is a metric $d : X \times X \to \mathbb{C}$ such that the unique arc joining two points is isometric to a closed interval of the real line, then (X,d) is called an $\mathbb{R}$-tree. We will say that a group G acts on an $\mathbb{R}$-tree X if it acts on it by isometries.

Observe that this definition is topological in nature since the underlying space is required to be arcwise connected. An alternative route to $\mathbb{R}$-trees can be taken by defining them as hyperbolic spaces with $\delta = 0$, i.e., every triangle is a *tripod*. These two definitions, although equivalent in spirit, are slightly different. Namely, a tree seen as a discrete set with the edge metric is a 0-hyperbolic metric space but not an $\mathbb{R}$-tree in our definition. This is not a problem since trees can still be seen as a subclass of $\mathbb{R}$-trees by treating them as *simplicial trees*, i.e. the one-dimensional simplicial complexes obtained from the incidence information of the tree. We also have that, given an $\mathbb{R}$-tree X, if the set of points whose complement has three or more connected components is discrete in X, then X is a simplicial tree. The first definition of $\mathbb{R}$-trees was given by Tits [85], then Morgan and Shalen [58], following earlier results of Alperin and Moss [1], drew attention to the theory of $\mathbb{R}$-trees by showing how to compactify a generalization of Teichmuller space for a finitely generated group using $\mathbb{R}$-trees. We refer the reader to [5] for more on $\mathbb{R}$-trees.

We define the following two models for a group acting on a UAC space. Let $G \curvearrowright X$ be a topological action and $x_0 \in X$ a selected point. We will say that a bounded measurable function $\varphi : X \setminus \{x_0\} \to \mathbb{C}$ is *constant along arcs* iff $\varphi(x) = \varphi(y)$ if there is an arc connecting x and y inside $X \setminus \{x_0\}$. This is equivalent to decomposing $X \setminus \{x_0\}$ as a union of arcwise connected subsets and imposing the function to be constant over those subsets.

Model 1. Let $G \curvearrowright X$ be a topological action on a UAC space, $x_0 \in X$ a point and $\widetilde{m} : X \to \mathbb{C}$ a bounded measurable function such that

(i) $\widetilde{m}$, restricted to $X \setminus \{x_0\}$, is constant along arcs.
(ii) $\widetilde{m}$ is invariant under the action $\mathrm{St}_{x_0} \curvearrowright X \setminus \{x_0\}$.

Then we define the multiplier $m : G \to \mathbb{C}$ as

$$m(g) = \widetilde{m}(g \cdot x_0)$$

and fix (G_0,G) as (St_{x_0},G).

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This definition has the drawback that the invariance under St_{x_0} of m can make the symbol constant outside St_{x_0} in some cases. We introduce the following, more involved, model.

Model 2. Let us fix two distinct constants $C_1,\ C_2 \in \mathbb{C}$. Let similarly $\mathrm{G} \curvearrowright X$ be a topological action on a UAC space and $x_0 \in X$ a point. Choose $X_0 \subseteq X \setminus \{x_0\}$ to be an arcwise connected subset. We define $m : \mathrm{G} \to \mathbb{C}$ to be

$$m(g) = \begin{cases} 0 & \text{when } g \in \mathrm{St}_{x_0} \text{ and } g \cdot X_0 = X_0 \\ C_1 & \text{when } g \in \mathrm{St}_{x_0} \text{ and } g \cdot X_0 \neq X_0 \\ C_1 & \text{when } g \notin \mathrm{St}_{x_0} \text{ and } g \cdot x_0 \notin X_0 \\ C_2 & \text{when } g \cdot x_0 \in X_0. \end{cases}$$

We will also fix G_0 to be $\mathrm{St}_{x_0} \cap \{g \in \mathrm{G} : g \cdot X_0 = X_0\}$.

Observe that Model 2 is a natural modification of Model 1 for the function $\widetilde{m} : X \to \mathbb{C}$ given by $\widetilde{m} = C_1 \mathbf{1}_{X \setminus (\{x_0\} \cup X_0)} + C_2 \mathbf{1}_{X_0}$. The main difference is that we extend the value of C_1 to a portion of the stabilizer.

Proposition 3.1. *Let $\mathrm{G} \curvearrowright X$ be an action as above.*

(i) *Let $m : \mathrm{G} \to \mathbb{C}$ and G_0 be like in Model 1. Then, T_m satisfies (Cotlar$_{\mathbb{E}_\perp}$) relative to G_0 and is left $\mathcal{L}\mathrm{G}_0$-modular.*

(ii) *Let $m : \mathrm{G} \to \mathbb{C}$ and G_0 be like in Model 2. Then, T_m satisfies (Cotlar$_{\mathbb{E}_\perp}$) relative to G_0 and is left $\mathcal{L}\mathrm{G}_0$-modular.*

The points above imply that T_m in both Models 1 and 2 are bounded in $L_p(\mathcal{L}\mathrm{G})$ for $1 < p < \infty$.

Proof. The statement in point (i) has already been proved in the introduction. Thus, we concentrate on point (ii). The fact that m is left invariant under the action of $\mathrm{G}_0 = \{g \in \mathrm{G} : g \cdot x_0 = x_0\} \cap \{g \in \mathrm{G} : g \cdot X_0 = X_0\}$ is immediate. Now, all we have to do is to verify ($\widehat{\text{Cotlar}}$). To that end, let us divide the group as a disjoint union $\mathrm{G} = \mathrm{G}_0 \cup \mathrm{G}_1 \cup \mathrm{G}_2 \cup \mathrm{G}_3$, where

$$\begin{aligned} \mathrm{G}_0 &= \{g \in \mathrm{G} : g \cdot x_0 = x_0\} \cap \{g \in \mathrm{G} : g \cdot X_0 = X_0\} \\ \mathrm{G}_1 &= \{g \in \mathrm{G} : g \cdot x_0 = x_0\} \cap \{g \in \mathrm{G} : g \cdot X_0 \neq X_0\} \\ \mathrm{G}_2 &= \{g \in \mathrm{G} : g \cdot x_0 \neq x_0\} \cap \{g \in \mathrm{G} : g \cdot x_0 \notin X_0\} \\ \mathrm{G}_3 &= \{g \in \mathrm{G} : g \cdot x_0 \neq x_0\} \cap \{g \in \mathrm{G} : g \cdot x_0 \in X_0\}. \end{aligned}$$

Observe also that, since m is both left and right G_0-invariant, it is enough to verify ($\widehat{\text{Cotlar}}$) for $g,h \in \mathrm{G} \setminus \mathrm{G}_0$. Assume that $m(g^{-1}) \neq m(h)$, otherwise we are done, our aim is to show that $m(gh) = m(g)$. We will proceed by cases. First, assume that $g \in \mathrm{G}_1$. This is equivalent to $g^{-1} \in \mathrm{G}_1$ and therefore $h \in \mathrm{G}_3$. But then, $gh \cdot x_0 \notin X_0$ and $gh \cdot x_0 \neq x_0$. Therefore $gh \in \mathrm{G}_2$ and we get $m(gh) = m(g)$. In the case of $g \in \mathrm{G}_2$ we have the whole range of possibilities and h can belong to G_1, G_2 or G_3. In the case of $g \in \mathrm{G}_2$ and $h \in \mathrm{G}_1$ we have that $gh \in \mathrm{G}_2$. Indeed, $gh \cdot x_0 = g \cdot x_0 \notin X_0$ and it is immediate that gh is not in the

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stabilizer of x_0. For the second case of $g \in \mathrm{G}_2$ and $h \in \mathrm{G}_2$ the condition $m(g^{-1}) \neq m(h)$ implies that $g^{-1} \cdot x_0$ and $h \cdot x_0$ live in distinct arcwise connected subsets of $X \setminus \{x_0\}$. But then there is a unique path connecting both points that passes through the root. Applying g to the whole arc gives that $g \cdot x_0$ and $gh \cdot x_0$ lie in the same arcwise connected subset and do not stabilize x_0, see Figure 1. Therefore $m(gh) = m(g)$. The third case is given by $g \in \mathrm{G}_2$ and $h \in \mathrm{G}_3$. Observe that if $g \in \mathrm{G}_2$, then g^{-1} can only be inside G_2 or G_3. But the case $g^{-1} \in \mathrm{G}_3$ can be easily discarded since it will contradict the assumption $m(g^{-1}) \neq m(h)$. But if $g^{-1} \in \mathrm{G}_2$ and $h \in \mathrm{G}_3$, then $g^{-1} \cdot x_0$ and $h \cdot x_0$ live in distinct arcwise connected subsets and we can proceed like in the previous case. It remains to check the case of $g \in \mathrm{G}_3$. We have that h can be in either G_1, G_2 or G_3. In the first case we deduce that $gh \in \mathrm{G}_3$. For the second one we have that if $g \in \mathrm{G}_3$ and $h \in \mathrm{G}_2$, then we can assume that $g^{-1} \in \mathrm{G}_3$, the only other choice being $g^{-1} \in \mathrm{G}_2$ which will contradict the assumption $m(g^{-1}) \neq m(h)$. But this implies that $g^{-1} \cdot x_0$ and $h \cdot x_0$ live in different arcwise connected subsets of $X \setminus \{x_0\}$ and we can apply the argument in Figure 1. Lastly, if $g \in \mathrm{G}_3$ and $h \in \mathrm{G}_3$ we obtain similarly that g^{-1} can only lie in G_2. The same path argument applies. Applying Theorem B, we get the L_p-boundedness of T_m. □

Remark 3.2. Recall also that, by Remark 2.5, when the subgroup St_{x_0} has empty interior, condition (2) in Model 1 can be dropped.

Observe that in the case of $\mathrm{G} = \mathbb{R}$, the Hilbert transform is also of weak type (1,1), i.e., $H : L_1(\mathbb{R}) \to L_{1,\infty}(\mathbb{R})$ and bounded between $L_\infty(\mathbb{R})$ and the space of *bounded mean oscillation functions* $\mathrm{BMO}(\mathbb{R})$. Both endpoint spaces give – by either complex or real interpolation with L_2 – the optimal order for the operator L_p norm of H. The following problem remains open.

Problem 3.3. Let G be a group and m a multiplier like in Model 1 or 2. Is it possible to construct spaces X_1 and X_∞, in place of $L_{1,\infty}$ and BMO, such that

(i) $\|T_m : L_1(\mathcal{L}\mathrm{G}) \to \mathrm{X}_1\| < \infty$ and $\|T_m : L_\infty(\mathcal{L}\mathrm{G}) \to \mathrm{X}_\infty\| < \infty$,

(ii) interpolation of X_1 or X_∞ with L_2 yields growth of $\max\{p,p'\}$ for the operator L_p norm of T_m?

This problem presents at least three challenges. The first difficulty comes from the fact that weak type (1,1) bounds are difficult to obtain for noncommutative singular integral type operators. They are known in some semicommutative examples [62, 8] but open in the case of Quantum Euclidean spaces [33] and in most group settings besides left-orderable groups [74]. The second challenge is that the specific endpoint space X_∞ to be used for m in Model 1 or Model 2 has to be defined in terms of the geometry of $\mathrm{G} \curvearrowright X$ and it cannot just be the usual noncommutative BMO space, see [43, 53, 54]. Indeed, there is a natural *unital and completely positive* semigroup $S_t : \mathcal{L}\mathbb{F}_2 \to \mathcal{L}\mathbb{F}_2$ in the free group algebra given by $S_t(\lambda_g) = e^{-t|g|}$, see [36]. This semigroup allows to construct a natural and interpolating semigroup BMO space $\mathrm{BMO}(\mathcal{L}\mathbb{F}_2)$, see [44]. But it is known that the multipliers (MR) are unbounded from $L_\infty(\mathcal{L}\mathbb{F}_2)$ to $\mathrm{BMO}(\mathcal{L}\mathbb{F}_2)$, see [56, Appendix A]. The third difficulty comes from the fact that the classical technique, employed by Kolmogorov [50], of comparing a singular integral operator with a maximal one is delicate in this context since some of

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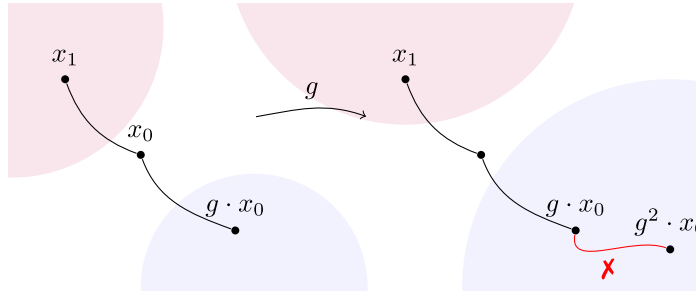


Figure 2. The action of g over the path connecting the global fixed point and $g \cdot x_0$.

the operators obtained are not positivity preserving, which makes interpolating maximal functions an open problem [47]. The technique of Kolmogorov has been used in the noncommutative case in [40]. We will also mention that some tentative progress in the direction of Problem 3.3 has been made. For instance, Gałązka and Osękowski [30] have lowered the operator L_p-norm of the Fourier multiplier $m : \mathbb{F}_2 \to \mathbb{C}$ that depends on the starting letter from $\mathrm{O}(p^\beta)$ to $\mathrm{O}(p \log p)$ as $p \to \infty$.

Now, we are going to study Models 1 and 2 in the context of $\mathbb{R}$-trees. Observe that, since any $\mathbb{R}$-tree action is an action of the underlying UAC topological space, Proposition 3.1 above works for $\mathbb{R}$-trees. Furthermore, in the case of group actions on $\mathbb{R}$-trees we have the following result connecting the existence of global fixed points with the form of the Fourier multiplier T_m. We are going to say that the multipliers m coming from Models 1 and 2 are *trivial* iff they are constant for any $g \in \mathrm{G} \setminus \mathrm{G}_0$.

Proposition 3.4. *Let X be an $\mathbb{R}$-tree and $\mathrm{G} \curvearrowright X$ an (isometric) action of a discrete group. The following holds:*

(i) *If the action $\mathrm{G} \curvearrowright X$ has a global fixed point, then for any choice of a root $x_0 \in X$ the multipliers in Model 1 and Model 2 are trivial.*

(ii) *If G is finitely generated, for any action on an $\mathbb{R}$-tree X, there is either a global fixed point or there exists $x_0 \in X$ such that the corresponding symbol given in Model 2 is nontrivial.*

Proof. We will prove first (i) for m as in Model 1. Assume that the action has a global fixed point $x_1 \in X$. If x_1 coincides with the root x_0 then $m = 0$ and there is nothing to prove. Therefore we assume $x_0 \neq x_1$. Similarly, we can assume without loss of generality that $\mathrm{St}_{x_0} \neq \mathrm{G}$, since otherwise $m = 0$. Pick $g \in \mathrm{G}$ and assume that $g \cdot x_0$ lies in an arcwise connected subset of $X \setminus \{x_0\}$ different from that of x_1. Then, there is a unique path joining x_1 and $g \cdot x_0$ that passes through the root x_0, see Figure 2. But since x_1 is fixed by any $g \in \mathrm{G}$, after applying g to the path we obtain a larger path, which contradicts the fact that the action is isometric.

This implies that $g^2 \cdot x_0 = g \cdot x_0$ and so $g \cdot x_0 = x_0$, which contradicts the assumptions. Therefore, $g \cdot x_0$ belongs to the same connected subset of x_1 for every g that do not stabilize the root x_0, that is, $m(g)$ is constant for any $g \in \mathrm{G} \setminus \mathrm{St}_{x_0}$.

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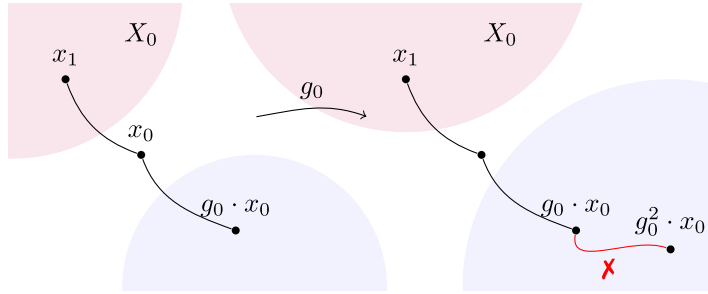


Figure 3. The action of g_0 over the path connecting x_1 and $g_0 \cdot x_0 \notin X_0$.

For the case of m as in Model 2, let $x_1 \in X$ be a global fixed point. We can again consider without loss of generality that $x_0 \neq x_1$ and that $\mathrm{St}_{x_0} \neq \mathrm{G}$. By those assumptions, there exists $g_0 \in \mathrm{G}$ with $g_0 \cdot x_0 \in X \setminus \{x_0\}$. We have two possibilities for g_0. If $g_0 \cdot x_0 \neq x_0$ and $g_0 \cdot x_0 \notin X_0$, then $m(g_0) = C_1$, while if $g_0 \cdot x_0 \in X_0$ then $m(g_0) = C_2$. If we are in the first case $g_0 \cdot x_0 \notin X_0$, then the multiplier m will be trivial unless there exists a $g_1 \in \mathrm{G}$ such that $g_1 \cdot x_0 \in X_0$. Let us obtain a contradiction. First, we claim that the global fixed point $x_1 \notin X_0$. Assume $x_1 \in X_0$. Then, there is a path joining x_1 and $g_0 \cdot x_0$ that passes through the root x_0, see Figure 3. But applying g_0 to the path gives that $g_0 \cdot x_0 = x_0$ which is a contradiction. Therefore $x_1 \notin X_0$, but since $g_1 \cdot x_0 \in X_0$, we can build a path from x_1 to $g_1 \cdot x_0$ that passes through the root x_0 and, repeating the same argument as before, obtain that $g_1 \cdot x_0 = x_0$, which is a contradiction. For the second case $g_0 \cdot x_0 \in X_0$. First, we notice that x_1 must live in X_0, if not, proceeding as before we will get that $g_0 \cdot x_0 = x_0$, which is a contradiction. In order for m to be nontrivial there should be a $g_1 \in \mathrm{G}$ such that either $g_1 \cdot x_0 = x_0$ and $g_1 \cdot X_0 \neq X_0$ or $g_1 \cdot x_0 \neq x_0$ and $g_1 \cdot x_0 \notin X_0$. For the first case, let us choose a point $x_2 \in X_0$. Then, $g_1 \cdot x_2 \notin X_0$ and similarly $g_1 \cdot x_2 \neq x_0$. Now, construct a path joining x_1 with $g_1 \cdot x_2$. Since $x_1 \in X_0$ but $g_1 \cdot x_2$ belongs to a different connected subset, the arc joining them passes through the root x_0. Let us apply g_1^{-1} to the whole arc. Since we have that $g_1^{-1} \cdot x_0 = x_0$, we obtain an arc that joins x_1 and x_2 and passes through the root x_0. But this is a contradiction with the fact that x_1 and x_2 both belong to X_0, see Figure 4, since X is a UAC space the arc joining two elements in the same connected subset of $X \setminus \{x_0\}$ cannot pass trough the root x_0. The remaining case is when $g_1 \cdot x_0 \neq x_0$ and $g_1 \cdot x_0 \notin X_0$. Then, the arc joining $x_1 \in X_0$ and $g_1 \cdot x_1$ passes through x_0. Applying g_1 gives a contradiction with the fact that the action is isometric and that $g_1 \cdot x_0 \neq x_0$. For the other point, if G is finitely generated, the converse statement is also true and it is a corollary of [6, Lemma 5.2]. This lemma tells us that when a finitely generated group G acts non-trivially on an $\mathbb{R}$-tree X, there exist a point y_0 and elements $g,h \in \mathrm{G}$ such that y_0 lies in the arc between $g \cdot y_0$ and $h \cdot y_0$ and y_0, $g \cdot y_0$ and $h \cdot y_0$ are all different. Therefore, if G does not have property (F$\mathbb{R}$), we can choose x_0 as y_0 and X_0 as the maximally connected subset of $g \cdot y_0$. Then, there is an $h \in \mathrm{G}$ such that $h \cdot x_0 \neq x_0$ and $h \cdot x_0 \notin X_0$. Thus, sending g and h to different values C_1 and C_2 gives a nontrivial multiplier within Model 2. □

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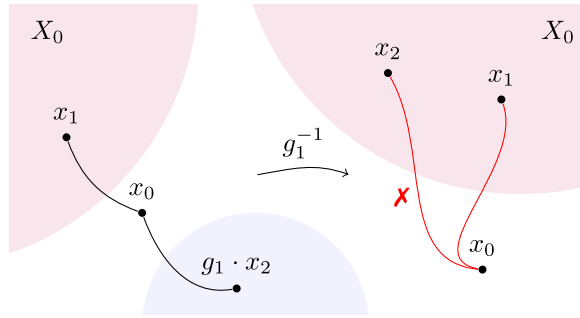


Figure 4. The action of g_1^{-1} over the path connecting x_1 and $g_1 \cdot x_2$.

Many groups we are familiar with admit actions on $\mathbb{R}$-trees. For example, every finitely generated hyperbolic group. Indeed, a finitely generated group is hyperbolic if and only if every asymptotic cone of the group is an $\mathbb{R}$-tree [35]. Then, the action of a hyperbolic group on its Cayley graph induces an action on the asymptotic cone of the Cayley graph. Moreover, any surface group having Euler characteristic less than -1 acts freely on an $\mathbb{R}$-tree [59].

On the other hand, there are many examples of groups for which any action on an $\mathbb{R}$-tree has a global fixed point. When this happens the group G is said to have property (F$\mathbb{R}$), see [5, 80]. The following corollary of Proposition 3.4 characterizes groups with property (F$\mathbb{R}$).

Corollary 3.5. G *has property (F$\mathbb{R}$) if and only if the symbol m in Model 2 is trivial for any $x_0 \in X$ in any $\mathbb{R}$-tree X on which* G *acts (isometrically).*

One natural question, that we leave open, is whether there are functions $m : \mathrm{G} \to \mathbb{C}$ satisfying (Cotlar) for $\mathrm{G}_0 = \{e\}$ and such that they do not lift to a function $\widetilde{m}$ on a UAC space like in Model 1 or 2 on which G acts. One way to prove the non-existence of such lift $\widetilde{m}$ would be a procedure to assemble from the group G and the function m a G-space X on which G acts naturally and a lift $\widetilde{m} : X \to \mathbb{C}$ satisfying hypothesis like those of Models 1 and 2. So far, this reverse construction has escaped us.

4. Left orderable groups

Recall that a *total* or *linear* order $\preceq$ is an order relation such that, given any two points x, y then it holds that $x \preceq y$ or $y \preceq x$. A *left-orderable group* is a group admitting a left-invariant total order, i.e., $(\mathrm{G}, \preceq)$ satisfies that $g \preceq h$ if and only if $kg \preceq kh$. Recall that, as we defined in the introduction, every left-orderable group has a *sign function* $\mathrm{sgn} : \mathrm{G} \to \mathbb{C}$ that assigns $+1$ or -1 depending on whether $e \prec g$ or $g \prec e$. We will prove that $H = T_{\mathrm{sgn}}$ is bounded.

Proof of Theorem C. First, let us tackle the case of G discrete. All that is required to do is to prove the identity (Cotlar) relative to $\mathrm{G}_0 = \{e\}$. Assume that $\mathrm{sgn}(g) \neq \mathrm{sgn}(gh)$,

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where both g and h are different from e. If both g and h had the same sign, so would gh, therefore the sign of h has to be different from that of g. But that implies that the signs of g and h^{-1} coincide. To see that the norm grows as $\mathrm{O}(p)$ as $p \to \infty$ just notice that $\mathrm{sgn}(g^{-1}) = -\mathrm{sgn}(g)$ and therefore $m(g)\overline{m(g^{-1})} = -1$, which allows to apply the same technique in Proposition 2.7 and Corollary 2.8. The continuous case follows by the same argument in view of Remark 2.5. □

Apart from the direct proof above, it is interesting to notice that these Hilbert transform type multipliers on left-orderable groups can be put into the framework of Model 1 and Theorem B. To do that we need to use the well-known characterization of countable left-orderable groups as order-preserving groups of homeomorphisms of the real line $\mathbb{R}$, which is a UAC space. The first reference we found for this is [39].

Proposition 4.1. *Every countable left-orderable group acts on the real line* $\mathbb{R}$ *by orientation preserving homeomorphisms and without global fixed point.*

Observe as well that, given a homeomorphism of the real line $f : \mathbb{R} \to \mathbb{R}$ it is either orientation preserving or orientation reversing and that the composition of two orientation reversing maps is orientation preserving. Therefore, there is a group homomorphism $\mathrm{Homeo}(\mathbb{R}) \to \mathbb{Z}_2 \cong \{1, -1\}$ that associate each homeomorphism of $\mathbb{R}$ with $+1$ or -1 depending on whether the homeomorphism is orientation preserving or reversing. As a consequence each discrete group G acting on $\mathbb{R}$ has an index 2 subgroup that is left-orderable. So, when the UAC space X in Model 1 is equal to $\mathbb{R}$, the group G is left-orderable and have an index 2 subgroup in which the multiplier m essentially coincides with the sign function.

The left invariant order of a left-orderable group is by no means unique. Thus, the sign functions associated to different orders may give different L_p-bounded Fourier multiplier. Similarly, left-orderable groups can have actions on UAC spaces other than $\mathbb{R}$ which will yield different multipliers still within our Model 1. An example of this will be that of Baumslag-Solitar groups $\mathrm{BS}(1,n)$, which both admit nontrivial actions on their Bass-Serre trees and are left-orderable.

Left orderable groups and subdiagonal algebras. Finally, let us mention that the Hilbert transform $H = T_{\mathrm{sgn}} : L_p(\mathcal{L}\mathrm{G}) \to L_p(\mathcal{L}\mathrm{G})$ of a left-orderable group can be considered as a particular case of the Hilbert transforms associated with *subdiagonal algebras* which have been studied in [74], see also [72, Section 8]. Such algebras were introduced by Arveson [2]. Indeed, let $(\mathcal{M},\tau)$ be a von Neumann algebra with a n.s.f. trace and let $\mathbb{E} : \mathcal{M} \to \mathcal{D} \subseteq \mathcal{M}$ be a τ-preserving conditional expectation onto a von Neumann subalgebra $\mathcal{D} \subseteq \mathcal{M}$. A *finite subdiagonal algebra* $\mathcal{H}^\infty(\mathcal{M}) \subseteq \mathcal{M}$ with respect to $\mathbb{E}$ is a weak-$*$ closed, non self-adjoint algebra such that

(i) $\mathbb{E}(fg) = \mathbb{E}(f)\mathbb{E}(g)$,
(ii) $\{f + g^* : f,g \in \mathcal{H}^\infty(\mathcal{M})\} = \mathcal{M}$ and
(iii) $\mathcal{H}^\infty(\mathcal{M}) \cap (\mathcal{H}^\infty(\mathcal{M}))^* = \mathcal{D}$.

The notation comes from the fact that, for the algebra $L_\infty(\mathbb{T})$ and the integral as expectation, the Hardy space $\mathcal{H}^\infty(\mathbb{T})$ gives a finite subdiagonal algebra. Other examples

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are the $n \times n$ upper triangular matrices, which are a finite subdiagonal algebra with respect to the diagonal subalgebra $\ell_\infty^n \subseteq M_n(\mathbb{C})$, and more generally the nest algebras [20] of totally ordered families of projections. In this setting any $f \in \mathcal{M}$ admits a unique decomposition as

$$f = g + \delta + h^*, \quad \text{with } g,h \in \mathcal{H}_0^\infty(\mathcal{M}),\ \delta \in \mathcal{D},$$

where $\mathcal{H}_0^\infty(\mathcal{M}) = \{f \in \mathcal{H}^\infty(\mathcal{M}) : \mathbb{E}(f) = 0\}$. The Hilbert transform associated to the finite subdiagonal algebra $\mathcal{H}^\infty(\mathcal{M})$ is thus $H(f) = -ig + ih^*$. For these family of operators Randrianantoanina [74] proved their weak type (1,1), which after interpolation gives the optimal constant in terms of p. In the case of a discrete left-orderable group $\mathcal{L}\mathrm{G}$ with the subalgebra $\mathbb{C}\mathbf{1} \subseteq \mathcal{L}\mathrm{G}$ and the conditional expectation given by $f \mapsto \tau(f)\mathbf{1}$, we have that

$$\mathcal{H}_0^\infty(\mathcal{L}\mathrm{G}) = \left\{ f = \sum_{e \prec g} \widehat{f}(g)\, \lambda_g : f \in \mathcal{L}\mathrm{G} \right\}$$

is a finite subdiagonal algebra whose Hilbert transform coincides with the multiplier $-iT_{\mathrm{sgn}}$ in Theorem C. Thus, by Corollary 2.8, Cotlar identities can be used to recover previously known results with optimal constant.

5. Multipliers from Bass-Serre theory

The theory of discrete groups acting on trees (without edge inversions) is very well understood due to the work of Serre. The interested reader can find more on this topic in [79]. Here we are going to briefly explain how it is possible to use this theory to build examples of multipliers satisfying Cotlar's identity and to understand previously known examples like free group multipliers from [55, 56] under a geometric lens.

Given a graph of groups $\mathbb{X}$, which we will assume is connected, its *fundamental group* $\pi_1(\mathbb{X})$ can be defined in two ways, with the second one being more handy when constructing the Bass-Serre tree of $\mathbb{X}$.

(D.i) The first construction, denoted by $\pi_1(\mathbb{X},x_0)$ and briefly explained in the introduction, consists of the subgroup of $F(\mathbb{X})$ given by words of type c, where c is a closed path in X that starts and ends at the base point $x_0 \in \mathrm{Vert}(X)$.

(D.ii) For the second construction fix a *spanning tree* $T \subseteq X$, that is a tree containing all vertices of X. The fundamental group $\pi_1(\mathbb{X};T)$ can be constructed as

$$\pi_1(\mathbb{X};T) = F(\mathbb{X})/\langle y : y \in \mathrm{Edge}(T)\rangle,$$

where $\langle y : y \in \mathrm{Edge}(T)\rangle$ is the normal subgroup generated by all the edges y in the spanning tree.

The first definition is independent of the choice of x_0, while the second is independent of the choice of spanning tree T. Both are isomorphic, see [79], and we will usually denote the resulting group just by $\pi_1(\mathbb{X})$. Recall from the introduction that a word g of type c is in *normal form* if, informally, it can not be shortened applying (1.3). More formally

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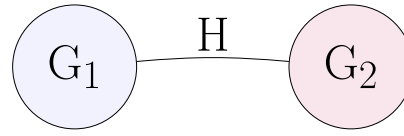


Figure 5. Amalgamated free product.

- Given a word $g = |c,r|$ of type c, it is in *normal form* if either $m = 0$ and $r_0 \neq e$ or, in the case in which $m \geq 1$, it holds that for every $1 \leq j \leq m-1$, $r_j \notin \mathrm{H}_{y_j}^{y_j}$ when $y_{j+1} = \overline{y}_j$.
- Let $|c,r|$ and $|c,\mu|$ be two words of type c with $r = (r_0, r_1, \ldots, r_m)$ and $\mu = (\mu_0, \mu_1, \ldots, \mu_m)$. They are said to be *equivalent* if $\mu_0 = r_0 a_1^{\overline{y}_1}$ and $\mu_j = (a_j^{y_j})^{-1} r_j a_{j+1}^{\overline{y}_{j+1}}$, where $a_j \in \mathrm{H}_{y_j}$ and $a_j^{y_j}$ and $a_j^{\overline{y}_j}$ are the corresponding images in $\mathrm{H}_{y_j}^{y_j}$ and $\mathrm{H}_{\overline{y}_j}^{\overline{y}_j}$.

We have that two words of type c in normal form are equivalent if and only if they represent the same element in $\pi_1(\mathbb{X},x_0)$. Observe also that any $g \in \pi_1(\mathbb{X},x_0)$ admits a unique normal form up to equivalence.

The fundamental group of a graph of groups unifies several constructions.

- **Homotopy groups of 1-dimensional simplicial complexes.** Any graph X is naturally a graph of groups by putting the trivial group in each vertex and edge. In this case $\pi_1(\mathbb{X})$ is naturally isomorphic to the (usual) fundamental group $\pi_1(X)$ of the associated 1-dimensional simplicial complex, see [77, Chapter 11].
- **Amalgamated free products.** Let X be a connected graph with two vertices x_1 and x_2 associated to groups G_1 and G_2 and connected by a single edge y to which a common subgroup H of G_1 and G_2 is attached. Then, the fundamental group of this graph of groups is the amalgamated free product $\pi_1(\mathbb{X}) = \mathrm{G}_1 *_{\mathrm{H}} \mathrm{G}_2$, see Figure 5, and when $\mathrm{H} = \{e\}$ the usual free product is obtained.
- **HNN extensions.** Let G be a group and H_1 and H_2 two isomorphic subgroups $\mathrm{H}_1, \mathrm{H}_2 \subseteq \mathrm{G}$. Fix an isomorphism $\theta : \mathrm{H}_1 \to \mathrm{H}_2$. The *HNN* extension of G relative to θ, see [77], is the smallest extension of G such that θ is implemented by an inner automorphism, i.e. $\theta(h) = t h t^{-1}$, for some t and every $h \in \mathrm{H}_1$. In the case of a group given by the presentation $\mathrm{G} = \langle S \,|\, R \rangle$, its HNN extension is

$$\big\langle S, t \,\big|\, R,\, t h t^{-1} = \theta(h) \big\rangle, \tag{5.1}$$

  where h runs through H_1. This construction can be obtained as the fundamental group of a graph of groups with a single vertex with group G and an edge y whose associated group is H_1 and the two inclusions are given by $\alpha_y(h) = h$ and $\alpha_{\overline{y}}(h) = \theta(h)$, see Figure 6.

Let $\mathbb{X}$ be a graph of groups whose underlying graph we will denote by X and let $\pi = \pi_1(\mathbb{X})$ be its fundamental group. We will recall the construction of the Bass-Serre tree $\widetilde{X}$ of $\mathbb{X}$ whose vertices are

$$\mathrm{Vert}(\widetilde{X}) = \bigsqcup_{x \in \mathrm{Vert}(X)} \pi/\mathrm{G}_x,$$

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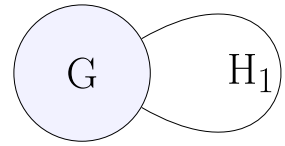


Figure 6. HNN extension.

while its edges are given by

$$\mathrm{Edge}_+(\widetilde{X}) = \bigsqcup_{y\in \mathrm{Edge}_+(X)} \pi/\mathrm{H}_{\bar{y}}^{\bar{y}} \quad \text{and} \quad \mathrm{Edge}_-(\widetilde{X}) = \bigsqcup_{y\in \mathrm{Edge}_-(X)} \pi/\mathrm{H}_{\bar{y}}^{\bar{y}},$$

where $\mathrm{Edge}_+(X)$ is a fixed orientation of the edges of X. In the definition above we are including only the edges of a fixed orientation of $\widetilde{X}$ induced by the orientation of X. We can define the endpoint of each edge as follows

$$\begin{array}{llll} \mathrm{o}(g\,\mathrm{H}_{\bar{y}}^{\bar{y}}) = g\mathrm{G}_{\mathrm{o}(y)}, & \mathrm{t}(g\mathrm{H}_{\bar{y}}^{\bar{y}}) = gg_y\mathrm{G}_{\mathrm{t}(y)} & \text{for } y\in \mathrm{Edge}_+(X),\, g\in\pi, \\ \mathrm{o}(g\mathrm{H}_{\bar{y}}^{\bar{y}}) = gg_y^{-1}\mathrm{G}_{\mathrm{o}(y)}, & \mathrm{t}(g\mathrm{H}_{\bar{y}}^{\bar{y}}) = g\mathrm{G}_{\mathrm{t}(y)} & \text{for } y\in \mathrm{Edge}_-(X),\, g\in\pi, \end{array} \tag{5.2}$$

where g_y is the image of y under the canonical projection from $F(\mathbb{X})$ to $\pi_1(\mathbb{X};T)$. The group π acts on $\widetilde{X}$ by left multiplication. The Bass-Serre tree construction also allows us to produce actions on trees for the fundamental group of any graph of groups. We will exploit this to obtain examples of Fourier multipliers satisfying Cotlar's identity.

We are ready to prove the main theorem of this section.

Proof of Theorem D. For (i), take $h\in \mathrm{G}_{x_0}$; then it is immediate that $m(gh)=m(g)$. Therefore, it is enough to prove the identity for $g,h\notin \mathrm{G}_{x_0}$. Assume that $m(gh)\neq m(g)$, since otherwise the identity holds. Let us write g in its normal form $g = r_0\,y_1\,r_1\,y_2\,r_2\cdots y_n\,r_n$ with $r_0\in g_0\cdot \mathrm{H}_{\bar{y}_1}^{\bar{y}_1}$. Since $m(gh)\neq m(g)$, h must admit a normal form $b = r_n^{-1}y_n^{-1}\cdots y_1^{-1}r$ with $r = r_0^{-1}\cdot r'$ for some $r'\in g_0'\cdot \mathrm{H}_{\bar{y}_1}^{\bar{y}_1}$. Here $g_0,g_0'\in \mathrm{G}_{x_0}$ and $g_0^{-1}g_0'\notin \mathrm{H}_{\bar{y}_1}^{\bar{y}_1}$. Moreover, since $r_n^{-1}y_n^{-1}\cdots y_1^{-1}r_0^{-1}$ is a normal form of g^{-1}, by the definition of m, it is clear that $m(g^{-1})=m(h)$. For (ii), notice that the left action $\mathrm{G}_{x_0}\curvearrowright \mathrm{G}_{x_0}/\mathrm{H}_{\bar{y}}^{\bar{y}}$ is transitive for each y, therefore a left G_{x_0}-invariant function has to be constant in each of the elements of the disjoint union W. □

Observe that, by the definition of the Bass-Serre tree of $\mathbb{X}$ in equation (5.2), the set of edges in $\widetilde{X}$ connected to the vertex $\widetilde{x}_0 = e\cdot \mathrm{G}_{x_0}\in \mathrm{Vert}(\widetilde{X})$ is given, up to orientation, by the disjoint union of $\mathrm{G}_{x_0}/\mathrm{H}_{\bar{y}}^{\bar{y}}$, where, again, y runs over every edge in X starting with x_0. This already establishes that the functions depending only on the initial segment and the functions that are constant on the connected components of $\widetilde{X}\setminus\{\widetilde{x}_0\}$ are in a natural and bijective correspondence. The following proposition, whose proof is immediate, asserts that m falls into Model 1. This provides an alternative proof of Theorem D via Proposition 3.1.(i).

Proposition 5.1. *Let $\pi=\pi_1(\mathbb{X},x_0)$ be the fundamental group of a connected graph of groups $\mathbb{X}$ and let $\widetilde{X}$ be its Bass-Serre tree.*

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(i) *Every symbol* $m : \pi \to \mathbb{C}$ *depending on the first segment satisfies*

$$m(g) = \widetilde{m}(g \cdot \widetilde{x}_0),$$

where $\widetilde{m} : \widetilde{X} \to \mathbb{C}$ *is the function constant over the connected components of* $\widetilde{X} \setminus \{\widetilde{x}_0\}$ *that is naturally associated to (1.6) and* $\widetilde{x}_0$ *is a vertex of* $\widetilde{X}$ *labelled by* G_{x_0}.

(ii) *If furthermore* m *depends only on the edge of the starting segment, then* $\widetilde{m} : \widetilde{X} \setminus \{\widetilde{x}_0\} \to \mathbb{C}$ *has the same value over any two connected components* $\widetilde{X}_\alpha$ *and* $\widetilde{X}_\beta$ *such that* $r \cdot \widetilde{X}_\alpha = \widetilde{X}_\beta$, *for some* $r \in \mathrm{G}_{x_0}$.

In the case in which G_{x_0} is Abelian, the more restrictive condition that $m : \pi \to \mathbb{C}$ depends on the initial edge can be removed and a multiplier theorem still holds for symbols depending on the starting segment. Let m be a symbol depending on the starting segment and $\widetilde{m} : W_{x_0} \to \mathbb{C}$ be its lift to the space of initial segments defined in (1.5). We will denote by $\widetilde{m}_y$ the restriction of $\widetilde{m}$ to $\mathrm{G}_{x_0}/\mathrm{H}_{\bar{y}}^{\bar{y}}$. We have the following result.

Theorem 5.2. *Let* $\pi = \pi_1(\mathbb{X}, x_0)$ *be the fundamental group of a graph of groups with* G_{x_0} *Abelian. In order to lighten the notation, let us denote* $\mathrm{G}_{x_0}/\mathrm{H}_{\bar{y}}^{\bar{y}}$ *by* G_y. *For each* $y \in \mathrm{Edge}(X)$ *with* $\mathrm{o}(y) = x_0$ *and each character* $\chi : \mathrm{G}_y \to \mathbb{T}$ *we define the multiplier* $\sigma_{y,\chi}$ *given by* $\sigma_{y,\chi}(g) = \chi(r_0 \mathrm{H}_{\bar{y}}^{\bar{y}})$, *whenever* g *is represented by a word starting by* $r_0\, y$, *and* 0 *otherwise. The following results hold*

(i) *The map* $\rho_y : L_p(\mathcal{L}\pi) \to L_p(\widehat{\mathrm{G}}_y; L_p(\mathcal{L}\pi))$, *defined by* $f \mapsto (T_{\sigma_{y,\chi}}(f))_{\chi \in \widehat{\mathrm{G}}_y}$ *satisfies*

$$\|f\|_p \le \big\|\rho_y(f)\big\|_{L_p(\widehat{\mathrm{G}}_y; L_p(\mathcal{L}\pi))} \lesssim \Big(\frac{p^2}{p-1}\Big)^\beta \|f\|_p, \quad \text{where } \beta = \log_2(1+\sqrt{2}),$$

(ii) *For a general symbol* $m : \pi \to \mathbb{C}$ *depending on the starting segment, it holds that*

$$\big\|T_m : L_p(\mathcal{L}\pi) \to L_p(\mathcal{L}\pi)\big\| \lesssim \Big(\frac{p^2}{p-1}\Big)^\beta \max_{\mathrm{o}(y)=x_0} \Big\{\big\|T_{\widetilde{m}_y} : L_p(\widehat{\mathrm{G}}_y) \to L_p(\widehat{\mathrm{G}}_y)\big\|_{\mathrm{cb}}\Big\}.$$

Proof. The proof is almost immediate. For (i) first notice that $m_{y,\chi}$ depends on the starting segment and that, given $a \in \mathrm{G}_{x_0}$, $\sigma_{y,\chi}$ satisfies that $\sigma_{y,\chi}(a\,g) = \chi(a)\,\sigma_{y,\chi}(g)$. Thus, since it satisfies ($\widehat{\text{Cotlar}}$) relative to G_{x_0}, by Remark 2.6 we have that it is a bounded multiplier in L_p for $1 < p < \infty$ satisfying the bound (1.1) in Theorem A. The fact that $\widehat{\mathrm{G}}_y$ is compact implies that ρ_y is bounded with the same bound as $T_{\sigma_{y,\chi}}$. For the lower bound, notice that, since the Haar measure of $\widehat{\mathrm{G}}_y$ is a probability measure, $\mathbf{1}_{\widehat{\mathrm{G}}_y}$ is of norm one on $L_{p'}(\widehat{\mathrm{G}}_y)$. But evaluating against it gives

$$\big\|\rho_y(f)\big\|_{L_p(\widehat{\mathrm{G}}_y; L_p(\mathcal{L}\pi))} \ge \Big\| \int_{\widehat{\mathrm{G}}_y} T_{m_{y,\chi}}(f)\, d\widehat{\mu}(\chi) \Big\|_{L_p(\mathcal{L}\pi)} = \|f\|_p.$$

This shows that ρ_y is a quasi-isometry, which gives (i). For (ii), just notice that if m is a symbol depending on the starting segment and that is 0 outside G_y, then the following

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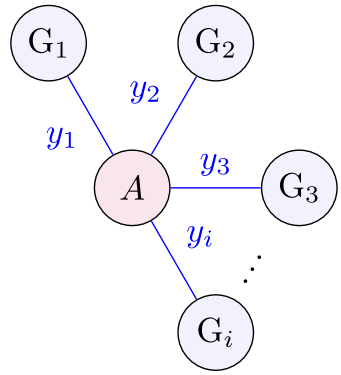


Figure 7. Graph of groups for the amalgamated free product.

diagram commutes

$$\begin{array}{ccc} L_p(\mathcal{L}\pi) & \xrightarrow{\quad\rho_y\quad} & L_p(\widehat{\mathrm{G}}_y; L_p(\mathcal{L}\pi)) \\ \Big\downarrow{\scriptstyle T_m} & & \Big\downarrow{\scriptstyle T_{\tilde{m}_y}\otimes \mathrm{id}} \\ L_p(\mathcal{L}\pi) & \xrightarrow{\quad\rho_y\quad} & L_p(\widehat{\mathrm{G}}_y; L_p(\mathcal{L}\pi)) \end{array}$$

and together with point (i), this yields the result. In the general case of m depending on the starting segment, we can decompose it as a sum of symbols $\widetilde{m}|_{\mathrm{G}_y}$ and build a direct sum map

$$\rho : L_p(\mathcal{L}\pi) \longrightarrow \bigoplus_{\mathrm{o}(y)=x_0} L_p(\widehat{\mathrm{G}}_y; L_p(\mathcal{L}\pi))$$

by taking ρ_y in each of the components. □

Now, we have all the tools required to illustrate our theory with the examples.

Examples: Amalgamated free products. This first example is already known, but our proof gives a concise geometric interpretation of it. Let $\{\mathrm{G}_i : i \geq 1\}$ be a family of groups and A a common subgroup. Let us denote the injective homomorphisms by $\alpha_i : A \to \mathrm{G}_i$, for $i \geq 1$. Now, consider the graph of groups $\mathbb{X}$ in Figure 7, whose underlying tree X connects the point x_0, with $\mathrm{G}_{x_0} = A$, with the points x_i, $i \geq 1$, whose groups are G_i. Meanwhile, the groups on the edges y_i are all isomorphic to A with the embedding into $\mathrm{G}_{x_0} = A$ being the identity map and the embedding into G_i being α_i.

In this case, we have that $\pi = \pi_1(\mathbb{X}, x_0)$ is the free product of $\{\mathrm{G}_i : i \geq 1\}$ with A as the amalgam $\pi = *_{i,A}\mathrm{G}_i$, see [79, Theorem 9]. Functions $m : \pi \to \mathbb{C}$ that depend on the starting edge in this context are given by the span of $\{L_i\}_{i\geq 1}$, where each L_i is the Fourier multiplier associated to the characteristic function of the words starting by $a\, s_i$, where $s_i \in \mathrm{G}_i$. These operators are an example of the *free Hilbert transforms* that have been studied in the pioneering work of Mei and Ricard [55]. Note that free groups can be obtained from other graphs of groups. Let us pick the free group of two generators $\mathbb{F}_2$

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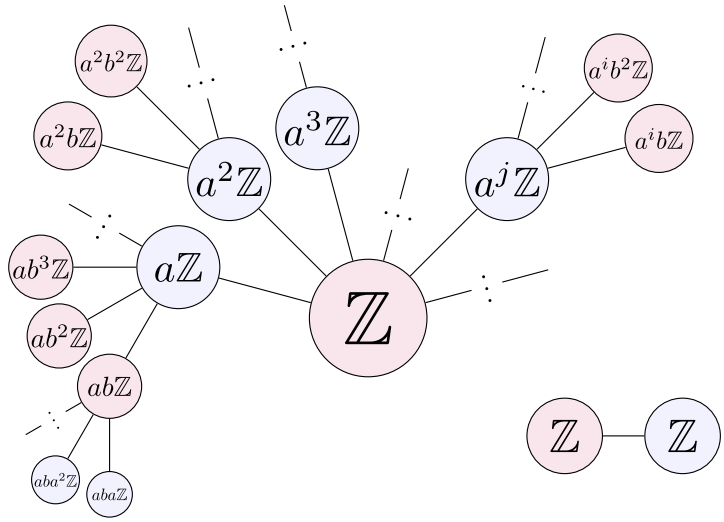


Figure 8. Graph of groups whose fundamental group is $\mathbb{F}_2$ (in the corner) and its Bass-Serre tree. For readability, we have omitted branches going from the root towards vertices labeled $a^k\mathbb{Z}$ with $k \leq 0$.

for simplicity. We can represent $\mathbb{F}_2$ as the fundamental group of the graph of groups $\mathbb{X}$ given by two vertices $\{x_0, x_1\}$ whose associated groups are isomorphic to $\mathbb{Z}$ and joined by a single edge with trivial group in it.

Let us denote the generator associated to x_0 by a and the generator associated to x_1 by b. Since $\mathbb{Z} = \langle a \rangle$ is Abelian, we can apply Theorem 5.2. Indeed, functions depending on the starting segment are of the form $m(s_1^{n_1} s_2^{n_2} \cdots s_r^{n_r}) = \widetilde{m}(n_1)$ if $s_1 = a$ and 0 otherwise. Changing the roles of x_0 and x_1 and combining the two multipliers obtained, we get symbols of the form

$$m\big(s_1^{n_1} s_2^{n_2} \cdots s_r^{n_r}\big) = \begin{cases} \widetilde{m}_1(n_1), & \text{if } \ s_1 = a \\ \widetilde{m}_2(n_1), & \text{if } \ s_1 = b. \end{cases} \tag{5.3}$$

Since m above is the sum of two symbols satisfying the hypothesis of Theorem 5.2(ii), we have that

$$\big\|T_m : L_p(\mathcal{L}\mathbb{F}_2) \to L_p(\mathcal{L}\mathbb{F}_2)\big\| \lesssim \big(p \cdot p'\big)^{\beta} \max_{i=1,2} \Big\{ \big\|T_{\widetilde{m}_1} : L_p(\mathbb{T}) \to L_p(\mathbb{T})\big\|_{\mathrm{cb}} \Big\},$$

where p' is the conjugate exponent $\frac{1}{p} + \frac{1}{p'} = 1$. With this, we partially recover the multiplier theorems for initial segments of the free group in [56]. Our approach has the advantage of giving a geometric interpretation as follows. Consider the Bass-Serre tree $\widetilde{X}$ of this graph of groups, see Figure 8; it is easy to see that group elements in $\mathbb{F}_2$ can only map pink vertices to pink vertices and blue ones to blue ones. Denote the set containing the blue vertices in the first layer and the pink vertex labeled by $\mathbb{Z}$ in the middle by B_1. It is straightforward to realize that any function on $\widetilde{X}$ that is constant along the connected components of $\widetilde{X} \setminus B_1$ induces a symbol of the form (5.3).

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The fact that the starting letter induces bounded multipliers suggests the enticing possibility of simultaneously generalizing our results for the starting segment in Theorem D and 5.2 and the main result in [56] to multipliers depending on the starting k-word of the normal form in the context of graphs of groups.

Examples: HNN extensions. Suppose H_1 and $\mathrm{H}_2 \subseteq \mathrm{G}$ are two subgroups isomorphic under $\theta : \mathrm{H}_1 \to \mathrm{H}_2$ and let π be the HNN-extension of G relative to θ, given in (5.1). Observe that the HNN-extension can be written as $\pi = H \rtimes T$, the semi-direct product of the infinite cyclic group T generated by t and the normal subgroup H generated by $t^n \mathrm{G} t^{-n}$, $n \in \mathbb{Z}$. In particular, π has a quotient isomorphic to $\mathbb{Z}$.

We have already seen, see Figure 6, that $\pi = \pi_1(\mathbb{X}, x_0)$ is the fundamental group of a graph of groups based on a single loop. By the definition of fundamental group, every element of π is represented by a word $r_0, t^{e_1} r_1 \cdots t^{e_k}, r_k$ with $k \geq 0$, $e_i = \pm 1$ and $r_i \in \mathrm{G}$. A word in this form will be reduced if it contains neither a substring of the form tat^{-1} with $a \in \mathrm{H}_1$ nor one of the form $t^{-1} b t$ with $b \in \mathrm{H}_2$. Fix coset representatives of G/H_1 and G/H_2; for any $g \in \pi$, there exists a unique reduced word such that

$$g = r_0 \, t^{e_1} \, r_1 \cdots t^{e_k} \, r_k \tag{5.4}$$

with $r_0 \in \mathrm{G}$, $r_i \in \mathrm{G}/\mathrm{H}_1$ if $e_i = 1$, and $r_i \in \mathrm{G}/\mathrm{H}_2$ if $e_i = -1$. We are going to give two algebraic forms for Fourier multipliers satisfying ($\mathrm{Cotlar}_{\mathbb{E}\perp}$), one falling within Model 1 and another within Model 2 by considering the action of the HNN-extension π on its Bass-Serre tree $\widetilde{X}$ by left multiplication. For the first one, we choose the root $\widetilde{x}_0$ to be the vertex labeled by G. Note that if we set the orientation $\mathrm{Edge}_+(X) = \{t\}$, then in the induced orientation of $\widetilde{X}$ there are $[\mathrm{G} : \mathrm{H}_2]$ many edges starting at $\widetilde{x}_0$ and $[\mathrm{G} : \mathrm{H}_1]$ many edges ending in $\widetilde{x}_0$. It is immediate that a function $m : \pi \to \mathbb{C}$ depends on the starting segment iff there is a function $\widetilde{m} : (\mathrm{G}/\mathrm{H}_1) \sqcup (\mathrm{G}/\mathrm{H}_2) \to \mathbb{C}$ such that

$$m(g) = \begin{cases} \widetilde{m}_1\big(r_0 \cdot \mathrm{H}_1\big) & \text{when } \ e_1 = +1 \\ 0 & \text{when } g \in \mathrm{G} \\ \widetilde{m}_2\big(r_0 \cdot \mathrm{H}_2\big) & \text{when } \ e_1 = -1, \end{cases} \tag{5.5}$$

where $\widetilde{m}_i$ is the restriction of $\widetilde{m}$ to G/H_i for $i \in \{1,2\}$ and g is equal to its normal form like in (5.4). By Proposition 5.1, we have that m is left G-invariant if and only if it depends only on the first edge in the normal form (5.4), which can only be t or t^{-1} in our setting, ie:

$$m(g) = C_1 \mathbf{1}_{\{e_1 = +1\}} + C_2 \, \mathbf{1}_{\{e_1 = +1\}} \tag{5.6}$$

Observe that, if G is Abelian, we can deal with the boundedness of (5.5).

Now, we can use the definition in Model 2 to the action $\pi \curvearrowright \widetilde{X}$. Choose $\widetilde{x}_0$ as the root and let $\widetilde{X}_0 \subseteq \widetilde{X}$ be the connected component separated from the root by the edge H_1. It is easy to check the vertices in the connected component of H_2 always take the form $g \cdot \mathrm{G}$ with the expression (5.4) of g starting with t. Hence we get the following symbol

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$\varphi : \pi \to \mathbb{C}$:

$$\varphi(g) = \begin{cases} 0 & \text{if } g \in \mathrm{H}_2 \subseteq \mathrm{G} \\ D_1 & \text{if } r_0 = e \text{ and } e_1 = 1 \\ D_2 & \text{otherwise,} \end{cases} \tag{5.7}$$

where D_1 and D_2 are two different constants.

Corollary 5.3. *Let π be the HNN extension of G with respect to θ as before.*

(i) *Let $m : \pi \to \mathbb{C}$ be like in (5.6). Since m depends on the starting edge, T_m is L_p-bounded for $1 < p < \infty$ and satisfies bound (1.1) by Theorem D.*

(ii) *If G is Abelian, and the symbol m in (5.5) lifts to $\widetilde{m} : (\mathrm{G}/\mathrm{H}_1) \sqcup (\mathrm{G}/\mathrm{H}_2) \to \mathbb{C}$, then, by Theorem 5.2, it holds that*

$$\big\|T_m : L_p(\mathcal{L}\pi) \to L_p(\mathcal{L}\pi)\big\| \lesssim \Big(\frac{p^2}{p-1}\Big)^{\beta} \max_{i=1,2} \Big\{ \big\|T_{m_i} : L_p(\widehat{\mathrm{G}/\mathrm{H}_i}) \to L_p(\widehat{\mathrm{G}/\mathrm{H}_i})\big\|_{\mathrm{cb}} \Big\}.$$

(iii) *Let $\varphi : \pi \to \mathbb{C}$ be as in (5.7). Then T_φ, by Theorem 2.4, satisfies (Cotlar$_{\mathbb{E}_\perp}$) relative to $\mathrm{H}_2 \subseteq \mathrm{G}$ and is left $\mathcal{L}\mathrm{H}_2$-modular. Thus T_φ is bounded in L_p with bound (1.1).*

The result above can be illustrated in the particular case of the Baumslag-Solitar group

$$\mathrm{BS}(n,m) = \big\langle r, t \,\big|\, t\, r^m\, t^{-1} = r^n \big\rangle, \text{ where } 1 < n \neq m \in \mathbb{N},$$

which can be seen as a HNN extension of $\mathbb{Z} = \langle r \rangle$ with respect to the map θ, which sends $r^{mk} \mapsto r^{nk}$ and establishes an isomorphism between the subgroups $m\mathbb{Z}$ and $n\mathbb{Z} \subseteq \mathbb{Z}$. It holds that $\mathbb{Z}/n\mathbb{Z} \cong \mathbb{Z}_n$ and $\mathbb{Z}/m\mathbb{Z} \cong \mathbb{Z}_m$. We can take representatives $\{0, 1, \dots n-1\}$ and $\{0, 1, \dots m-1\}$ of the quotients. The unique normal form of g is given by

$$g = r^{k_0} t^{e_1} r^{k_1} \cdots t^{e_m} r^{k_m} \quad \text{ with } k_0 \in \mathbb{Z}, e_1, \dots e_j \in \{\pm 1\} \tag{5.8}$$

if $e_j = 1$, then $k_j \in \mathbb{Z}/m\mathbb{Z}$ for $j \geq 1$. If $e_j = -1$, then $k_j \in \mathbb{Z}/n\mathbb{Z}$ for $j \geq 1$. Multipliers that depend on the starting segment $\varphi : \pi \to \mathbb{C}$ lift to a function $\widetilde{\varphi} : (\mathbb{Z}/n\mathbb{Z}) \sqcup (\mathbb{Z}/m\mathbb{Z}) \to \mathbb{C}$ as in (5.5). Using the fact that $\mathbb{Z}$ is Abelian and Corollary 5.3 gives

$$\big\|T_\varphi : L_p(\mathcal{L}\mathrm{BS}(n,m)) \to L_p(\mathcal{L}\mathrm{BS}(n,m))\big\| \lesssim \max\{n,m\}^{\left|\frac{1}{2} - \frac{1}{p}\right|} \big(p \cdot p'\big)^{\beta} \|\varphi\|_\infty,$$

where we have also used the fact that the multiplier norm of $\varphi : \mathbb{Z}/\ell\mathbb{Z} \to \mathbb{C}$ on $L_p(\mathbb{Z}/\ell\mathbb{Z})$ is bounded by $\ell^{\left|\frac{1}{2} - \frac{1}{p}\right|} \|\varphi\|_\infty$. Now, let us explore multipliers on $\mathrm{BS}(n,m)$ coming from Model 2 as in (5.7). Let $\mathbb{X}$ be the single loop graph of groups associated with $\mathrm{BS}(n,m)$ and $\widetilde{X}$ be its Bass-Serre tree depicted in Figure 9. The two directions of the loop give us the two subgroups $m\mathbb{Z}, n\mathbb{Z}$, and by definition of the Bass-Serre tree you get that $\widetilde{X} \setminus \{\widetilde{x}_0\}$ has $n+m$ connected components, see Figure 9. Now, we select the connected component of the vertex $t\mathrm{G}$. The symbol obtained has a value that depends on whether the normal form (5.8) starts with t or not.

Observe that $\mathrm{BS}(n,m)$ has an Abelian quotient $q : \mathrm{BS}(n,m) \to \mathbb{Z}$ given by killing the 'base group' $\langle r \rangle$, i.e. $q(t) = 1$ and $q(r) = 0$. Composing q with a sign gives an example of a multiplier satisfying Cotlar's identity. Nevertheless, the multipliers studied above,

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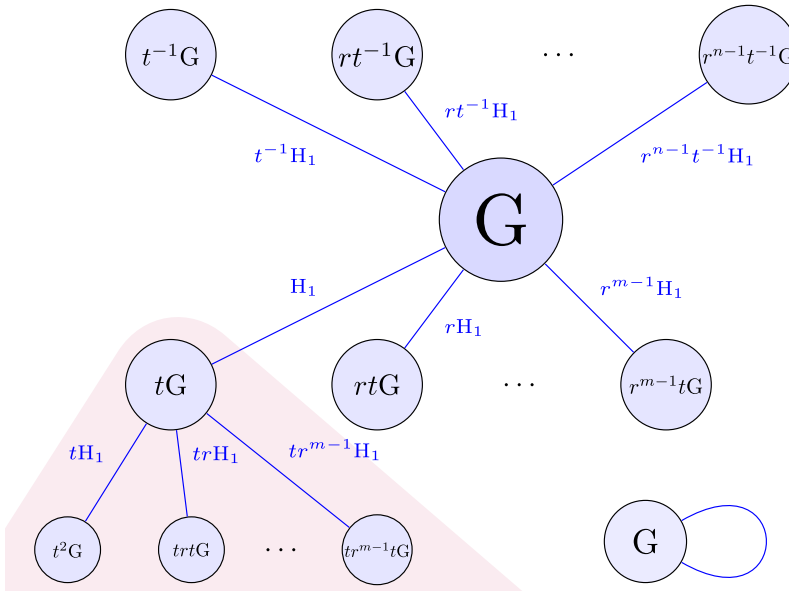


Figure 9. Graph of groups whose fundamental group is the Baumslag-Solitar group (in the corner) and its Bass-Serre tree. Here $\mathrm{G} = \mathbb{Z}$ and $\mathrm{H}_1 = n\mathbb{Z} \subseteq \mathbb{Z}$.

(5.5) and (5.7) do not arise from the signs of this Abelian quotient $\mathbb{Z}$. The following was pointed out to us by the referee.

Remark 5.4. When m and n have a common factor $d \in \mathbb{Z}$, $\mathrm{BS}(n,m)$ can be written as an amalgamated free product

$$\mathrm{BS}(n,m) = \mathrm{BS}\Big(\frac{n}{d}, \frac{m}{d}\Big) *_{\mathbb{Z}} \mathbb{Z},$$

where $\mathrm{BS}(\frac{n}{d}, \frac{m}{d}) = \langle s,t \,\big|\, ts^{\frac{m}{d}}t^{-1} = s^{\frac{n}{d}} \rangle$, the amalgam $\mathbb{Z}$ is identified with $\langle s \rangle$ in $\mathrm{BS}(\frac{n}{d}, \frac{m}{d})$ and with $d\mathbb{Z}$ in $\mathbb{Z}$. Then in this case we also have a free Hilbert transform on $\mathrm{BS}(n,m)$ whose symbol takes the form:

$$m(g) = C_1 \mathbf{1}_{\{k_0 \equiv 0 \pmod d\}} + C_2 \mathbf{1}_{\{k_0 \not\equiv 0 \pmod d\}},$$

which is different from the multiplier formulas in (5.6) and (5.7).

Examples with Serre's Property (FA). We will present here the examples of groups having multipliers satisfying Cotlar's identity and fitting into our Model 1 while having Serre's property (FA). Recall that a group has Serre's property (FA) – the A stemming from the French word for tree, *arbre* – if and only if every orientation-preserving isometric action on a simplicial tree has global fixed point. This property admits a closed characterization. Indeed, a countable group G has property (FA) if and only if the following conditions are satisfied: G is not an amalgamated free product, G has no quotient isomorphic to $\mathbb{Z}$ and G is finitely generated [79, Theorem 15].

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First, we will give an example of a left-orderable group with property (FA). This gives an example for which Cotlar's identity was previously unknown. On the other hand, its associated Hilbert transform can be proven to be of weak type (1,1) due to the theory of Hilbert transforms on finite subdiagonal algebras [74].

From the presentation of the von Dyck group $D(2,3,7) = \langle x,y \,|\, x^2 = y^3 = (xy)^7 = 1\rangle$, it is easy to see that $D(2,3,7)$ is a quotient of the modular group $\mathrm{PSL}(2,\mathbb{Z})$. Since it is isomorphic to a discrete subgroup of $\mathrm{PSL}(2,\mathbb{R})$, we may consider its lifting to the universal cover of $\mathrm{PSL}(2,\mathbb{R})$. We denote this lifting by Γ, and it has the presentation

$$\Gamma = \langle x,y,z \,|\, x^2 = y^3 = z^7 = xyz\rangle.$$

This group was studied in [4] as the first example of an orderable group which is non-locally indicable.

Proposition 5.5. *The group* Γ *has property* (FA) *and is left-orderable. Therefore, its sign Hilbert transform* $H = T_{\mathrm{sgn}}$ *satisfies (Cotlar*$_{\mathbb{E}\perp}$*) and thus* $\|H : L_p(\mathcal{L}\Gamma) \to L_p(\mathcal{L}\Gamma)\| < \infty$ *for* $1 < p < \infty$.

Proof. Since the kernel of the covering homomorphism is isomorphic to $\mathbb{Z}$, we have a short exact sequence

$$1 \to \mathbb{Z} \to \Gamma \to D(2,3,7) \to 1.$$

Note that Γ is perfect [4], i.e., it does not contain any nontrivial Abelian quotient; then [16, Proposition 3.2] tells us that Γ has property (FA) if and only if $D(2,3,7)$ has property (FA). Since $D(2,3,7)$ has property (FA), see [79, p. 61], we deduce Γ also has property (FA). Moreover, the action of $\mathrm{PSL}(2,\mathbb{R})$ by Möbius transformations on the circle lifts to an action of $\widetilde{\mathrm{PSL}}(2,\mathbb{R})$ on $\mathbb{R}$ by orientation-preserving homeomorphisms, so in this way, $\widetilde{\mathrm{PSL}}(2,\mathbb{R})$ embeds into $\mathrm{Homeo}_+(\mathbb{R})$. Therefore, Γ is also left-orderable. Applying Theorem C, we conclude that Γ admits a nontrivial Hilbert transform. □

Recall that this gives a new example of a group multiplier satisfying Cotlar's identity, but not a new example of a group multiplier being L_p bounded since it falls within the theory described in [74]. In order to obtain new examples outside both the classical theory, the theory of free Hilbert transforms [55] and the theory of subdiagonal algebras, we will need an example of a group with property (FA) acting without global fixed points on an $\mathbb{R}$-tree other than $\mathbb{R}$ itself. There are a few examples in that direction in the literature; see [57], but the complexity of the constructions makes finding explicit formulas for the multipliers more involved.

6. Hilbert transforms over lattices of $\mathrm{SL}_2(\mathbb{R})$ and $\mathrm{SL}_2(\mathbb{C})$

The modular group $\mathrm{PSL}_2(\mathbb{Z})$ is a discrete subgroup of $\mathrm{PSL}_2(\mathbb{R})$. Over $\mathrm{PSL}_2(\mathbb{R})$ there is a very natural multiplier symbol, playing a role analogous to that of the Hilbert transform in $\mathbb{R}$. This is given by equation (1.7). In the statement of the following theorem we will abuse our notation and write the elements in $\mathrm{PSL}_2(\mathbb{R})$, which are classes of matrices up

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to a sign, simply by matrices. We will also denote by S and T the matrices

$$S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \quad \text{and} \quad T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}.$$

It holds that $S^2 = -\mathrm{id}$ and $(ST)^3 = -\mathrm{id}$, which in the quotient group gives that $S^2 = \mathrm{id}$ and $(ST)^3 = \mathrm{id}$. We have the following result.

Proposition 6.1. *Let $m|_{\mathrm{PSL}_2(\mathbb{Z})} : \mathrm{PSL}_2(\mathbb{Z}) \to \mathbb{C}$ be the restriction of (1.7) to the modular group. Then m satisfies (Cotlar$_{\mathbb{E}_\perp}$) with respect to the subgroup $\mathrm{G}_0 = \{\mathrm{id}, S\}$ and as a consequence*

$$\big\| T_m : L_p(\mathcal{L}\mathrm{PSL}_2(\mathbb{Z})) \to L_p(\mathcal{L}\mathrm{PSL}_2(\mathbb{Z})) \big\| \lesssim \Big(\frac{p^2}{p-1} \Big)^{\beta} \quad \text{where } \beta = \log_2(1+\sqrt{2}).$$

Proof. First notice that $m : \mathrm{PSL}_2(\mathbb{R}) \to \mathbb{C}$ is right K-invariant, with $K = \mathrm{PSO}(2)$. Similarly, when G_0 acts on the left of m we have that

$$m(S^k g) = (-1)^k m(g), \quad \text{for } g \in \mathrm{PSL}_2(\mathbb{Z}), k \in \mathbb{Z}$$

which means that m is left G_0-invariant with respect to a character $\mathrm{G}_0 \to \mathbb{T}$ and Remark 2.6 applies. We need to prove that $m(gh) = m(g)$ or $m(g^{-1}) = m(h)$ for $g \in \mathrm{PSL}_2(\mathbb{Z}) \setminus \mathrm{G}_0$ and $h \in \mathrm{PSL}_2(\mathbb{Z}) \setminus \{\mathrm{id}\}$. Without loss of generality, we can assume that h lives in the larger group $\mathrm{PSL}_2(\mathbb{R})$ and use the right K-invariance of m to assume that

$$h = \begin{pmatrix} \sqrt{y} & \frac{x}{\sqrt{y}} \\ 0 & \sqrt{y}^{-1} \end{pmatrix} \quad \text{and} \quad g = \begin{pmatrix} a & b \\ c & d \end{pmatrix},$$

for some $x, y \in \mathbb{R}$ with $y > 0$ and $a, b, c, d \in \mathbb{Z}$. We are going to use the following elementary identities

$$\begin{aligned} m(g) &= \mathrm{sgn}(ac + bd), \\ m(g^{-1}) &= \mathrm{sgn}(-dc - ab), \\ m(h) &= \mathrm{sgn}(x), \\ m(gh) &= \mathrm{sgn}\big(ac(x^2+y^2) + (ad+bc)x + bd\big). \end{aligned}$$

If $m(g^{-1}) = m(h)$, then Cotlar's identity is already fulfilled. Thus, we consider the case when $m(g^{-1}) \neq m(h)$, which is equal to saying that $x(-ab - dc) < 0$. On the other hand, since $ad - bc = 1$ and $a, b, c, d \in \mathbb{Z}$, we get $abcd = bc + (bc)^2 \geq 0$. We also get that $x(ab(d^2 + c^2) + dc(a^2 + b^2)) \geq 0$. This observation gives us the following result.

$$\begin{aligned} &\big(ac(x^2+y^2) + (ad+bc)x + bd\big)(ac+bd) \\ &\quad = (a^2c^2 + abcd)(x^2+y^2) + \big(ab(d^2+c^2) + dc(a^2+b^2)\big)x + abcd + b^2d^2 \geq 0. \end{aligned}$$

This translates to $m(gh) = m(g)$, which gives the result. □

The proposition above also admits a geometric proof. As discussed in the introduction, $\mathrm{PSL}_2(\mathbb{R})$ acts faithfully and transitively by isometries on the Poincaré plane $\mathbb{H}$ via Möbius transformations. As a lattice of $\mathrm{PSL}_2(\mathbb{R})$, $\mathrm{PSL}_2(\mathbb{Z})$ naturally acts on $\mathbb{H}$. Fix two points

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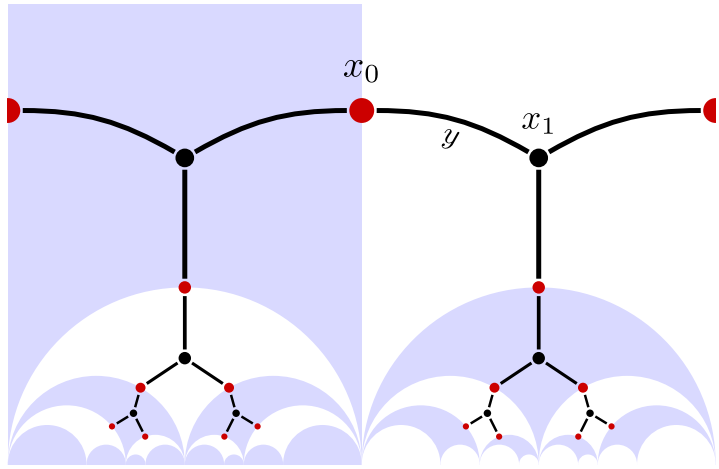


Figure 10. Bass-Serre tree of $\mathrm{PSL}_2(\mathbb{Z})$ inside the hyperbolic plane.

$x_0 = i$ and $x_1 = e^{\frac{\pi}{3}i} = (1+\sqrt{-3})/2$ on $\mathbb{H}$, and let y be the geodesic arc joining them. Since $\mathrm{PSL}_2(\mathbb{Z})$ acts on both arcs and points, we will define X to be the union of all translates of $g \cdot y$ for $g \in \mathrm{PSL}_2(\mathbb{Z})$. This union of segments forms a tree, see Figure 10. It is easy to see that the stabilizer of x_0 is isomorphic to $\mathbb{Z}_2$ and that of x_1 is isomorphic to $\mathbb{Z}_3$. Indeed, the stabilizer of x_0 is generated by the order 2 inversion $z \mapsto -1/z$, while the stabilizer of x_1 is generated by the order 3 map $z \mapsto (z-1)/z$. An elementary computation shows that no nontrivial Möbius transformation with integer coefficients sends x_0 into x_1. Therefore, the stabilizer of the edge y is trivial. By the Bass-Serre theory reviewed in Section 5 it is immediate that $\mathrm{PSL}_2(\mathbb{Z}) \cong \mathbb{Z}_2 * \mathbb{Z}_3$. Given that $X \setminus \{x_0\}$ admits two connected components, one lying in the left quadrant and the other in the right, it follows that the multiplier defined in (1.7), when restricted to $\mathrm{PSL}_2(\mathbb{Z})$, fits within Model 1, with the tree given in Figure 10.

The case of $\mathrm{PSL}_2(\mathcal{O}_{-1})$. The purpose of this subsection is to prove Theorem E. Recall that $\mathrm{PSL}_2(\mathcal{O}_{-1}) \subseteq \mathrm{PSL}_2(\mathbb{C})$ is a lattice; we will denote such group by Γ_1. Recall as well that $\mathrm{PSL}_2(\mathbb{C})$ is semisimple and that any element admits a unique KAN-decomposition with $K = \mathrm{PSU}(2)$, A the Abelian group of real diagonal matrices and N the nilpotent, in fact Abelian, group of upper triangular complex matrices with ones on the diagonal. More concretely, we have that

$$\underbrace{\begin{pmatrix} a & b \\ c & d \end{pmatrix}}_{g} = \underbrace{\begin{pmatrix} \alpha & \beta \\ -\bar{\beta} & \bar{\alpha} \end{pmatrix}}_{\in K} \underbrace{\begin{pmatrix} s & 0 \\ 0 & s^{-1} \end{pmatrix}}_{\in A} \underbrace{\begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}}_{\in N}, \quad \text{where } s = \sqrt{|a|^2+|c|^2} \text{ and } t = \frac{\bar{a}b+\bar{c}d}{|a|^2+|c|^2}.$$

Let $g \in \Gamma_1$ be a matrix of the form

$$g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a_1 + i\,a_2 & b_1 + i\,b_2 \\ c_1 + i\,c_2 & d_1 + i\,d_2 \end{pmatrix}. \tag{6.1}$$

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The condition that the determinant equals 1 implies that

$$a_1d_1 + b_2c_2 - (b_1c_1 + a_2d_2) = 1, \tag{6.2}$$

$$a_1d_2 + a_2d_1 - b_1c_2 - b_2c_1 = 0. \tag{6.3}$$

Using the two identities above, we can prove elementarily the following key inequality.

Lemma 6.2. *Let $g \in \Gamma_1$ be as in (6.1). It holds that*

$$0 \le \big(a_1d_1 + b_2c_2\big)\big(b_1c_1 + a_2d_2\big) \le \operatorname{Re}(a\bar{c})\operatorname{Re}(b\bar{d}) = \big(a_1c_1 + a_2c_2\big)\big(b_1d_1 + b_2d_2\big)$$

Proof. For the first inequality, let $A = a_1d_1 + b_2c_2$ and $B = b_1c_1 + a_2d_2$. By equation (6.2) we have $A = 1 + B$. But this implies that $AB = B^2 + B$. Since B is an integer $AB \ge 0$. For the second inequality, it is enough to show that $0 \le (\mathrm{I}) - (\mathrm{II})$, where $(\mathrm{I}) = (a_1c_1 + a_2c_2)(b_1d_1 + b_2d_2)$ and $(\mathrm{II}) = (a_1d_1 + b_2c_2)(b_1c_1 + a_2d_2)$. We apply (6.3) to obtain

$$\begin{aligned}(\mathrm{I}) &= a_1b_1c_1d_1 + a_1b_2c_1d_2 + a_2b_1c_2d_1 + a_2b_2c_2d_2\\ &= a_1b_1c_1d_1 + b_1b_2c_1c_2 + \big(b_2c_1 - a_2d_1\big)b_2c_1 + a_2b_1c_2d_1 + a_2b_2c_2d_2,\\ (\mathrm{II}) &= a_1b_1c_1d_1 + a_1a_2d_1d_2 + b_1b_2c_1c_2 + a_2b_2c_2d_2\\ &= a_1b_1c_1d_1 + a_2b_1c_2d_1 + \big(b_2c_1 - a_2d_1\big)a_2d_2 + b_1b_2c_1c_2 + a_2b_2c_2d_2.\end{aligned}$$

Thus $(\mathrm{I}) - (\mathrm{II}) = \big(b_2c_1 - a_2d_1\big)b_2c_1 - \big(b_2c_1 - a_2d_1\big)a_2d_2 = (X - Y)X - (X - Y)Y = (X - Y)^2 \ge 0$, where $X = b_2c_1$ and $Y = a_2d_1$. □

The next lemma follows by elementary calculations.

Lemma 6.3. *Let $g \in \Gamma_1$ as in (6.1).*

(i) *If we assume that $\operatorname{Re}\{a\bar{c}\} \neq 0$ and $\operatorname{Re}\{b\bar{d}\} \neq 0$, then we get $\operatorname{sgn}\big(\operatorname{Re}\{a\bar{c}\}\big) = \operatorname{sgn}\big(\operatorname{Re}\{b\bar{d}\}\big)$.*

(ii) *On the other hand, $\operatorname{Re}\{a\bar{c}\}\operatorname{Re}\{b\bar{d}\} = 0$ if and only if $(a_1d_1 + b_2c_2)(b_1c_1 + a_2d_2) = 0$ and $b_2c_1 = a_2d_1$.*

(iii) *If $\operatorname{Re}\{\bar{a}b\} \neq 0$ and $\operatorname{Re}\{\bar{c}d\} \neq 0$, then $\operatorname{sgn}(\operatorname{Re}\{\bar{a}b\}) = \operatorname{sgn}(\operatorname{Re}\{\bar{c}d\})$.*

(iv) *$\operatorname{Re}\{\bar{a}b\}\operatorname{Re}\{\bar{c}d\} = 0$ if and only if $\big(a_1d_1 + b_2c_2\big)\big(b_1c_1 + a_2d_2\big) = 0$ and $b_1c_2 = a_2d_1$.*

(v) *It holds that $\operatorname{Im}\big\{b\bar{c} - a\bar{d}\big\}^2 - 4\operatorname{Re}\{a\bar{c}\} \cdot \operatorname{Re}\{b\bar{d}\} \le 0$.*

Proof. We will just given an sketch. Observe that, by Lemma 6.2, we have that $\operatorname{Re}(a\bar{c})\operatorname{Re}(b\bar{d}) \ge 0$. Therefore, if both terms are different from zero, they have the same sign. The point (ii) follows similarly by Lemma 6.2 and its proof. Points (iii) and (iv) are a reiteration of the previous two points but changing rows by columns. Lastly, for v, we start by noticing that

$$\operatorname{Im}\{b\bar{c}\} - \operatorname{Im}\{a\bar{d}\} = b_2c_1 - b_1c_2 - a_2d_1 + a_1d_2 = 2\big(a_1d_2 - b_2c_1\big),$$

where we have used (6.3). Now, we have that

$$\begin{aligned}&\big(b_2c_1 - a_2d_1\big)^2 - \big(a_1c_1 + a_2c_2\big)\big(b_1d_1 + b_2d_2\big)\\ &= b_2^2c_1^2 + a_2^2d_1^2 - 2a_2b_2c_1d_1 - a_1b_1c_1d_1 - a_1b_2c_1d_2 - a_2b_1c_2d_1 + a_2b_2c_2d_2\end{aligned} \tag{6.4}$$

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$$= a_2d_1\big(a_2d_1 - b_2c_1 - b_1c_2\big) - a_1b_1c_1d_1 - b_1b_2c_1c_2 - a_2b_2c_2d_2 \tag{6.5}$$

$$\begin{aligned} &= -a_1a_2d_1d_2 - a_1b_1c_1d_1 - b_1b_2c_1c_2 - a_2b_2c_2d_2 \\ &= -\big(a_2d_2 + b_1c_1\big)\big(a_1d_1 + b_2c_2\big) \qquad (6.6) \\ &= -X(1+X) \;\le\; 0, \end{aligned}$$

where $X = a_2d_2 + b_1c_1$. We have used identity (6.3) in (6.4), and (6.5). For (6.6) we use (6.2). The last term is negative since X is an integer and $X^2 + X$ is always positive when $X \in \mathbb{Z}$. □

Now let us consider the symbol given in (1.8). When restricted to Γ_1, it is 0 over a large subset. We are going to show that this subset is in fact a subgroup $\Gamma_0 \subseteq \Gamma_1$. The following proposition follows from the previous lemma.

Lemma 6.4. *Let $g \in \Gamma_1$ be a matrix with coefficients like those in (6.1) and such that $\mathrm{Re}\{a\bar{c} + b\bar{d}\} = 0$. Then $g \in \Gamma_0^+ \cup \Gamma_0^-$, where*

$$\Gamma_0^+ = \left\{ \begin{pmatrix} a_1 & b_2 i \\ c_2 i & d_1 \end{pmatrix} : a_1, b_2, c_2, d_1 \in \mathbb{Z}, a_1d_1 + b_2c_2 = 1 \right\}$$

$$\Gamma_0^- = \left\{ \begin{pmatrix} a_2 i & b_1 \\ c_1 & d_2 i \end{pmatrix} : a_2, b_1, c_1, d_2 \in \mathbb{Z}, -a_2d_2 - b_1c_1 = 1 \right\}$$

It is clear that Γ_0^+ and $\Gamma_0 = \Gamma_0^+ \cup \Gamma_0^-$ are subgroups of Γ_1 with $[\Gamma_0 : \Gamma_0^+] = 2$.

Proof. Observe that, if $\mathrm{Re}\{a\bar{c}\} \neq 0$ and $\mathrm{Re}\{b\bar{d}\} \neq 0$, by Lemma 6.3.(i), $\mathrm{Re}\{a\bar{c}\}$ and $\mathrm{Re}\{b\bar{d}\}$ have the same sign, therefore if $\mathrm{Re}\{a\bar{c} + b\bar{d}\} = 0$ that is because $\mathrm{Re}\{a\bar{c}\} = \mathrm{Re}\{b\bar{d}\} = 0$. Assume that a_1 and a_2 are coprime and that b_1 and b_2 are also coprime. Since $\mathrm{Re}\{a\bar{c}\} = a_1c_1 + a_2c_2 = 0$, we have that (a_1, a_2) and (c_1, c_2) are perpendicular vectors with integer coordinates. But, since a_1 and a_2 are coprime $(c_1, c_2) = \ell(a_2, -a_1)$ for some $\ell \in \mathbb{Z}$. Similarly $(d_1, d_2) = m(b_2, -b_1)$ for some $m \in \mathbb{Z}$. Computing the determinant gives

$$\det \begin{pmatrix} a_1 + ia_2 & b_1 + ib_2 \\ \ell a_2 - i\ell a_1 & mb_2 - imb_1 \end{pmatrix} = \big(m - \ell\big)\big(a_2b_1 + a_1b_2\big) + i\big(m - \ell\big)\big(a_2b_2 - a_1b_1\big) = 1.$$

But this implies that $m - \ell = \pm 1$ and $a_2b_1 + a_1b_2 = \pm 1$. Since the imaginary part has to be 0 and $m - \ell = \pm 1$, it follows that $a_2b_2 = a_1b_1$. But since a_1 and a_2, like b_1 and b_2, are coprime, we have that $a_2 = b_1$ and $a_1 = b_2$. Therefore $a_1^2 + a_2^2 = 1$ and $b_1^2 + b_2^2 = 1$. Since they are integers, one of a_1, a_2 and of b_1 and b_2 has to be 0. So we have either $a_2 = b_1 = 0$ or $a_1 = b_2 = 0$. The case of non-coprime a_1 and a_2 can be proved similarly, by noticing that $a_1 + ia_2 = k\alpha_1 + ik\alpha_2$ with $k = \gcd(a_1, a_2)$ and α_1 and α_2 coprime. In that case every perpendicular vector to (a_1, a_2) with integer coordinates is of the form $(\ell\alpha_2, -\ell\alpha_1)$. □

Observe that Γ_0^+ is isomorphic to the subgroup $\mathrm{PSL}_2(\mathbb{Z})$ in Γ_1. Indeed, if we take the natural embedding $\mathrm{PSL}_2(\mathbb{Z}) \subseteq \Gamma_1$, we have that

$$\Gamma_0^+ \;=\; J\,\mathrm{PSL}_2(\mathbb{Z})\,J^{-1}, \quad \text{where } J = \begin{pmatrix} e^{\frac{i\pi}{4}} & 0 \\ 0 & e^{-\frac{i\pi}{4}} \end{pmatrix}.$$

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The group $\Gamma_0 = \Gamma_0^+ \cup \Gamma_0^-$ is generated by Γ_0^+ and the diagonal matrix with eigenvalues i and $-i$. It is trivial to check that Γ_0 is isomorphic to $\mathrm{PSL}_2(\mathbb{Z}) \rtimes \mathbb{Z}_2$, with the order two automorphism given by conjugation.

Now, we introduce the following modification of the symbol in (1.8).

Definition 6.5. Let us define the symbol $m_2 : \Gamma_1 \to \mathbb{C}$ by

$$m_2(g) = \begin{cases} \operatorname{sgn}\big(\operatorname{Re}\{a\bar{c} + b\bar{d}\}\big) & \text{when } g \notin \Gamma_0 \\ 1 & \text{when } g \in \Gamma_0^- \\ 0 & \text{when } g \in \Gamma_0^+. \end{cases}$$

Proposition 6.6. *The function m_2 given in Definition 6.5 is both left and right Γ_0^+-invariant.*

Proof. We first prove that m_2 is left Γ_0^+-invariant. Let $g \in \Gamma_0^+$. If $h \in \Gamma_0^+$, since Γ_0^+ is a subgroup of Γ_1, it is obvious that $m_2(gh) = m_2(h) = 0$. If $h \in \Gamma_0^-$, it is easy to check $gh \in \Gamma_0^-$, thus $m_2(gh) = m_2(h) = 1$. Since m_2, when restricted to $\Gamma_1 \setminus \Gamma_0$, is just the hermitian product of its rows, it is clearly right K-invariant. Thus, if $h \notin \Gamma_0$, we have $m_2(h) = \operatorname{Re}(t)$, where

$$h = \begin{pmatrix} s & st \\ 0 & s^{-1} \end{pmatrix} k \quad \text{for some } k \in K.$$

Moreover, we have $gh \notin \Gamma_0$ and

$$m_2(gh) = m_2\left(\begin{pmatrix} a_1 & b_2 i \\ c_2 i & d_1 \end{pmatrix} \begin{pmatrix} s & st \\ 0 & s^{-1} \end{pmatrix} \right) = \big(a_1 d_1 + b_2 c_2\big) \operatorname{Re}(t) = \operatorname{Re}(t) = m_2(h).$$

Now we show m_2 is right Γ_0^+-invariant. Similarly to the case above, if $h \in \Gamma_0^+$, $m_2(hg) = m_2(h) = 0$; if $h \in \Gamma_0^-$, we have $hg \in \Gamma_0^-$ and $m_2(hg) = m_2(h) = 1$. If $h \notin \Gamma_0$, let

$$h = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

Then $hg \notin \Gamma_0$. Therefore we have

$$\begin{aligned} m_2(hg) &= m_2\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} u & uv \\ 0 & u^{-1} \end{pmatrix} \right) \\ &= \operatorname{sgn}\Big[\operatorname{Re}(a\bar{c})\, u^2 (1 + \operatorname{Im} v^2) + \operatorname{Im}(b\bar{c} - a\bar{d}) \operatorname{Im} v + \operatorname{Re}(b\bar{d})\, u^{-2} \Big], \end{aligned}$$

where

$$u = \sqrt{d_1^2 + c_2^2} \neq 0 \quad \text{and} \quad v = \frac{(b_2 d_1 - a_1 c_2) i}{\sqrt{d_1^2 + c_2^2}}$$

Recall that $m_2(h) = \operatorname{sgn}(\operatorname{Re}(a\bar{c} + b\bar{d}))$. If $\operatorname{Re}(a\bar{c}) \neq 0$ and $\operatorname{Re}(b\bar{d}) \neq 0$, then by Lemmas 6.3 and v, it is easy to see $m_2(hg) = m_2(h)$. If $\operatorname{Re}(a\bar{c}) = 0$ and $\operatorname{Re}(b\bar{d}) \neq 0$, we have $m_2(hg) = \operatorname{sgn}[\operatorname{Re}(b\bar{d})\, u^{-2}]$ and $m_2(h) = \operatorname{sgn}(\operatorname{Re}(b\bar{d}))$. Thus $m_2(hg) = m_2(h)$. If $\operatorname{Re}(a\bar{c}) \neq 0$ and $\operatorname{Re}(b\bar{d}) = 0$, we have $m_2(hg) = \operatorname{sgn}[\operatorname{Re}(a\bar{c})\, u^2 (1 + \operatorname{Im} v^2)]$ and $m_2(h) = \operatorname{sgn}(\operatorname{Re}(a\bar{c}))$, which implies $m_2(hg) = m_2(h)$. □

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Proposition 6.7. *The function m_2 given in Definition 6.5 satisfies the Cotlar identity (Cotlar$_{\mathbb{E}\perp}$) relative to Γ_0^+, i.e., for any $g \in \Gamma_1 \setminus \Gamma_0^+$ and $h \in \Gamma_1$, it holds that*

$$\big(m_2(g^{-1}) - m_2(h)\big)\big(m_2(gh) - m_2(g)\big) = 0.$$

Proof. Since m_2 is right Γ_0^+-invariant by the Proposition 6.6, it suffices to prove the Cotlar identity for g and h in $\in \Gamma_1 \setminus \Gamma_0^+$. That set decomposes as $\Gamma_1 \setminus \Gamma_0^+ = (\Gamma_1 \setminus \Gamma_0) \sqcup \Gamma_0^-$ and thus we have 4 potential cases. When $g \in \Gamma_1 \setminus \Gamma_0$, then for every $h \in \Gamma_0$ it holds that $m_2(gh) = m_2(h)$. In the case in which $g \in \Gamma_0^-$ and $h \in \Gamma_0^-$ we have that both $g, h^{-1} \in \Gamma_0^-$ and as such $m_2(g) = m_2(h^{-1})$. Thus, we can assume that $g \in \Gamma_1 \setminus \Gamma_0^+$ and $h \in \Gamma_1 \setminus \Gamma_0$. First, let us tackle the case of $g \in \Gamma_0^-$. In this case we have $m_2(g) = m_2(g^{-1}) = 1$. If $m_2(g^{-1}) = m_2(h)$, then the identity is satisfied. If $m(g^{-1}) \neq m_2(h)$, this implies $m_2(h) = -1$, and then $h, gh \notin \Gamma_0$. By the right K-invariance of m_2 on $\Gamma_1 \setminus \Gamma_0$, we get

$$m_2(gh) = m_2\left(\begin{pmatrix} a_2 i & b_1 \\ c_1 & d_2 i \end{pmatrix}\begin{pmatrix} s & st \\ 0 & s^{-1} \end{pmatrix}\right) = (a_2 d_2 + b_1 c_1)\operatorname{Re} t = -\operatorname{Re} t = -m_2(h) = 1.$$

Therefore, we get $m_2(gh) = m_2(g)$, so the Cotlar identity is satisfied. Now let us focus on the case of $g \notin \Gamma_0$. Let $g \notin \Gamma_0$, i.e., $\operatorname{Re}(a\bar{c} + b\bar{d}) \neq 0$ and

$$h = \begin{pmatrix} s & st \\ 0 & s^{-1} \end{pmatrix} k \notin \Gamma_0$$

with $k \in \mathrm{PSU}(2)$. We get the following expressions for the four terms appearing in the Cotlar identity:

$$\begin{aligned} m_2(g) &= \operatorname{sgn}(\operatorname{Re}(a\bar{c} + b\bar{d})), \\ m_2(g^{-1}) &= -\operatorname{sgn}(\operatorname{Re}(\bar{a}b + \bar{c}d)), \\ m_2(h) &= \operatorname{sgn}(\operatorname{Re} t), \\ m_2(gh) &= m_2\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} s & st \\ 0 & s^{-1} \end{pmatrix}\right) \\ &= \operatorname{sgn}\Big[\operatorname{Re}(a\bar{c})\, s^2(1 + |t|^2) + \operatorname{Re}(a\bar{d} + b\bar{c})\operatorname{Re} t + \operatorname{Im}(b\bar{c} - a\bar{d})\operatorname{Im} t + \operatorname{Re}(b\bar{d})\, s^{-2}\Big]. \end{aligned}$$

In the identities for $m_2(h)$ and $m_2(gh)$, we have used the property that m_2 is right K-invariant on $\Gamma_1 \setminus \Gamma_0$. Now we prove the Cotlar identity. If $m_2(g^{-1}) = m_2(h)$, then the identity is satisfied. If not, that means

$$\operatorname{Re} t \cdot \operatorname{Re}(\bar{a}b + \bar{c}d) > 0. \tag{6.7}$$

Then we need to show $m(g) = m(gh)$, or in other words,

$$\begin{aligned} &\Big[\operatorname{Re}(a\bar{c})\, s^2(1 + |t|^2) + \operatorname{Re}(a\bar{d} + b\bar{c})\operatorname{Re} t + \operatorname{Im}(b\bar{c} - a\bar{d})\operatorname{Im} t + \operatorname{Re}(b\bar{d})\, s^{-2}\Big] \cdot \operatorname{Re}(a\bar{c} + b\bar{d}) \\ &= \operatorname{Re}(a\bar{c} + b\bar{d})\operatorname{Re}(a\bar{c})\, s^2(1 + (\operatorname{Re} t)^2) + (\operatorname{Re}(a\bar{c} + b\bar{d})\operatorname{Re}(a\bar{d} + b\bar{c})\operatorname{Re} t \\ &\quad + \operatorname{Re}\,(a\bar{c} + b\bar{d}) \cdot \big[\operatorname{Re}(a\bar{c})\, s^2(\operatorname{Im} t)^2 + \operatorname{Im}(b\bar{c} - a\bar{d})\operatorname{Im} t + \operatorname{Re}(b\bar{d})\, s^{-2}\big] > 0. \end{aligned}$$

We first assume that $\operatorname{Re}(a\bar{c}) \neq 0$ and $\operatorname{Re}(b\bar{d}) \neq 0$. By Lemma 6.3, we have

$$\operatorname{Re}(a\bar{c} + b\bar{d})\operatorname{Re}(a\bar{c}) > 0.$$

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Moreover, Lemma 6.3.v implies that the determinant of the quadratic function of $\operatorname{Im} t$ is negative; together with the inequality above, we see that

$$\operatorname{Re}(a\bar{c}+b\bar{d})\cdot\left[\operatorname{Re}(a\bar{c})\,s^2(\operatorname{Im} t)^2+\operatorname{Im}(b\bar{c}-a\bar{d})\operatorname{Im} t+\operatorname{Re} b\bar{d}\,s^{-2}\right]\geq 0$$

Moreover, we claim that

$$\operatorname{Re}(a\bar{c}+b\bar{d})\operatorname{Re}(a\bar{d}+b\bar{c})\operatorname{Re} t\geq 0.$$

If the claim is true then we will obtain $m(g)=m(gh)$. So now it remains to prove the claim. Notice that by (6.7) and Lemma 6.3, it is enough to show

$$\operatorname{Re}(\bar{a}b)\operatorname{Re}(a\bar{c})\operatorname{Re}(a\bar{d}+b\bar{c})\geq 0.$$

According to (6.2), $\operatorname{Re}(a\bar{d}+b\bar{c})=a_1d_1+a_2d_2+b_1c_1+b_2c_2=2(b_1c_1+a_2d_2)+1$. On the other hand, (6.2) and (6.3) imply that

$$\begin{aligned}\operatorname{Re}(\bar{a}b)\operatorname{Re}(a\bar{c})&=(a_1b_1+a_2b_2)(a_1c_1+a_2c_2)\\&=a_1a_2(b_1c_2+b_2c_1)+a_1^2b_1c_1+a_2^2b_2c_2\\&=a_1a_2(a_1d_2+a_2d_1)+a_1^2b_1c_1+a_2^2b_2c_2\\&=a_1^2(b_1c_1+a_2d_2)+a_2^2(a_1d_1+b_2c_2)\\&=(b_1c_1+a_2d_2)(a_1^2+a_2^2)+a_2^2.\end{aligned}$$

Let $X=b_1c_1+a_2d_2$, $A=a_1^2+a_2^2$ and $B=a_2^2$. Then $\operatorname{Re}(\bar{a}b)\operatorname{Re}(a\bar{c})\operatorname{Re}(a\bar{d}+b\bar{c})=(AX+B)(2X+1)$. Since $X\in\mathbb{Z}$ and $|\frac{B}{A}|\leq 1$, we get $(AX+B)(2X+1)\geq 0$, which proves the claim.

Now we deal with the case when $\operatorname{Re}(a\bar{c})\operatorname{Re}(b\bar{d})=0$. By Lemma 6.3, it is equivalent to saying $(a_1d_1+b_2c_2)(b_1c_1+a_2d_2)=0$ and $b_2c_1=a_2d_1$. Without loss of generality, we assume that $b_1c_1+a_2d_2=0$. Then by (6.2), $a_1d_1+b_2c_2=1$. Since $b_2c_1=a_2d_1$, (6.3) tells us that $b_1c_2=a_1d_2$ and $\operatorname{Im}(b\bar{c}-a\bar{d})=2(b_2c_1-a_2d_1)=0$. This implies that

$$a_1b_1=a_1b_1(a_1d_1+b_2c_2)=a_1^2(b_1d_1+b_2d_2)=a_1^2\operatorname{Re}(b\bar{d}). \tag{6.8}$$

Similarly, we have

$$c_2d_2=c_2^2\operatorname{Re}(b\bar{d}),\ \ a_2b_2=b_2^2\operatorname{Re}(a\bar{c})\ \ \text{and}\ \ c_1d_1=d_1^2\operatorname{Re}(a\bar{c}). \tag{6.9}$$

Since $\operatorname{Re}(a\bar{c}+b\bar{d})\neq 0$, $\operatorname{Re}(a\bar{c})$ and $\operatorname{Re}(b\bar{d})$ can not be zero at the same time. Suppose $\operatorname{Re}(a\bar{c})=0$ and $\operatorname{Re}(b\bar{d})\neq 0$. Then $m(g)=\operatorname{sgn}\operatorname{Re}(b\bar{d})$, $m(gh)=\operatorname{sgn}\left[\operatorname{Re}(a\bar{d}+b\bar{c})\operatorname{Re} t+\operatorname{Re}(b\bar{d})\,s^{-2}\right]=\operatorname{sgn}\left[\operatorname{Re} t+\operatorname{Re}(b\bar{d})\,s^{-2}\right]$. Applying (6.8) and (6.9), we get $\operatorname{Re}(\bar{a}b+\bar{c}d)=(a_1^2+c_2^2)\operatorname{Re}(b\bar{d})$. Recall that we have $\operatorname{Re} t\cdot\operatorname{Re}(\bar{a}b+\bar{c}d)>0$ by the assumption $m(g^{-1})\neq m(h)$. Therefore, we have $\operatorname{Re} t\cdot\operatorname{Re}(b\bar{d})>0$, which implies $m(g)=m(gh)$. For the case $\operatorname{Re}(a\bar{c})\neq 0$ and $\operatorname{Re}(b\bar{d})=0$, we omit the proof since it is similar to the previous case. □

Now, we can prove Theorem E by just reducing the L_p-boundedness of m to that of m_2.

Proof of Theorem E. Notice that $m:\Gamma_1\to\mathbb{C}$ satisfies that

$$m(g)=m_2(g)-\mathbf{1}_{\Gamma_0}+\mathbf{1}_{\Gamma_0^+}.$$

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But Fourier multipliers with symbols being characteristic functions of open subgroups are bounded in L_p for every $1 \leq p \leq \infty$. Therefore, up to a finite constant smaller than 2, the operator L_p-norm of T_m is bounded by that of T_{m_2}. □

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