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Decay estimate for subcritical semilinear damped wave equations with slowly decreasing data‡

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We study the decay properties of non-negative solutions to the one-dimensional defocusing damped wave equation in the Fujita subcritical case under a specific initial condition. Specifically, we assume that the initial data are positive, satisfy a condition ensuring the positiveness of solutions, and exhibit polynomial decay at infinity. To show the decay properties of the solution, we construct suitable supersolutions composed of an explicit function satisfying an ordinary differential inequality and the solution of the linear damped wave equation. Our estimates correspond to the optimal ones inferred from the analysis of the heat equation.

Keywords: damped wave equations; decay estimate; Fujita critical exponent; power-type nonlinearity; Fujita subcritical case

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1. Introduction

We consider the pointwise decay behaviour of solutions to the Cauchy problem for the following damped wave equations under certain initial condition:

$$\begin{cases} \partial_t^2 u + \partial_t u - \Delta u + u^p = 0, & t \in (0,\infty),\ x \in \mathbb{R} \\ u(0,x) = u_0(x) & x \in \mathbb{R}, \\ \partial_t u(0,x) = u_1(x) & x \in \mathbb{R}. \end{cases} \tag{1}$$

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We assume that $p > 1$ and the initial data u_0, u_1 are sufficiently regular and satisfy the conditions

$$\inf_{x\in\mathbb{R}} u_0(x), \quad \inf_{x\in\mathbb{R}} (u_1(x) + \frac{1}{2}u_0(x)) \geq 0. \tag{2}$$

We call u a mild solution to (1) if u is non-negative and it satisfies the integral form of (1) given by

$$u(t,x) = S(t)(u_0+u_1) + \partial_t S(t)u_0 - \int_0^t S(t-\tau)u^p(\tau)d\tau, \tag{3}$$

where $S(t)$ denotes the solution operator defined by

$$S(t)f(x) = \frac{e^{-t/2}}{2}\int_{-t}^{t} I_0\left(\frac{\sqrt{t^2-y^2}}{2}\right) f(x-y)dy$$

for locally integrable function f. Here, I_0 denotes the modified Bessel function of the first kind. For details, see [1, Chapter 6, Section 5, Part 7]. Namely, $S(t)f$ solves the Cauchy problem:

$$\begin{cases} (\partial_t^2 + \partial_t - \Delta)S(t)f = 0, & t \in (0,\infty),\ x \in \mathbb{R}, \\ S(0)f = 0, & x \in \mathbb{R}, \\ \partial_t S(0)f = f, & x \in \mathbb{R}. \end{cases}$$

To investigate the existence of mild solutions to (1), we review known results for the Cauchy problem of the damped wave equation:

$$\begin{cases} \partial_t^2 u - \Delta u + \partial_t u = \lambda |u|^{p-1}u, & t \in (0,\infty),\ x \in \mathbb{R}, \\ u(0,x) = u_0(x), & x \in \mathbb{R}, \\ \partial_t u(0,x) = u_1(x), & x \in \mathbb{R} \end{cases} \tag{4}$$

with $\lambda \in \mathbb{R}$ and $p > 1$. A mild solution, allowing sign changes, can be constructed by applying a standard contraction mapping principle and the estimates for the solution operator given by Marcati–Nishihara [11]. In particular, they showed the asymptotic equivalence between $S(t)$ and the heat kernel $e^{t\Delta}$.

The existence of mild-solutions to (1) is basically guaranteed by the argument above and the a priori estimates of Li and Zhou [13] under the initial condition (2). Indeed, if $u_0, u_1 \in C^\infty$, then the following a priori estimate holds for any x and t, provided that u is a mild solution of (1):

$$\begin{aligned} &e^{-t/2}\frac{u_0(x+t)+u_0(x-t)}{2} + e^{-t/2}\int_{x-t}^{x+t} u_1(y) + \frac{1}{2}u_0(y)dy \\ &\leq u(t,x) \leq S(t)(u_0+u_1)(x) + \partial_t S(t)u_0(x). \end{aligned} \tag{5}$$

Combining the a priori estimate (5) and contraction mapping argument discussed above, mild solutions to (1) can be constructed. In the case where $(u_0, u_1) \in W^{1,1} \cap$

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$W^{1,\infty}(\mathbb{R}) \times L^1 \cap L^\infty(\mathbb{R})$, (5) has also been established in [5]. See also Proposition A.3 in the Appendix.

On the other hand, it is known that (5) is not sufficient to describe the decay rate of the solution u. Indeed, by rewriting $-u$ as $\widetilde{u}$, we have

$$\begin{cases} \partial_t^2 \widetilde{u} + \Delta \widetilde{u} - \Delta \widetilde{u} = |\widetilde{u}|^p, \\ \widetilde{u}(0,x) = -u_0(x), \\ \partial_t \widetilde{u}(0,x) = -u_1(x). \end{cases}$$

By integrating the equation above formally with respect to x and t,

$$\int_{\mathbb{R}} \widetilde{u}(t,x) + \partial_t \widetilde{u}(t,x) dx = -\int_{\mathbb{R}} u_0(x) + u_1(x) dx + \|\widetilde{u}\|^p_{L^p(0,t;L^p(\mathbb{R}))}.$$

Then u belongs to $L^p(0,\infty;L^p(\mathbb{R}))$ as long as it exists globally. Indeed, if $u \notin L^p(0,\infty;L^p(\mathbb{R}))$, then there exists T_0 such that

$$\int_{\mathbb{R}} \widetilde{u}(T_0,x) + \partial_t \widetilde{u}(T_0,x) dx > 0.$$

However, Li and Zhou [13] showed that there exists $T_1 > T_0$ such that

$$\liminf_{t \to T_1} \inf_{|y| \le \sqrt{t}} \sqrt{t} |\widetilde{u}(t,y)| = \infty,$$

which contradicts the global existence of u. Since $S(t)$ is asymptotically equivalent to $e^{t\Delta}$, if $S(t)f \in L^p(0,\infty;L^p(\mathbb{R}))$ for any $f \in L^1$, then the estimate $p > 3$ is necessary. Therefore, when $p \le 3$(in the Fujita critical and subcritical cases), solutions u must decay faster than free solutions for some initial data. Namely, the asymptotic behaviour of the solution is primarily governed by nonlinear effects rather than perturbative effects. This nonlinear effect has attracted attention.

In order to study the nonlinear effect of (1) in the subcritical case, the decay behaviour of solutions to the corresponding semilinear heat equation has been considered as an indicator. Let us recall the known results for the Cauchy problem of the following semilinear heat equation:

$$\begin{cases} \partial_t v - \Delta v + v^p = 0, \\ v(0,x) = v_0(x), \end{cases} \tag{6}$$

where v_0 and v are non-negative functions. Roughly speaking, if v_0 decays sufficiently fast, then the behaviour of the solution v is approximated as $t \to \infty$ as follows

$$v(t,x) \sim \begin{cases} t^{-1/(p-1)} h(|x|/t^{1/2}) & \text{if} \quad 1 < p < 3, \\ \theta (\log t)^{-1/2} t^{-1/2} e^{-x^2/4t} & \text{if} \quad p = 3, \\ \theta t^{-1/2} e^{-x^2/4t} & \text{if} \quad p > 3. \end{cases} \tag{7}$$

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Here θ is a constant determined by the initial data and $h : [0, \infty) \to [0, \infty)$ solves the ODE:

$$\begin{cases} -\ddot{h} - \frac{r}{2}\dot{h} + h^p = \frac{1}{p-1}h, \quad r \in (0, \infty), \\ \dot{h}(0) = 0, \end{cases} \tag{8}$$

and $t^{-1/2}e^{-x^2/4t}$ is the fundamental solution of the heat equation. Therefore, in the subcritical case, solutions v are asymptotically equivalent to self-similar function. For details, see [2, 3, 6]. Moreover, Herraiz [9, Theorem 1] showed that the solution v behaves like

$$v(t,x) \sim \begin{cases} t^{-1/(p-1)}\Big((p-1)^{p-1} - Ct^{1-\frac{\rho(p-1)}{2}}h(|x|/t^{1/2})\Big) & \text{if } |x| \lesssim t^{1/2}, \\ \Big((p-1)t + A^{1-p}|x|^{\rho(p-1)}\Big)^{-1/(p-1)} & \text{if } t^{1/2} \lesssim |x| \lesssim t^{1/\rho(p-1)}, \\ A|x|^{-\rho} & \text{if } t^{1/\rho(p-1)} \lesssim |x|. \end{cases} \tag{9}$$

when $1 < p < 3$ and v_0 decays slower than h above, namely

$$v_0(x) \sim A|x|^{-\rho}$$

with $A > 0$ and $0 < \rho < 2/(p-1)$ as $|x| \to \infty$. We note that $g(t, \varepsilon) = ((p-1)t + \varepsilon^{1-p})^{-1/(p-1)}$ solves the ODE:

$$\dot{g} + g^p = 0$$

and

$$g(t, A|x|^{-\rho}) = \Big((p-1)t + A^{1-p}|x|^{\rho(p-1)}\Big)^{-1/(p-1)}.$$

Therefore, in the subcritical case, the behaviour of solutions to (6) may be summarized from the viewpoint of the decay rate of L^q-norm of the solutions as follows: if $v_0 = \varepsilon\langle x\rangle^{-\rho}$, where ε is small enough and $\langle x\rangle = (1 + |x|^2)^{1/2}$, then

$$\|v(t)\|_{L^q(\mathbb{R})} \sim \begin{cases} t^{\frac{1}{2q} - \frac{1}{p-1}} & \text{if} \quad \rho > \frac{2}{p-1}, \\ t^{\frac{1}{q\rho(p-1)} - \frac{1}{p-1}} & \text{if} \quad 1 < \rho < \frac{2}{p-1}. \end{cases} \tag{10}$$

for $q \in [1, \infty]$, ensuring that $v \in L^p(0, \infty; L^p(\mathbb{R}))$.

Comparing to (6), the behaviour of solutions to (1) is much less understood. Hayashi, Kaikina, and Naumkin [8] showed that an asymptotic equivalence similar to (7) holds for the solution u to (1), when $(u_0, u_1) \in (H^2 \cap W^{2,1} \cap \langle x\rangle^a L^1) \times (H^1 \cap W^{1,1} \cap \langle x\rangle^a L^1)$ with $a > 0$, where h is replaced appropriately. However, in the subcritical case, the asymptotic equivalence is shown when $3 - \varepsilon^3 < p < 3$, where $\varepsilon > 0$ represents the size of initial data and is small enough. Later, Nishihara and Zhao [12] and Ikehata, Nishihara, and Zhao [10] showed that in the subcritical

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case, solutions u enjoy a weighted energy estimate guaranteeing the estimate

$$\|u(t)\|_{L^q(\mathbb{R})} \lesssim t^{\frac{1}{2q}-\frac{1}{p-1}}$$

for any $q \in [1,\infty]$, when the following initial condition is satisfied:

$$(1+|x|)^{\frac{2}{p-1}-\frac{1-\delta}{2}}\left(u_0, \nabla u_0, u_1, |u_0|^{\frac{p+1}{2}}\right) \in L^2$$

with some small positive δ. We note that $u_0, u_1 \sim \langle x\rangle^{-\rho}$ with $\rho > \frac{2}{p-1}$, satisfies the condition above. We also note that Wakasugi [14] generalized the results of [10]. However, there is still a gap between the known decay rate for the solutions to (1) and (9). We remark that L^∞ control corresponds to (9) is known to hold under an initial regularity condition and (2) in [4].

The aim of this manuscript is to show a pointwise decay estimate of solutions to (1) comparable to (9) under a certain initial condition. The main result is as follows:

THEOREM 1.1 *Let $(u_0,u_1) \in W^{1,1}\cap W^{1,\infty}(\mathbb{R}) \times L^1 \cap L^\infty(\mathbb{R})$. Let $\phi \in W^{1,1}\cap W^{1,\infty}(\mathbb{R};(0,\infty))$. We assume that there exists a constant C such that the estimates*

$$\phi(x) \le C \inf_{y\in(x-1,x+1)} \phi(y), \tag{11}$$

$$\phi(x) \le C \inf_{|y|<2|x|} \phi(y), \tag{12}$$

$$|\dot\phi(x)| \le C\phi(x) \tag{13}$$

hold for almost every $x \in \mathbb{R}$. We further assume that for any $\delta > 0$, there exists a positive constant C_δ such that

$$e^{-\delta|x|} \le C_\delta \phi(x) \tag{14}$$

holds for almost every $x \in \mathbb{R}$. Then there exist a positive constant C, $T_0 \ge 0$ large enough, and $\varepsilon_0 > 0$ small enough such that if u_0 and u_1 satisfy

$$0 \le u_0(x), u_1(x) + \frac{1}{2}u_0(x) \le C\frac{\varepsilon_0}{T_0^{p-1/2}}\phi(x)$$

for almost every $x \in \mathbb{R}$, then solutions u to (1) satisfy

$$0 \le u(t,x) \le C\left(\frac{u_L(t+T_0,x)^{p-1}}{(t+T_0)u_L(t+T_0,x)^{p-1}+1}\right)^{\frac{1}{p-1}}, \tag{15}$$

for almost every $x \in \mathbb{R}$ and $t \ge 0$, where $u_L = \varepsilon_0 S(t)\phi + \varepsilon_0 \partial_t S(t)\phi$.

We note that $\phi(x) = \langle x\rangle^{-\rho}$ with $\rho > 1$ satisfies the condition of Theorem 1.1. Then we have the following corollary:

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COROLLARY 1.2. *Let $1 < p < 3$. Let $\varepsilon > 0$ be small enough and $1 < \rho < \frac{2}{p-1}$. Let $0 \le u_0, u_1 + u_0/2 \le \varepsilon \langle x \rangle^{-\rho}$. Then, u enjoys the estimate*

$$\|u(t)\|_{L^q} \lesssim t^{\frac{1}{q\rho(p-1)} - \frac{1}{p-1}}$$

for $q \in [1, \infty]$. Especially $u \in L^p(0, \infty; L^p(\mathbb{R}))$.

Therefore, we conclude that, under the initial condition of Corollary 1.2, the solution u to (1) decays as fast as the solution to (6) with the same initial condition.

We note that Theorem 1.1 holds for any $\rho > 1$ but the sharp upperbound (10) can be obtained by Theorem 1.1 only when $\rho < \frac{2}{p-1}$. Indeed, a logarithmic loss occurs when one estimate the L^q norm of the RHS of the second estimate of (15) with $\phi(x) = \langle x \rangle^{-\rho}$ with $\rho \ge \frac{2}{p-1}$. For details, see Proposition A.2 in the Appendix. This logarithmic loss also implies that solutions to (1) cannot be estimated from below by a function taking the form of the RHS of the second estimate of (15) at least when $\rho \ge \frac{2}{p-1}$.

Our approach showing Theorem 1.1 is to compare solution u with supersolutions. For the definition of supersolution, see Section 4. We assume the initial condition (2) to apply a comparison argument. For details, see Appendix A.3. We construct supersolutions by combining ODE supersolutions and free solutions. To illustrate the idea, we reconsider (6). The ODE corresponding to (6) is

$$\begin{cases} \dot{g} + g^p = 0, \\ g(0) = f \ge 0. \end{cases} \tag{16}$$

Here, $g(t) = G(t, f) = ((p-1)t + f^{1-p})^{-1/(p-1)}$. Then, we claim that $G^*(t, x) = G(t, e^{t\Delta}\phi(x))$ is a supersolution to (6) when ϕ is a non-negative and non-trivial function. Indeed, we compute that

$$\partial_f G(t, f) = G^p f^{-p}, \quad \partial_f^2 G(t, f) = p G^p f^{-2p} (G^{p-1} - f^{p-1}),$$

and therefore $\partial_f^2 G < 0$ if $f > 0$. Then we further compute

$$\begin{aligned} &\partial_t G^* - \Delta G^* + (G^*)^p \\ &= (\partial_t G + G^p)^* + (\partial_f G)^* (\partial_t e^{t\Delta}\phi - \Delta e^{t\Delta}\phi) - (\partial_f^2 G)^* (\partial_x \phi)^2 > 0. \end{aligned}$$

This implies that the pointwise estimate $v(t, x) \le G^*(t, x)$ holds. We remark that if $\phi(x) = \langle x \rangle^{-\rho}$ with $1 < \rho < \frac{2}{p-1}$, then the estimate

$$\|G^*(t)\|_{L^q} \lesssim t^{\frac{1}{q\rho(p-1)} - \frac{1}{p-1}}$$

holds for any $q \in [1, \infty]$. Therefore, the upper estimate of (10) is obtained by this comparison argument for small ρ. For details, see Proposition A.1 in the Appendix.

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We note that, since $\ddot{g}(t) = pg^{2p-1}$, g is a supersolution to the ODE:

$$\ddot{f} + \dot{f} + f^p = 0, \tag{17}$$

which corresponds to (1). We remark that a comparison argument for (17) holds under an initial condition

$$f(0) \geq 0, \quad \dot{f}(0) + \frac{1}{2}f(0) \geq 0.$$

One can check this fact by a similar argument to Proposition A.3. On the other hand, $G^*(t,x) = G(t, u_L(t,x))$ may not be a supersolution to (1), where $u_L(t) = S(t)\phi + \partial_t S(t)\phi$ solving

$$\begin{cases} \partial_t^2 u_L + \partial_t u_L - \Delta u_L = 0, \\ u_L(0) = \phi, \\ \partial_t u_L(0) = 0. \end{cases} \tag{18}$$

Indeed, by putting $G^*(t,x) = G(t, u_L(t,x))$, we compute

$$\begin{aligned} &\partial_t^2 G^* + \partial_t G^* - \Delta G^* + (G^*)^p \\ &= ((\partial_t^2 G + \partial_t G + G^p)^* + (\partial_f G)^*(\partial_t^2 + \partial_t - \Delta)u_L \\ &\quad + 2(\partial_t \partial_f G)^* \partial_t u_L + (\partial_f^2 G)^*((\partial_t u_L)^2 - (\partial_x u_L)^2) \\ &= p(G^{2p-1})^* - 2p(G^{2p-1})^* u_L^{-p} \partial_t u_L + p\Big((G^*/u_L)^{2p-2} - (G^*/u_L)^{p-1}\Big) \\ &\quad \times G^* u_L^{-2}\Big((\partial_t u_L)^2 - (\partial_x u_L)^2\Big). \end{aligned}$$

We note that $G^* \leq u_L$ holds because $G(t, u_L(t)) \leq G(0, u_L(t)) = u_L(t)$. Then in the above computation, it is unclear whether G^* is a supersolution to (1). In this manuscript, we overcome this difficulty by using a modified supersolution to (18).

This manuscript is organized as follows: In Section 2, we discuss a supersolution to the ODE related with (1). In Section 3, we show some estimates related with linear solutions to (1). In Section 4, we prove Theorem 1.1 and Corollary 1.2.

2. ODE related with (1)

In this section, we consider a supersolution to (17). The construction of supersolutions is the basis to show the decaying estimate Theorem 1.1.

Proposition 2.1. *Let $p > 1$ and $\varepsilon > 0$ small. Let $\alpha \in (0,1]$. Let*

$$g(t,\varepsilon) = \left(\frac{p(p+1)\varepsilon^{p-1}}{(p-1)^2\varepsilon^{p-1}t + p^2 + p}\right)^{1/(p-1)}, \quad w(t) = \left(\frac{1}{2} + \frac{1}{(2^{1/\alpha} + t)^\alpha}\right)^{-1/(p-1)}. \tag{19}$$

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Then $H(t,\varepsilon) = w(t)g(t,\varepsilon)$ enjoys the following properties with a positive constant $C_{p,\alpha}$ depending only on p and α:

$$\begin{cases} \ddot{H} + \dot{H} + H^p \geq \dfrac{C_{p,\alpha}}{(1+t)^{\alpha+1}} H + C_{p,\alpha} g^{p-1} H, & t > 0 \\ \dot{H} + \dfrac{1}{2} H \geq 0, & t \geq 0 \\ H(0) = \varepsilon, \\ \dot{H}(0) = \dfrac{\alpha}{(p-1)2^{(\alpha+1)/\alpha}} \varepsilon - \dfrac{p-1}{p(p+1)} \varepsilon^p. \end{cases} \tag{20}$$

Proof. For simplicity, we abbreviate $g(t,\varepsilon)$ as $g(t)$. We compute

$$g(0) = \varepsilon, \quad \dot{g}(t) = -\frac{p-1}{p(p+1)} g(t)^p, \qquad \ddot{g}(t) = \frac{(p-1)^2}{p(p+1)^2} g(t)^{2p-1} > 0$$

and

$$\begin{aligned} \dot{w}(t) &= -\frac{1}{(p-1)} w(t)^p \frac{d}{dt} \left(\frac{1}{2} + \frac{1}{(2^{1/\alpha} + t)^\alpha} \right) \\ &= \frac{\alpha}{(p-1)} \frac{1}{(2^{1/\alpha} + t)^{\alpha+1}} w^p(t). \end{aligned}$$

These identities above imply that $H = wg$ satisfies

$$H(0) = \varepsilon, \quad \dot{H}(0) = g(0)\dot{w}(0) + w(0)\dot{g}(0) = \frac{\alpha}{(p-1)2^{(\alpha+1)/\alpha}} \varepsilon - \frac{p-1}{p(p+1)} \varepsilon^p.$$

We also estimate

$$\ddot{w}(t) + \dot{w}(t) = \left(\frac{p\alpha}{(p-1)} \frac{1}{(2^{1/\alpha} + t)^{\alpha+1}} w^{p-1}(t) + \frac{2^{1/\alpha} - \alpha - 1 + t}{2^{1/\alpha} + t} \right) \dot{w}(t) \geq 0. \tag{21}$$

Here we have used the fact that $2^{1/\alpha} - \alpha - 1$ is strictly decreasing for positive α and is 0 at $\alpha = 1$. We further estimate

$$\dot{w}(t) \leq \frac{\alpha}{(p-1)2^{1/\alpha}} w(t)$$

and

$$w^{p-1}(t) - \frac{p-1}{p+1} \geq 1 - \frac{p-1}{p+1} \geq \frac{2}{p+1}.$$

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Then the estimates above imply that

$$\begin{aligned}\ddot{H}+\dot{H}+H^p &= w\ddot{g}+(\ddot{w}+\dot{w})g+(2\dot{w}+w)\dot{g}+w^pg^p\\ &\geq \left(w^p-\frac{p-1}{p(p+1)}w-\frac{2(p-1)}{p(p+1)}\dot{w}\right)g^p+(\ddot{w}+\dot{w})g\\ &\geq \left(w^p-\frac{p-1}{(p+1)}w-\frac{2\alpha}{p(p+1)2^{1/\alpha}}w\right)g^p+(\ddot{w}+\dot{w})g\\ &\geq \left(\frac{2}{p+1}-\frac{2\alpha}{p(p+1)2^{1/\alpha}}\right)wg^p+\frac{C}{(1+t)^{\alpha+1}}H.\end{aligned}$$

Here we have used the estimate $(\ddot{w}+\dot{w})g\gtrsim(1+t)^{-\alpha-1}H$. Moreover, we have

$$\dot{H}+\frac{1}{2}H=\dot{w}g+\frac{1}{2}wg+w\dot{g}\geq\left(\frac{1}{2}-\frac{p-1}{p(p+1)}g^{p-1}\right)g\geq 0,$$

where we have used the assumption that ε is small and

$$\max_{1\leq p\leq 3}\frac{p-1}{p(p+1)}=\frac{p-1}{p(p+1)}\bigg|_{p=1+\sqrt{2}}\sim 0.17<\frac{1}{2}.$$

□

REMARK 2.2. Summarizing the relations used in the proof of the previous Lemma, we can write

$$\begin{cases}\dot{w}(t)=\dfrac{\alpha}{(p-1)}\dfrac{1}{(2^{1/\alpha}+t)^{\alpha+1}}w^p(t),\\ \ddot{w}(t)=\left(\dfrac{p\alpha}{(p-1)}\dfrac{1}{(2^{1/\alpha}+t)^{\alpha+1}}w^{p-1}(t)-\dfrac{\alpha+1}{2^{1/\alpha}+t}\right)\dot{w}(t).\end{cases}\tag{22}$$

We also have the following relations for the function $g(t,\varepsilon)$ defined in (19);

$$g(t,\varepsilon)^{p-1}=\frac{p(p+1)\varepsilon^{p-1}}{(p-1)^2\varepsilon^{p-1}t+p^2+p},\tag{23}$$

so we have the following partial derivatives up to order 2 in t

$$\begin{cases}\partial_t g=-\dfrac{p-1}{p(p+1)}g^p,\\ \partial_t^2 g=\dfrac{(p-1)^2}{p(p+1)^2}g^{2p-1}.\end{cases}\tag{24}$$

Similarly, from (23) for the partial derivatives in ε we have

$$\begin{cases}\partial_\varepsilon g=g^p\varepsilon^{-p},\\ \partial_\varepsilon^2 g=(-p\varepsilon^{-1}+pg^{p-1}\varepsilon^{-p})\partial_\varepsilon g=(-p\varepsilon^{-1}+pg^{p-1}\varepsilon^{-p})g^p\varepsilon^{-p}.\end{cases}\tag{25}$$

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Finally, differentiating the first relation in (25) in t and using and the first relation in (24), we conclude that the mixed derivative is

$$\partial_t \partial_\varepsilon g = -\frac{p-1}{p+1} g^{2p-1} \varepsilon^{-p}.$$

The above remark leads to the following.

COROLLARY 2.3. *For $\varepsilon > 0$ and $t > 0$, we have the relations*

$$\begin{cases} \partial_t H = \left(\dfrac{\alpha w^{p-1}}{(p-1)(2^{1/\alpha}+t)^{\alpha+1}} - \dfrac{p-1}{p(p+1)} g^{p-1} \right) H, \\ \partial_\varepsilon H = H g^{p-1} \varepsilon^{-p}, \\ \partial_t \partial_\varepsilon H = \left(\dfrac{\alpha w^{p-1} g^{p-1}}{\varepsilon^p (p-1)(2^{1/\alpha}+t)^{\alpha+1}} - \dfrac{(p-1) g^{2p-2}}{(p+1)\varepsilon^p} \right) H, \\ \partial_\varepsilon^2 H = \left(p g^{2p-2} \varepsilon^{-2p} - p g^{p-1} \varepsilon^{-p-1} \right) H \le 0. \end{cases} \tag{26}$$

Proof. The last estimate holds because $g(t, \varepsilon) \le \varepsilon$. □

3. Linear solutions to damped wave equation

In this section, we collect the estimate for kernel parts of $S(t)$ and its derivatives. For simplicity, we denote

$$\omega = \sqrt{t^2 - y^2}.$$

We discuss a pointwise lower bound of I_0 by using the representation

$$I_0(x) = \frac{1}{\pi} \int_{-x}^{x} \frac{e^\eta}{\sqrt{x^2 - \eta^2}} d\eta. \tag{27}$$

For the detail of (27), see [7, 8.431].

LEMMA 3.1.

$$I_0(x) \ge \begin{cases} 1 & \text{if} \quad 0 < x \le 1, \\ \frac{5}{6\pi} \frac{e^x}{\sqrt{x}} & \text{if} \quad x \ge 1. \end{cases}$$

Proof. Let $x > 0$. Changing variables in (27) as $\eta = x \cos\theta$, we have

$$I_0(x) = \frac{1}{\pi} \int_0^\pi e^{x\cos\theta} d\theta = \frac{2}{\pi} \int_0^{\frac{\pi}{2}} \cosh(x\cos\theta)\, d\theta.$$

If $x \le 1$, then we get

$$I_0(x) \ge \frac{2}{\pi} \int_0^{\frac{\pi}{2}} 1 d\theta = 1.$$

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We next consider the case where $x \geq 1$. We note that

$$1 - \frac{\theta^2}{2} \leq \cos\theta \leq 1 - \frac{\theta^2}{\pi}$$

for any $\theta \in [0, \pi/2]$. By using the above inequalities, we obtain

$$\begin{aligned} I_0(x) &\geq \frac{1}{\pi}\int_0^{\frac{\pi}{2}} e^{x\cos\theta} d\theta \\ &\geq \frac{1}{\pi} e^x \int_0^{\frac{1}{\sqrt{x}}} e^{-\frac{x\theta^2}{2}} d\theta \\ &= \frac{\sqrt{2}}{\pi}\frac{e^x}{\sqrt{x}}\int_0^{\frac{1}{\sqrt{2}}} e^{-\rho^2} d\rho \\ &\geq \frac{\sqrt{2}}{\pi}\frac{e^x}{\sqrt{x}}\int_0^{\frac{1}{\sqrt{2}}} \left(1-\rho^2\right) d\rho \\ &= \frac{5}{6\pi}\frac{e^x}{\sqrt{x}}. \end{aligned}$$

This completes the proof of Lemma 3.1. □

Next, we show pointwise upper bounds of $\partial_t S(t)$ and $\partial_t^2 S(t)$. We note $\partial_t S(t)$ and $\partial_t^2 S(t)$ are represented as follows:

$$\partial_t S(t) f(x) = e^{-t/2}\frac{f(x+t)+f(x-t)}{2} + \widetilde{\mathcal{K}}_1(t) * f(x), \tag{28}$$

$$\begin{aligned} \partial_t^2 S(t) f(x) = {} & e^{-t/2}\frac{f'(x+t)-f'(x-t)}{2} \\ & + e^{-t/2}\left(\frac{t}{16}-\frac{1}{2}\right)(f(x+t)+f(x-t)) + \widetilde{\mathcal{K}}_2(t) * f(x), \end{aligned} \tag{29}$$

where

$$\begin{aligned} \mathcal{K}_1(t,y) &= \frac{1}{4}e^{-t/2}\left(\frac{t}{\omega}I_1(\frac{\omega}{2}) - I_0(\frac{\omega}{2})\right), \\ \mathcal{K}_2(t,y) &= e^{-t/2}\left(\frac{t^2}{16\omega^2}I_2(\frac{\omega}{2}) - \left(\frac{t}{4\omega}+\frac{y^2}{4\omega^3}\right)I_1(\frac{\omega}{2}) + \left(\frac{1}{8}+\frac{t^2}{16\omega^2}\right)I_0(\frac{\omega}{2})\right), \end{aligned}$$

I_1 and I_2 are first kind modified Bessel functions of first and second order, and

$$\widetilde{\mathcal{K}_j}(t,y) = \begin{cases} \mathcal{K}_j(t,y) & \text{if} \quad |y| \leq t, \\ 0 & \text{if} \quad |y| \geq t \end{cases}$$

for $j = 1, 2$. For the detail of $\mathcal{K}_1$ and $\mathcal{K}_2$, we refer the reader [5] and reference therein, for example.

Then we have the following estimates:

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LEMMA 3.2. *Let $\phi \in L^1 \cap L^\infty(\mathbb{R};[0,\infty))$ satisfy (11)–(14). Then $\partial_t S(t)\phi$ enjoys the pointwise estimate*

$$|\partial_t S(t)\phi(x)| \le C_\sigma t^{-2(1-\sigma)} S(t)\phi(x)$$

for any $t \ge 2$, $x \in \mathbb{R}$, and $\sigma \in (1/2,1)$, where C_σ is a positive constant depending on σ and ϕ.

Proof. We first note that Lemma 3.1 and the assumption (11) imply that the following estimates hold:

$$S(t)\phi(x) \ge \int_{-1}^{1} e^{\frac{\omega-t}{2}} \frac{\phi(x-y)}{\sqrt{\langle\omega\rangle}} dy \ge Ct^{-\frac{1}{2}}\phi(x). \tag{30}$$

We claim that the estimate

$$e^{-t/2}\frac{\phi(x+t)+\phi(x-t)}{2} \le Ce^{-t/4}\phi(x)$$

holds. Indeed, if $|x| \le 2t$, then the estimates

$$e^{-t/2}\frac{\phi(x+t)+\phi(x-t)}{2} \le Ce^{-t/2} \le Ce^{-t/4}\phi(t) \le Ce^{-t/4}\phi(x).$$

follows from (14) and (12). Moreover, if $|x| \ge 2t$, (12) implies that

$$e^{-t/2}\frac{\phi(x+t)+\phi(x-t)}{2} \le e^{-t/2}\phi(x).$$

This and (30) imply the assertion for the first term in (28).

Now we show

$$\int_{-t}^{t} \mathcal{K}_1(t,y)\phi(x-y)dy \le Ct^{-\frac{1}{2}-2(1-\sigma)}\phi(x).$$

We separate the integral domain into the cases:

$$\begin{aligned} D_1 &= \{y \in [-t,t] \mid \ \omega \le 1\},\\ D_2 &= \{y \in [-t,t] \mid \ \omega \ge 1,\ |y| \le t^\sigma\},\\ D_3 &= \{y \in [-t,t] \mid \ \omega \ge 1,\ |y| \ge t^\sigma\}. \end{aligned}$$

On D_2, we use the estimate

$$\mathcal{K}_1(t,y) \le C\omega^{-3/2}e^{\frac{\omega-t}{2}} + C\left(\frac{t}{\omega}-1\right)\omega^{-1/2}e^{\frac{\omega-t}{2}}. \tag{31}$$

For the detail of (31), we refer [11, Proof of Proposition 2.1] for instance. We note that the estimate

$$\frac{t}{\omega}-1 = \frac{y^2}{\omega(t+\omega)} \le t^{-2(1-\sigma)}.$$

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holds on D_2 because $\omega \sim t$ on D_2. Then the estimate above, (31), and (30) imply that the following estimates hold on D_2:

$$\int_{D_2} \mathcal{K}_1(t,y)\phi(x-y)dy \le Ct^{-2(1-\sigma)} \int_{D_2} \frac{e^{(\omega-t)/2}}{\omega^{1/2}}\phi(x-y)dy \le Ct^{-2(1-\sigma)}S(t)\phi(x).$$

To estimate the integrals on D_1 and D_3, we separate the cases where $|x| \le 2t$ and $|x| \ge 2t$.

Assume $|x| \le 2t$. The estimate $\mathcal{K}_1(t,y) \le Ce^{-t/2}$ holds on D_1 from the analyticity of $\mathcal{K}_1(t,y)$ with respect to ω. Then (14) and (12) again imply the estimate

$$\int_{D_1} \mathcal{K}_1(t,y)\phi(x-y)dy \le Cte^{-t/2} \le Ct^{-\frac{1}{2}-2(1-\sigma)}\phi(x).$$

On D_3, the estimate $e^{\frac{\omega-t}{2}} \le Ce^{-t^{2\sigma-1}/4}$ holds. Therefore, we have

$$\int_{D_3} \mathcal{K}_1(t,y)\phi(x-y)dy \le Ce^{-t^{2\sigma-1}/4}$$

and this implies the assertion on D_3.

In the case where $|x| \ge 2t$, we estimate the integral on D_1 and D_3 similarly by using the estimate

$$\phi(x-y) \le C\phi(x)$$

following from $|y| \le t$ and (12). □

LEMMA 3.3. *Let ϕ satisfy the assumption in Theorem 1.1. Then $\partial_t^2 S(t)\phi$ enjoys the pointwise estimate*

$$|\partial_t^2 S(t)\phi(x)| \le C_\sigma t^{-4(1-\sigma)} S(t)\phi(x)$$

for any $t \ge 2$, $x \in \mathbb{R}$, and $\sigma \in (1/2, 1)$, where C_σ is a positive constant depending on σ.

Proof. We note that the estimate

$$e^{-t/2}\left|\frac{\dot\phi(x+t) - \dot\phi(x-t)}{2}\right| + e^{-t/2}\left(\frac{t}{16} - \frac{1}{2}\right)(\phi(x+t) + \phi(x-t)) \le Ce^{-t/4}\phi(x).$$

is shown similarly to the proof of Lemma 3.2 thanks to (13).

Next, we separate the integral domain into three cases as in the proof of Lemma 3.2. The estimate

$$\int_{D_1 \cup D_3} \mathcal{K}_2(t,y)\phi(x-y)dy \le Ct^{-\frac{1}{2}-4(1-\sigma)}\phi(x).$$

follows similarly to the proof of Lemma 3.2. In order to treat the integral on D_2, we use the following estimate:

$$\mathcal{K}_2(t,y) \le C\left(\frac{1}{\omega^2} + \frac{y^2}{\omega^3} + \frac{y^4}{\omega^2(t+\omega)^2}\right)\frac{e^{\frac{\omega-t}{2}}}{\sqrt{\omega}} \tag{32}$$

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holding for $\omega \geq 1$. We note that (32) is similarly shown to (31). On D_2, we further estimate

$$\left(\frac{1}{\omega^2} + \frac{y^2}{\omega^3} + \frac{y^4}{\omega^2(t+\omega)^2}\right) \leq C(t^{-2} + t^{-3+2\sigma} + t^{-4(1-\sigma)}).$$

Then the estimate above, (32), and (30) imply that

$$\int_{D_2} \mathcal{K}_2(t,y)\phi(x-y)dy \leq Ct^{-4(1-\sigma)}S(t)\phi(x)$$

because the relations $-4+4\sigma > -3+2\sigma$ and $-4+4\sigma > -2$ are equivalent to $\sigma > 1/2$. Therefore, the assertion holds. □

4. Proof of main statements

First, we give the definition of the supersolution to (1);

Definition 4.1. *A function $H^* \in C^2((0,\infty)\times\mathbb{R}) \cap C([0,\infty); L^1 \cap L^\infty)$ is called supersolution if*

$$\begin{cases} \partial_t^2 H^* - \Delta H^* + \partial_t H^* + (H^*)^p \geq 0, & (t,x) \in (0,\infty)\times\mathbb{R}, \\ H^*(0,x) \geq 0, & x \in \mathbb{R}, \\ \partial_t H^*(0,x) + \frac{1}{2}H^*(0,x) \geq 0, & x \in \mathbb{R}. \end{cases}$$

Indeed, if such H^* exists, under initial condition

$$0 \leq u_0 \leq H^*(0), \quad 0 \leq u_1 + \frac{1}{2}u_0 \leq \partial_t H^*(0) + \frac{1}{2}H^*(0),$$

the comparison principle holds: the solution u to (1) enjoys the pointwise estimate

$$0 \leq u(t,x) \leq H^*(t,x)$$

for almost every $t > 0$ and $x \in \mathbb{R}$. For details, see Proposition A.3 in the Appendix.

4.1. Proof of Theorem 1.1

In this subsection, we construct a supersolution to (1) and show the decay estimate of the solution u.

In order to construct a supersolution, we use (19) and replacing ε by $f \in \mathbb{R}$ in (19) we put

$$g(t,f) = \left(\frac{p(p+1)f^{p-1}}{(p-1)^2 f^{p-1}t + p^2 + p}\right)^{1/(p-1)}$$

and $H(t,f) = w(t)g(t,f)$. Further we define the pull back of H with $f = u_L(t,x)$, i.e.

$$H^*(t,x) = H(t,u_L(t,x))$$

where $u_L(t,x) = \varepsilon S(t)\phi(x) + \varepsilon\partial_t S(t)\phi(x)$. We note that $u_L(t,x) > 0$ holds for any $t > 0$ and $x \in \mathbb{R}$ because its integral kernel is shown to be positive. We shall show

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that $H^*(t,x)$ is a supersolution for $t \geq T_0$ with sufficiently large T_0 and with ε small.

Direct computations imply the following identities,

LEMMA 4.2. *We have*

$$\begin{aligned}
&H^* = wg^*, \quad \partial_t(H^*) = (\partial_t H)^* + (\partial_f H)^* \partial_t u_L, \\
&\partial_t^2 H^* = (\partial_t^2 H)^* + 2(\partial_t \partial_f H)^* \partial_t u_L + (\partial_f^2 H)^* (\partial_t u_L)^2 + (\partial_f H)^* (\partial_t^2 u_L) \\
&\partial_x H^* = (\partial_f H)^* \partial_x u_L, \\
&\Delta H^* = (\partial_f^2 H)^* (\partial_x u_L)^2 + (\partial_f H)^* \Delta u_L.
\end{aligned} \tag{33}$$

Proof.

$$H^* = wg^*, \quad \partial_t(H^*) = g^* \partial_t w + w(\partial_t g)^* + w(\partial_f g)^* \partial_t u_L = (\partial_t H)^* + (\partial_f H)^* \partial_t u_L, \tag{34}$$

Moreover,

$$\begin{aligned}
&\partial_t^2 H^* = g^* \partial_t^2 w + 2\partial_t w (\partial_t g)^* + w(\partial_t^2 g)^* + 2\partial_t w (\partial_f g)^* \partial_t u_L \\
&+ 2w(\partial_t \partial_f g)^* \partial_t u_L + w(\partial_f^2 g)^* (\partial_t u_L)^2 + w(\partial_f g)^* \partial_t^2 u_L \\
&= (\partial_t^2 H)^* + 2\partial_t w (\partial_f g)^* \partial_t u_L + 2w(\partial_t \partial_f g)^* \partial_t u_L + w(\partial_f^2 g)^* (\partial_t u_L)^2 + w(\partial_f g)^* \partial_t^2 u_L \\
&= (\partial_t^2 H)^* + 2(\partial_t \partial_f H)^* \partial_t u_L + (\partial_f^2 H)^* (\partial_t u_L)^2 + (\partial_f H)^* (\partial_t^2 u_L)
\end{aligned} \tag{35}$$

and

$$\begin{aligned}
&\partial_x H^* = w(\partial_f g)^* \partial_x u_L = (\partial_f H)^* \partial_x u_L, \\
&\Delta H^* = w(\partial_f^2 g)^* (\partial_x u_L)^2 + w(\partial_f g)^* \Delta u_L \\
&\quad = (\partial_f^2 H)^* (\partial_x u_L)^2 + (\partial_f H)^* \Delta u_L.
\end{aligned} \tag{36}$$

This completes the proof. □

Further we have

$$\begin{aligned}
&\partial_t^2 H^* + \partial_t H^* - \Delta H^* + (H^*)^p \\
&= ((\partial_t^2 H)^* + (\partial_t H)^* + (H^p)^*) + (\partial_f H)^* (\partial_t^2 u_L + \partial_t u_L - \Delta u_L) \\
&+ 2(\partial_t \partial_f H)^* \partial_t u_L + (\partial_f^2 H)^* \left((\partial_t u_L)^2 - (\partial_x u_L)^2 \right)
\end{aligned} \tag{37}$$

Using the fact that $(\partial_t^2 + \partial_t - \Delta) u_L = 0$, we get

$$\begin{aligned}
&\partial_t^2 H^* + \partial_t H^* - \Delta H^* + (H^*)^p \\
&= ((\partial_t^2 H)^* + (\partial_t H)^* + (H^p)^*) \\
&+ 2(\partial_t \partial_f H)^* \partial_t u_L + (\partial_f^2 H)^* \left((\partial_t u_L)^2 - (\partial_x u_L)^2 \right)
\end{aligned} \tag{38}$$

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Using the estimate (20) of Proposition 2.1, we can write

$$\begin{aligned}
&\partial_t^2 H^* + \partial_t H^* - \Delta H^* + (H^*)^p \\
&\geq \frac{C_{p,\alpha}}{(2+t)^{1+\alpha}} H^* + \frac{C_{p,\alpha}}{p+1} (g^*)^{p-1} H^* + 2(\partial_t \partial_f H)^* \partial_t u_L \\
&\quad + (\partial_f^2 H)^* ((\partial_t u_L)^2 - (\partial_x u_L)^2) \\
&\geq \frac{C_{p,\alpha}}{(2+t)^{1+\alpha}} H^* + \frac{C_{p,\alpha}}{p+1} (g^*)^{p-1} H^* + 2(\partial_t \partial_f H)^* \partial_t u_L + (\partial_f^2 H)^* (\partial_t u_L)^2 \qquad (39)
\end{aligned}$$

due to Corollary 2.3. Further, we quote Corollary 2.3 and estimate the terms

$$2(\partial_t \partial_f H)^* \partial_t u_L, (\partial_f^2 H)^* (\partial_t u_L)^2$$

from below. Indeed, from Corollary 2.3, we see that $H \leq u_L$, $g \leq u_L$ so we get

$$\begin{aligned}
&|(\partial_t \partial_f H)^* \partial_t u_L| \\
&= \left| \left(\frac{H^*}{u_L} \right)^{p-1} \frac{\alpha}{(p-1)} \frac{1}{(2^{1/\alpha} + t)^{\alpha+1}} H^* - \left(\frac{g^*}{u_L} \right)^{p-1} \frac{p-1}{(p+1)} (g^*)^{p-1} H^* \right| \left| \frac{\partial_t u_L}{u_L} \right| \\
&\leq C \left(\frac{H^*}{\langle t \rangle^{\alpha+1}} + |g^*|^{p-1} H^* \right) \left| \frac{\partial_t u_L}{u_L} \right| \\
&\leq C \left(\frac{H^*}{\langle t \rangle^{\alpha+1}} + (g^*)^{p-1} H^* \right) \frac{1}{\langle t \rangle^{2(1-\sigma)}} \leq C \frac{H^*}{\langle t \rangle^{2(1-\sigma)+\alpha+1}} + C (g^*)^{p-1} H^* \frac{1}{\langle t \rangle^{2(1-\sigma)}}. \qquad (40)
\end{aligned}$$

In a similar way, we find

$$\begin{aligned}
\left| (\partial_f^2 H)^* (\partial_t u_L)^2 \right| &= \left| p w (g^*)^{2p-1} u_L^{-2p} - p w (g^*)^p u_L^{-p-1} \right| (\partial_t u_L)^2 \\
&\leq C \left| H^* (g^*)^{p-1} u_L^{-p-1} \right| (\partial_t u_L)^2 \\
&= C \left(\frac{g^*}{u_L} \right)^{p-1} H^* \left(\frac{\partial_t u_L}{u_L} \right)^2 \leq C H^* \langle t \rangle^{-4(1-\sigma)}.
\end{aligned}$$

Hence,

$$\begin{aligned}
&|(\partial_t \partial_f H)^* \partial_t u_L| + \left| (\partial_f^2 H)^* (\partial_t u_L)^2 \right| \\
&\quad \leq \frac{C H^*}{\langle t \rangle^{2(1-\sigma)+1+\alpha}} + \frac{C H^*}{\langle t \rangle^{4(1-\sigma)}} + C (g^*)^{p-1} H^* \frac{1}{\langle t \rangle^{2(1-\sigma)}}, \qquad (41)
\end{aligned}$$

and we can continue (39) as follows

$$\begin{aligned}
&\partial_t^2 H^* + \partial_t H^* - \Delta H^* + (H^*)^p \\
&\geq \frac{C_{p,\alpha}}{(2+t)^{1+\alpha}} H^* - \frac{C H^*}{\langle t \rangle^{2(1-\sigma)+\alpha+1}} - \frac{C H^*}{\langle t \rangle^{4(1-\sigma)}} \\
&\quad + C_{p,\alpha} (g^*)^{p-1} H^* - C (g^*)^{p-1} H^* \frac{1}{\langle t \rangle^{2(1-\sigma)}} \qquad (42)
\end{aligned}$$

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The positiveness of the term

$$\frac{C_{p,\alpha}H^*}{(2+t)^{1+\alpha}} - \frac{CH^*}{\langle t\rangle^{2(1-\sigma)+\alpha+1}} - \frac{CH^*}{\langle t\rangle^{4(1-\sigma)}} \tag{43}$$

for large t is guaranteed by

$$1+\alpha < 4(1-\sigma) \Leftrightarrow 3-4\sigma > \alpha.$$

We have also

$$C_{p,\alpha}(g^*)^{p-1}H^* - C(g^*)^{p-1}H^*\frac{1}{\langle t\rangle^{2(1-\sigma)}} > 0$$

for large t. Therefore, choosing first $\sigma \in (1/2, 3/4)$, and then $\alpha \in (0, 3-4\sigma)$ we see that there exists $T_0 \geq 0$ with

$$\partial_t^2 H^*(t,x) + \partial_t H^*(t,x) - \Delta H^*(t,x) + (H^*(t,x))^p > 0$$

for $t \geq T_0$.

We note that

$$H(T_0,x) \geq \frac{1}{T_0^{p-1}}u_L(T_0,x) \geq \varepsilon\frac{1}{T_0^{p-1/2}}\phi(x).$$

Moreover, we compute

$$\begin{aligned}\partial_t H(T_0) + \frac{1}{2}H(T_0) \geq \\ -\frac{p-1}{p(p+1)}g(T_0)^{p-1}H(T_0) + \frac{g(T_0)^{p-1}}{u_L(T_0)^{p-1}}H(T_0)\frac{\partial_t u_L(T_0)}{u_L(T_0)} + \frac{1}{2}H(T_0).\end{aligned}$$

Since we assume ε small, we have $\partial_t H(T_0) + H(T_0)/2 \geq 0$.

REMARK 4.3. In the argument above, it is crucial to cancel $(\partial_f^2 H)^*(\partial_x u_L)^2$ by its sign. Indeed, a similar argument of Corollary 2.3 may imply that

$$(\partial_f^2 H)^*(\partial_x u_L)^2 \leq \frac{CH^*}{\langle t\rangle^{1-\delta}}$$

with some small number δ. This implies that our argument is insufficient to control $(\partial_f^2 H)^*(\partial_x u_L)^2$ by $CH^*\langle t\rangle^{-1-\alpha}$. We also note that our approach is insufficient to construct subsolutions in a similar manner because $(\partial_f^2 H)^*(\partial_x u_L)^2$ has to be treated in this case. On the other hand, this appears to be a technical problem but is in fact an essential one. Indeed, when we examine the decay estimate for the linear solution, we see that for $\rho > \frac{2}{p-1}$, the estimate (15) does not correspond to the estimate (10). As shown by Ikehata, Nishihara, and Zhao [10], the solution decays faster than the estimate (15) suggests. Indeed, it can be seen from Proposition A.2 that the estimate (15) is insufficient for large ρ.

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4.2. Proof of Corollary 1.2

Corollary 1.2 follows from the pointwise estimate

$$S(t)\phi \le Ce^{t\Delta}\phi$$

holding for integrable non-negative function ϕ. Therefore, we have

$$H^*(t,x) \le CG^*(t,x)$$

for any t and x. This and Proposition A.1 imply the assertion.

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References

1. R. Courant and D. Hilbert. *Methods of Mathematical Physics: Partial Differential Equations*, Methods of Mathematical Physics. Vol. 2, ((Wiley), 1989).
2. M. Escobedo, O. Kabian and H. Matano. Large time behavior of solutions of a dissipative semilinear heat equation. *Communications in Partial Differential Equations*. **20** (1995), 1427–1452.
3. M. Escobedo and O. Kavian. Asymptotic behaviour of positive solutions of a nonlinear heat equation. *Houston J. Math*. **14** (1988), 39–50.
4. K. Fujiwara and V. Georgiev. Decay of defocusing damped wave equation with nonlinearity below the Fujita exponent. *Mathematical Methods in the Applied Sciences*. *(14)* **48** (2025), 13799–13807. https://doi.org/10.1002/mma.11143.
5. K. Fujiwara and V. Georgiev. On extended lifespan for 1D damped wave equation. *Mathematical Methods in the Applied Sciences*. *(11)* **48** (2025), 10802–10808.
6. V. A. Galaktionov, S. P. Kurdyumov and A. A. Samarskll. On asymptotic "eigenfunctions" of the Cauchy problem for a nonlinear parabolic equation. *Math. USSR Sbornik TOM*. **54** (1986), 421. https://doi.org/10.1070/SM1986v054n02ABEH002979.
7. S.I Gradshteyn and M.I Ryzhik. *Table of integrals, series, and products*. seventh edition, (Elsevier/Academic Press, Amsterdam, 2007) Translated from the Russian, Translation edited and with a preface by Alan Jeffrey and Daniel Zwillinger, With one CD-ROM (Windows, Macintosh and UNIX).
8. N. Hayashi, E.I Kaikina and P.I Naumkin. Damped wave equation in the subcritical case. *Journal of Differential Equations*. **207** (2004), 161–194.
9. L. Herraiz. Asymptotic behaviour of solutions of some semilinear parabolic problems. *Annales de l'Institut Henri Poincaré C, Analyse non linéaire*. **16** (1999), 49–105.
10. R. Ikehata, K. Nishihara and H. Zhao. Global asymptotics of solutions to the Cauchy problem for the damped wave equation with absorption. *Journal of Differential Equations*. **226** (2006), 1–29.

https://doi.org/10.1017/prm.2025.10119 Published online by Cambridge University Press

11. P. Marcati and K. Nishihara. The Lp-Lq estimates of solutions to one-dimensional damped wave equations and their application to the compressible flow through porous media. *J. Differential Equations*. **191** (2003), 445–469.
12. K. Nishihara and H. Zhao. Decay properties of solutions to the Cauchy problem for the damped wave equation with absorption. *Journal of Mathematical Analysis and Applications*. **313** (2006), 598–610.
13. L. Ta Tsien and Y. Zhou. Breakdown of solutions to $\Box u + u_t = |u|^{1+\alpha}$. *Discrete Contin. Dynam. Systems*. **1** (1995), 503–520.
14. Y. Wakasugi. Decay property of solutions to the wave equation with space-dependent damping, absorbing nonlinearity, and polynomially decaying data. *Mathematical Methods in the Applied Sciences*. **46** (2023), 7067–7107.

Appendix

PROPOSITION A.1. *Let $1 < p < 3$ and $1 < \rho < 2/(p-1)$. Then the following estimate holds as $t \to \infty$:*

$$\left\| \left((p-1)t + (e^{t\Delta}\langle\cdot\rangle^{-\rho})^{1-p} \right)^{-1/(p-1)} \right\|_{L^q} = O(t^{\frac{1}{q\rho(p-1)} - \frac{1}{p-1}}).$$

Proof. We note that the estimates

$$e^{t\Delta}\langle\cdot\rangle^{-\rho}(x) \le C \begin{cases} 1 & \text{if} \quad |x| \le t^{\frac{1}{\rho(p-1)}}, \\ |x|^{-\rho} & \text{if} \quad |x| \ge t^{\frac{1}{\rho(p-1)}} \end{cases},$$

$$e^{t\Delta}\langle\cdot\rangle^{-\rho}(x) \ge C \begin{cases} 0 & \text{if} \quad |x| \le t^{\frac{1}{\rho(p-1)}}, \\ |x|^{-\rho} & \text{if} \quad |x| \ge t^{\frac{1}{\rho(p-1)}} \end{cases}$$

follow from the assumption $1 < \rho < 2/(p-1)$. Indeed, when $|x| \ge t^{\frac{1}{\rho(p-1)}}$, we estimate the upper bound as follows:

$$\begin{aligned} e^{t\Delta}\langle\cdot\rangle^{-\rho}(x) &= \frac{1}{\sqrt{4\pi t}} \int_{|y| \le |x|/2} e^{-y^2/4t} \langle x-y\rangle^{-\rho} dy \\ &\quad + \frac{1}{\sqrt{4\pi t}} \int_{|y| \ge |x|/2} e^{-y^2/4t} \langle x-y\rangle^{-\rho} dy \\ &\le C\langle x\rangle^{-\rho} + Ce^{-|x|^{2-\rho(p-1)}} |x|^{-\frac{1}{2\rho(p-1)}} \int_{|y| \ge |x|/2} \langle y\rangle^{-\rho} dy \le C\langle x\rangle^{-\rho}. \end{aligned}$$

We note that the second term on the right-hand side of the inequality above is negligible when $|x|$ is large enough because it decays exponentially as $|x| \to \infty$. Then we estimate

$$\begin{aligned} \int \left((p-1)t + (e^{t\Delta}\langle\cdot\rangle^{-\rho})^{1-p} \right)^{-q/(p-1)} dx &\le C \int_{|x| \le t^{\frac{1}{\rho(p-1)}}} t^{-\frac{q}{p-1}} dx \\ &\quad + C \int_{|x| \ge t^{\frac{1}{\rho(p-1)}}} \langle x\rangle^{-q\rho} dx \le Ct^{\frac{1}{\rho(p-1)} - \frac{q}{p-1}}. \end{aligned}$$

The lower bound is estimated similarly. □

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PROPOSITION A.2. *Let $1 < p < 3$ and $\rho \geq 2/(p-1)$. Then the following estimate holds for sufficiently large t:*

$$\left\| \left((p-1)t + (e^{t\Delta}\langle\cdot\rangle^{-\rho})^{1-p} \right)^{-1/(p-1)} \right\|_{L^q} \geq Ct^{\frac{1}{2q}-\frac{1}{p-1}} \sqrt{\log(t)}.$$

Proof. We note that for $x \geq 2$, we estimate

$$e^{t\Delta}\langle\cdot\rangle^{-\rho}(x) \geq \frac{1}{\sqrt{4\pi t}} \int_{|y|\leq 1} e^{-(x-y)^2/4t} \langle y\rangle^{-\rho} dy \geq \frac{C}{\sqrt{t}} e^{-x^2/2t}$$

with some constant $C > 0$. We also have

$$\left(\frac{1}{\sqrt{t}} e^{-x^2/2t} \right)^{1-p} \leq t$$

for $|x| \leq \sqrt{\frac{2t}{p-1} \log(t^{\frac{3-p}{2}})} =: r(t)$. Therefore, we estimate

$$\begin{aligned} \left\| \left((p-1)t + (e^{t\Delta}\langle\cdot\rangle^{-\rho})^{1-p} \right)^{-1/(p-1)} \right\|_{L^q} &\geq Ct^{-1/(p-1)} \|1\|_{L^q(2\leq x\leq r(t))} \\ &\geq Ct^{\frac{1}{2q}-\frac{1}{p-1}} \sqrt{\log(t)}. \end{aligned}$$

□

PROPOSITION A.3. *Let $\overline{u}, \underline{u} \in C([0,\infty); L^1 \cap L^p(\mathbb{R}; [0,\infty)))$ be mild solutions to the following inequality:*

$$\begin{cases} \partial_t^2 \overline{u} + \partial_t \overline{u} - \Delta \overline{u} + \overline{u}^p \geq 0, & (t,x) \in (0,\infty) \times \mathbb{R} \\ \partial_t \underline{u} + \partial_t \underline{u} - \Delta \underline{u} + \underline{u}^p \leq 0, \end{cases}$$

under the initial conditions

$$\overline{u}(0,x) \geq \underline{u}(0,x), \quad \partial_t \overline{u}(0,x) + \frac{1}{2}\overline{u}(0,x) \geq \partial_t \underline{u}(0,x) + \frac{1}{2}\underline{u}(0,x).$$

We further assume that we have

$$\sup_{\tau,y} \left(\overline{u}(\tau,y) + \underline{u}(\tau,y) \right)^{p-1} \leq \frac{1}{4p}. \tag{A.1}$$

Then the following inequality holds:

$$\overline{u}(t,x) \geq \underline{u}(t,x)$$

for any $t > 0$ and $x \in \mathbb{R}$.

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Proof. Let $\overline{w} = e^{t/2}\overline{u}$ and $\underline{w} = e^{t/2}\underline{u}$. Then we have

$$\partial_t^2 \overline{w} - \Delta \overline{w} \geq \frac{1}{4}\overline{w} + \overline{u}^{p-1}\overline{w},$$
$$\partial_t^2 \underline{w} - \Delta \underline{w} \leq \frac{1}{4}\underline{w} + \underline{u}^{p-1}\underline{w}.$$

The inequalities above imply that by putting $W = \overline{w} - \underline{w}$, the following inequality holds:

$$W(t,x) \geq \frac{1}{2}\int_0^t \int_{\tau-t}^{t-\tau} \frac{1}{4}W(\tau, x+y) + R(\tau, x+y)dyd\tau,$$

where

$$R(\tau, y) = e^{t/2}\Big(\overline{u}^p(\tau, y) - \underline{u}^p(\tau, y)\Big).$$

The estimate

$$|R(\tau, y)| \leq p\Big(\overline{u}(\tau, y) + \underline{u}(\tau, y)\Big)^{p-1}|W(\tau, y)|$$

follows from a direct computation. Now we put $W_-(\tau, y) = \max\{0, W(\tau, y)\}$. Then, the assumption (A.1) and the discussion above implies that

$$\begin{aligned}\inf_{x\in\mathbb{R}} W_-(t,x) &\geq C\int_0^t \int_{\tau-t}^{t-\tau} \inf_{x\in\mathbb{R}} W_-(\tau, y)dyd\tau \\ &\geq Ct\int_0^t \inf_{x\in\mathbb{R}} W_-(\tau, y)d\tau\end{aligned}$$

with some constant $C > 0$. Therefore, we see that $W_-(t,x) \geq 0$ holds for any $t > 0$ and $x \in \mathbb{R}$. □

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