Mathematika 58 (2012) 1–9 © University College London
doi:10.1112/S0025579311001999

GRAPHS ON EUCLIDEAN SPACES DEFINED USING TRANSCENDENTAL DISTANCES

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Abstract. Given a set D of positive real numbers, let $X_n(D)$ denote the graph with $\mathbb{R}^n$ as the vertex set such that two points are joined if their distance is in D. Bukh conjectured in [Measurable sets with excluded distances. *Geom. Funct. Anal.* **18** (2008), 668–697] that if D is algebraically independent, then $\mathrm{Chr}(X_n(D))$, the chromatic number of $X_n(D)$, is finite. Here we prove that $\mathrm{Chr}(X_n(D))$ is countable and that, if $n = 2$, even the coloring number is countable. Furthermore, we prove that $\mathrm{Chr}(Y)$ is countable, where Y is the following graph on $\mathbb{C}^n$: let $\mathbb{F}$ be a countable subfield of $\mathbb{C}$ and let $D \subseteq \mathbb{C}$ be algebraically independent over $\mathbb{F}$; join $\mathbf{a}, \mathbf{b} \in \mathbb{C}^n$ if there is some $p(\mathbf{x}, \mathbf{y}) \in \mathbb{F}[\mathbf{x}, \mathbf{y}]$ such that $p(\mathbf{x}, \mathbf{x})$ is identically zero and $p(\mathbf{a}, \mathbf{b}) \neq 0$ is algebraic over some $d \in \mathbb{F} \cup D$.

If D is a set of positive real numbers and $1 \leq n < \omega$, we consider the graph $X_n(D)$ on $\mathbb{R}^n$ where two points are joined if their (Euclidean) distance belongs to D. We are interested in the chromatic number $\mathrm{Chr}(X_n(D))$ of $X_n(D)$. For finite D this is a finite problem, but even the exact determination of $\mathrm{Chr}(X_2(\{1\}))$ is notoriously difficult and has been the topic of countless research papers.

It was probably Paul Erdős who first investigated, in [2], the chromatic number of $X_2(D)$ for an infinite D, specifically $D = \mathbb{Q}$. He noticed that $X_2(\mathbb{Q})$ does not contain the complete bipartite graph K_{2,ω_1}, and so $\mathrm{Chr}(X_2(\mathbb{Q}))$ is countable by the results of [3]. The same proof works for the coloring number and for any countable D. Erdős raised the question of whether $\mathrm{Chr}(X_n(D))$ is countable for any n and countable D. It was clear that for $n \geq 3$ the same argument would not work, as the coloring number of $X_3(\mathbb{Q})$ is the continuum. An affirmative answer to Erdős's question was established in [4] for $n = 3$ and in [6] for general n. Later, Schmerl replaced the proof in [6] with a shorter argument (unpublished, but see [8]).

Here we consider the question of whether $\mathrm{Chr}(X_n(D))$ can be small (countable) while D is large (uncountable). In this form, the question can easily be answered: if D is an interval $[a, b]$ with $0 < a < b < \infty$, then $\mathrm{Chr}(X_n(D))$ is finite—one only has to decompose $\mathbb{R}^n$ into identical hypercubes with diagonal less than a and color them in such a way that the distance between any two cubes of the same color is greater than b; clearly, this can be done with finitely many colors, whereas the cardinality of D is the continuum.

Received 13 November 2010, published online 14 June 2011.
MSC (2000): 03E05 (primary), 51M05 (secondary).
The first author's research was supported by the Centre Interfacultaire Bernoulli (École Polytechnique Fédérale de Lausanne) and the Hungarian National Research Grant OTKA K 81121.

https://doi.org/10.1112/S0025579311001999 Published online by Cambridge University Press

To make the question less trivial and more interesting, let us assume that D is an algebraically independent set of real numbers, i.e. there are no elements $a_1, \dots, a_n \in D$ such that $p(a_1, \dots, a_n) = 0$ holds for some non-zero polynomial $p(x_1, \dots, x_n)$ with rational coefficients. A motivation for making this assumption is that the one-dimensional case becomes easy, similar to the case of countable D: there $X_1(D)$ is countably chromatic, as it is a graph with all connected components countable; in our case, if D is linearly independent over $\mathbb{Q}$, then $X_1(D)$ is bipartite (see Proposition 1). We prove that the coloring number of $X_2(D)$ is countable (Theorem 1) and that the chromatic number of $X_n(D)$ (for $n = 3, 4, \dots$) is also countable (Theorem 2). Finally, we prove a result which is a common extension of the case where D is countable and the case where it is algebraically independent (Theorem 3).

In [1], Bukh conjectured that if D is an algebraically independent set of real numbers, then $\mathrm{Chr}(X_n(D))$ is finite. Needless to say, our methods are insufficient for proving this conjecture.

Notation and definitions. Let $\mathbb{R}^n$ denote n-dimensional Euclidean space. We use boldface letters to represent points of $\mathbb{R}^n$. The ith coordinate of $\mathbf{x} \in \mathbb{R}^n$ will be denoted by x_i. The Euclidean distance between two points $\mathbf{x}$ and $\mathbf{y}$ in $\mathbb{R}^n$ is denoted by $d(\mathbf{x}, \mathbf{y})$. A *hyperspace* is a translated copy of some subspace of $\mathbb{R}^n$, and a *sphere in a hypersphere* Z is the set $\{\mathbf{x} \in Z : d(\mathbf{x}, \mathbf{z}) = r\}$ for some $r > 0$ and $\mathbf{z} \in Z$.

A *graph* is a pair (V, E) consisting of a set V of *vertices* and a set E of *edges*, an arbitrary collection of two-element subsets of V. Two vertices x and y are *joined* if $\{x, y\} \in E$. The *chromatic number* $\mathrm{Chr}(V, E)$ of a graph (V, E) is the smallest cardinal μ such that there is a *good coloring* with μ colors, i.e. a coloring of the vertices with μ colors in such a way that no vertices of the same color are joined. The *coloring number* $\mathrm{Col}(V, E)$ of a graph (V, E) is the smallest cardinal μ such that the vertex set V has a well ordering $<$ whereby each vertex is joined with fewer than μ smaller vertices. In this case, a straightforward transfinite recursion gives a good coloring with μ colors, so that $\mathrm{Chr}(V, E) \leq \mathrm{Col}(V, E)$ always holds.

PROPOSITION 1. *If D is linearly independent over $\mathbb{Q}$, then*

$$\mathrm{Chr}(X_1(D)) = 2.$$

Proof. It suffices to show that there is no odd circuit in $X_1(D)$. Suppose, to the contrary, that $v_0, v_1, \dots, v_{2n+1} = v_0$ are the consecutive vertices of an odd circuit. Then $d_i = |v_{i+1} - v_i| \in D$, and so $\lambda_0 d_0 + \lambda_1 d_1 + \cdots + \lambda_{2n} d_{2n} = 0$ where each λ_i is either 1 or -1. Notice that $\lambda_0 + \lambda_1 + \cdots + \lambda_{2n}$ is an odd integer. It is possible that there are repeated numbers among the $d_0, d_1, \dots, d_{2n}$. If the distinct elements are $d_0^*, \dots, d_s^*$, then the above equality contracts to $\lambda_0^* d_0^* + \cdots + \lambda_s^* d_s^* = 0$. Some of the coefficients $\lambda_0^*, \dots, \lambda_s^*$ may be zero, but not all of them, since

$$\lambda_0^* + \cdots + \lambda_s^* = \lambda_0 + \lambda_1 + \cdots + \lambda_{2n}$$

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and the latter expression, an odd integer, cannot be zero. We have thus obtained a non-trivial linear combination of elements of D whose value is zero, contradicting the assumption of linear independence. □

LEMMA 2. *Suppose that a and b are natural numbers, X and Y are sets with $|X| = 2a$, $|Y| = b^{2^{2a}}$ and $X \cap Y = \emptyset$, and $f : X \times Y \to V$ for some set V. Then there exist $X' \subseteq X$ and $Y' \subseteq Y$ with $|X'| = a$ and $|Y'| = b$ such that one of the following holds:*

(1) $f(x, y) = f(x, y')$ *for all* $x \in X'$ *and* $y, y' \in Y'$;
(2) $f(x, y) \neq f(x, y')$ *for all* $x \in X'$ *and* $y \neq y' \in Y'$.

Proof. We use the following version of the pigeonhole principle. If $u_1, \ldots, u_{k^2}$ are elements of V (not necessarily distinct), then there is a set $s \subseteq \{1, \ldots, k^2\}$ with $|s| = k$ such that the u_i, $i \in s$, are either all equal or pairwise distinct.

Enumerate X as $X = \{x_i : 1 \leq i \leq 2a\}$. Using the above principle repeatedly, we obtain sets $Y = Y_0 \supseteq Y_1 \supseteq \cdots \supseteq Y_{2a}$ such that $|Y_i| = b^{2^{(2a-i)}}$, and so $|Y_i| = |Y_{i+1}|^2$ holds for $i < 2a$ and the $f(x_i, y)$, $y \in Y_i$, are either all equal or all different; set $\varepsilon_i = 0$ or 1 according to which case applies. Choose $X' \subseteq X$ with $|X'| = a$ such that ε_i is the same for $x_i \in X'$. Then X' and $Y' = Y_{2a}$ are as required. □

As was pointed out by the referee, Lemma 2 can be deduced quickly from Ramsey's theorem.

LEMMA 3. *If k is a natural number, then there is a natural number $g(k)$ such that the following holds. Suppose that X and Y are sets with $|X| = |Y| = g(k)$ and that $f : X \times Y \to \mathbb{R}$. Then there exist $X^* \subseteq X$ and $Y^* \subseteq Y$ with $|X^*| = |Y^*| = k$ such that one of the following holds:*

(1A) $f(x, y) = f(x, y')$ *for all* $x \in X^*$ *and* $y, y' \in Y^*$;
(1B) $f(x, y) \neq f(x, y')$ *for all* $x \in X^*$ *and* $y \neq y' \in Y^*$;

and one of the following holds:

(2A) $f(x, y) = f(x', y)$ *for all* $x, x' \in X^*$ *and* $y \in Y^*$;
(2B) $f(x, y) \neq f(x', y)$ *for all* $x \neq x' \in X^*$ *and* $y \in Y^*$.

Proof. Set

$$g(k) = 2k^{2^{2k^{2^{2k}}}}.$$

If $|X| = |Y| = g(k)$ and $f : X \times Y \to \mathbb{R}$, apply Lemma 2 to obtain $X' \subseteq X$ and $Y' \subseteq Y$ with $|X'| = k^{2^{2k}}$ and $|Y'| = 2k$, such that either (1A) or (1B) holds. Then apply Lemma 2 again, this time to X', Y' and f, with Y' and X' taking on the roles of X and Y, respectively; this gives $X^* \subseteq X'$ and $Y^* \subseteq Y'$ with $|X^*| = |Y^*| = k$ such that either (2A) or (2B) holds. □

LEMMA 4. *Suppose that k is a natural number, $|X| = |Y| = k^3$, and $f : X \times Y \to \mathbb{R}$ is such that:*

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(1) $f(x, y) \neq f(x, y')$ *for all* $x \in X$ *and* $y \neq y' \in Y$*; and*
(2) $f(x, y) \neq f(x', y)$ *for all* $x \neq x' \in X$ *and* $y \in Y$.

Then there exist $X' \subseteq X$ *and* $Y' \subseteq Y$ *with* $|X'| = |Y'| = k$ *such that the values* $f(x, y)$*, for* $x \in X'$ *and* $y \in Y'$*, are all different.*

Proof. We use induction on k. The $k = 1$ base case is obvious. Assume that we have already established the statement for some k and we are given X, Y and f as described, with $|X| = |Y| = (k+1)^3$. By the inductive hypothesis, we can choose $X^* \subseteq X$ and $Y^* \subseteq Y$ with $|X^*| = |Y^*| = k$ such that the values $f(x, y)$, $x \in X^*$ and $y \in Y^*$, are all different. Set $F = \{f(x, y) : x \in X^*, y \in Y^*\}$; then, clearly, $|F| = k^2$. Define

$$X^+ = \{x \in X : f(x, y) \in F \text{ for some } y \in Y^*\}.$$

By condition (2), we have $|X^+| \leq k^3$, so we can choose $x^* \in X - X^+$. Next, set $F^* = F \cup \{f(x^*, y) : y \in Y^*\}$; then it is obvious that $|F^*| = k(k+1)$. Let

$$Y^+ = \{y \in Y : f(x, y) \in F^* \text{ for some } x \in X^* \cup \{x^*\}\}.$$

A straightforward calculation gives $|Y^+| \leq k(k+1)^2$, so we can pick $y^* \in Y - Y^+$, and then $X' = X^* \cup \{x^*\}$ and $Y' = Y^* \cup \{y^*\}$ are as required. □

THEOREM 1. *If* D *is algebraically independent, then* $\mathrm{Col}(X_2(D)) \leq \omega$.

Proof. If $\mathrm{Col}(X_2(D)) > \omega$, then, by the results of Erdős and Hajnal in [3], it contains arbitrarily large finite complete bipartite graphs. Hence there is a complete bipartite graph with bipartition classes X and Y such that $|X| = |Y| = m$, where $m = g(125)$ from Lemma 3. Applying Lemma 3 to X, Y and d, the Euclidean distance, we obtain $X' \subseteq X$ and $Y' \subseteq Y$ with $|X'| = |Y'| = 125$. A circle in the plane has only one center, so there cannot exist distinct x and x' such that $d(x, y) = d(x, y')$ and $d(x', y) = d(x', y')$ both hold for all y and y' in Y'. This implies that we cannot have (1A). Similarly, we cannot have (2A). In other words, (1B) and (2B) hold for X', Y' and d. Using Lemma 4, we can choose $X'' \subseteq X'$ and $Y'' \subseteq Y'$ with $|X''| = |Y''| = 5$ such that all 25 distances $d(x, y)$ for $x \in X''$ and $y \in Y''$ are distinct. These are 25 different elements of D, which must therefore be algebraically independent. This, however, is not possible, as these elements are all in the algebraic closure of the field $\mathbb{Q}(A)$, where A consists of the 20 coordinates of the points in $X'' \cup Y''$, and, in general, there cannot be more than $|A|$ algebraically independent elements of the algebraic closure of $\mathbb{Q}(A)$; see, for instance, [5]. □

Toward the proof of Theorem 2, let $n \geq 3$ be a natural number and fix $M = g((2n+1)^3)$, where g is as in Lemma 3. We say that $U \subseteq \mathbb{R}^n$ is *closed* if the following conditions hold.

(α) If $1 \leq d \leq n-1$, $Z \subseteq \mathbb{R}^n$ is a d-dimensional hyperspace and $\mathbf{x}_1, \ldots, \mathbf{x}_M \in U \cap Z$ are in general position, i.e. any $d+1$ of them span Z and no $d+2$ of them are in a sphere of Z, then

$$\left\{\mathbf{y} \in Z : d(\mathbf{x}_i, \mathbf{y}) \in D(1 \leq i \leq M)\right\} \subseteq U.$$

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(β) If $Z \subseteq \mathbb{R}^n$ is a sphere in a $(d+1)$-dimensional hyperspace and $\mathbf{x}_1, \ldots, \mathbf{x}_M \in Z \cap U$ are in general position, i.e. any $d+2$ of them span Z and no $d+2$ of them are on a $(d-1)$-dimensional sphere, then

$$\left\{\mathbf{y} \in Z : d(\mathbf{x}_i, \mathbf{y}) \in D(1 \le i \le M)\right\} \subseteq U.$$

LEMMA 5. *If $X = \{\mathbf{x}_1, \ldots, \mathbf{x}_M\}$ and Z are as in (α) or (β) above, then*

$$\left|\left\{\mathbf{y} \in Z : d(\mathbf{x}_i, \mathbf{y}) \in D(1 \le i \le M)\right\}\right| < M.$$

Proof. Suppose, by way of contradiction, that $\mathbf{y}_1, \ldots, \mathbf{y}_M$ are different elements of the above sets. Apply Lemma 3 to get $X' \subseteq X$ and $Y' \subseteq \{\mathbf{y}_1, \ldots, \mathbf{y}_M\}$ with $|X'| = (2n+1)^3$ and $|Y'| = (2n+1)^3$. If case (1A) holds, then $d(\mathbf{x}, \mathbf{y}) = d(\mathbf{x}, \mathbf{y}')$ for any $\mathbf{x} \in X'$ and $\mathbf{y} \neq \mathbf{y}' \in Y'$. So X' is in a hyperspace of Z orthogonal to $\mathbf{y} - \mathbf{y}'$, contradicting our hypothesis on the elements of X being in general position in Z.

If case (2A) holds, then if $\mathbf{y} \in Y'$, we have that $d(\mathbf{x}, \mathbf{y}) = d(\mathbf{x}', \mathbf{y})$ holds for any $\mathbf{x} \neq \mathbf{x}' \in X'$; this means that the elements of X' are on a sphere in Z, which contradicts our assumption that no $d+2$ elements of X are on a sphere with center in Z.

We therefore deduce that (1B) and (2B) hold. We now apply Lemma 4 to X' and Y', obtaining $X'' \subseteq X'$ and $Y'' \subseteq Y'$ with $|X''| = 2n+1$ and $|Y''| = 2n+1$ such that the distances $d(\mathbf{x}, \mathbf{y})$, $\mathbf{x} \in X''$ and $\mathbf{y} \in Y''$, are all distinct. But this is impossible: the coordinates of $X'' \cup Y''$ form a set A with

$$|A| \le n\left|X'' \cup Y''\right| = 2n(2n+1),$$

while the distances $\{d(\mathbf{x}, \mathbf{y}) : \mathbf{x} \in X'', \mathbf{y} \in Y''\}$ form a set of $(2n+1)^2$ different elements of D; therefore these distances are algebraically independent elements of a field algebraically generated by A, a set with $2n(2n+1) < (2n+1)^2$ elements. □

LEMMA 6.

(a) *If α is a limit ordinal and $\{U_\beta : \beta < \alpha\}$ is an increasing sequence of closed sets in $\mathbb{R}^n$, then $U = \bigcup\{U_\beta : \beta < \alpha\}$ is also closed.*

(b) *If $U \subseteq \mathbb{R}^n$, then there is a closed V such that $U \subseteq V \subseteq \mathbb{R}^n$ and $|V| \le |U| + \omega$.*

(c) *If $U \subseteq \mathbb{R}^n$ is closed and $\kappa = |U| > \omega$, then there is an increasing, continuous decomposition $U = \bigcup\{U_\alpha : \alpha < \kappa\}$ such that U_α is closed for each $\alpha < \kappa$.*

Proof. Define

$$F(\mathbf{x}_1, \ldots, \mathbf{x}_M) = \{\mathbf{y} \in Z : d(\mathbf{x}_i, \mathbf{y}) \in D(1 \le i \le M)\},$$

where Z is the hyperspace or sphere-in-hyperspace spanned by the points $\mathbf{x}_1, \ldots, \mathbf{x}_M$, with the $\mathbf{x}_1, \ldots, \mathbf{x}_M$ satisfying either (α) or (β) in the definition of "closed". Note that $U \subseteq \mathbb{R}^n$ is closed if and only if it is closed under the operation F.

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For (a), observe that if $\mathbf{x}_1, \dots, \mathbf{x}_M \in U$, then there is some $\beta < \alpha$ such that $\mathbf{x}_1, \dots, \mathbf{x}_M \in U_\beta$ and thus $F(\mathbf{x}_1, \dots, \mathbf{x}_M) \subseteq U_\beta \subseteq U$.

For (b), let $V = V_0 \cup V_1 \cup \cdots$ where $V_0 = U$ and, for $t = 0, 1, \dots$,

$$V_{t+1} = V_t \cup \bigcup \{F(\mathbf{x}_1, \dots, \mathbf{x}_M) : \mathbf{x}_1, \dots, \mathbf{x}_M \in V_t\}.$$

Clearly, V is closed; moreover, $|V_{t+1}| \leq |V_t| + \omega$ for $t = 0, 1, \dots$ and $|V| \leq |U| + \omega$. Thus V is the smallest closed set containing U; it is the so-called *closed hull* of U.

For (c), enumerate U as $\{\mathbf{x}_\alpha : \alpha < \kappa\}$, and then let U_α be the closed hull of $\{\mathbf{x}_\beta : \beta < \alpha\}$. □

THEOREM 2. *If D is algebraically independent, then* $\mathrm{Chr}(X_n(D)) \leq \omega$ *for all* $n \geq 3$.

Proof. We shall prove that there is a well ordering $\prec$ of $\mathbb{R}^n$ such that

$$\varepsilon(\mathbf{x}) = \inf\big(\{d(\mathbf{y}, \mathbf{x}) : \mathbf{y} \prec \mathbf{x},\ d(\mathbf{x}, \mathbf{y}) \in D\}\big) > 0$$

for every $\mathbf{x} \in \mathbb{R}^n$. This suffices to establish the theorem, as we can define a coloring $c(\mathbf{x})$ as follows: let $c(\mathbf{x}) = \langle k, \mathbf{y}\rangle$, where $1 \leq k < \omega$ is such that $\varepsilon(\mathbf{x}) > 1/k$ and $\mathbf{y} \in \mathbb{Q}^n$ with $d(\mathbf{x}, \mathbf{y}) < 1/(2k)$. We claim that c is a good coloring of $X_n(D)$ with countably many colors. The range of c, i.e. $(\omega - \{0\}) \times \mathbb{Q}^n$, is clearly countable. Assume that $c(\mathbf{x}) = c(\mathbf{y}) = \langle k, \mathbf{z}\rangle$ and, say, $\mathbf{y} \prec \mathbf{x}$. Then $d(\mathbf{x}, \mathbf{y}) \leq d(\mathbf{x}, \mathbf{z}) + d(\mathbf{z}, \mathbf{y}) < 1/k < \varepsilon(\mathbf{x})$ by the triangle inequality, and so $\mathbf{x}$ and $\mathbf{y}$ are not joined in $X_n(D)$ by the definition of $\varepsilon(\mathbf{x})$.

Rather than proving that a well ordering as above exists, we shall prove that for every $U \subseteq \mathbb{R}^n$ there is a well ordering $\prec$ of U such that

$$\inf\big(\{d(\mathbf{x}, \mathbf{y}) : \mathbf{y} \in U,\ \mathbf{y} \prec \mathbf{x},\ d(\mathbf{x}, \mathbf{y}) \in D\}\big) > 0$$

for each $\mathbf{x} \in U$. Our statement is then the special case where $U = \mathbb{R}^n$.

We prove the claim by induction on $\kappa = |U|$. If $\kappa \leq \omega$, the statement is obvious, because in this case we can have a well ordering into some ordinal less than or equal to ω. As then each element is preceded by only finitely many elements, the above infimum is always positive.

Assume now that $\kappa > \omega$ and we are given a set U with $|U| = \kappa$. By Lemma 6(b) we can assume that U is closed. Using Lemma 6(c), we decompose U into the increasing, continuous union $U = \bigcup\{U_\alpha : \alpha < \kappa\}$, where each U_α is closed and $|U_\alpha| < \kappa$. We can also assume that $U_0 = \emptyset$. By the inductive hypothesis, there is a well ordering $\prec_\alpha$ as claimed on $U_{\alpha+1} - U_\alpha$. We define $\prec$ on U as follows: if $\mathbf{x}, \mathbf{y} \in U$, then set $\mathbf{x} \prec \mathbf{y}$ if either $\mathbf{x} \in U_{\alpha+1} - U_\alpha$ and $\mathbf{y} \in U_{\beta+1} - U_\beta$ for some $\alpha < \beta$, or $\mathbf{x}, \mathbf{y} \in U_{\alpha+1} - U_\alpha$ for some α and $\mathbf{x} \prec_\alpha \mathbf{y}$.

It is easy to see that $\prec$ is a well ordering of U. In order to prove the required property, assume that $\mathbf{x} \in U$ and

$$\inf\big(\{d(\mathbf{y}, \mathbf{x}) : \mathbf{y} \in U,\ \mathbf{y} \prec \mathbf{x},\ d(\mathbf{x}, \mathbf{y}) \in D\}\big) = 0.$$

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Let $\alpha < \kappa$ be the unique ordinal for which $\mathbf{x} \in U_{\alpha+1} - U_\alpha$ holds. Choose the elements $\mathbf{x}_0, \mathbf{x}_1, \ldots$ such that $\mathbf{x}_n \prec \mathbf{x}$, $d(\mathbf{x}_n, \mathbf{x}) \in D$ and $d(\mathbf{x}_n, \mathbf{x}) \to 0$. In what follows, we repeatedly shrink this sequence to infinite subsequences. As $\prec_\alpha$ satisfies the required property, $\mathbf{x}_n \in U_{\alpha+1} - U_\alpha$ holds for only finitely many n; therefore $\mathbf{x}_n \in U_\alpha$ for all sufficiently large n, and we can assume without loss of generality that this is true for all n.

Let d be the smallest number such that there is a hyperspace of dimension d containing infinitely many of the $\mathbf{x}_0, \mathbf{x}_1, \ldots$. Upon shrinking the sequence again, we assume that $\mathbf{x}_0, \mathbf{x}_1, \ldots \in Z$ where Z is a d-dimensional hyperspace. Assume first that no sphere in Z contains infinitely many elements of $\mathbf{x}_0, \mathbf{x}_1, \ldots$. By the minimality of d, for each $i_1 < \cdots < i_{d'} < \omega$ (with $2 \le d' \le d$) there are only finitely many elements of the sequence that are contained in the hyperspace (of dimension at most $d' - 1$) spanned by $\mathbf{x}_{i_1}, \ldots, \mathbf{x}_{i_{d'}}$. Similarly, given any sphere S in Z determined by $d + 1$ members of the sequence, only finitely many terms of the sequence are contained in S.

Using these observations, we can choose an infinite subsequence $\mathbf{y}_0, \mathbf{y}_1, \mathbf{y}_2, \ldots$ of $\mathbf{x}_0, \mathbf{x}_1, \mathbf{x}_2, \ldots$ such that any $d' + 1$ elements span a d'-dimensional sub-hyperspace of Z (for $2 \le d' \le d$) and no $d + 2$ of them are in a sphere in Z.

If, however, there is a sphere S in Z that contains infinitely many elements of the sequence, we shrink the sequence to that subsequence. Notice that no infinite subsequence is contained in a sphere whose center is in S, as that would then be an infinite subsequence contained in a hyperspace of dimension less than d. Now we inductively select an infinite subsequence $\mathbf{y}_0, \mathbf{y}_1, \ldots$ of $\mathbf{x}_0, \mathbf{x}_1, \ldots$ such that any $d + 2$ points span Z and no $d + 2$ points are on a sphere whose center is in S. Finally, we rename S to Z.

In either case, Z is (topologically) closed, so, as $d(\mathbf{x}, \mathbf{y}_i) \to 0$, we have $\mathbf{x} \in Z$. We now observe that $\{\mathbf{y}_1, \ldots, \mathbf{y}_M\}$ and Z satisfy Lemma 4; then, by the definition of closed sets, we deduce that $\mathbf{x} \in U_\alpha$, which is a contradiction. □

Finally, we give an extension of Theorem 2, where two points are joined if their distance is algebraic over some $d \in D$. Again, a simple one-dimensional observation, given below, may serve as motivation for the result.

PROPOSITION 7. *For a set $D \subseteq \mathbb{R}$ that is linearly independent over $\mathbb{Q}$, let Y be the graph on $\mathbb{R}$ which has x and y joined if $x - y = \lambda d$ for some $\lambda \in \mathbb{Q} - \{0\}$ and $d \in D$. Then* $\mathrm{Chr}(Y) \le \omega$.

Proof. We may assume that D is a basis of $\mathbb{R}$ over $\mathbb{Q}$. Let $F : \mathbb{R} \to \mathbb{Q}$ be the following coloring: if $x \in \mathbb{R}$, let $x = \lambda_0 d_0 + \cdots + \lambda_{n-1} d_{n-1}$ be the unique decomposition of x into a linear combination of elements of D; then set $F(x) = \lambda_0 + \cdots + \lambda_{n-1}$. Obviously, if $F(x) = F(y)$ and $x - y = \lambda d$ for some $d \in D$, then $\lambda = F(x) - F(y) = 0$, which is a contradiction. □

THEOREM 3. *Let $\mathbb{F}$ be a countable subfield of $\mathbb{C}$ and let $D \subseteq \mathbb{C}$ be algebraically independent over $\mathbb{F}$. Define a graph Y_n on $\mathbb{C}^n$ by joining points $\mathbf{a}$ and $\mathbf{b}$ if there is some $p(\mathbf{x}, \mathbf{y}) \in \mathbb{F}[\mathbf{x}, \mathbf{y}]$ such that $p(\mathbf{x}, \mathbf{x})$ is identically zero and $p(\mathbf{a}, \mathbf{b}) \neq 0$ is algebraic over some $d \in D \cup \mathbb{F}$. Then* $\mathrm{Chr}(Y_n) \le \omega$.

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Proof. The coloring will be a slight modification of the one used in [9], which the reader may consult for more details about this proof. We let $\mathbb{C} = (\mathbb{C}, +, \cdot, 0, 1)$ be the field of complex numbers and, without loss of generality, assume that $\mathbb{F}$ is an algebraically closed subfield. We say that $U \subseteq \mathbb{C}$ is a *rational disk* if there are $a, b, r \in \mathbb{Q}$ such that $r > 0$ and

$$U = \{z \in \mathbb{C} : |a + b\mathrm{i} - z| < r\}.$$

We can assume that D is a transcendence basis over $\mathbb{F}$ (by extending it if necessary). If $R \subseteq \mathbb{C}^n$ and $X \subseteq \mathbb{C}$, we say that *R is X-definable* if it is definable in the structure $(\mathbb{C}, \mathbb{R}, \mathrm{i})$ by a first-order formula in which we allow parameters from $X \cup \mathbb{F}$.

If $\mathbf{a} \in \mathbb{C}^n$, then the *support of* $\mathbf{a}$, $\mathrm{supp}(\mathbf{a})$, is the (necessarily finite) smallest $D_0 \subseteq D$ such that $\{\mathbf{a}\}$ is D_0-definable.

Finally, let $\lhd$ be some linear order of $\mathbb{C}$.

We now define the good coloring F of $\mathbb{C}^n$ with countably many colors. Pick $\mathbf{a} \in \mathbb{C}^n$ with $\mathrm{supp}(\mathbf{a}) = \{t_0, t_1, \ldots, t_{m-1}\}$ where $t_0 \lhd t_1 \lhd \cdots \lhd t_{m-1}$. Let $F(\mathbf{a})$ be an $\emptyset$-definable analytic function $\mathbf{f} : U \to \mathbb{C}^n$ such that the following hold:

(1) $U = U_0 \times U_1 \times \cdots \times U_{m-1}$ where the U_i are pairwise disjoint rational disks;
(2) $\mathbf{t} = \langle t_0, t_1, \ldots, t_{m-1}\rangle \in U$;
(3) $\mathbf{f}(\mathbf{t}) = \mathbf{a}$;
(4) if $j < n$ and $I \subseteq m$ are such that $\mathrm{supp}(a_j) = \{t_i : i \in I\}$, then f_j is such that whenever $\mathbf{x}, \mathbf{y} \in U$ and $x_i = y_i$ for all $i \in I$, we have $f_j(\mathbf{x}) = f_j(\mathbf{y})$.

It is obvious that F is a coloring of $\mathbb{C}^n$ with countably many colors. In order to show that it is a good coloring, assume for contradiction that $F(\mathbf{a}) = F(\mathbf{b}) = \mathbf{f}$, $p(\mathbf{x}, \mathbf{y}) \in \mathbb{F}[\mathbf{x}, \mathbf{y}]$ is a polynomial such that $p(\mathbf{x}, \mathbf{x})$ is identically 0, and $p(\mathbf{a}, \mathbf{b}) \neq 0$ is algebraic over $\mathbb{F} \cup \{d\}$ where $d \in D$. Thus, there is an irreducible, non-zero polynomial $q(u, v) \in \mathbb{F}[u, v]$ such that $q(p(\mathbf{a}, \mathbf{b}), d) = 0$. Let $\mathrm{supp}(\mathbf{a}) = \{s_0, s_1, \ldots, s_{m-1}\}$ and $\mathrm{supp}(\mathbf{b}) = \{t_0, t_1, \ldots, t_{m-1}\}$, where $\mathbf{s} = \langle s_0, s_1, \ldots, s_{m-1}\rangle \in U$ and $\mathbf{t} = \langle t_0, t_1, \ldots, t_{m-1}\rangle \in U$. Hence

$$q\big(p(\mathbf{f}(\mathbf{s}), \mathbf{f}(\mathbf{t})), d\big) = 0.$$

Without loss of generality, we can assume that if $d \in \mathrm{supp}(\mathbf{a})$, then $d \in \mathrm{supp}(\mathbf{b})$. Thus, exactly one of the following holds:

- $d \notin \mathrm{supp}(\mathbf{a}) \cup \mathrm{supp}(\mathbf{b})$;
- $d \in \mathrm{supp}(\mathbf{b}) - \mathrm{supp}(\mathbf{a})$;
- $d = s_j = t_j$ for some $j < m$.

For each of these possibilities, it follows that $q(p(\mathbf{f}(\mathbf{s}), \mathbf{f}(\mathbf{t})), d) = 0$. As a representative example, we consider the first case. Let $g(\mathbf{x}, \mathbf{y}) = q(p(\mathbf{f}(\mathbf{x}), \mathbf{f}(\mathbf{y})), d)$, which is a $\{d\}$-definable analytic function such that $g : U^2 \to \mathbb{C}$. Since $g(\mathbf{s}, \mathbf{t}) = 0$ and $\mathbf{s}, \mathbf{t} \in (D \backslash \{d\})^m$, we get that $g(\mathbf{s}, \mathbf{s}) = 0$. Thus, $q(p(\mathbf{a}, \mathbf{a}), d) = g(0, d) = 0$. Since d is transcendental over $\mathbb{F}$, this implies that $q(0, v)$ is identically 0, contradicting the fact that $q(u, v)$ is irreducible and non-zero. □

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Acknowledgement. We are grateful to the referee for several suggestions which greatly helped to improve the exposition.

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