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Cubic polynomials with a parabolic point

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Abstract. We consider cubic polynomials with a simple parabolic fixed point of multiplier 1. For those maps, we prove that the boundary of the immediate basin of attraction of the parabolic point is a Jordan curve (except for the polynomial $z + z^3$ where it consists in two Jordan curves). Moreover, we give a description of the dynamics and obtain the local connectivity of the Julia set under some assumptions.

1. *Introduction*

Every polynomial of degree three is conjugate (by an affine map) to a polynomial of the form $\lambda z + az^2 + z^3$ with $(a, \lambda) \in \mathbf{C}^2$. We restrict ourself to the case $\lambda = 1$, i.e. to the cubic polynomials of the form $z + az^2 + z^3$ possessing a parabolic fixed point of multiplier 1 at $\mathbf{0}$.

Let f be such a polynomial. The basin of attraction of $\mathbf{0}$, i.e. $\tilde{B} := \{z \mid f^n(z) \to \mathbf{0}\}$, has exactly one connected component containing $\mathbf{0}$ in its boundary, except for $f(z) = z + z^3$ when the parabolic fixed point $\mathbf{0}$ is double and is in the closure of exactly two connected components. Denote by B the union of these components; it is called the *immediate basin*.

THEOREM 1. *The boundary of the immediate basin B of* $\mathbf{0}$ *is a Jordan curve, or the union of two Jordan curves in the particular case of the polynomial* $z^3 + z$.

This theorem is the generalization to the parabolic case of the results obtained in the super-attracting case in [**Ro1**] (see also the appendix of [**Ro2**]). The principal novelty here is that we deal with difficulties arising from the presence of a parabolic point (see also [**PeRo**] for a different approach of this problem). Apart from this, the proof of Theorem 1 essentially uses the classical theory of Yoccoz puzzles and does not need any 'refinement of nests', as recently developed in the case of several critical points (see [**KSS, QY, RoYi, TaYi2**]).

Theorem 1 is also the first step in understanding the parameter space of cubic polynomials with such a parabolic fixed point (see [**Ro3**]). Therefore, we give here also a description of the dynamics (Theorem 2), as well as a discussion of local connectivity for the whole Julia set (Theorems 3 and 4).

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FIGURE 1. The Julia set.

Denote by $K(f) := \{z \mid f^n(z) \text{ is bounded}\}$ the *filled-in Julia set* of f and by $J = J(f)$ the corresponding *Julia set*: $J(f) := \partial K(f)$ (see Figure 1). Provided it is connected, the Julia set $J(f)$ is known to be locally connected in the *geometrically-finite* case, i.e. if the closure of the orbit critical points intersects $J(f)$ under a finite set, (see [**TaYi1**]).

In general, as in the quadratic case, one has to consider the renormalizations of f. The map f is called *renormalizable* if there exist an integer $k > 0$ and two open topological disks U, V with $\overline{U} \subset V$ such that $f^k : U \to V$ is holomorphic, proper of degree two and $K(f^k) := \bigcap_n f^{-kn}(U)$ is connected. We say then that f is k-renormalizable and $K(f^k)$ is called the filled-in Julia set of the renormalization.

The parabolic point can generate more complicated dynamics called 'weakly quadratic-like' maps (see [**Go-Mi**]). We give a description of the dynamics in terms of limbs attached to $\overline{B}$ as follows.

THEOREM 2. *Let K_B be the connected component of $K(f)$ containing B. Then,*

$$K_B = \overline{B} \cup \bigcup_{t \in \mathbf{R}/\mathbf{Z}} L(t) \quad \textit{where } \operatorname{card}(L(t) \cap \overline{B}) = 1.$$

A limb $L(t)$ is not reduced to a point if and only if there exists $n \geq 0$ such that $2^n t = t_0$, where $L(t_0)$ contains a critical point.

Concerning the local connectivity of the whole Julia set, we obtain the following results.

THEOREM 3. *If $L(\mathbf{0}) = \{\mathbf{0}\}$ and if f is not renormalizable, the Julia set $J(f)$ is locally connected provided it is connected.*

THEOREM 4. *If f has a connected Julia set and has $L(\mathbf{0}) = \{\mathbf{0}\}$, then the Julia set is locally connected if any one of the following is true.*

(1) *f is geometrically-finite.*
(2) *f is of 'Yoccoz type', i.e. is not infinitely renormalizable and has only repelling cycles except from $\mathbf{0}$.*

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(3) *f is of 'Petersen–Zakeri type', i.e. has an irrational indifferent periodic point whose rotation number satisfies the Petersen–Zakeri condition (see* [**PeZa**, Theorem A]*, so that the corresponding Julia set (see Proposition 7.6) is locally connected.*

Note that the limb $L(\mathbf{0})$ containing the parabolic point $\mathbf{0}$ plays a special role. Handling the case $L(\mathbf{0}) \neq \mathbf{0}$ needs other ideas coming from [**PeRo**]. This is the object of a coming paper (see [**Ro3**]).

The paper develops as follows. In §2, Proposition 2.4, we prove the local connectivity of ∂B at the parabolic fixed point (and therefore at all its iterated inverse images) reformulating an argument of Kiwi (see [**Ki**]). Theorem 1 is proved in §5. The proof involves Yoccoz' results for puzzles 'jigsawed' by graphs containing 'periodic' accesses. These accesses are constructed in §3. Finally, we describe the dynamics of f in §7 and prove Theorems 3 and 4 in §8. All this is based on a partial description of $K(f)$ obtained in §4. For backgrounds in dynamics we recommend the articles [**D-H1, Mi1, Ca-Ga, St**].

1.1. *Background.* First of all, note that it is enough to prove that the boundary of B is locally connected. Indeed, B is conformally equivalent to the unit disk (or two the union of two such disks for $a = 0$), as this is the case for every bounded component of $K(f)$ for polynomials. Therefore, by a theorem of Carathéodory, the boundary of B is a curve. Applying the maximum principle, one sees easily that it is a Jordan curve.

The map f has two critical points. One of them is in the immediate basin B, call it c_0 and the other one c. It is classical that $J(f)$ is connected if and only if c is not in the basin of infinity $B(\infty) := \{z \in \mathbf{C} \mid f^n(z) \to \infty\}$. Note that if c is in $B(\infty)$, the map f is geometrically finite and by the work of Tan Lei and Yin Yongcheng, the boundary of B is locally connected, so that it is a Jordan curve.

Remark that if $a = 0$, f is also geometrically finite. Indeed, the parabolic point is double since $f(z) = z^3 + z$ and B has two connected components symmetric by $z \mapsto -z$. Therefore the boundary of B is now the union of two Jordan curves.

We ignore now these two particular cases and their 'extensions' which are clearly geometrically finite cases, by taking for the rest of the article the following assumptions.

Assumption. We assume that the orbit of the free critical point c does not intersect the union $B \cup \{\mathbf{0}\} \cup B(\infty)$.

Remark 1.1. $K(f)$ and B are connected.

Coordinates at ∞. Let ϕ_∞ be the Böttcher coordinate at infinity. It is a conformal bijection from $B(\infty)$ onto $\mathbf{C}\backslash\overline{\mathbf{D}}$ and satisfies $\phi_\infty \circ f(z) = (\phi_\infty(z))^3$. Denote the *equipotential* of level $v > 0$ by $E_\infty(v) := \phi_\infty^{-1}(\{e^{v+2\pi i\theta}, \theta \in \mathbf{R}/\mathbf{Z}\})$ and by $R_\infty(\theta) := \phi_\infty^{-1}(\{re^{2\pi i\theta}, r > 1\})$ the *external ray* of angle $\theta \in \mathbf{R}/\mathbf{Z}$. Recall that every ray with rational angle lands at a (eventually) periodic point which is either parabolic or repelling (see [**D-H1, Mi1**]). Conversely, since $J(f)$ is connected one has (see [**Mi1, Pe**]).

PROPOSITION 1.2. (Douady–Yoccoz) *For every repelling or parabolic (eventually) periodic point of f there exists a ray $R_\infty(\theta)$ landing at it. Moreover, the angle θ is rational.*

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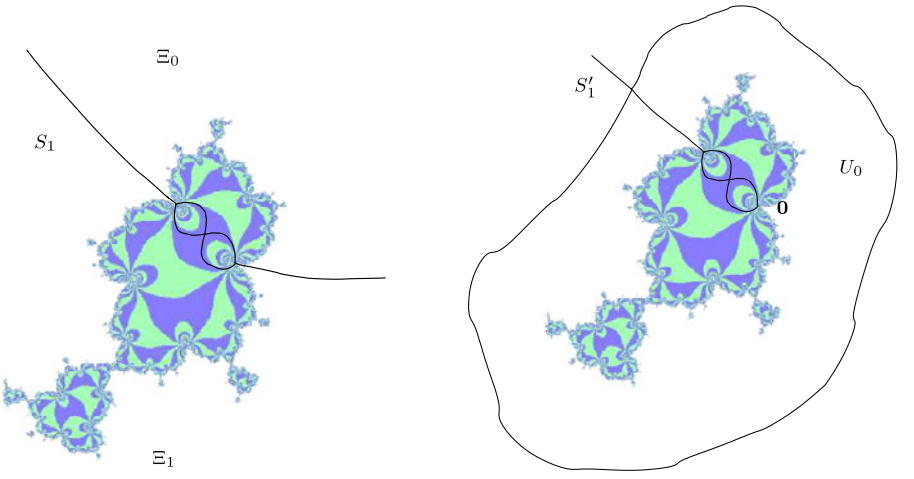


FIGURE 2. The graph S_1, S_1' and the domains Ξ_0, Ξ_1.

In particular, the parabolic fixed point $\mathbf{0}$ is the landing point of an external fixed ray (since $f'(\mathbf{0}) = 1$, see [**Go-Mi, Pe**]). Thus, up to changing the Böttcher coordinate ϕ_∞ by $-\phi_\infty$, we can assume that $R_\infty(0)$ lands at the parabolic fixed point $\mathbf{0}$.

Coordinates at $\mathbf{0}$. Let ϕ_0 be the Fatou coordinate at the parabolic fixed point $\mathbf{0}$, defined on a topological disk $U \subset B$ (an attracting petal) such that $\phi_0 : U \to \{z \mid \Re e(z) > 0\}$ is biholomorphic, satisfies $\phi_0(f(z)) = \phi_0(z) + 1$ with $\phi_0(f(c_0)) = 1$ and ∂U, which contains $\mathbf{0}$ and c_0, is smooth outside $\mathbf{0}$ and c_0. It extends to a semi-conjugacy $\overline{\phi_0} : B \to \mathbf{C}$, defined up to the post-addition of a constant. The 'lines' $E_{\mathbf{0}}(k) = \overline{\phi_0}^{-1}(\{k + it, t \in \mathbf{R}\})$ will play the role in the basin B of the equipotentials of the usual attracting case. (Compare with [**PeRo**] for a different approach.)

2. *Local connectivity at the parabolic point* $\mathbf{0}$

We define a sequence of neighbourhoods of the parabolic fixed point $\mathbf{0}$ and prove that the diameters tend to zero or some part of $K(f)$ is attached there (Proposition 2.4).

Definition 2.1. (See Figure 2.) Let S_1 be the connected component of $f^{-1}(S_0)$ containing c_0, where $S_0 = E_{\mathbf{0}}(1) \cup R_\infty(0) \cup \{\mathbf{0}\}$ and let S_1' be $E_\infty(1) \cup (S_1 \backslash R_\infty(0))$. Denote by U_0 the connected component of $\mathbf{C} \backslash S_1'$ containing $\mathbf{0}$ in its boundary and not included in B. We define recursively U_n as the connected component of $f^{-1}(U_{n-1})$ containing $\mathbf{0}$ in its boundary. Denote the intersection $\bigcap_{n \geq 0} \overline{U_n}$ by X.

Remark 2.2. There is only one connected component of $f^{-1}(U_{n-1})$ containing $\mathbf{0}$ in its boundary, otherwise $\mathbf{0}$ has to be critical. So the sequence $(U_n)_{n \in \mathbf{N}}$ is well-defined.

Remark 2.3. By construction, the sets $\overline{U}_n \cap \partial B$ are connected neighbourhoods of $\mathbf{0}$ in ∂B. Moreover, $U_{n+1} \subset U_n$ and the map $f : U_{n+1} \to U_n$ is a (ramified) covering of degree $d \leq 2$ between topological disks.

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Proof. The critical point c_0 is not in U_n so the degree d is less than or equal to two and U_n are disks by induction. □

PROPOSITION 2.4. *Either X reduces to the fixed point $\{\mathbf{0}\}$ or the rays $R_\infty(0)$ and $R_\infty(1/2)$ converge to $\mathbf{0}$ and $\gamma = R_\infty(0) \cup R_\infty(1/2) \cup \{\mathbf{0}\}$ separates $X\backslash\{\mathbf{0}\}$ from $\overline{B}\backslash\{\mathbf{0}\}$.*

COROLLARY 2.5. *The boundary ∂B is locally connected at the parabolic point $\mathbf{0}$ and at all its iterated inverse images.*

Proof of Corollary 2.5. If X reduces to the fixed point $\mathbf{0}$, the sequence $(\overline{U}_n \cap \partial B)$ is a basis of connected neighbourhoods of $\mathbf{0}$ in ∂B. For the second case of Proposition 2.4, let V be the connected component of $\mathbf{C}\backslash\gamma$ containing B and take $V_n = V \cap U_n$. The sequence $(\overline{V_n})$ has diameter shrinking to zero since $\bigcap_n \overline{V}_n \subset \overline{V} \cap (\bigcap_n \overline{U}_n) = \overline{V} \cap X = \{\mathbf{0}\}$. Thus $(\overline{V}_n \cap \partial B)$ is a basis of connected neighbourhoods of $\mathbf{0}$ in ∂B since each $\overline{V_n}$ intersects ∂B under a connected set (by construction).

For an iterated inverse image $x \in \partial B$ of $\mathbf{0}$, there exists i such that $f^i(x) = \mathbf{0}$. Consider W_n the connected components of $f^{-i}(V_n)$ (or $f^{-i}(U_n)$) containing x in its closure. Then $(\overline{W}_n \cap \partial B)_{n\in\mathbf{N}}$ forms a basis of connected neighbourhoods of x in ∂B. □

Proof of Proposition 2.4. The proof is by contradiction. Assume that X is not a point. Therefore it is a non-trivial full connected compact set (the intersection of a decreasing sequence of closed disks). Hence, we can consider a conformal representation Φ from $\mathbf{C}\backslash X$ onto $\mathbf{C}\backslash\overline{\mathbf{D}}$.

Step 1. There exist A_N, A_{N-1} symmetric neighbourhoods of $\mathbf{S}^1$ such that $\Phi \circ f \circ \Phi^{-1}$ is defined on $A_N\backslash\overline{\mathbf{D}}$ and admits an extension $F\colon A_N \to A_{N-1}$ which is a holomorphic covering of degree $d \leq 2$ (see Figure 3).

Proof. By definition the sequence (U_n) is decreasing to X, so there exists N such that $c \notin U_N\backslash X$. Therefore $f^{-1}(X) \cap U_N = X$, noting that the set X is fixed by f (since $f(U_n) = U_{n-1}$ for all $n > 0$). Thus the map $F = \Phi \circ f \circ \Phi^{-1}$ is well defined from $\Phi(U_N\backslash X)$ to $\Phi(U_{N-1}\backslash X)$. The open set $\Phi(U_N\backslash X)$ is not a neighbourhood of $\mathbf{S}^1$ since $X \cap \partial U_N$ contains the iterated inverse images of the parabolic fixed point $\mathbf{0}$. Therefore, consider $\widetilde{U}_{N-1}$ a small enlargement of U_{N-1} near the points of $X \cap \partial U_{N-1}$ (by a small ball for instance) such that $f(c) \notin \widetilde{U}_{N-1}\backslash X$ and let $\widetilde{U}_N$ be the connected component of $f^{-1}(\widetilde{U}_{N-1})$ containing $\mathbf{0}$. Thus, $f\colon \widetilde{U}_N \to \widetilde{U}_{N-1}$ is a (ramified) covering of degree $d \leq 2$, and, if $W_i = \Phi(\widetilde{U}_i\backslash X)$, the map $F\colon W_N \to W_{N-1}$ is then a covering of degree d. Now, let A_N be $W_N \cup \mathbf{S}^1 \cup \tau W_N$ where τ is the reflection in the circle; F extends to a covering $F\colon A_N \to A_{N-1}$ (of degree $d \leq 2$) using the Schwarz reflection principle.

Step 2. The restriction $g = F_{|_{\mathbf{S}^1}}$ is a degree d covering which preserves the orientation. Every fixed point of f in X with a fixed access from $\mathbf{C}\backslash X$ gives a fixed point for g.

Proof. The map $F_{|_{\mathbf{S}^1}}$ is a degree d covering which preserves the orientation since $F\colon A_N \to A_{N-1}$ is a holomorphic covering of degree d preserving $\mathbf{S}^1$. Let $\delta \subset \mathbf{C}\backslash X$ be a fixed access converging to a fixed point $x \in X$. The access $\Phi(\delta)$ is fixed by F and converges (see [**Ca-Ga, Po**]) to some point $p \in \mathbf{S}^1$ which is necessary fixed by g.

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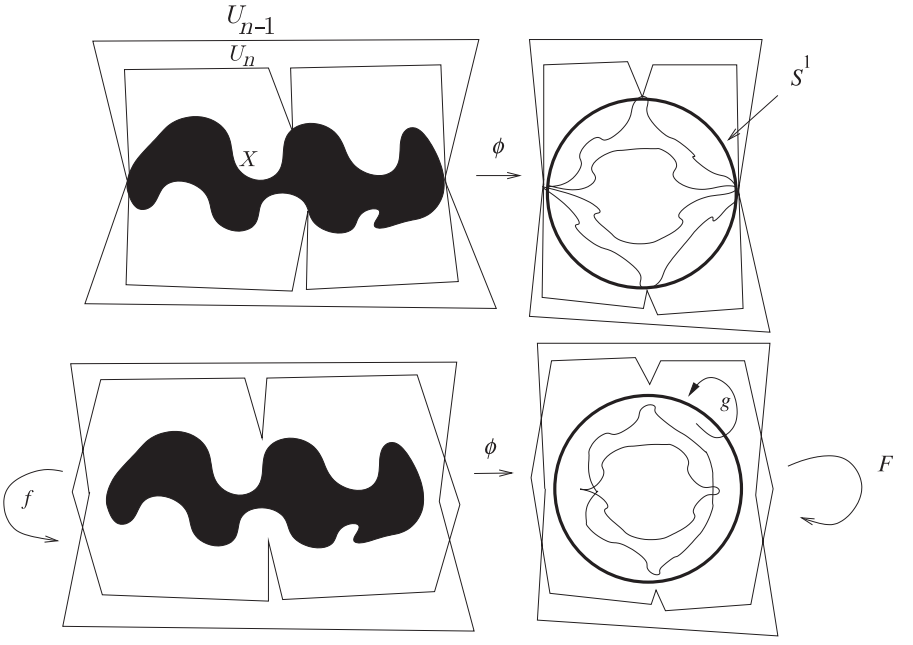


FIGURE 3. The neighbourhood U_n with a schematic representation of g and X.

Step 3. Every fixed point p of g is weakly repelling, i.e. satisfies, for every $x \neq p$, $|g(x) - p|_{\mathbf{S}^1} > |x - p|_{\mathbf{S}^1}$. Thus g has exactly $d - 1$ fixed points.

Proof. Assume, to get a contradiction, that p is attracting on one side (at least). Then p is an attracting or a parabolic fixed point for the holomorphic map F so that there exists an arc $\alpha \subset W_N$ from p to some point $q \in \mathbf{S}^1$ bounding an open set Ω_1 in $\mathbf{C}\backslash\overline{\mathbf{D}}$ such that $F(\Omega_1) \subset \Omega_1$ (in the attracting domain of p). Let Ω be $\Phi^{-1}(\Omega_1)$. It satisfies $f(\Omega) \subset \Omega$, thus the family (f^n) is normal on Ω. Morally this gives a 'neighbourhood' of points of J on which f is normal. We prove that this is impossible. There is a fixed Fatou component $\widetilde{\Omega} \neq B(\infty)$ containing Ω and (by Denjoy–Wolff's theorem) $\overline{\widetilde{\Omega}}$ contains a fixed point attracting every point of $\widetilde{\Omega}$. Moreover $\partial\Omega \cap X \subset \partial\widetilde{\Omega}$ contains more than one point. So there is a cross cut c of X in $\widetilde{\Omega}$. Indeed, the map Φ^{-1} admits limit points at almost every $\theta \in (p, q)$ by Fatou's theorem and these limit points are not all equal (see for instance Koebe lemma page 31 of [**Go**]). This cross cut c bounds a domain U which is the trace in $\widetilde{\Omega}$ of some open set U' intersecting J (i.e. $U = U' \cap \widetilde{\Omega}$). There is some $i > 0$ such that $f^i(U') \supset J$ and therefore $f^i(\partial U \cap \partial\widetilde{\Omega}) = \partial\widetilde{\Omega}$. Every point of $\partial U \cap \partial\widetilde{\Omega}$ is in X (the filled in Julia set of a polynomial is full). Moreover the map f on $\widetilde{\Omega}$ is conjugate to a Blashcke product $b : \mathbf{D} \to \mathbf{D}$ with Julia set $\partial\mathbf{D}$. It follows that at most one point of $\widetilde{\Omega}$ is not in $\bigcup_{n\geq 1} f^n(U)$. Hence $\partial\widetilde{\Omega} \subset X$. This is not possible since X is a full compact connected set and $\widetilde{\Omega}$ is a topological disk.

We prove now that g has $d - 1$ fixed points. Let $\widetilde{g}\colon \mathbf{R} \to \mathbf{R}$ with $\widetilde{g}(0) = 0$ be a lift of g; it is an increasing homeomorphism of $\mathbf{R}$. Since every fixed point of g is weakly repelling,

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the graph of $\widetilde{g}$ has to cross the diagonal and its translates from below to above. Therefore the graph crosses exactly $d-1$ translates of the diagonal (including the diagonal) exactly once. The result follows. □

Step 4. The rays $R_\infty(0)$ and $R_\infty(1/2)$ converge to the fixed point $\mathbf{0}$ and γ separates $X\backslash\{\mathbf{0}\}$ from $\overline{B}\backslash\{\mathbf{0}\}$.

Proof. The map g has at least one fixed point on $\mathbf{S}^1$, it corresponds to the ray $R_\infty(0)$ by Step 2. Therefore, $d=2$ and g has only one fixed point. Indeed, by Step 3 the number f of fixed points of g satisfies $1 \le f \le d-1$ and $d \le 2$ by Remark 2.3.

It remains to prove that $R_\infty(1/2)$ enters every U_n for $n>0$. Indeed, this implies that $R_\infty(1/2)$ lands at a fixed point in X that corresponds to the unique fixed point of g, i.e. $\Phi(R_\infty(0))$ and $\Phi(R_\infty(1/2))$ land at the same point. Thus $R_\infty(0)$ and $R_\infty(1/2)$ both land at $\mathbf{0}$ (see [**Po**]).

Since $f: U_n \to U_{n-1}$ is of degree $d=2$ for $n>N$, there is an inverse image, say $R_\infty(\eta)$, of $R_\infty(0)$ converging to a point of X, so entering every U_n for $n>0$. On the other side, the boundary of U_n contains portions of external rays $R_\infty(\theta_n)$, $R_\infty(\theta'_n)$ landing at points of ∂B, say $\theta'_n < \eta < \theta_n$ (for the cyclic order on $\mathbf{R}/\mathbf{Z}$). Moreover, these external rays satisfy $f(R_\infty(\theta_n)) = R_\infty(\theta_{n-1})$, $f(R_\infty(\theta'_n)) = R_\infty(\theta'_{n-1})$ and $0<\theta_n<\theta_{n-1}<\theta'_{n-1}<\theta'_n<1$ (by definition of U_n). Therefore, the sequences (θ_n) and (θ'_n) converge to limit angles θ and θ', which are distinct since $\theta' \le \eta \le \theta$ and since η is prefixed while θ, θ' are fixed by multiplication by 3. Thus one of θ, θ' is 0 and the other one $1/2$, so that $R_\infty(1/2)$ enters every U_n for $n>0$.

To prove that γ separates B from X, consider V and W the two connected components of $\mathbf{C}\backslash\gamma$ with $B \subset V$. For degree reasons, c belongs to W, so $X \cap W \neq \emptyset$. Let $I = X \cap V$, it is the limit of the decreasing sequence $V_n = U_n \cap V$. Since there is no critical points in V_n, the map $f: V_1 \to V_0$ is an homeomorphism with inverse h. We can suppose that V_0 and V_1 are topological disks up to taking the corresponding connected component (there are two). Hence we can apply Denjoy–Wolff's theorem (see [**St**]), which says that there exists a fixed point x in $\overline{V}_0$ to which h^n converges uniformly on every compact set of V_0. Let P be an open repelling petal near $\mathbf{0}$. By definition for any some small $\epsilon>0$, there exists some $N>0$, such that $h^n(P)$ is in the ϵ neighbourhood of $\mathbf{0}$ for any $n>N$. Now, since h^n converges uniformly to x on the compact set $I\backslash P$, there exists some $M>0$ so that for $n>M$, $h^M(I\backslash P)$ is in the ϵ neighbourhood of x. This contradicts the fact that $h(I)=I$ since $h^n(I) = h^n(P) \cup h^n(I\backslash P)$ is in the ϵ neighbourhood of $\{\mathbf{0}\} \cup \{x\}$. □

3. *Parabolic access to the boundary of B*

We construct periodic accesses to points of ∂B. They lie in $B\backslash\Omega$, where Ω is the attracting petal of $\mathbf{0}$ containing the critical value $f(c_0)$ in its boundary: $\Omega = \phi_0^{-1}(\{x+it,\ x>1,\ t\in\mathbf{R}\})$. Recall that $f(\Omega) \subset \Omega$.

Definition 3.1. A k-periodic access of angle θ is a simple curve $\gamma(\theta) \subset B\backslash\Omega$ such that the following hold.

(1) $f^i(\gamma(\theta)) \cap (B\backslash\Omega)$ are simple and disjoint curves for $0 \le i \le k-1$.
(2) $f^k(\gamma(\theta)) \cap (B\backslash\overline{\Omega}) = \gamma(\theta)$.
(3) $f^i(\gamma(\theta))$ intersects $E_{\mathbf{0}}(1)$ at exactly one point denoted $y_i(\theta)$ for $0 \le i \le k-1$.

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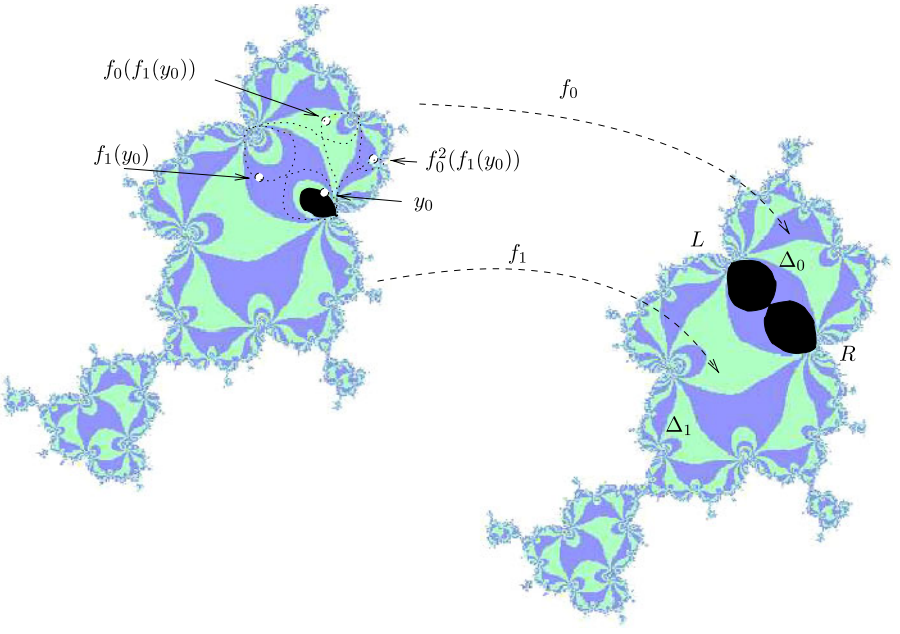


FIGURE 4. The construction of a k-periodic access.

Note that the points $y_0(\theta), \ldots, y_{k-1}(\theta)$ are in the same cyclic order on the line $E_{\mathbf{0}}(1)$ as the points $z_j = e^{2i\pi\theta 2^j}$ for $0 \le j \le k-1$.

The k-periodic accesses can be thought of as internal rays (in the case of an attracting basin). In particular, there exist periodic accesses for any angle θ periodic by multiplication by 2 but for our purpose we do not need this construction in general. We construct for angles of the form $\pm 1/(2^m - 1)$ the accesses using the itinerary with respect to the following partition.

Definition 3.2. Let Ξ_0, respectively Ξ_1, be the unbounded connected component of $\mathbf{C}\backslash S_1$ which is on the left, respectively on the right, of $R_\infty(0)$ when one goes from $\mathbf{0}$ to ∞. (Recall that S_1 is the connected component of $f^{-1}(E_{\mathbf{0}}(1) \cup R_\infty(0) \cup \{\mathbf{0}\})$ containing c_0. See Figure 2.)

LEMMA 3.3. *There exist k-periodic accesses of angle $\theta_\pm = \pm 1/(2^k - 1)$, with $k \ge 0$. They land at k-periodic points of ∂B.*

Proof. We do the construction for θ_+, the argument is similar for θ_-. The proof is illustrated by Figures 4–7.

Let us decompose $B\backslash E_{\mathbf{0}}(0)$ as follows: $\Omega' := f^{-1}(\Omega) := R \sqcup L$ where R is the connected component containing $\mathbf{0}$ in its closure, then $B\backslash\overline{\Omega'} = \Delta_0 \sqcup \Delta_1$ where $\Delta_i \subset \Xi_i$. The map $f\colon B\backslash\overline{\Omega'} \to B\backslash\overline{\Omega}$ admits two inverse branches $f_0\colon B\backslash\overline{\Omega} \to \Delta_0$ and $f_1\colon B\backslash\overline{\Omega} \to \Delta_1$ which extend continuously to the boundary of Ω by $f_0\colon B\backslash\Omega \to \overline{\Delta}_0$ and $f_1\colon B\backslash\Omega \to \overline{\Delta}_1$.

Let $y_0(\theta)$ be a point on $E_{\mathbf{0}}(1)$ with $\Im m(\phi_0(y_0(\theta))) > 0$ and denote by y_0' the point $g(y_0(\theta))$ where $g = f_0^{k-1} \circ f_1$ is defined on $B\backslash\Omega$. Consider the set $U = \Delta \cup f_0(\Delta) \cup \cdots \cup f_0^{k-1}(\Delta)$ on which the extension of the Fatou coordinate $\overline{\phi}_0$ is an homeomorphism

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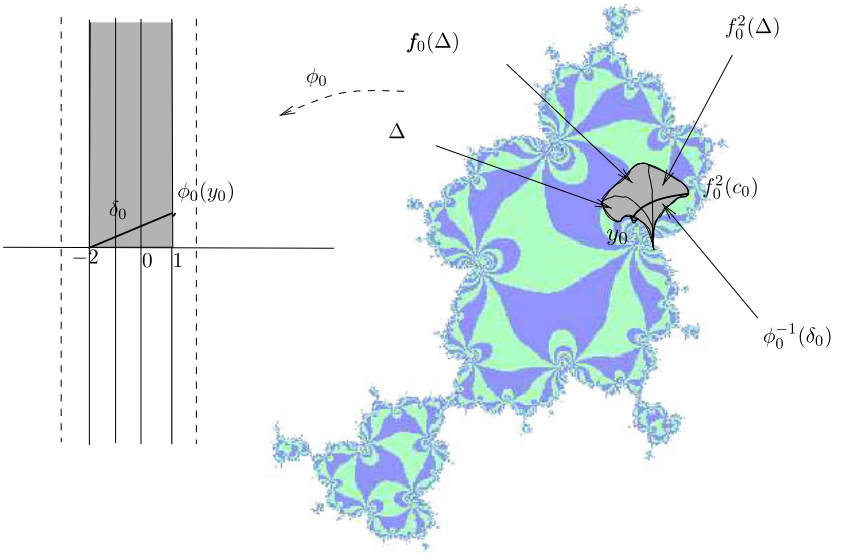


FIGURE 5. Starting the construction of a k-periodic access.

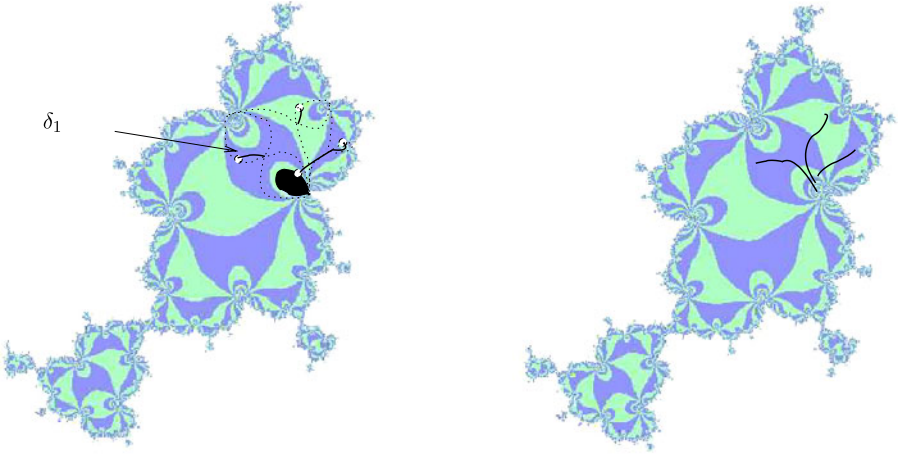


FIGURE 6. Details of the construction of a k-periodic access.

to $[-(k-1), 1] \times \mathbf{R}^+$, with $\Delta = \overline{\phi_0^{-1}((0, 1] \times \mathbf{R}^+)}$. Let δ_0 be the straight line joining $\phi_0(y_0(\theta))$ and $-(k-1) = \phi_0(f_0^{k-1}(c_0))$. Thus $\phi_0^{-1}(\delta_0)$ joins $y_0(\theta)$ with $f_0^{k-1}(c_0)$, the iterates $f^j(\phi_0^{-1}(\delta_0))$ are disjoints and the points $y_j(\theta) = f^j(\phi_0^{-1}(\delta_0)) \cap E(1)$ are, in the Fatou coordinate, the intersection of $\delta_0 + j$ with the line $x = 1$, so are ordered like the points z_j. Let δ_1 be any closed arc in L joining $f_1(y_0(\theta))$ and c_0. Consider the curve $\gamma(\theta) = \bigcup_{i \geq 0} g^i(\delta)$ where $\delta = \phi_0^{-1}(\delta_0) \cup f_0^{k-1}(\delta_1)$.

By construction, $\gamma(\theta)$ is a curve since $\overline{\delta}$ joins $y_0(\theta)$ and $g(y_0(\theta))$. It has no self-intersection. There are none on δ, by choice of δ, and any point of $\delta \cap \overline{g^i(\delta)}$ would give,

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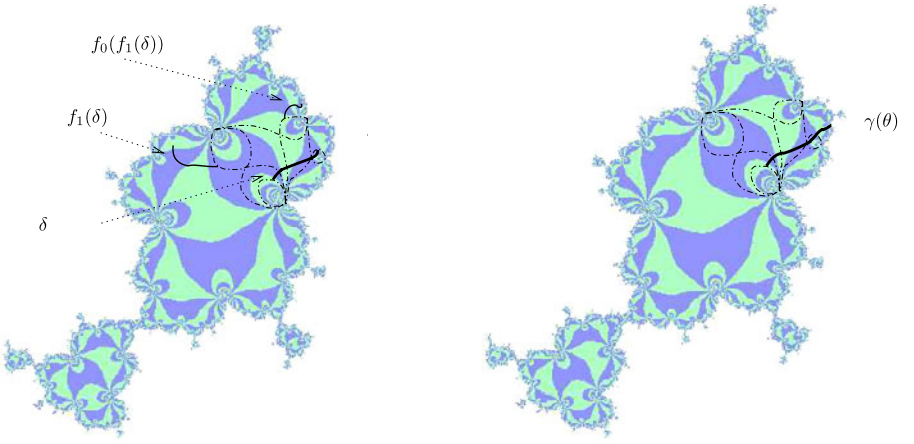


FIGURE 7. End of the construction of a k-periodic access.

after composition by f^{ik}, a point in $\delta \cap \overline{\Omega}$ which is empty. For the same reason, since the $f^i(\overline{\delta})$ are disjoint, the iterates $f^i(\overline{\gamma}(\theta)) \cap B \backslash \Omega$ are disjoint. Finally, since the cyclic order on $E_{\mathbf{0}}(1)$ is given by the order on the line $\{x = 1\}$ in the Fatou coordinate, $\gamma(\theta)$ is a k-periodic access of angle θ.

There exists a point $x \in \partial B$ such that the iterates of $g\colon B\backslash\overline{\Omega} \to B\backslash\overline{\Omega}$ converge uniformly to $\{x\}$ on every compact set of $B\backslash\overline{\Omega}$ (by the Denjoy–Wolff theorem). Applying this to $C = g(\delta)$, it follows that $\gamma(\theta)$ converges to x. By construction $f^i(\gamma(\theta)) \subset \Xi_0$ for $0 \le i \le k-2$, $f^{k-1}(\gamma) \subset \Xi_1$ and $f^k(\gamma(\theta)) \subset \gamma \cup \overline{\Omega}$ so x is exactly k-periodic. □

4. *Wakes attached to B*

The different regions outside B are dynamically characterized by the decomposition in wakes. This is a standard way (see [**Mi2**]) to localize points with respect to internal accesses.

LEMMA 4.1. *There exists a monotone (i.e. with connected fibers) map* $\Theta\colon \partial B \to \mathbf{R}/\mathbf{Z}$ *such that* $\exp(2i\pi\Theta)\colon \partial B \to \mathbf{S}^1$ *is a semi-conjugacy between* f *and* $z \mapsto z^2$. *Moreover* $\Theta(z)$ *is dyadic if and only if* z *is an iterated inverse image of* $\{\mathbf{0}\}$.

Proof. We use the notations of Definition 3.2. For $z \in \partial B \backslash \bigcup_i f^{-i}(\mathbf{0})$, let $\Theta(z)$ be the sum $\sum_{k\ge 0}(\epsilon_k/2^k)$ modulo 1 where $\epsilon_k = 0$, respectively 1, if $f^k(z) \in \Xi_0$, resp Ξ_1. Remark that $\Theta(z)$ is not dyadic, otherwise some $f^i(z)$ would be in $\partial B \cap \bigcap_n U_n = \{\mathbf{0}\}$ (where U_n is defined in Definition 2.1). Now, set $\Theta(\mathbf{0}) = 0 \bmod 1$ and $\Theta(\beta) = 1/2 \bmod 1$ where β is the first inverse image of $\mathbf{0}$ (on B). For any iterated inverse image z of β, with $f^k(z) = \beta$, the sequence ϵ_k depends on the position of $f^i(z)$ relatively to Ξ_0 and Ξ_1 for $i < k$ and continues with the sequence $1\overline{0}$ or $\overline{1}$. Thus, the map $\exp(2i\pi\Theta)$ is well defined on ∂B and semi-conjugates f to $z \mapsto z^2$. □

LEMMA 4.2. *The map* Θ *extends continuously to* $\Theta\colon \mathbf{C}\backslash B \to \mathbf{R}/\mathbf{Z}$. *This map is constant along external rays and throughout any region of* $\mathbf{C}\backslash B$ *which is bounded by two external rays landing at a common point.*

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FIGURE 8. Two wakes attached to B (different from $W(0)$) and $R_\infty(0)$.

Proof. Let z be a point of $\mathbf{C}\backslash\overline{B}$. If z lies on a ray $R_\infty(\theta)$ with θ triadic and landing at a point $u \in \partial B$, define $\Theta(z) = \Theta(u)$. Otherwise, z belongs to a connected component $T_n(z)$ of $\mathbf{C}\backslash S_n$, where S_n is the connected component, containing $\mathbf{0}$, of $f^{-n}(S_0)$ (Definition 2.1). Each $\overline{T_n(z)}$ is connected and has connected intersection with ∂B that is mapped onto either $\Xi_0 \cap \partial B$ or $\Xi_1 \cap \partial B$ by f^{n-1}. Thus Θ is constant on $T(z) \cap \partial B$ where $T(z) = \bigcap_n \overline{T_n(z)}$ (since such points all have the same itinerary). Therefore we can set $\Theta(z) = \Theta(u)$ where $u \in \partial B$ belongs to $\overline{T_n(z)}$ for every $n \geq 0$. The last statement follows from the construction. □

Definition 4.3. Denote by $W(z)$ the *wake* of $z \in \mathbf{C}\backslash B$ which is by definition the connected component of $\Theta^{-1}(\Theta(z))$ containing z. (We will also use the notation $W(t)$ with $t = \Theta(z)$.) Denote by $L(z) := W(z) \cap K(f)$ the *limb* of $z \in \mathbf{C}\backslash B$ (it can also be denoted by $L(t)$). (See Figure 8.)

By definition we have $\mathbf{C} = B \cup \bigcup_{t \in \mathbf{R}/\mathbf{Z}} W(t)$.

Remark 4.4. Note that wakes and limbs are closed subsets of $\mathbf{C}$.

LEMMA 4.5. *If $c \notin W(t)$, the image $f(W(t))$ is the wake $W(2t)$. If $c \in W(t)$ then the image $f(W(t))$ is the whole plane. Moreover $f(c)$ belongs to $W(2t)$.*

Proof. If $c \notin W(t)$, the result follows from the definition. Let $W(s)$ be the wake containing the critical value $f(c)$. Assume (to get a contradiction) that neither $W(s/2)$ nor $W((s+1)/2)$ contains c. Then, it follows by previous remark that $W(s/2)$ and $W((s+1)/2)$ are mapped onto $W(s)$ so that each one contains a preimage of $f(c)$. Since there are exactly two preimages, c is contained in one of them, which contradicts the

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assumption. Let $W(t)$ be the wake containing c. If $c \notin \partial B$, $W(t)$ contains also a preimage of $W(s)$. Since f is injective on $\partial W(t)$ but not on $W(t)$, it follows from the properness of f that $f(W(t)) = \mathbf{C}$. If $c \in \partial B$, the preimage B' of B is in $W(t)$ and therefore there is a preimage of every $W(\xi)$ for $\xi \in \mathbf{R}/\mathbf{Z}$ which is attached to B'; therefore f maps $W(t)$ onto $\mathbf{C}$ since $\mathbf{C} = \overline{B} \cup \bigcup_{\xi \in \mathbf{R}/\mathbf{Z}} W(\xi)$. □

Remark 4.6. The map $\exp(2i\pi\Theta)\colon \mathbf{C}\backslash B \to \mathbf{S}^1$ is a semi-conjugacy between f and $z \mapsto z^2$ outside the critical wake $W(t_0)$.

LEMMA 4.7. *For any wake W not reduced to the closure of an external ray, there exists a smallest $n \geq 0$ such that the closure of $f^n(W)$ contains the critical point c.*

Proof. The size of a wake $W(z)$ is defined to be the limit of the difference of the angles of the external rays of $T_n(z)$. If a wake does not contain the critical point in its closure, its size is multiplied by 3 at each iteration. Hence we deduce from Lemma 4.5 that the closure of $f^i(W)$ contains the critical point c for some $i \geq 0$. □

LEMMA 4.8. *If $c \notin \partial B$, there exists some $t \in \mathbf{Q}/\mathbf{Z}$, k-periodic by multiplication by 2, such that $W(t)$ contains c.*

Proof. Let $W(t)$ denote the wake containing the critical point. Since c belongs to $K(f)$, $W(t)$ is not reduced to the closure of an external ray. The critical value $f(c)$ belongs to the wake $W(2t)$. If $t \neq 0$, $W(2t)$ does not contain the critical point c so there is some $n \geq 1$ such that $f^n(W(2t))$ contains c in its closure (Lemma 4.7). Therefore $f^n(W(2t)) = W(t)$ and since $f^n(W(2t))$ is a wake of angle $2^n(2t)$, we get by Lemma 4.2 that $2^{n+1}t = t \bmod 1$. □

Note that Lemmas 4.5, 4.7 and 4.8 hold with the limbs instead of the wakes and $K(f)$ instead of $\mathbf{C}$.

COROLLARY 4.9. $c \notin W(\mathbf{0}) \iff L(\mathbf{0}) = \{\mathbf{0}\}$.

Proof. If $c \notin W(\mathbf{0})$ by Lemma 4.5 $f^n(W(\mathbf{0})) = W(\mathbf{0})$. However, Lemma 4.7 implies that if $L(\mathbf{0}) \neq \{\mathbf{0}\}$ there is an $n \geq 0$ such that $c \in f^n(W(\mathbf{0}))$. This gives a contradiction. □

As a consequence we obtain the following.

Remark 4.10. If $L(1/2)$ is not reduced to one point, either $c \in L(\mathbf{0})$ or $c \in L(1/2) \cap \partial B$.

5. *Puzzles and Yoccoz' theory*

We will now use Yoccoz result in our context, that is with graphs containing periodic accesses in B (defined in §3). For any $k > 0$, let θ be one of the angles $\{\theta_+, \theta_-\}$ where $\theta_\pm = \pm 1/(2^k - 1)$ and denote by $x(\theta)$ the landing point of $\gamma(\theta)$. There is an external ray $R_\infty(\zeta)$ converging to $x(\theta)$ (Proposition 1.2) it has period exactly k since the point $x(\theta)$ has rotation number 1 (see **[Go-Mi, Pe]**). Consider $\gamma_i(\theta) = f^i(\gamma(\theta)) \cap (B\backslash\Omega)$ for $0 \leq i \leq k-1$.

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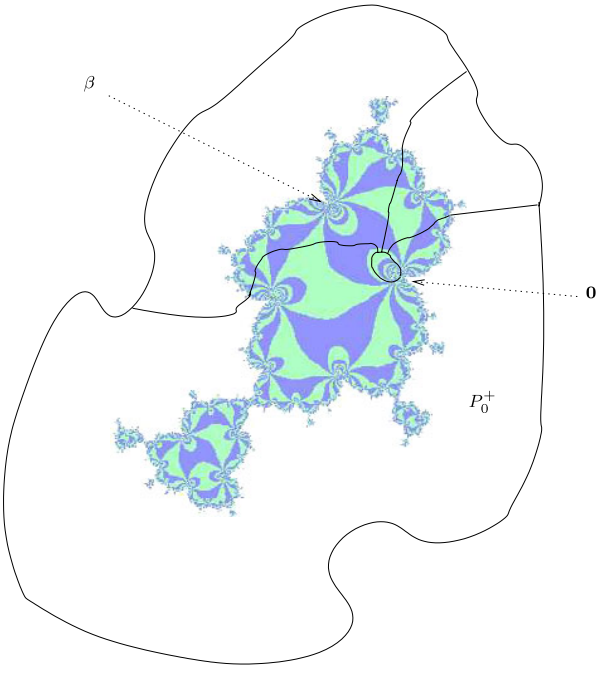


FIGURE 9. The graph Γ.

Definition 5.1. The puzzle of depth n (associated to this θ) is the collection of the bounded connected components of $\mathbf{C}\backslash\Gamma_n$, called *puzzle pieces of depth n*, where $\Gamma_n = f^{-n}(\Gamma(\theta))$ and

$$\Gamma(\theta) = E_\infty(1) \cup E_{\mathbf{0}}(1) \cup \bigcup_{0\le i\le k-1} (\gamma_i(\theta) \cup f^i(R_\infty(\zeta) \cup x(\theta))).$$

Denote by $P_n(z)$ the puzzle piece of depth n containing z and by $A_n(z)$ the annulus of depth n surrounding z: $A_n(z) := P_n(z)\backslash\overline{P_{n+1}(z)}$. It is said to be *non-degenerate* if and only if $\overline{P_{n+1}(z)} \subset P_n(z)$. (See Figure 9.)

Remark 5.2.

(1) The map $f : P_{n+1}(z) \to P_n(f(z))$ is proper and $P_{n+1}(z) \subset P_n(z)$.

(2) The annulus $A_N(f^j(z))$ is non-degenerate if and only if $A_{N+j}(z)$ is non-degenerate.

The theorem of Yoccoz, as presented below, involves the following condition: a point z has property ($\diamondsuit$) if there exist a graph Γ (of the form of Definition 5.1) and a sequence (n_i) such that $A_{n_i}(z)$ (defined by Γ) is non-degenerate.

PROPOSITION 5.3. *Let z be a point of $J(f)$. If there is a graph Γ for which z satisfies property ($\diamondsuit$) and for which $A_N(c)$ is non-degenerate for some $N \ge 0$, then either $\cap P_n(z) = \{z\}$ or there exists $j > 0$ such that f is j-renormalizable.*

More precisely, in the last case $f^j : P_{n+j}(c) \to P_n(c)$ is a renormalization and $\cap P_n(z)$ is (an iterated inverse image of) the filled-in Julia set $K(f^j) = \cap P_n(c)$ of the renormalization.

Note that the choice of the graph Γ depends on the point z and on c.

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Proof of Proposition 5.3. The proposition follows from the work of Yoccoz. We refer here to the exposition of [**Ro1**]. It is written in the context of rational-like maps, to get non-degenerate annuli, but this condition is not necessary if we have property ($\diamond$) (which provides the non-degenerate annuli). It corresponds to [**Ro1**, Theorem 1.10]. The argumentation is exactly the same, since it consists in combinatorial considerations on non-degenerate annuli. The goal is to get a sequence of annuli surrounding z such that the sum of the moduli is infinite. We recall this now briefly without proof; the interested reader should consult the original paper (an English version can be found in the appendix of [**Ro2**]). Recall that z is said to be *non-recurrent* to c if there exists N such that $c \notin P_n(f^i(z))$ for every $i \geq 0$ and for every $n \geq N$. In this case the annuli $A_{n_i}(z)$ provided by ($\diamond$) are clearly all conformally equivalent to some annulus of depth N. Hence, in the non-recurrent case the sum of the moduli of the $A_{n_i}(z)$ is infinite since there is only a finite number of annuli at depth N. Assume that z is non-persistently recurrent to c, i.e. there exist r and an increasing sequence r_i such that $c \notin f^t(P_{r_i}(z))$ for $0 < t \leq r_i - r$. Then for any $p \geq r$ there exists a sequence k_i such that $A_{k_i}(f(z))$ is conformal to $A_p(c)$. If there is a non-degenerate critical annulus of depth $p \geq r$ the result follows. Else, one obtains a non-degenerate annulus after passing through a ramified covering (an iterate of f) but of bounded degree (see [**Ro1**, Lemma 1.25]). Finally, from [**Ro1**, Lemma 1.27], we obtain that in the last case, called persistently recurrent case, it is enough to consider the nest of the puzzle pieces around the critical point c. Lemma 1.30 of [**Ro1**] considers this case: $(P_n(c))_{n\in\mathbf{N}}$ is not 'periodic' and c is persistently recurrent. The tree of the good children is built to get annuli around c such that the sum of the moduli is infinite. Lemma 1.36 of [**Ro1**] proves that in the 'periodic' case some iterate is renormalizable (we have a non-degenerate annulus between two critical puzzle pieces). □

6. *Proof of Theorem 1*

In this section we show that all points z in the set $S = \partial B \setminus \bigcup_{i\geq 0} f^{-i}(\mathbf{0})$ satisfy condition ($\diamond$). Combined with the results of §2, Theorem 1 follows.

LEMMA 6.1. *Any point z of $S = \partial B \setminus \bigcup_{i\geq 0} f^{-i}(\mathbf{0})$ satisfies ($\diamond$) for one of the graphs $\Gamma(\theta_\pm)$ with $\theta_\pm = \pm 1/(2^k - 1)$ and every large $k > 0$.*

Proof. Note that S is invariant by f. Recall that Δ_1 is defined in the proof of Lemma 3.3 as follows. The part of the basin $B\setminus\Omega$ decomposes as $\Delta_0 \sqcup \Delta_1$ where $\Delta_i \subset \Xi_i$. The component Ξ_i is the unbounded connected component of $\mathbf{C}\setminus S_1$ which is on the left for Ξ_0 and on the right for Ξ_1, of $R_\infty(0)$ when one goes from $\mathbf{0}$ to ∞. (Recall that S_1 is the connected component of $f^{-1}(E_{\mathbf{0}}(1) \cup R_\infty(0) \cup \{\mathbf{0}\})$ containing c_0. See Figure 2.)

Step 1. Let $k > 0$, $\theta = \theta_+$ and $\Gamma = \Gamma(\theta)$. Let P_0^+ be the puzzle piece containing $\mathbf{0}$ in its closure and let P_1^+ be one of the puzzle pieces of depth 1 intersecting Δ_1 and containing neither $f^{k-1}(\gamma(\theta))$ nor $\mathbf{0}$ in its closure. Then the annulus $P_0^+ \setminus \overline{P}_1^+$ is non-degenerate. (See Figure 10.)

Proof. Denote by T the union of accesses $\{f^i(\gamma(\theta)), i \in \{1, \ldots, k-1\}\}$. By construction, the puzzle piece P_1^+ is bounded in B by two accesses γ_i, γ_j of $f^{-1}(T)\setminus(T \cup \gamma(\theta))$ and a portion of $E_{\mathbf{0}}(0)$ (see Figure 10). The curves γ_i, γ_j do not intersect ∂P_0^+

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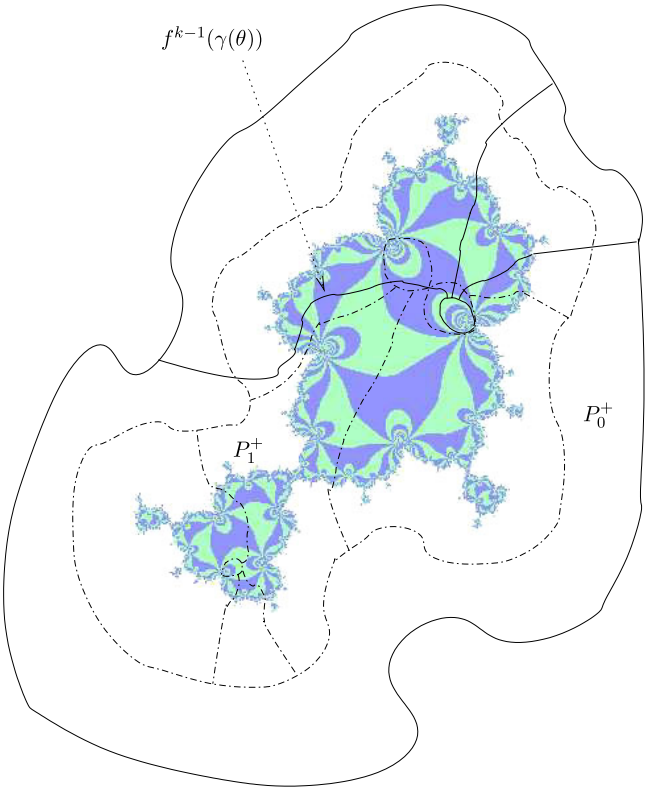


FIGURE 10. The graph Γ and its inverse image Γ_1.

since, on one hand, any intersection with an element of T would give (after iterations) an intersection between two elements of T, which is excluded; and on the other hand, γ_i, γ_j start on $E_{\mathbf{0}}(0)$ and stay in $B\backslash(L \cup R)$ so cannot cut $E_{\mathbf{0}}(1)$. Remark that $E_{\mathbf{0}}(0)$ intersects $E_{\mathbf{0}}(1)$ at $\{\mathbf{0}\}$ but this point is not in the closure of P_1^+. Now, the portion of $E_{\mathbf{0}}(0)$ in ∂P_1^+ do not intersect ∂P_0^+. Indeed, the points $y'_{k-1} = f^{k-1}(\gamma(\theta)) \cap E_{\mathbf{0}}(0)$, $\gamma_i \cap E_{\mathbf{0}}(0)$, $\gamma_j \cap E_{\mathbf{0}}(0)$ are on ∂L in the same cyclic order as $y_0(\theta)$, $y_i(\theta)$, $y_j(\theta)$ (where $y_i(\theta) = f(\gamma_i \cap E_{\mathbf{0}}(0))$), so the part of $E_{\mathbf{0}}(0)$ on the boundary of P_1^+ is included in $\partial\Delta_1 \cap E_{\mathbf{0}}(0)$ after y'_{k-1} so cannot intersect Γ. Thus $\overline{P_1^+} \subset P_0^+$.

The analogue statement for θ_- guaranties that the annulus $P_0^-\backslash\overline{P_1^-}$ is non-degenerate.

Step 2. The landing point $x_j(\theta)$ of the access $f^j(\gamma(\theta))$ satisfies $\Theta(x_j(\theta)) = 2^j\theta \bmod 1$.

Proof. The map $\exp(2i\pi\Theta)$ semi-conjugates f to $z \mapsto z^2$, so that

$$\exp(2i\pi\Theta(f^k(x_0(\theta)))) = \exp(2^k 2i\pi\Theta(x_0(\theta))).$$

Since $x_0(\theta)$ is k-periodic, $f^k(x_0(\theta)) = x_0(\theta)$, so that $\Theta(x_0(\theta)) = 2^k\Theta(x_0(\theta)) \bmod 1$ and $\Theta(x_0(\theta))$ is of the form $l/(2^k - 1)$. The points $x_0(\theta), \ldots, f^{k-2}(x_0(\theta))$ belong to Ξ_0 and $f^{k-1}(x_0(\theta))$ is in Ξ_1, so it is easy to see that $\Theta(x_0(\theta)) = 1/(2^k - 1)$ and the result follows.

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Step 3. For any point $z \in S$, there exist $k > 0$ and $\theta \in \{\theta_\pm = \pm 1/(2^k - 1)\}$ such that the orbit of z enters infinitely many times in one of the puzzle pieces, $P_1^\pm$, considered in Step 1 (for the graph $\Gamma(\theta_\pm)$).

Proof. For $k > 0$ sufficiently large, the orbit of z is not on $\Gamma(\theta_\pm)$ for $\theta_\pm \in \{\pm 1/(2^k - 1)\}$. Indeed, if z is not eventually periodic, any $\theta_\pm$ is fine; if z is eventually periodic, we chose $k > 0$ large enough so that the period of $\theta_\pm$ is larger than the one of the iterated image of z. We use the map Θ to localize the orbit of z. Let $x'_j(\theta_\pm)$ be the inverse image of $x_{j+1}(\theta_\pm)$ different from $x_j(\theta_\pm)$ for $0 \le j \le k-2$. By Lemma 4.1 and by Step 2, it is easy to see that $\Theta(x'_j(\theta_\pm)) = 2^j\theta_\pm + 1/2 \bmod 1$. Hence,

$$\begin{aligned}\Theta(x_{k-1}(\theta_+)) < \Theta(x'_0(\theta_+)) &= \theta_+ + \frac{1}{2} \bmod 1 < \Theta(x'_1(\theta_+)) < \cdots < \Theta(x'_{k-2}(\theta_+)) \\ &= \frac{3}{4} + \frac{\theta_+}{4} < 1 \qquad (*) \\ 0 < \Theta(x'_{k-2}(\theta_-)) &= \frac{1}{4} + \frac{\theta_-}{4} \bmod 1 < \cdots < \Theta(x'_0(\theta_-)) \\ &= \theta_- + \frac{1}{2} \bmod 1 < \Theta(x_{k-1}(\theta_-)). \qquad (**)\end{aligned}$$

Assume that there exists a subsequence (n_j) such that $2^{n_j}\Theta(z) \bmod 1$ tends to $1/2$, then $2^{(n_j-1)}\Theta(z) \bmod 1$ admits a sub-subsequence that tends to $1/4$ or to $3/4$. Therefore the corresponding sub-subsequence of points $f^{(n_{j_l}-1)}(z)$ belongs to P_1^- (in the first case), respectively to P_1^+ (in the second case). P_1^- (respectively P_1^+) is the puzzle piece containing the point on ∂B of $f^{-1}(\beta) \cap \Xi_0$ (respectively of $f^{-1}(\beta) \cap \Xi_1$) by the inequality $(**)$ (respectively $(*)$) with $\beta = (f^{-1}(\mathbf{0}) \backslash \{\mathbf{0}\}) \cap \partial B$.

Assume now that there exists $\epsilon > 0$ such that for every $n \ge 0$ the angles $2^n\Theta(z) \bmod 1$ avoid the interval $(1/2 - \epsilon, 1/2 + \epsilon)$ containing $1/2$. Since $\Theta(z)$ is not dyadic, there exists a subsequence (n_j) such that $2^{n_j}t \bmod 1$ belongs to the interval $(1/2, 3/4)$ (there exist infinitely many occurrences of 10 in its dyadic expansion). For k large enough, $\theta_+ = 1/(2^k - 1) < \epsilon$ so $\Theta(f^{n_j}(z)) \bmod 1$ belongs to the interval $(1/2 + \theta_+, 3/4)$. Therefore, by the inequality $(*)$, the sequence $f^{n_j}(z)$ belongs to one of the puzzle pieces P_1^+ considered in Step 1.

Step 4. Property $(\diamondsuit)$ is satisfied for the points of S.

Proof. Let (n_j) be one of the sequences obtained in Step 3 such that $f^{n_j}(z)$ belongs to $P_1^\pm$. Since $A_0(f^{n_i}(z))$ is non-degenerate, the annulus $A_{n_j}(z)$ is also non-degenerate by Remark 5.2. □

LEMMA 6.2. *One of the following holds.*

- *There exists k_0 and N such that $\overline{P_{N+1}(c)} \subset P_N(c)$ where the pieces are defined by the two graphs $\Gamma(\theta_\pm)$ with $\theta_\pm = \pm 1/(2^k - 1)$ for any $k \ge k_0$.*
- *The orbit of c stays in $W(0)$.*

Proof. Let t_0 be the angle such that $c \in W(t_0)$.

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Step 1. For any puzzle piece of depth 0 not intersecting $W(2t_0)$, its preimage in $W(t_0)$ is compactly contained in the corresponding puzzle piece of depth 0.

Proof. This puzzle piece of depth 1 will be bounded by accesses in B' (the preimage of B) and external rays stemming from B'. Therefore there is no intersection point between the boundaries.

For the rest of the proof, we assume that the orbit of c does not stay in $W(0)$, so that there exists a first $p \geq 0$ such that $f^p(c) \in W(t)$ with $t \neq 0$. We prove now that there exist N and k_0 such that $\overline{P_{N+1}(c)} \subset P_N(c)$ for the pieces defined by the two graphs $\Gamma(\theta_\pm)$ with $\theta_\pm = \pm 1/(2^k - 1)$ and $k \geq k_0$. The proof depends on the value of p.

Step 2. The case where $p > 0$, i.e. $f^j(c) \in W(0)$ for $0 \leq j < p$.

Proof. If $t \in (0, 1/2]$, there exists k_0 such that for $k \geq k_0$ and Γ defined by θ_+, the puzzle piece $P_0(f^p(c))$ (containing $W(t)$) does not intersect $L(2t_0) = L(0)$, since θ_+ tends to 0 with k so that $0 < \theta_+ < t$ for large k. Therefore $P_1(f^{p-1}(c))$ is compactly contained in $P_0(c) = P_0(f^{p-1}(c))$ (by previous step). Hence, $A_{p-1}(c)$ is non-degenerate (by Remark 5.2).

If $t \in (1/2, 1)$, there exists $q \geq 0$ such that $2^q t \in (1/2, 3/4]$. Then Lemma 6.1 gives some k_0 (depending on t) such that for $k \geq k_0$ and Γ defined by θ_+, the puzzle piece $P_1(f^{p+q}(c))$ (containing $W(2^q t)$ since $2^q t \neq 0 \bmod 1$) is compactly contained in $P_0(f^{p+q}(c))$. Therefore $A_{p+q}(c)$ is non-degenerate (Remark 5.2).

The argument is similar for θ_-.

Step 3. The case where $c \notin W(0)$, i.e. $p = 0$.

Proof. Note that $f(c) \in W(2t_0) \neq W(t_0)$ (since $t_0 \neq 0$). Let u be a point of $W(2t_0) \cap \partial B$. For k large enough, there exists i_0, i_1 such that $2^{i_0} t_0 \in (1/2 + \theta_+ \bmod 1, 3/4 + (\theta_+/4) \bmod 1)$ and also $2^{i_1} t_0 \in (1/4 + (\theta_-/4) \bmod 1, \theta_- + 1/2 \bmod 1)$, since t_0 is not dyadic so that there are occurrences of 01 and of 10 in the dyadic expansion of t_0 (see Step 3 of Lemma 6.1). We take the smallest such integers. Hence the points $f^j(u)$ and $f^j(f(c))$ belong to the same puzzle of depth 1 which is compactly contained in the corresponding puzzle piece of depth 0 for $j = i_0, i_1$ and the result follows from Remark 5.2. □

LEMMA 6.3. *We assume that the orbit of c stays in $W(0)$. For every point $z \in S$, there exist $k_0 > 0$ and $\theta \in \{\theta_\pm = \pm 1/(2^k - 1)\}$, such that the sequence of puzzle pieces $(P_n(z))$ defined by any of these graphs with $k \geq k_0$ shrinks to $\{z\}$.*

Proof. Let z be a point of S. By Step 3 of the proof of Lemma 6.1, there exist $k_0 > 0$ and $\theta \in \{\theta_\pm = \pm 1/(2^k - 1)\}$ such that the orbit of z enters infinitely many times in $P_1^\pm$, one of the puzzle pieces of depth 1 considered in Step 1 of the proof of Lemma 6.1 with $k \geq k_0$.

If the orbit of c is included in $W(0)$, the diameter of the iterated inverse images of $P_1^\pm$ shrinks to 0. Indeed, $\overline{P_1^\pm}$ do not intersect the closure of the post-critical set so that there is a strong contraction. We can for instance apply the so called 'shrinking lemma' of Lyubich and Minsky (see [**Ly-Mi**, Appendix 2]). The result follows. □

Now we can prove the Theorem 1 summarized as follows.

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THEOREM 1. *The boundary ∂B is locally connected. It is a Jordan curve.*

Proof. The boundary ∂B is locally connected at every iterated inverse image of the parabolic fixed point $\mathbf{0}$ by Corollary 2.5. For $z \in S$, Lemma 6.1, Proposition 5.3 and Lemma 6.3 imply that the intersection $\cap P_n(z)$ is equal either to $\{z\}$ or to an iterated inverse image (say by some f^j) of the filled in Julia set of some renormalization $f^l\colon P_{n+l}(c) \to P_n(c)$. Note that the intersection $\partial P_n(z) \cap \partial B$ is clearly a connected set. Therefore, ∂B is locally connected at z in the first case.

In the second case, the puzzle pieces $P_n(c)$ intersect B since $f^j(z) \in P_n(c)$ for every $n \geq 0$ (and $z \in \partial B$). Let $R_\infty(\zeta_n)$, $R_\infty(\zeta'_n)$ be the two external rays of $\partial P_n(c)$ that land on ∂B. Since $P_{n+1}(c) \subset P_n(c)$, the angles (ζ_n) and (ζ'_n) form monotone and bounded sequences, they converge to limit angles ζ and ζ' respectively. Moreover, the periodicity property $f^l(P_{n+l}(c)) = P_n(c)$ implies that the limits ζ and ζ' are periodic by multiplication by 3. Hence the rays $R_\infty(\zeta)$ and $R_\infty(\zeta')$ are fixed by f^l so converge to fixed points, with rotation number 1. They belongs to $K(f^l) = \cap P_n(c)$ since $R_\infty(\zeta)$ and $R_\infty(\zeta')$ enter every puzzle piece $P_n(c)$ for $n \geq 0$. There is only one fixed point of rotation number 1 for the renormalized map f^l in $K(f^l)$ (since f^l is conjugated to a quadratic polynomial), denoted $\beta(f^l)$. Therefore the two rays $R_\infty(\zeta)$ and $R_\infty(\zeta')$ both converge to $\beta(f^l)$.

Let V_n, respectively W_n, be the connected components of $P_n(c)\backslash(R_\infty(\zeta) \cup R_\infty(\zeta') \cup \{\beta(f^l)\})$ intersecting B, respectively not intersecting B. We show that $(\overline{V}_n)_{n\in\mathbf{N}}$ forms a basis of neighbourhoods of $\beta(f^l) = \partial B \cap K(f^l)$ in ∂B. For this we prove that $V_n \cap K(f^l) = \emptyset$. We only need to consider the case where $\beta(f^l) \neq \mathbf{0}$, because of Proposition 2.4. In this case, note that V_n is connected and that the wake W_n contains the critical point c (otherwise it would be mapped by f^l homeomorphically to itself which contradicts Lemma 4.7). Hence, the map f^l admits an inverse branch on V_n, denoted $g\colon V_n \to V_{n+l}$. There is some fixed point on ∂V_n such that every orbit under g^n converges to it (by Denjoy–Wolff's theorem). This implies that no point of $K(f^l)$ is in V_n. Hence, the curve $R_\infty(\zeta) \cup R_\infty(\zeta') \cup \{\beta(f^l)\}$ separates B from $K(f^l)\backslash\{\beta(f^l)\}$. Therefore $\beta(f^l) = f^j(z)$. Let Q_n be the pull-back of V_n by the corresponding branch of f^j. The sequence $(\overline{Q}_n)$ forms a basis of neighbourhoods of z in ∂B.

In both cases we get that $N_n = \partial P_n(z) \cap \partial B$ form a basis of connected neighbourhoods of z in ∂B (since $\cap N_n = \{z\}$). Thus, ∂B is locally connected. The classical theorem of Carathéodory implies that ∂B is a curve (since B is a topological disk). Finally, this curve is simple since $K(f)$ is a full compact set (by the maximum principle). □

COROLLARY 6.4. *The map* $\exp(2i\pi\Theta)$ *defined in Lemma 4.1 is a conjugacy between f on ∂B and $z \mapsto z^2$ on* $\mathbf{S}^1$.

Proof. The map $\exp(2i\pi\Theta)$ is a semi-conjugacy between f and $z \mapsto z^2$. Two points with the same image have the same itinerary so are in the same iterated inverse image of Ξ_0 and Ξ_1 and thus in the same puzzle piece. Since they are on ∂B they coincide. □

7. *Global description*

LEMMA 7.1. *For every point z of $\mathbf{C}\backslash B$ there exists a unique $u \in \partial B$ such that z belongs to the wake $W(u)$.*

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Proof. This follows directly from the local connectivity or Corollary 6.4. □

LEMMA 7.2. *For any point z of ∂B there are at most two external rays converging to it.*

Proof. Suppose first that $c \notin \partial B$, so that $c \in W(t_0)$ with t_0 some k-periodic angle. Let y be the point on $L(t_0) \cap \partial B$. Since it is k-periodic and has rotation number 1 (it admits a 'k-fixed' access from B) there is only a finite number of external rays landing at y and all of rotation number 1. Denote by R_1 and R_2 the two external rays bounding a component W containing c and not containing any other external ray landing at y.

Assume now, to get a contradiction, that there are at least three rays (R, R', R'') landing at z. They cut $\mathbf{C}$ in three connected components. Denote by W_1 and W_2 the closures of the two connected components that do not contain B. Assume, up to changing the rays, that neither W_1 nor W_2 contains in its interior an external ray landing at z. By the similar argument as in Lemma 4.7 there exist integers n_1 and n_2 (with say $n_1 \leq n_2$) such that $c \in f^{n_1}(W_1)$ and also $c \in f^{n_2}(W_2)$ (else the map is an homeomorphism and the angular size grows too much). Assume that n_1 and n_2 are the least integers satisfying this property. If W_1 is not itself the critical wake (i.e. $n_1 \neq 0$), it is mapped homeomorphically to W by f^{n_1} (as in Lemma 4.5) and its boundary rays, say R, R', are mapped to R_1, R_2 respectively. The same property holds for W_2 so that its boundary rays, say R', R'', are mapped to R_1, R_2 also. Therefore, R_2 is mapped to R_1. Indeed, the wake $f^{n_1}(W_2)$ lies between $f^{n_1}(R') = R_2$ and $f^{n_1}(R'')$ (due to the cyclic order that is preserved around $f^{n_1}(z)$) so that the wake $f^{n_2}(W_2)$ lies between $f^{n_2-n_1}(R_2)$ and $f^{n_2}(R'')$ which are R_1 and R_2 respectively (because of the cyclic order around $f^{n_2}(z)$). But this contradicts the fact that R_1 and R_2 have rotation number 1.

We consider now the case where $c \in \partial B$. Assume that two rays R, R' land at the critical value $f(c)$. By Lemma 4.7, the wake W they define is mapped homeomorphically by some f^m to the wake containing the critical point c. Hence, $f^m(f(c)) = c$ which is impossible. Hence, there is at most on external ray landing at the critical value and so at most two at the iterated preimages. □

COROLLARY 7.3. *If $c \in \partial B$, for any $z \in \partial B$ there is exactly one external ray converging to it except if there exists $j \geq 0$ such that $f^j(z) = c$ in this case there are two external rays landing at z. Moreover, in this case $f^j : W(z) \to W(c)$ is a homeomorphism, $W(c)$ contains the inverse image of B. As a consequence every Fatou component is an iterated inverse image of B.*

Proof. This follows directly from Lemma 4.7. □

COROLLARY 7.4. *If $c \notin \partial B$, for any $z \in \partial B$ there is exactly one external ray converging to it, except if there exists $j \geq 0$ such that $\Theta(f^j(z)) = t = \Theta(c)$ in this case $f^j : W(z) \to W(t)$ is a homeomorphism.*

Proof. It follows from the local connectivity and from Lemma 4.7 □

Remark 7.5. The proof of Theorem 2 follows from these results and §4.

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PROPOSITION 7.6. *If the map f has a non-repelling k-periodic point x with $k > 1$, then f is renormalizable near x. Moreover, $f^k : P_{n+k}(c) \to P_n(c)$ is the renormalization (for any of the graphs considered).*

Proof. The sequence $(P_n(x))_{n\in\mathbf{N}}$ of puzzle pieces containing x is well defined, for any of the graphs considered. Indeed x is not on the graphs since it is neither eventually repelling and nor equal to $\mathbf{0}$. The periodicity of x implies that there exists $k \geq 1$ such that $f^k(P_{n+k}(x)) = P_n(x)$ for any large n. Take the minimal k such that this is true. The critical point c belongs to some piece $P_n(f^i(x))$, for every $n \geq 0$. Otherwise the map $f^k\colon P_{n+k}(x) \to P_n(x)$ would be invertible and its inverse $g\colon P_n(x) \to P_n(x)$ would have an attracting fixed point (by the Schwarz lemma) since $P_{n+k}(x) = g(P_n(x)) \neq P_n(x)$. This is not possible since $|g'(x)| \geq 1$. Note that i is independent of n since the pieces $P_n(f^j(c))$ are disjoint for $j < k$ (if $k > 1$). We prove now that if $k > 1$, the map $f^k\colon P_{n+k}(f^i(x)) \to P_n(f^i(x))$ is renormalizable. It is enough to prove that $\overline{P_{n+k}(f^i(x))} \subset P_n(f^i(x))$, i.e. that the critical point is surrounded by a non-degenerate annulus. If $c \notin W(0)$, this follows from Lemma 6.1. If $c \in W(0)$ and if there is k such that $f^k(c)$ is not in $W(0)$ then Lemma 6.2 implies the result. Now, consider the case where the orbit of c stays in $W(0)$. The rays $R(0)$ and $R(1/2)$ bound $W(0)$ and converge to $\mathbf{0}$. The map f has three fixed points in $\mathbf{C}$, one double at $\mathbf{0}$, so that there is a fixed point α in $W(0)$ (since it contains c, see [**Go-Mi**]). This point is repelling since there cannot be more than two non-repelling periodic points (there are already the parabolic fixed point and another one by hypothesis), so there is a cycle of external rays converging to it, say $R(3^j\zeta)$ with $0 \leq j \leq m$. Consider the connected component U of $W(0)\backslash(R_\infty(1/6) \cup R_\infty(1/3) \cup R_\infty(2/3) \cup R_\infty(5/6) \cup E_\infty(1))$ containing the critical point. We define a new puzzle in U. Let Γ be the graph formed in U by the cycle $R(3^j\zeta)$ with $0 \leq j \leq m$ together with ∂U. Denote by Y_n the puzzle pieces. Since x is periodic by f, there is some minimal l such that f^l maps $Y_{n+l}(x)$ onto $Y_n(x)$. Moreover this map has degree two, otherwise by the Schwarz lemma the fixed point x would be repelling for f^l. Since α is repelling, one can thicken near α the puzzle pieces Y_n to get larger puzzle pieces $\tilde{Y}_n$, as done in Milnor's article [**Mi3**], so that $f^l\colon \tilde{Y}_{n+l}(x) \to \tilde{Y}_n(x)$ is renormalizable. The result follows. □

COROLLARY 7.7. *Assume that f possesses a periodic point p of period greater than* 1 *and with multiplier $\lambda = e^{2i\pi\theta}$, such that $\theta \in \mathbf{R}\backslash\mathbf{Q}$. Then f is linearizable near p if and only if $\theta \in \mathcal{B}$. Moreover, if $\theta \notin \mathcal{B}$ there exist periodic cycles in any neighbourhood of p.*

Here $\mathcal{B}$ denotes the set of *Brjuno* numbers: an irrational θ of convergents p_n/q_n (rational approximations obtained by the continued fraction development) is a *Brjuno* number, if $\sum_{n=1}^{\infty}(\log q_{n+1})/q_n$ is finite.

Proof. The map f is renormalizable by Proposition 7.6. Hence there is a homeomorphism that conjugates f^k to a quadratic polynomial $z^2 + c$ on a neighbourhood of the corresponding Julia set (see [**D-H2**]). The multiplier at the fixed points are the same by the Naĭshul' theorem (see [**Nai**]). So the result follows from Yoccoz and Brjuno's work (see [**Yo**]). □

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8. *Proof of Theorem 3 and Theorem 4*

We modify the proof of Theorem 1 to apply the Yoccoz theory (i.e. Proposition 5.3) at every point of the Julia set $J(f)$. To prove Theorem 1, we have followed the orbit of every point of ∂B not eventually mapped to $\mathbf{0}$ using the itinerary. We have thus obtained property ($\diamondsuit$) (in Lemma 6.1). Note that the condition on the critical point was proved in Lemma 6.2 and will be used here also. The itinerary of the orbit of a point of ∂B was easy to obtain since the map Θ semi-conjugates f to $z \mapsto z^2$ on ∂B. A point of $J(f)$ can be followed by the itinerary of its limb, except that when it goes through the critical limb one can not deduce the itinerary of the image of this point, it can come back at any place of $K(f)$. So, condition ($\diamondsuit$) is much more tricky to verify for points of $J(f)$ than for points of ∂B.

Denote by $L(t_0)$ the limb containing the critical point c. Recall that if $c \notin \partial B$ then t_0 is periodic. We assume in this section that $t_0 \neq 0$. Recall that in this case the limb $L(\mathbf{0})$ reduces to $\{\mathbf{0}\}$.

8.1. *Verification of property ($\diamondsuit$).*

LEMMA 8.1. *Let $z \in J(f) \setminus \bigcup_{i\geq 0} f^{-i}(\mathbf{0})$. If there exists $j_0 \geq 0$ such that for $j \geq j_0$ the iterates $f^j(z)$ avoid $L(t_0)$, then the point z satisfies condition ($\diamondsuit$).*

Proof. Let u be the root of the limb $L(t)$ containing $f^{j_0}(z)$ (i.e. $u = \partial B \cap L(t)$). For every $i \geq 0$, the points $f^i(f^{j_0}(z))$ and $f^i(u)$ belong to the limb $L(2^i t) \neq L(t_0)$ and so are in the same puzzle piece of depth 1. Lemma 6.1 gives a subsequence n_i such that $f^{n_i}(u)$, and therefore $f^{n_i}(f^{j_0}(z))$, is surrounded by a non-degenerate annulus of depth 0. The result follows. □

LEMMA 8.2. *Let $z \in J(f) \setminus \bigcup_{i\geq 0} f^{-i}(\mathbf{0})$ be such that there exists a sequence n_i of iterations with $f^{n_i}(z) \in L(t_0)$, then z satisfies condition ($\diamondsuit$).*

Proof. First of all we should note that Steps 1 and 2 of Lemma 6.1 can be stated more generally as follows.

Step 1. We consider the angle $\theta = \theta_+$. Let P_0^+ be the puzzle piece containing $\mathbf{0}$ in its closure. Let P_1^+ be one of the puzzle pieces of depth 1 included in Ξ_1 and containing neither $f^{k-1}(\gamma(\theta))$ nor $\mathbf{0}$ in its closure. The annulus $P_0^+ \setminus \overline{P}_1^+$ is non-degenerate. The analogue statement holds for θ_-.

Proof. If the puzzle piece P_1^+ does not intersect B it is clearly compactly contained in the corresponding piece of depth 0 since neither equipotentials nor rays can touch.

Step 2. Let z be a point such that $\Theta(z)$ belongs to $[1/4, 1/2 - \epsilon] \cup [1/2 + \epsilon, 3/4]$, then there exists $k_0 > 0$ such that, for the graph Γ defined by the angle $\theta \in \{\theta_\pm = \pm 1/(2^k - 1)\}$ with $k \geq k_0$, the annulus $A_0(z)$ is non-degenerate.

Proof. For $k > 0$ sufficiently large, $2^i\Theta(z) \neq \theta_\pm$ for every $i \geq 0$. Let $x'_j(\theta_\pm)$ be the inverse image of $x_{j+1}(\theta_\pm)$ different from $x_j(\theta_\pm)$ for $0 \leq j \leq k-2$ (see Step 2 of Lemma 6.1). By Lemma 4.1 and Step 2 of Lemma 6.1, it is easy to see that $\Theta(x'_j(\theta_\pm)) = 2^j\theta_\pm + 1/2 \bmod 1$.

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Hence,

$$\Theta(x_{k-1}(\theta_+)) < \Theta(x'_0(\theta_+)) = \theta_+ + \frac{1}{2} \bmod 1 < \Theta(x'_1(\theta_+)) < \cdots < \Theta(x'_{k-2}(\theta_+))$$
$$= \frac{3}{4} + \frac{\theta_+}{4} < 1 \qquad (*)$$
$$0 < \Theta(x'_{k-2}(\theta_-)) = \frac{1}{4} + \frac{\theta_-}{4} \bmod 1 < \cdots < \Theta(x'_0(\theta_-))$$
$$= \theta_- + \frac{1}{2} \bmod 1 < \Theta(x_{k-1}(\theta_-)). \qquad (**)$$

Hence the result follows from Step 1 since $\theta_\pm$ tends to 0 when k goes to ∞.

Step 3. If $t_0 \in (1/4, 1/2)$ then there exist $k_0 > 0$ and a graph Γ defined by an angle of the form $\theta_- = -1/(2^k - 1)$ with $k \geq k_0$, such that for any point z of $L(t_0)$ the annulus $A_0(z)$ is non-degenerate.

Proof. Any point z of $W(t_0)$ will satisfy $\Theta(z) \in [1/4, 1/2 - \epsilon]$ where ϵ is less than $1/2 - t_0$. So the result follows from previous step.

Step 4. If $t_0 \in (1/2^{n+1}, 1/2^n)$ with $n > 0$, then there exist three graphs Γ defined by an angle of the form $\theta_\pm$ such that for any point z of $L(t_0)$ one of the annuli $A_i(z)$ is non-degenerate for $0 \leq i \leq n$.

Proof. Note that $2t_0 \in (1/2^n, 1/2^{n-1})$. For any $\epsilon > 0$, there exists $k'_0 > 0$ such that for $k \geq k'_0$ and for θ_+ the puzzle piece of depth 0 containing $L(2t_0)$ does not contain the limbs $L(t)$ for $t \in [1/2^{n-1} + \epsilon, 1] \cup [0, 1/2^n + \epsilon/2]$ (since $2^{k-i}\theta_+$ tends to $1/2^i$ when k tends to ∞). Hence for any point z of $L(t_0)$ that after one iteration falls in this domain, the annulus $A_0(z)$ is non-degenerate. We will take $\epsilon < 3/2^n$. Now we consider a point z of $L(t_0)$ that falls under one iteration in $L(t)$ with $t \in [1/2^n + \epsilon/2, 1/2^{n-1} + \epsilon]$. There exists $k''_0 > 0$ such that for $k \geq k''_0$ and for θ_-, the puzzle pieces of depth 1 containing $L(t)$ with $t \in [1/4, 1/2 + \theta_-]$ define a non-degenerate annulus of depth 0. Now $L(t)$ for $t \in [1/2^n, 1/2^{n-1} + \theta_-/2^{n-2}]$ is sent by f^{n-2} onto $L(2^{n-2}t)$ with $2^{n-2}t \in [1/4, 1/2 + \theta_-]$. Finally, we chose $\theta'_- < 2\theta_-$ (i.e. $k' < k$) so that $[1/4 + \theta_-/2, 1/4 + 2^{n-3}\epsilon] \subset [1/4 + \theta'_-/4, 3/8]$ since $\epsilon < 3/2^n$. The interval $[1/2^{n-1} + \theta_-/2^{n-2}, 1/2^{n-1} + \epsilon]$ is sent by multiplication by 2^{n-3} onto $[1/4 + \theta_-/2, 1/4 + 2^{n-3}\epsilon]$. Hence, for any $z \in L(t_0)$ sent by one iteration in $L(t)$ with $t \in [1/2^{n-1} + \theta_-/2^{n-2}, 1/2^{n-1} + \epsilon]$, the annulus $A_{n-2}(z)$ is non-degenerate. The conclusion follows with $k_0 = \max(k'_0, k''_0 + 1)$.

Step 5. For any $t_0 \in (0, 1/2)$, one can find a finite number of angles θ belonging to the set $\{\theta_\pm = \pm 1/(2^k - 1), k \geq 1\}$ such that $A_{n_i}(z)$ is non-degenerate for one of the graphs Γ defined by these angles.

Proof. Let N be such that $t_0 \in (1/2^N, 1/2^{N+1})$. By assumption, there exists a subsequence (n_i) such that $f^{n_i}(z) \in L(t_0)$. By previous step, there exists a finite number of graphs such that $A_j(f^{n_i}(z))$ is non-degenerate for some $0 \leq j \leq N$. Therefore, some of the annuli $A_{n_i+j}(z)$ are non-degenerate for $0 \leq j \leq N$. So z satisfies property ($\diamond$).

The argument is similar with symmetric angles for $t_0 \in (1/2, 1)$.

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8.2. *Proof of Theorem 3.*

LEMMA 8.3. *For any puzzle piece P_n, the intersection of $\overline{P}_n$ with $J(f)$ is a connected set.*

Proof. The graph is formed by equipotentials, external rays and internal accesses that cut the Julia set only at eventually periodic points on the boundary of iterated preimages of B. Therefore it is equivalent that $\overline{P}_n \cap J(f)$ is connected and that $J(f)$ is connected. □

Proof of Theorem 3. Since $L(\mathbf{0}) = \{\mathbf{0}\}$, the angle t_0 of the critical limb is non-zero. By Lemmas 6.2, 8.1 and 8.2, the hypothesis of Proposition 5.3 are satisfied. The local connectivity at points of $J(f) \setminus \bigcup_{i\geq 0} f^{-i}(\mathbf{0})$ follows since the map is not renormalizable. The local connectivity at $\mathbf{0}$ (and its preimages) follows from the fact that the intersection of the puzzle pieces containing $\mathbf{0}$ reduces to $\{\mathbf{0}\}$ (since $L(\mathbf{0}) = \{\mathbf{0}\}$). □

8.3. *Proof of Theorem 4.*

LEMMA 8.4. *Assume that f satisfies $L(\mathbf{0}) = \{\mathbf{0}\}$ and that f is p-renormalizable around c for some p. The Julia set $J(f)$ is locally connected if it is connected and if $K(f^p)$ (the filled Julia set of the renormalization) is locally connected, where p is the smallest renormalization period.*

Proof. First, note that if f is p-renormalizable around c, then $K(f^p) = \bigcap_n \overline{P_n(c)}$ for any of the graphs considered in §5, excepted a finite number described below. Indeed, $K(f^p)$ intersects $\overline{B}$ in at most one point since the dynamics on $\overline{B}$ is well understood. Therefore, for any of the graphs considered except the ones passing through this probable attached point, $K(f^p)$ is included in $\overline{P}_0(c)$. Using the dynamics it follows that $K(f^p)$ is included in $\overline{P}_{kp}(c)$ for $k \geq 0$ so in every $\overline{P}_n(c)$ with $n \geq 0$. Hence $K(f^p) \subset \bigcap_n \overline{P_n(c)}$. Now, $f^p : P_{n+p}(c) \to P_n(c)$ is clearly a degree-two ramified covering that will be a renormalization if we know that $\overline{P_{n+p}(c)} \subset P_n(c)$ for some n. The boundary of puzzle pieces intersect only at pre-periodic points, so avoiding p-periodic point in the starting graph (see §5) we obtain that $\overline{P_{n+p}(c)} \subset P_n(c)$ for every large n. Since p is minimal it follows that $K(f^p) = \bigcap_n \overline{P_n(c)}$.

Let z be a point of $J(f)$. If z is not an iterated inverse image of $\mathbf{0}$, it satisfies property ($\diamond$), by §8.2. Therefore Proposition 5.3 guaranties that either $\bigcap_n \overline{P_n(z)} = \{z\}$ or $\bigcap_n \overline{P_n(z)}$ is a preimage of $\bigcap_n \overline{P_n(c)}$ which is the filled Julia set of the renormalization $f^p : P_{n+p}(c) \to P_n(c)$.

It is enough to prove the local connectivity of $J(f)$ at the point z of $K(f^p)$: since $\bigcap_n \overline{P_n(\mathbf{0})} = \{\mathbf{0}\}$ $(= L(\mathbf{0}))$, the result for the iterated inverse images will follow.

We use the local connectivity of the small Julia set by pulling back neighbourhoods of the quadratic filled-in Julia set to $K(f^p)$ as follows. By the classical theory of polynomial-like mappings of Douady and Hubbard (see [**D-H2**]), there exists $c \in \mathbf{C}$ and a quasi-conformal homeomorphism $\phi : P_N(c) \to D$ that conjugates f to $Q_c(z) = z^2 + c$ for some large N and where D is some neighbourhood of the filled-in Julia set K_c of Q_c. Since $J_c := \partial K_c$ is locally connected, for each point $z \in K(f^p)$, the image $y = \phi(z)$ admits a sequence U_n of connected neighbourhoods in J_c whose intersection is $\{\phi(z)\}$. Moreover,

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we can chose the sequence such that U_n is the intersection of J_c with a puzzle piece defined by external rays, internal rays or accesses (in the case of Siegel disks) and 'equipotentials'. Therefore the corresponding landing points (image by ϕ^{-1}) for the map f will be also landing point of rays or accesses. Let $V_n := \phi^{-1}(U_n)$. The set $W_n := P_n(c) \cap V_n$ is a neighbourhood of z (since it is open), and it intersects $J(f)$ under a connected set since the Julia set is cut only at landing points of accesses and rays. Therefore, the sequence $(W_n)_{n\in\mathbf{N}}$ defines a basis of connected neighbourhoods for the point z. The local connectivity follows. □

COROLLARY 8.5. *Assume that f satisfies $L(\mathbf{0}) = \{\mathbf{0}\}$. The Julia set is locally connected, provided it is connected in the following cases.*

(1) *f is of 'Yoccoz type', i.e. is not infinitely renormalizable and has only repelling cycles, except from* **0**.
(2) *f is geometrically finite.*
(3) *f is of 'Petersen–Zakeri type', i.e. has an indifferent periodic point whose rotation number satisfies the Petersen–Zakeri condition (see* [**PeZa**, Theorem A]*) so that the corresponding Julia set (see Proposition 7.6) is locally connected.*

Proof. The second case follows from [**TaYi1**]. The first case goes as follows. If f is not renormalizable, Theorem 3 applies here. Else, consider the smallest renormalization period p of f around c. The map f^p is not infinitely renormalizable and has only repelling cycles. The map f^p is quasi-conformally conjugate to a quadratic polynomial Q_c that is not infinitely renormalizable and has only repelling cycles. By the theorem of Yoccoz (see [**Mi3**]), $K(Q_c)$ is locally connected. Since $K(f^p)$ is homeomorphic to $K(Q_c)$, it is also locally connected. Therefore we can apply Lemma 8.4.

The last case goes the same way. Remark first that $L(\mathbf{0}) = \{\mathbf{0}\}$ implies $p > 1$. Indeed, $K(f^p)$ belongs to some limb $L(t_0)$ with t_0 periodic, and the critical value $f(c)$ belongs to $L(2t_0)$. Therefore, if $p = 1$, $f(c) \in f(K(f)) = K(f) \subset L(t_0)$ so that $t_0 = 0$ and $L(\mathbf{0}) \neq \{\mathbf{0}\}$. Hence we can apply Proposition 7.6 to get a renormalization. The renormalized map has a periodic point that satisfies Petersen–Zakeri condition (by [**Nai**]) so its Julia set is locally connected (by [**PeZa**]) and the result follows as before applying Lemma 8.4. □

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