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Direct topological factorization for topological flows

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Abstract. This paper considers the general question of when a topological action of a countable group can be factored into a direct product of non-trivial actions. In the early 1980s, D. Lind considered such questions for $\mathbb{Z}$-shifts of finite type. In particular, we study direct factorizations of subshifts of finite type over $\mathbb{Z}^d$ and other groups, and $\mathbb{Z}$-subshifts which are not of finite type. The main results concern direct factors of the multidimensional full n-shift, the multidimensional 3-colored chessboard and the Dyck shift over a prime alphabet. A direct factorization of an expansive $\mathbb{G}$-action must be finite, but an example is provided of a non-expansive $\mathbb{Z}$-action for which there is no finite direct-prime factorization. The question about existence of direct-prime factorization of expansive actions remains open, even for $\mathbb{G} = \mathbb{Z}$.

1. *Introduction*

In this paper we study the notion of 'direct factorization' for topological dynamical systems. Other concepts of 'factorizations' and 'disjointness' in topological dynamics and ergodic theory have numerous, diverse and deep applications in mathematics, going back at least to Furstenberg's influential paper [**8**].

Throughout this paper, $\mathbb{G}$ will denote a countable group. By a $\mathbb{G}$-*topological dynamical system* or $\mathbb{G}$-*flow* we mean a pair (X, T), where X is a Hausdorff compact topological space, and T is an action of the group $\mathbb{G}$ on X by homeomorphisms. In other words, the map $g \mapsto T^g$ is a homomorphism from $\mathbb{G}$ to the group of self-homeomorphisms of X. A $\mathbb{G}$-flow (Y, S) is a *factor* of another $\mathbb{G}$-flow (X, T) if there exists a continuous surjective map $\pi : X \to Y$ which is equivariant, meaning $S^g \circ \pi = \pi \circ T^g$ for all $g \in \mathbb{G}$. The map π is called a *factor map* or *semi-conjugacy*. A $\mathbb{G}$-flow is called *prime* if its only proper factor is the *trivial* $\mathbb{G}$-*flow*, which is the trivial action on a singleton.

A *direct topological factorization* of a $\mathbb{G}$-flow (X, T) is a *topological conjugacy* or *isomorphism* of the form

$$(X, T) \cong (Y_1 \times \cdots \times Y_r, S_1 \times \cdots \times S_r).$$

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In other words, a direct topological factorization corresponds to a homeomorphism $\phi : X \to \prod_{i=1}^{r} Y_i$ so that $\phi(T^g) = (S_1 \times \cdots \times S_r)^g \phi(x)$ for all $x \in X$, $g \in \mathbb{G}$. We say each (Y_i, S_i), as above, is a *direct factor* of (X, T). Call a $\mathbb{G}$-flow (X, T) *topologically direct prime* if it does not admit a non-trivial direct topological factor. By a *direct-prime factorization* (DPF) we mean a direct factorization of (X, T) into direct-prime flows. Any direct factor of a $\mathbb{G}$-flow is indeed a factor. It follows that any prime $\mathbb{G}$-flow is direct-prime, but the converse is false.

The existence of a non-trivial factorization for a flow (X, T) immediately implies that the topological space X is homeomorphic to a non-trivial product $Y \times Z$. Thus, for instance any flow on the unit interval $X = [0, 1]$ or the one-dimensional sphere $X = \mathbb{R}/\mathbb{Z}$ is direct-prime, for 'purely topological reasons', having nothing to do with the group action.

Most of our study will concern *symbolic flows* or *subshifts*. These are *expansive* flows on totally disconnected compact metrizable topological spaces. One of the motivations for our choice to focus on symbolic systems is the attempt to avoid 'purely topological obstructions' as above.

Here is an outline of the rest of the paper.

In §2 we consider the following question: given a $\mathbb{G}$-flow, is it isomorphic to a finite product of direct-prime $\mathbb{G}$-flows? In the expansive case, the question remains open. An example of a $\mathbb{Z}$-action for which there is no finite direct-prime factorization is described.

In §3 we consider direct factorizations for *subshifts of finite type* (SFTs). We review results about $\mathbb{Z}$-SFTs and discuss factorization for $\mathbb{Z}^d$-SFTs, where much less is known. We present a result regarding direct factorization of $\mathbb{Z}^d$-full shifts. We also obtain a partial result which provides a weak form of a conjecture of Kari (Theorem 3.10).

In the remaining sections, we study direct factorizations for specific systems. In §4 we consider specific $\mathbb{Z}^d$-subshift of finite types: we prove that the d-dimensional 3-colored chessboard is topologically direct-prime for any $d \geq 1$. In §5 we consider *Dyck shifts*. These are $\mathbb{Z}$-subshifts which are not of finite type. In both cases we establish that the systems are topologically direct-prime. Our methods involve specific combinatorial and algebraic structure of the systems. To obtain our result on the 3-colored chessboards we rely on the cohomology of the system. For the Dyck shifts we rely on lack of intrinsic ergodicity, and the structure of the measures maximal entropy. In both cases we exploit information about periodic points of the system.

2. *On the existence of finite direct topological factorizations*

A $\mathbb{G}$-flow (X, T) is called *expansive* if there exists a finite open cover $\mathcal{U} = \{U_1, \ldots, U_L\}$ so that for any function $F : \mathbb{G} \to \mathcal{U}$ we have $|\bigcap_{g \in \mathbb{G}} T^{-g}[F(g)]| \leq 1$. Informally, this means that points can be separated by finite-precision measurements along the orbit. To slightly simplify the proofs, we assume below that the topological spaces involved are metrizable: let d be a compatible metric on X, then (X, T) is expansive if and only if there exists an $\epsilon > 0$ with the property that for every pair (x, y) of distinct points in X there exists a $g \in \mathbb{G}$ with $d(T^g(x), T^g(y)) \geq \epsilon$ [**9**, Lemma 17.10]. Such ϵ is called an expansive constant for (X, T), with respect to the metric d. We note that the assumption that X is metrizable is not essential for any of the results below.

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The following question remains open, in particular when $\mathbb{G} = \mathbb{Z}$.

Question 2.1. Is any expansive $\mathbb{G}$-flow isomorphic to a finite product of direct-prime $\mathbb{G}$-flows?

The following simple observation is useful for the study of direct factorizations of expansive systems.

PROPOSITION 2.2. *Any direct factor of an expansive $\mathbb{G}$-flow is expansive.*

Proof. Suppose $(X, T) \cong (Y_1 \times Y_2, S_1 \times S_2)$ is expansive. Let d_i be a compatible metric on Y_i. Identifying X with $Y_1 \times Y_2$, it follows that $d = d_1 + d_2$ is a compatible metric on X. Let $\epsilon > 0$ be an expansive constant for (X, T) with respect to the metric d. fix distinct points $y_1, \tilde{y}_1$ in Y_1 and $y_2 \in Y_2$, and consider $x = (y_1, y_2)$ and $\tilde{x} = (\tilde{y}_1, y_2)$ as points in X, whose projections to Y_2 coincide. By expansiveness of (X, T) there exists $g \in \mathbb{G}$ so that $d(T^g(x), T^g(\tilde{x})) > \epsilon$. Now,

$$d(T^g(x), T^g(\tilde{x})) = d_1(S_1^g(y_1), S_1^g(\tilde{y}_1)) + d_1(S_2^g(y_2), S_2^g(y_2)) = d_1(S_1^g(y_1), S_1^g(\tilde{y}_1)).$$

It follows that ϵ is also an expansive constant for (Y_1, S_1) with respect to the metric d_1. □

The following remark was kindly brought to the author's attention by the anonymous referee.

Remark 2.3. A factor of an expansive $\mathbb{G}$-flow might not be expansive. For instance, an irrational rotation is a non-expansive factor of the corresponding Sturmian shift, which is expansive. Even when (Y, S) is a factor of (X, T) and the factor map $\pi : X \to Y$ is open, expansivity of (X, T) does not in general imply expansivity of (Y, S). An example for an expansive algebraic action of the free group on two generators admitting a non-expansive open factor via an algebraic map is described in [**7**, Remark 3.4]. As remarked in [**7**], such algebraic examples are impossible in the case $\mathbb{G} = \mathbb{Z}^d$ by [**19**, Corollary 3.11].

LEMMA 2.4. *An infinite product of non-trivial systems is not expansive. Namely, if $(Y_i, S_i)_{i\in\mathbb{N}}$ are a sequence of non-trivial flows, then their product $\prod_{i=1}^{\infty}(Y_i, S_i)$ is not expansive.*

Proof. Suppose $(X, T) = \prod_{i=1}^{\infty}(Y_i, S_i)$ is expansive. As in the proof of Proposition 2.2, suppose d_i is a compatible metric on Y_i. Since Y_i is compact, the diameter of Y_i with respect to d_i is bounded, so by rescaling the distance d_i we can assume $\sup\{d_i(y, z) : y, z \in Y_i\} \leq 1$ for all $i \in \mathbb{N}$. Then $d = \sum_{i=1}^{\infty} 2^{-i} d_i$ is a compatible metric on X. If Y_i is non-trivial for infinitely many i, then for any $n \in \mathbb{N}$ we can find distinct $x, \tilde{x} \in X$, with $x = (y_i)_{i\in\mathbb{N}}$ and $\tilde{x} = (\tilde{y}_i)_{i\in\mathbb{N}}$ so that $y_i = \tilde{y}_i$ for $i \leq n$. It follows that $0 < d(T^g(x), T^g(\tilde{x})) < 2^{-n}$ for all $g \in \mathbb{G}$, so (X, T) is not expansive. □

LEMMA 2.5. *If (X, T) does not admit a finite DPF then (X, T) admits a factor which is isomorphic to an infinite product of non-trivial flows.*

Proof. By induction on $n \in \mathbb{N}$, construct sequences $\{(Y_n, S_n)\}_{n=1}^{\infty}$ and $\{(X_n, T_n)\}_{n=1}^{\infty}$ of non-trivial $\mathbb{G}$-flows so that (X_1, T_1) and $(X_n, T_n) \cong (Y_n \times X_{n+1}, S_n \times T_{n+1})$ and

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(X_n, T_n) does not admit a finite DPF. Indeed, assuming (X_n, T_n) does not admit a DPF, in particular it is not direct-prime, so it is a product of two non-trivial $\mathbb{G}$-flows, at least one of which does not admit a finite DPF.

Let $\pi_n : X \to Y_n$ be a factor map from (X, T) to (Y_n, S_n). The map $\pi : X \to \prod_{n=1}^{\infty} Y_n$ is continuous and equivariant, and the image is dense in $\prod_{n=1}^{\infty} Y_n$. Because X is compact, it follows that $\pi(X) = \prod_{n=1}^{\infty} Y_n$. So $(\prod_{n=1}^{\infty} Y_n, \prod_{n=1}^{\infty} S_n)$ is a factor of (X, T). □

From Lemma 2.5 we obtain the following corollary.

COROLLARY 2.6. *If (X, T) is a $\mathbb{G}$-flow such that X is countable set, then (X, T) admits a finite DPF.*

Question 2.7. What $\mathbb{G}$-flows admit a finite DPF?

Example 2.9 below is a system which does not admit a finite DPF, but is isomorphic to an infinite product of non-trivial direct-prime systems. We do not know if being isomorphic to an infinite product of non-trivial systems precludes the possibility of a finite DPF.

We conclude this section by discussing direct factorizations of group rotations, providing an example for a (non-expansive) system which does not admit a finite DPF.

2.1. *Direct factorization of group rotations.* A $\mathbb{Z}$-dynamical system (X, T) is called a *group rotation* if X admits a commutative group structure compatible with the given compact topology and $T(x) = x + x_0$ for some $x_0 \in X$. For a group rotation (X, T), if the orbit of some (hence any) $x \in X$ is dense then (X, T) is uniquely ergodic, where the unique invariant measure is Haar measure.

Recall that a $\mathbb{G}$-action is *equicontinuous* if for every $\epsilon > 0$ there exists $\delta > 0$ such that $d(x_1, x_2) < \delta$ implies $d(T^g(x), T^g(y)) < \epsilon$ for every $g \in \mathbb{G}$.

For a group Γ a direct group factorization is a group isomorphism $\Gamma \cong \Gamma_1 \times \Gamma_2$.

PROPOSITION 2.8. *The direct topological factorizations of a minimal group rotation (X, T) are in bijection with the direct factorizations of X as a topological group.*

Proof. A well-known characterization of minimal group rotations states that a minimal $\mathbb{Z}$-dynamical (X, T) system is a group rotation if and only if it is equicontinuous. Furthermore, any automorphism of (X, T) as a $\mathbb{Z}$-flow respects the group structure on X [**9**, Theorem 1.8]. Since a factor of an equicontinuous system is also equicontinuous, it follows that any factor of a minimal group rotation is a minimal group rotation. □

Example 2.9. Let $X_{\mathbb{P}} = \prod_{p \in \mathbb{P}} \mathbb{Z}/p\mathbb{Z}$ where p ranges over all primes $\mathbb{P}$, and let $T : X \to X$ be the map given by $(Tx)_p = x_p + 1 \bmod p$. This is a minimal compact group rotation, uniquely ergodic with rational pure-point spectrum. Such systems are often called 'Odometers'.

PROPOSITION 2.10. *The Odometer $(X_{\mathbb{P}}, T)$ does not admit a finite direct factorization into direct primes.*

Proof. By Corollary 2.8, the direct-topological factorizations of $(X_{\mathbb{P}}, T)$ correspond to the direct-group factorizations of the group $X_{\mathbb{P}} = \prod_{p \in \mathbb{P}} \mathbb{Z}/p\mathbb{Z}$, but this group does not admit a

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finite direct-group factorization into direct-prime groups. To see this, note that in any direct factorization $X_{\mathbb{P}} = \prod_{k=1}^{n} Y_k$ of $X_{\mathbb{P}}$, for each prime p there is a unique $k \in \{1, \ldots, n\}$ such that Y_k has non-trivial p-torsion, and the p-torsion of Y_k is isomorphic to $\mathbb{Z}/p\mathbb{Z}$. It follows that each Y_k is isomorphic as a group to $\prod_{p \in A_k} \mathbb{Z}/p\mathbb{Z}$, where $\bigcup_{k=1}^{n} A_k = \mathbb{P}$, and thus at least one direct factor Y_k is not prime. □

3. *Direct factorizations for shifts of finite type*

A *subshift* over a countable group $\mathbb{G}$ is characterized as an expansive $\mathbb{G}$-flow (X, T) of a totally-disconnected compact metrizable space X. A more concrete description is the following: a $\mathbb{G}$-system (X, T) is a subshift if and only if it is isomorphic to a subsystem of $(A^{\mathbb{G}}, \sigma)$ where A is a finite set, called 'the alphabet' or 'spins', and σ is the *shift-action*, given by $(\sigma^g(x))_h = x_{g^{-1}h}$. It is well known and easy to check that any subsystem of $(A^{\mathbb{G}}, \sigma)$ can be specified by a countable set $\mathcal{F} \subset \bigcup_{F \Subset \mathbb{G}} A^F$ of 'forbidden configurations' as follows:

$$X_{\mathcal{F}} = \{x \in A^{\mathbb{G}} : (\sigma^g x)_F \notin \mathcal{F}, \forall g \in \mathbb{G}, F \Subset \mathbb{G}\}, \tag{1}$$

where $x_F \in A^F$ denotes the restriction of $x \in A^{\mathbb{G}}$ to F, and $F \Subset \mathbb{G}$.

The system $(A^{\mathbb{G}}, \sigma)$ is called the $\mathbb{G}$-*full-shift* over the alphabet A.

PROPOSITION 3.1. (See [**17**, §6]) *Any direct factor of a $\mathbb{G}$-subshift is a $\mathbb{G}$-subshift.*

Proof. By Proposition 2.2 any direct factor of an expansive $\mathbb{G}$-action is expansive.

Both compactness and metrizability pass to continuous images of topological spaces.

A subspace of a totally disconnected space is totally disconnected. Since Y is homeomorphic to $Y \times \{z_0\}$ for $z_0 \in Z$, it follows that, in the case when $Y \times Z$ is totally disconnected, Y and Z must also be. Thus, whenever a totally-disconnected compact metrizable space X is homeomorphic to $Y \times Z$, Y and Z are also totally disconnected, metrizable and compact. □

A $\mathbb{G}$-subshift is of *finite type* (abbreviated $\mathbb{G}$-SFT) if it is isomorphic to a subsystem $(X_{\mathcal{F}}, \sigma)$ of the form (1) with $|\mathcal{F}| < \infty$. Equivalently, a $\mathbb{G}$-SFT is a $\mathbb{G}$-subshift which is not isomorphic to a strictly decreasing countable intersection of subshifts.

We record the following observation.

PROPOSITION 3.2. *Any direct factor of a $\mathbb{G}$-SFT is a $\mathbb{G}$-SFT.*

Proof. Suppose $X \cong Y \times Z$ and Z is not an SFT. Then there exist subshifts

$$\cdots \subset Z_n \subset Z_{n-1} \subset \cdots \subset Z_1$$

such that $Z = \bigcap_{n=1}^{\infty} Z_n$ and each inclusion is strict. It follows that

$$Y \times Z = \bigcap_{n=1}^{\infty} (Y \times Z_n),$$

so X is not an SFT. □

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Remark 3.3. A proof of Proposition 3.2 above for the particular case $\mathbb{G} = \mathbb{Z}$ appears in [**17**, §6], using an argument involving 'canonical coordinates' in the sense of Bowen [**3**]. It is not clear if there is a meaningful extension of this notion for subshifts over general groups.

Remark 3.4. A subshift factor of an SFT is called a *sofic shift*, and is not in general an SFT. When $\mathbb{G} = \mathbb{Z}$ there are several useful equivalent conditions for a subshift to be sofic. For a general group $\mathbb{G}$ (or even $\mathbb{G} = \mathbb{Z}^d$ with $d > 1$), sofic shifts are generally not so well understood.

Direct factorizations for $\mathbb{Z}$-SFTs were considered and studied in [**16**, **17**]. For $\mathbb{Z}$-SFTs, it turns out that direct factorizations are intimately related to a numerical invariant called the *topological entropy*.

Recall that a countable group $\mathbb{G}$ is called *amenable* if there exists a sequence $F_1, F_2, \ldots, F_n, \ldots \subset \mathbb{G}$ of finite sets satisfying $\lim_{n\to\infty} |gF_n \triangle F_n|/|F_n| = 0$ for all $g \in \mathbb{G}$. A sequence as above is called a *Følner sequence* in $\mathbb{G}$. This is one of many equivalent definitions for amenability.

The topological entropy of a $\mathbb{G}$-subshift (X, T) over an amenable group $\mathbb{G}$ is given by

$$h(X, T) = \lim_{n\to\infty} \frac{\log |\{x_{F_n} : x \in X\}|}{|F_n|}, \tag{2}$$

where $(F_n)_{n=1}^{\infty}$ is a Følner sequence. When $\mathbb{G} = \mathbb{Z}^d$, balls with radius increasing to infinity form a Følner sequence, for instance,

$$F_n = \{v \in \mathbb{Z}^d : \|v\|_\infty \le n\}.$$

The limit in (2) is equal to the infimum of the sequence inside the limit and does not depend on the particular choice of Følner sequence [**22**, Theorem 4.9]. More importantly, the topological entropy of (X, T) is invariant under isomorphism, and does not depend on the representation of (X, T).

In [**16**], Lind formulated a characterization of the numbers which can be realized as entropies of $\mathbb{Z}$-SFT's and of topologically mixing (or aperiodic) SFTs. Following [**16**], an algebraic integer $\lambda \in \mathbb{R}_+$ is called a *Perron number* if λ is greater than the absolute value of any one of its algebraic conjugates.

THEOREM 3.5. [**16**, Theorem 1] *For any Perron number λ there exists a topologically mixing $\mathbb{Z}$-SFT (X, T) such that $h(X, T) = \log(\lambda)$. Conversely, the topological entropy of any mixing $\mathbb{Z}$-SFT (X, T) is of the form $\log(\lambda)$ for a Perron number.*

For $\mathbb{Z}$-SFTs which are not necessarily topologically mixing, the class of entropy numbers consists of logarithms of nth roots of Perron numbers.

A Perron number λ is called *irreducible* if it impossible to write $\lambda = \alpha\beta$ with $\alpha, \beta > 1$ Perron numbers.

THEOREM 3.6. [**17**, Theorem 4] *Any Perron number admits a finite number of factorizations into a finite number of irreducible Perron numbers. There are only a finite number of such, but factorization is not always unique.*

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It follows that a mixing $\mathbb{Z}$-SFT (X, T) with $h(X, T) = \log(\lambda)$ with λ an irreducible Perron number is direct prime. There are additional obstructions to factorization of $\mathbb{Z}$-SFTs. For instance, a non-trivial direct factorization can be detected by the dimension-module, which is a certain ordered abelian group along with an order preserving automorphism [**4**].

For $\mathbb{Z}^d$ with $d \geq 2$, there is no analogous condition on the entropy of $h(X, T)$ of an SFT which guarantees that (X, T) is direct-prime: the class of numbers which occur as the topological entropy for a $\mathbb{Z}^d$-SFT is the class of non-negative, right recursively enumerable numbers [**11**].

Question 3.7. Does every $\mathbb{Z}^d$-SFT admit only a finite number of DPFs?

For $n \in \mathbb{N}$ we denote the first n positive integers by

$$[n] = \{1, \ldots, n\}.$$

For $d = 1$ the following result appears as [**17**, Theorem 7].

THEOREM 3.8. *For any $d \geq 1$, the full-shift $([n]^{\mathbb{Z}^d}, \sigma)$ is direct-prime if and only if n is a prime number.*

In fact, in [**17**], three different proofs are provided for this fact for $d = 1$. One proof is based on factorizations and does not extend to $d \geq 1$, since for every $d > 1$ and a every prime number p there exists a $\mathbb{Z}^d$ shift of finite type X with topological entropy $h(X) = \log p$ which is not topologically prime. Furthermore, such examples can be constructed with 'good mixing properties' [**5**]. Another proof involves '$\otimes$-factorization' of ζ-functions and factorizations over $\mathbb{C}[t]$, and does not seem to extend to higher dimensions. Yet another proof, attributed to Hansel, uses only periodic point counts, and extends to any dimension with minor modification.

The essence of Hansel's proof for Theorem 3.8 will serve us in §5 as a component in the proof that the Dyck shifts with a prime number of brackets are direct-prime. We can deduce Theorem 3.8 using Theorem 3.10 below.

If $n = p_1 \cdot \cdots \cdot p_k$ is a factorization of n, it is clear that

$$([n]^{\mathbb{G}}, \sigma) \cong \prod_{i=1}^{k} ([p_i]^{\mathbb{G}}, \sigma)$$

is a direct topological factorization, where $\mathbb{G}$ can be any discrete group.

The following question seems to be open, even for $\mathbb{G} = \mathbb{Z}$.

Question 3.9. For which groups $\mathbb{G}$ and natural numbers n does $([n]^{\mathbb{G}}, \sigma)$ have a unique direct prime factorization up to reordering?

It is known that any direct factor of $([n]^{\mathbb{Z}}, \sigma)$ is *shift-equivalent* to $([m]^{\mathbb{Z}}, \sigma)$ with $m|n$ (see [**12**, Lemma 2.1]). Recall that $\mathbb{Z}$-shifts of finite type (Y, σ) and (Z, σ) are shift-equivalent if and only if they are *eventually conjugate*, which means that (Y, σ^k) is topologically conjugate to (Z, σ^k) for all but a finite number of ks.

The following is a natural generalization of 'eventual conjugacy': say that $\mathbb{Z}^d$-flows (X, T) and (Y, S) are eventually conjugate if the subactions obtained from T and S by

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passing to a finite-index subgroup $L < \mathbb{Z}^d$ are topologically conjugate for all but a finite number of subgroups L. In [**12**], Kari conjectured that, for any $d \geq 1$, $n \in \mathbb{N}$ and any direct factorization $([n]^{\mathbb{Z}^d}, \sigma) \cong (X \times Y, T \times S)$, there exists a finite-index subgroup $L < \mathbb{Z}^d$ so that the restriction of the actions T and S to L are both topologically conjugate to $\mathbb{Z}^d$-full shifts.

Call $\mathbb{G}$-actions (X, T) and (Y, S) *periodically equivalent* if for any finite-index normal subgroup $\mathbb{H} \lhd \mathbb{G}$, the number of $\mathbb{H}$-fixed points in X is equal to the number of $\mathbb{H}$-fixed points in Y.

THEOREM 3.10. *For any $d \geq 1$, any direct factor of a $\mathbb{Z}^d$ full-shift is periodically equivalent to a $\mathbb{Z}^d$ full-shift.*

Specifically, up to reordering of the terms, any direct factorization of the $\mathbb{Z}^d$ full-shift $([n]^{\mathbb{Z}^d}, \sigma)$ into direct-primes is of the form

$$([n]^{\mathbb{Z}^d}, \sigma) \cong \prod_{i=1}^{k} (Y_i, \sigma_i),$$

where (Y_i, σ) is periodically equivalent to the full-shift $([m_i]^{\mathbb{Z}^d}, \sigma)$ $n = \prod_{i=1}^{k} m_i$.

Proof. We will prove the statement by induction on d. The base cases $d = 1$ follows from $\otimes$-factorizations of the ζ function of the full shift as in [**17**, §7], or using the fact that shift equivalence determines periodic-equivalence as in [**12**, Lemma 2.1].

We now assume $d > 1$. Given a $\mathbb{Z}^d$-dynamical system (X, T) and an infinite subgroup $L < \mathbb{Z}^d$, we denote by $X^{(L)}$ the fixed points of the L-subaction of T,

$$X^{(L)} := \{x \in X : T^n(x) = x \ \forall n \in L\}.$$

It follows that $(X^{(L)}, T)$ is a sub-system of (X, T). Furthermore, $\mathbb{Z}^d/L$ acts on $X^{(L)}$ via T, so we interpret $(X^{(L)}, T)$ as a $(\mathbb{Z}^d/L)$-flow.

Suppose $([n]^{\mathbb{Z}^d}, \sigma) \cong \prod_{i=1}^{k} (Y_i, T_i)$. Let m_i be the number of fixed points of (Y_i, T_i). It follows that $n = \prod_{i=1}^{k} m_i$. We need to show that for any finite-index subgroup $L < \mathbb{Z}^d$, we have $|Y_i^{(L)}| = m_i^{[\mathbb{Z}^d : L]}$.

Let L be a finite-index subgroup of $\mathbb{Z}^d$. There exist $v_1, \ldots, v_d \in \mathbb{Z}^d$ so that $L = \bigoplus_{i=1}^{d} \mathbb{Z} v_i$ and so that $\{v_1, \ldots, v_d\}$ is a basis for $\mathbb{Q}^d$ as a vector space over $\mathbb{Q}$. Let $K := \mathrm{span}_{\mathbb{Z}}\{v_1, \ldots, v_{d-1}\}$, $H := \mathrm{span}_{\mathbb{Q}}\{v_1, \ldots, v_{d-1}\}$ and $\tilde{K} := H \cap \mathbb{Z}^d$.

Check that $\mathbb{Z}^d/\tilde{K}$ is a torsion free quotient of $\mathbb{Z}^d$, and $\dim_{\mathbb{Q}}(\mathbb{Q}^d/H) = 1$. It follows that $\mathbb{Z}^d/\tilde{K} \cong \mathbb{Z}$. Let $w_d \in \mathbb{Z}^d$ be such that $w_d + \tilde{K}$ generates the group $\mathbb{Z}^d/\tilde{K}$. We see that $\mathbb{Z}^d = \tilde{K} \oplus \mathbb{Z} w_d$.

By the discussion above, the group $\mathbb{Z}^d/\tilde{K}$ acts on $(Y_i)^{(\tilde{K})}$. This is a direct factor of $X^{(\tilde{K})}$, which can be viewed as a $\mathbb{Z}$-full shift because $\mathbb{Z}^d/\tilde{K} \cong \mathbb{Z}$. Observe that the points in $(Y_i)^{(\tilde{K})}$ that are fixed by the shift action of $\mathbb{Z}^d/\tilde{K}$ are precisely the fixed point of Y_i under the $\mathbb{Z}^d$ action. Viewing $(Y_i)^{(\tilde{K})}$ as a direct factor of $X^{(\tilde{K})}$, it follows by the induction hypothesis (case $d = 1$) that $((Y_i)^{(\tilde{K})})^{(w_d)}$ is periodically equivalent to a $\mathbb{Z}$-full shift of the form $[m_i]^{\mathbb{Z}}$. In particular,

$$|((Y_i)^{(\tilde{K})})^{(\langle v_d \rangle)}| = m_i^{[(\mathbb{Z}^d/\tilde{K}) : \langle v_d + \tilde{K} \rangle]}, \tag{3}$$

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where $\langle v_d + \tilde{K}\rangle$ is the subgroup of $\mathbb{Z}^d/\tilde{K}$ spanned by $v_d + \tilde{K}$. Viewing $Y_i^{(\langle v_d\rangle)}$ as a subshift with respect to the action of $\tilde{K} \cong \mathbb{Z}^{d-1}$ we have

$$|((Y_i)^{(\langle v_d\rangle)})^{(\tilde{K})}| = |((Y_i)^{(\tilde{K})})^{(\langle v_d+\tilde{K}\rangle)}| = m_i^{[(\mathbb{Z}^d/\tilde{K}):\langle v_d+\tilde{K}\rangle]}, \tag{4}$$

where the second inequality follows from (3).

As both $Y_i^{(\langle v_d\rangle)}$ and $X^{(\langle v_d\rangle)}$ are $\tilde{K} \cong \mathbb{Z}^{d-1}$-subshifts, by induction hypothesis, $((Y_i)^{(\langle v_d\rangle)})$ is periodically equivalent to a $\tilde{K}$-full-shift. It follows that

$$|Y_i^{(L)}| = |((Y_i)^{(v_d)})^{(K)}| = |((Y_i)^{(v_d)})^{(\tilde{K})}|^{[\tilde{K}:K]} = m_i^{[(\mathbb{Z}^d/\tilde{K}):\langle v_d+\tilde{K}\rangle]\cdot[\tilde{K}:K]}.$$

Since $L \cong K \oplus \langle v_d\rangle$ we have $[(\mathbb{Z}^d/\tilde{K}) : \langle v_d + \tilde{K}\rangle] \cdot [\tilde{K} : K] = [\mathbb{Z}^d : L]$, and so

$$|Y_i^{(L)}| = m_i^{[\mathbb{Z}^d:L]}. \qquad \square$$

Problem. Do Theorems 3.8 and 3.10 above extend to other countable groups?

The following example shows that this is not the case for finite cyclic groups. I thank the referee for highlighting the relation to algebra.

Example 3.11. Consider the full shift with 5 symbols over the finite cyclic group $\mathbb{G} = \mathbb{Z}/4\mathbb{Z}$. The orbit counts determine a finite $\mathbb{G}$-flow up to isomorphism. The given system has 5 fixed points, 10 orbits of length 2 and 150 orbits of length 4. This $\mathbb{G}$-action lifts to a $\mathbb{Z}$-action given by multiplication by x on $L = \mathbb{F}_5[x]/\langle x^4 - 1\rangle$. As an $\mathbb{F}_5[x]$-module this system is a direct sum of two submodules, corresponding to the factorization $x^2 - 1 = (x^2 - 1)(x^2 + 1)$. Explicitly, we have $L = M \oplus N$ where $M = (x^2 - 1)L$ and $N = (x^2 + 1)L$. Here M has 1 fixed point and 10 orbits of length 4, while N has 5 fixed points and 10 orbits of length 2.

4. *Direct-primeness for the* 3*-colored chessboard*

In this section we prove that the 'd-dimensional 3-colored chessboard', denoted by C_3^d, is direct-prime. The $\mathbb{Z}^d$-subshift $C_3^d \subset \{0, 1, 2\}^{\mathbb{Z}^d}$ is a subshift of finite type which consists of proper 3-colorings of $\mathbb{Z}^d$, where we consider $\mathbb{Z}^d$ as the vertices of the Cayley graph with respect to the usual generators. Namely,

$$C_3^d := \{x \in \{0, 1, 2\}^{\mathbb{Z}^d} : x_n \neq x_m \text{ whenever } \|n - m\|_1 = 1\},$$

where $\|m\|_1 = \sum_{i=1}^d |m_i|$ is the l^1 norm of $m = (m_1, \ldots, m_d) \in \mathbb{Z}^d$.

It is useful to interpret C_3^d in the context of *graph homomorphisms*. We introduce the following notation.

Given graphs $G = (V_G, E_G)$ and $H = (V_H, E_H)$, we let $\mathrm{GHom}(G, H)$ denote the space of graph homomorphisms from G to H.

$$\mathrm{GHom}(G, H) := \{x \in (V_H)^{V_G} : (g_1, g_2) \in E(G) \Longrightarrow (x_{g_1}, x_{g_2}) \in E_H\}.$$

When H is finite, we consider $\mathrm{GHom}(G, H)$ as a compact topological space, with the topology induced from the product topology on $(V_H)^{V_G}$.

We identify $\mathbb{Z}^d$ with the vertices of the Cayley graph of $\mathbb{Z}^d$ with respect to the natural set of generators, and interpret $A \subset \mathbb{Z}^d$ as the vertex set of the induced graph from the

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Cayley graph of $\mathbb{Z}^d$. Let $\mathrm{GHom}(A, H)$ denote the set of graph homomorphisms from A to H. The restriction $\mathrm{res} : \mathrm{GHom}(\mathbb{Z}^d, H) \to \mathrm{GHom}(A, H)$ given by $\mathrm{res}(x) = x|_A$ is thus well defined.

With this notation,

$$C_3^d = \mathrm{GHom}(\mathbb{Z}^d, \mathbb{Z}/3\mathbb{Z}).$$

Consider the space $\mathrm{GHom}(\mathbb{Z}^d, \mathbb{Z}) \subset \mathbb{Z}^{\mathbb{Z}^d}$ of graph homomorphisms from the Cayley graph of $\mathbb{Z}^d$ to the Cayley graph of $\mathbb{Z}$, both with respect to the standard generators. Namely,

$$\mathrm{GHom}(\mathbb{Z}^d, \mathbb{Z}) = \{x \in \mathbb{Z}^{\mathbb{Z}^d} \, | x_n - x_m| = 1 \text{ whenever } \|n - m\|_1 = 1\}.$$

Since the Cayley graph of $\mathbb{Z}$ covers the Cayley graph $\mathbb{Z}/3\mathbb{Z}$, it follows that $\mathrm{GHom}(\mathbb{Z}^d, \mathbb{Z})$ projects to C_3^d via the following continuous shift-equivariant map $\pi : \mathrm{GHom}(\mathbb{Z}^d, \mathbb{Z}) \to C_3^d$ defined by

$$\pi(x)_n := x_n \mod 3. \tag{5}$$

The following observation is classical (see for instance [**20**, §4.3]).

PROPOSITION 4.1. *The map $\pi : \mathrm{GHom}(\mathbb{Z}^d, \mathbb{Z}) \to C_3^d$ is surjective. Furthermore, if $x, y \in \mathrm{GHom}(\mathbb{Z}^d, \mathbb{Z})$ satisfy $\pi(x) = \pi(y)$ then there exists $m \in 3\mathbb{Z}$ so that $x_n = y_n + m$ for all $n \in \mathbb{Z}^d$.*

Our goal in this section is to prove the following result.

THEOREM 4.2. *For any $d \geq 1$, the $\mathbb{Z}^d$ 3-colored C_3^d chessboard is topologically direct prime.*

The case $d = 1$ easily follows from Lind's work [**16**]: note that $h(C_3^1) = \log(2)$ and 2 is a prime number (in particular an irreducible Perron number). Now observe that C_3^1 is a mixing $\mathbb{Z}$-SFT and conclude it must be direct-prime.

For $d \geq 2$ our argument is based on the cohomology of shift action on C_3^d. We briefly recall definitions to make our exposition reasonably self-contained. See [**20**] for various results and applications regarding cohomology of $\mathbb{Z}^d$ subshifts of finite type.

Definition 4.3. Let (X, T) be a $\mathbb{Z}^d$-dynamical system.

(1) A continuous cocycle (X, T) (abbreviated T-cocycle, or cocycle when T is clear for the context) is a continuous function $c : X \times \mathbb{Z}^d \to \mathbb{R}$ satisfying

$$c(x, n + m) = c(x, m) + c(T^m x, n) \quad \text{for all } x \in X, m, n \in \mathbb{Z}^d.$$

(2) A *trivial* cocycle is of the from $c(x, n) = \alpha(n)$, where $\alpha \in \mathrm{Hom}(\mathbb{Z}^d, \mathbb{R})$.

(3) A T-coboundary is a cocycle $b : X \times \mathbb{Z}^d \to \mathbb{R}$ of the form $b(x, n) = f(T^n x) - f(x)$, where $f \in C(X)$.

(4) The *cohomology group* of (X, T) is the group of cocycles modulo the subgroup of coboundaries, where the group operation is pointwise addition. We denote the cohomology group by $H(X, T)$. When X is a subshift and T the shift action, we abbreviate this by $H(X)$.

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By Proposition 4.1 the map $\mathrm{ht} : C_3^d \times \mathbb{Z}^d \to \mathbb{R}$ given by

$$\mathrm{ht}(x, n) = \hat{x}_0 - \hat{x}_n, \tag{6}$$

where $\hat{x} \in \mathrm{GHom}(\mathbb{Z}^d, \mathbb{Z})$ satisfies $\pi(\hat{x}) = x$ and is a well-defined cocycle. It is not difficult to check that ht is not cohomologous to a trivial cocycle (see [**21**]). We refer to ht as the *height cocycle*.

In the following we consider $\mathbb{Z}^{d-1}$ as a subgroup of $\mathbb{Z}^d$, using the embedding

$$(n_1, \ldots, n_{d-1}) \mapsto (n_1, \ldots, n_{d-1}, 0).$$

The subshift C_3^{d-1} is embedded as a sub-system of a $\mathbb{Z}^{d-1}$ sub-action of C_3^d as follows. Consider the subspace

$$\tilde{C}_3^d = \{x \in C_3^d : x_{n+e_d} = x_n + 1 \bmod 3 \ \forall n \in \mathbb{Z}^d\}. \tag{7}$$

The map $x \mapsto x|_{\mathbb{Z}^{d-1}}$ is a $\mathbb{Z}^{d-1}$-equivariant homeomorphism from $\tilde{C}_3^d$ to C_3^{d-1}.

LEMMA 4.4. *For any $x \in C_3^d$ and $N \in \mathbb{Z}$ there exist $y \in C_3^d$ and $z \in \tilde{C}_3^d$ so that*

$$y_{n+ke_d} = \begin{cases} x_{n+ke_d} & k \geq N, \\ z_{n+ke_d} & k \leq N, \end{cases} \quad n \in \mathbb{Z}^{d-1}. \tag{8}$$

Proof. For $x \in C_3^d$, define $z \in \tilde{C}_3^d$ by

$$z_{n+ke_d} := x_{n+Ne_d} + k - N \bmod 3, \quad n \in \mathbb{Z}^{d-1}, k \in \mathbb{Z}. \tag{9}$$

Check that indeed $z \in \tilde{C}_3^d$ and that the unique $y \in \{0, 1, 2\}^{\mathbb{Z}^d}$ given by (8) is a well-defined point in C_3^d. □

The following is an adaptation of a result from [**21**].

PROPOSITION 4.5. *For any $d \geq 2$, any continuous locally constant cocycle on C_3^d is cohomologous to a sum of a trivial cocycle and a multiple of the height cocycle ht. To be precise, for any locally constant cocycle $c : C_3^d \times \mathbb{Z}^d \to \mathbb{R}$ there exist $a \in \mathbb{R}$, $\alpha \in \mathrm{Hom}(\mathbb{Z}^d, \mathbb{R})$ and a locally constant function $f : C_3^d \to \mathbb{R}$ so that*

$$c(x, n) = a \cdot \mathrm{ht}(x, n) + \alpha(n) + f(\sigma^n(x)) - f(x).$$

Proof. For $d = 2$ this follows from [**21**, Theorem 7.1], which identifies the so called 'fundamental cocycle' for the two-dimensional 3-colored chessboard $C_3^{(2)}$. Instead of trying to extend the arguments of [**21**] to higher dimensions, we will proceed by induction on $d \geq 2$, using [**21**, Theorem 7.1] to start the induction.

Let $c : C_3^d \times \mathbb{Z}^d \to \mathbb{R}$ be a locally constant cocycle. Hence, there exists a finite set $F \Subset \mathbb{Z}^d$ so that $x|_F$ determines $c(x, e_i)$ for all $x \in C_3^d$ and $i = 1, \ldots, d$. It follows that the restriction of $x|_{F+\mathbb{Z}^{d-1}}$ determines $c(x, n)$ for all $n \in \mathbb{Z}^{d-1}$. Choose $x \in C_3^d$. Apply Lemma 4.4 to find $y \in C_3^d$ and $z \in \tilde{C}_3^d$ so that $y|_{F+\mathbb{Z}^{d-1}} = x|_{F+\mathbb{Z}^{d-1}}$, and so that, for all $k \geq N$,

$$\sigma^{ke_d}(y)|_{F+\mathbb{Z}^{d-1}} = \sigma^{ke_d}(z)|_{F+\mathbb{Z}^{d-1}}.$$

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It follows that for $n \in \mathbb{Z}^{d-1}$

$$c(x, n) = c(y, n) = c(y, 2Ne_d) + c(\sigma^{2Ne_d}(y), n) + c(\sigma^{2Ne_d+n}(y), -2Ne_d),$$

so

$$c(x, n) = c(y, 2Ne_d) + \tilde{c}(z, n) - c(\sigma^n y, 2Ne_d), \tag{10}$$

where $\tilde{c} : \tilde{C}_3^d \times \mathbb{Z}^{d-1} \to \mathbb{R}$ is the restriction of $c : C_3^d \times \mathbb{Z}^d \to \mathbb{R}$. Since the $\mathbb{Z}^{d-1}$ shift action on $\tilde{C}_3^d$ is isomorphic to C_3^{d-1}, by induction on $d \geq 2$, there exist $a \in \mathbb{R}$, $\alpha \in \mathrm{Hom}(\mathbb{Z}^{d-1}, \mathbb{R})$ and a locally constant function $\tilde{f} : \tilde{C}_3^d \to \mathbb{R}$ so that

$$\tilde{c}(z, n) = a \,\mathrm{ht}(z, n) + \alpha(n) + \tilde{f}(\sigma^n(z)) - \tilde{f}(z). \tag{11}$$

From (9) we see that $\mathrm{ht}(z, n) = \mathrm{ht}(\sigma^{Ne_d}(x), n)$ for $n \in \mathbb{Z}^{d-1}$. Let $f(y) := c(y, 3Ne_d)$. It follows from (10) and (11) that for any $n \in \mathbb{Z}^{d-1}$ and any $x \in C_3^d$

$$c(x, n) = a \cdot \mathrm{ht}(\sigma^{Ne_d}(x), n) + \alpha(n) + \tilde{f}(\sigma^n(x)) - \tilde{f}(x) + f(\sigma^n(x)) - f(x). \tag{12}$$

Now, because

$$\mathrm{ht}(x, n) - \mathrm{ht}(\sigma^{Ne_d}(x), n) = \mathrm{ht}(x, Ne_d) - \mathrm{ht}(\sigma^n(x), Ne_d),$$

we see that indeed that there exists $a_1 \in \mathbb{R}$, $\alpha_1 \in \mathrm{Hom}(\mathbb{Z}^d, \mathbb{R})$ and a locally constant function $f_1 : C_3^d \to \mathbb{R}$ so that, for any $x \in C_3^d$,

$$c(x, e_i) = a_1 \,\mathrm{ht}(x, e_i) + \alpha_1(e_i) + f_1(\sigma^{e_i} x) - f_1(x) \tag{13}$$

for all $1 \leq i \leq d-1$.

Repeat the above argument, this time embedding $\mathbb{Z}^{d-1}$ in $\mathbb{Z}^d$ via

$$(n_1, \ldots, n_{d-1}) \mapsto (0, n_1, \ldots, n_{d-1})$$

to conclude that there exists $a_2 \in \mathbb{R}$, $\alpha_2 \in \mathrm{Hom}(\mathbb{Z}^d, \mathbb{R})$ and a locally constant function $f_2 : C_3^d \to \mathbb{R}$ so that, for any $x \in C_3^d$,

$$c(x, e_i) = a_2 \,\mathrm{ht}(x, e_i) + \alpha_2(e_i) + f_2(\sigma^{e_i} x) - f_2(x) \tag{14}$$

for all $2 \leq i \leq d$. The proof will be complete once we show that we can choose $a_1 = a_2$ $\alpha_1 = \alpha_2$ and $f_1 = f_2$.

Since $d \geq 3$, we choose $2 \leq i \leq d-1$, and conclude from (13) and (14) that $a_1 = a_2$ $\alpha_1(e_i) = \alpha_2(e_i)$ and $f_1(\sigma^{ke_i} x) - f_1(x) = f_2(k\sigma^{ke_i} x) - f_2(x)$ for all $k \in \mathbb{N}$ and $x \in C_3^d$. We can thus define $\alpha \in \mathrm{Hom}(\mathbb{Z}^d, \mathbb{R})$ as follows:

$$\alpha(e_i) = \begin{cases} \alpha_1(e_i), & 1 \leq i \leq d-1, \\ \alpha_2(e_i), & i = d. \end{cases}$$

Because $C_3^{(d)}$ is topologically mixing, it follows that f_1 and f_2 differ by a constant, and so indeed the cocycle $c : C_3^d \times \mathbb{Z}^d \to \mathbb{R}$ is cohomologous to $a \cdot \mathrm{ht} + \alpha$. □

The following lemma is a slight refinement of the fact that C_3^d has a dense set of periodic points.

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LEMMA 4.6. *For any infinite $A \subset \mathbb{N}$, the set of points $x \in C_3^d$ whose stabilizer contains $k\mathbb{Z}^d$ for some $k \in A$ is dense in C_3^d.*

Proof. By Proposition 4.1 the statement will follow once we prove that a corresponding set of points is dense in $\mathrm{GHom}(\mathbb{Z}^d, \mathbb{Z})$.

Indeed, for any $k \in \mathbb{N}$, any $\hat{x} \in \mathrm{GHom}([-k, k]^d, \mathbb{Z})$ is the restriction of a $\hat{y} \in \mathrm{GHom}([-2k, 2k], \mathbb{Z})$ with 'flat boundary', that is $|\hat{y}_n - \hat{y}_m| \leq 1$ for any $n, m \in \partial[-k, k]^d$, where $\partial[-k, k]^d := \{m = (m_1, \ldots, m_d) \in \mathbb{Z}^d : \max_i |m_i| = k\}$. A proof of can be found, for instance, in [**6**, Lemma 7.3]. □

The next lemma says that there only 3 points in C_3^d of 'maximal slope'.

LEMMA 4.7. *There exist precisely* 3 *points $x \in C_3^d$ which satisfy*

$$\mathrm{ht}(x, m) = m_1 + m_2 + \cdots + m_d \tag{15}$$

for any $m = (m_1, \ldots, m_d) \in \mathbb{Z}^d$.

Furthermore, if $x \in C_3^d$ has finite orbit and satisfies (15) for all m in the stabilizer of x, then x satisfies (15) for all $m \in \mathbb{Z}^d$.

Proof. Suppose $\hat{x} \in \mathrm{GHom}(\mathbb{Z}^d, \mathbb{Z})$ and $(m_1, \ldots, m_d) \in \mathbb{Z}^d$ are such that

$$\hat{x}_{(m_1,\ldots,m_d)} - \hat{x}_0 = m_1 + \cdots + m_d. \tag{16}$$

Since $\hat{x}_{n+e_i} - \hat{x}_n \leq 1$ for all $n \in \mathbb{Z}^d$, it follows that $\hat{x}_{n+e_i} = \hat{x}_n + 1$ whenever n and $n + e_i$ are on a shortest path connecting 0 and $(m_1, \ldots, m_d)$ in the Cayley graph of $\mathbb{Z}^d$. Now suppose x satisfies (15) for all $m = (m_1, \ldots, m_d) \in \mathbb{Z}^d$ and let $\hat{x} \in \mathrm{GHom}(\mathbb{Z}^d, \mathbb{Z})$ be such that $\pi(\hat{x}) = x$. It follows that $\hat{x}_{n+e_i} = \hat{x}_n + 1$ for all $n \in \mathbb{Z}^d$ and $i = 1, \ldots, d$. Thus, the value of $x_0 \in \mathbb{Z}/3\mathbb{Z}$ uniquely determines x. This proves the first part of the lemma.

Now, if x has finite orbit, L is the stabilizer of x and (15) holds for all $m \in L$ and $\hat{x} \in \mathrm{GHom}(\mathbb{Z}^d, \mathbb{Z})$ satisfies $\pi(\hat{x}) = x$, then $\hat{x}_{n+e_i} = \hat{x}_n + 1$ for all $n \in \mathbb{Z}^d$, because there exists some $m \in L$ for which n and $n + e_i$ are on some shortest path connecting 0 and m. This proves the second part of the lemma. □

Recall that an automorphism ψ of a $\mathbb{Z}^d$-topological dynamical system (X, T) is a homeomorphism $\psi : X \to X$ satisfying $\psi(T^n(x)) = T^n(\psi(x))$ for all $x \in X$, $n \in \mathbb{Z}^d$. We denote the group of automorphisms of (X, T) by $\mathrm{Aut}(X, T)$.

LEMMA 4.8. *For any $\psi \in \mathrm{Aut}(C_3^d, \sigma)$, there exists $u_\psi \in \{\pm 1\}$ such that the cocycle ht_ψ defined by $\mathrm{ht}_\psi(x, n) := \mathrm{ht}(\psi(x), n)$ is cohomologous to $u_\psi \cdot \mathrm{ht}$.*

Proof. By Proposition 4.5 above, there exists $a_\psi \in \mathbb{R}$ so that ht_ψ is cohomologous to $a_\psi \cdot \mathrm{ht} + \alpha_\psi$, where $\alpha_\psi \in \mathrm{Hom}(\mathbb{Z}^d, \mathbb{R})$ is a trivial cocycle. Note that for any $n = (n_1, \ldots, n_d) \in \mathbb{Z}^d$ such that $\|n\|_1$ is even

$$\max_{x \in C_3^d} \mathrm{ht}(x, n) = 2\|n\|_1 \quad \text{and} \quad \min_{x \in C_3^d} \mathrm{ht}(x, n) = -2\|n\|_1.$$

So

$$\max_{x \in C_3^d} \mathrm{ht}_\psi(x, n) = 2|a_\psi| \cdot \|n\|_1 + \alpha_\psi(n) \quad \text{and} \quad \min_{x \in C_3^d} \mathrm{ht}_\psi(x, n) = -2|a_\psi| \cdot \|n\|_1 + \alpha_\psi(n).$$

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On the other hand, since ψ is surjective,

$$\max_{x\in C_3^d} \mathrm{ht}(x,n) = \max_{x\in C_3^d} \mathrm{ht}_\psi(x,n) \quad \text{and} \quad \min_{x\in C_3^d} \mathrm{ht}(x,n) = \min_{x\in C_3^d} \mathrm{ht}_\psi(x,n).$$

It follows that $a_\psi \in \{\pm 1\}$ and $\alpha_\psi(n) = 0$. □

Concluding the proof of Theorem 4.2: Suppose $C_3^d \cong Y \times Z$ is a non-trivial direct factorization.

By Lemma 4.6 there exists a finite-index subgroup $L \lhd \mathbb{Z}^d$ for which $(C_3^d)^L := \bigcap_{n\in L}\{x \in C_3^d : \sigma^n x = x\}$ is non-empty. Furthermore, we can choose L so that $[\mathbb{Z}^d : L]$ is odd. It follows that $Z^L := \bigcap_{n\in L}\{z \in Z : \sigma^n z = z\}$ is also non-empty. Choose $\hat{z} \in Z^L$, and a finite set $F_L \subset \mathbb{Z}^d$ of representatives for $\mathbb{Z}^d/L$. Define a cocycle $c : Y \times \mathbb{Z}^d \to \mathbb{R}$ by

$$c(y,n) := \sum_{m\in F_L} \mathrm{ht}((y, \sigma^m \hat{z}), n),$$

where we naturally identify the pair $(y, \sigma^m \hat{z})$ as an element of C_3^d. To check that c is indeed a cocycle note that, for any $n \in \mathbb{Z}^d$,

$$\{\sigma^m \tilde{z} : m \in F_L\} = \{\sigma^{m+n} \tilde{z} : m \in F_L\},$$

because both are equal to the orbit of $\tilde{z}$ under σ. Thus, for any $n_1, n_2 \in \mathbb{Z}^d$

$$\begin{aligned} c(y, n_1 + n_2) &= \sum_{m\in F_L} \mathrm{ht}((y, \sigma^m \hat{z}), n_1) + \sum_{m\in F_L} \mathrm{ht}((\sigma^{n_1} y, \sigma^{m+n_1} \hat{z}), n_2) \\ &= c(y, n_1) + c(\sigma^{n_1} y, n_2). \end{aligned}$$

Because (Y, σ) is a factor of (C_3^d, σ), c naturally lifts to a locally constant cocycle on (C_3^d, σ).

From Proposition 4.5 it follows that there exists $\phi = \phi_{\hat{z}} \in \mathrm{Hom}(\mathbb{Z}^d, \mathbb{R})$ and $\alpha_{\hat{z}} \in \mathbb{R}$ (both *a priori* depending on $\hat{z}$) so that $c : C_3^d \times \mathbb{Z}^d \to \mathbb{R}$ is cohomologous to $\alpha_{\hat{z}}$ ht $+\phi$.

Observe that for any $m \in \mathbb{Z}^d$, the map $(y, z) \mapsto (y, \sigma^m(z))$ is an automorphism of C_3^d. It follows from Lemma 4.8 that for any $m \in \mathbb{Z}^d$ either $\mathrm{ht}((y, z), n) = \mathrm{ht}((y, \sigma^m(z)), n)$ for all $y \in Y$, $z \in Z$ $n \in \mathbb{Z}^d$ or $\mathrm{ht}((y, z), n) = -\mathrm{ht}((y, \sigma^m(z)), n)$.This means that there is homomorphism $s \in \mathrm{Hom}(\mathbb{Z}^d, \mathbb{Z}/2\mathbb{Z})$ so that $\mathrm{ht}((y, z), n) = (-1)^{s(m)} \mathrm{ht}((y, \sigma^m(z)), n)$. Because $[\mathbb{Z}^d : L]$ is odd and Z^L non-empty there is some $C \in \mathbb{Z}\backslash\{0\}$ and coboundary $b : C_3^d \times \mathbb{Z}^d \to \mathbb{R}$ so that, for any $z \in Z$ and $n \in \mathbb{Z}^d$,

$$\sum_{m\in F_L} \mathrm{ht}((y, \sigma^m z), n) = C \cdot \mathrm{ht}((y, z), n) + b((y, z), n).$$

First, suppose that there *exist* $L < \mathbb{Z}^d$ and $\hat{z} \in Z^L$ as above so that $\alpha_{\hat{z}} \neq 0$. It follows that $\mathrm{ht} : (Y \times Z) \times \mathbb{Z}^d \to \mathbb{R}$ is cohomologous to a cocycle which only depends on Y, and so $\phi_{\hat{z}} = 0$ for all $\hat{z}$. Thus, $\alpha_z = \alpha_{\hat{z}} \neq 0$ for *any* point $z \in C_3^d$ with finite orbit.

Choose $k \in 3\mathbb{Z}$, and let $\tilde{L} = k\mathbb{Z}^d$. Let $\tilde{y} \in Y, \tilde{z} \in Z$ be such that $(\tilde{y}, \tilde{z}) \cong x \in C_3^d$ satisfies $\sigma_{ke_i} x = x$ and $\mathrm{ht}(x, ke_i) = k$ for $i = 1, \ldots, d$, as in Lemma 4.7. Since ht is cohomologous to a cocycle that does not depend on z, it follows that $\mathrm{ht}((\tilde{y}, z), ke_i) = k$ for all $z \in Z^{\tilde{L}}$. From Lemma 4.7 we conclude that there are at most 3 points in $\bigcup_L Z^L$, where the union is over all finite-index subgroups L such that $[\mathbb{Z}^d : L]$ is odd. By Lemma 4.6 $\bigcup_L (C_3^d)^L$ is

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a dense in C_3^d. It follows that $\bigcup_L Z^L$ is dense in Z, so Z must be finite. Because C_3^d is topologically mixing, it has only trivial finite factors. This implies Z is a trivial one-point system.

Otherwise, $\alpha_{\hat{z}} = 0$ for all $\hat{z} \in Z^L$ with $[\mathbb{Z}^d : L]$ odd. In this case it follows that ht is cohomologous to a cocycle which only depends on Z. Replacing the roles of Y and Z, we conclude using Lemma 4.6 as in the previous case that Y is finite, hence trivial.

5. *Direct-primeness for Dyck shifts*

Dyck shifts are a one parameter class of non-sofic $\mathbb{Z}$-subshifts. They were introduced by Krieger in [**13**], as a counterexample to a conjecture of Weiss, and appeared in various papers in the literature since. We now recall a definition of the Dyck shifts.

Let $N > 1$ be a natural number. Write $\Sigma_N = (\{\alpha_1, \ldots, \alpha_N\} \cup \{\beta_1, \ldots, \beta_N\})$. Consider the monoid M generated by $\Sigma_N \cup \{0\}$ subject to the following relations:

(1) $\alpha_i \beta_i = 1$ for $i \in \{1, \ldots, N\}$;

(2) $\alpha_i \beta_j = 0$ for $i \neq j$, $i, j \in \{1, \ldots, N\}$.

The N-Dyck shift is defined by

$$D_N = \{x \in \Sigma_N^{\mathbb{Z}} : x_n \cdot x_{n+1} \cdot \cdots \cdot x_{n+k} \neq 0 \ \forall n \in \mathbb{Z}, k \in \mathbb{N}\}.$$

Informally, if we think of $\{\alpha_1, \ldots, \alpha_N\}$ as N types of 'left brackets' and of $\{\beta_1, \ldots, \beta_N\}$ as N corresponding 'right brackets', D_N consists of all bi-sequences with no 'mismatching pairs of brackets'.

In this section we prove the following.

THEOREM 5.1. *For any prime number N, the N-Dyck shift D_N is topologically direct-prime.*

The assumption that N is prime seems to be an artifact of the proof method.

Let us introduce auxiliary definitions. Following [**13**], define two continuous shift-commuting maps $\pi_+, \pi_- : D_N \to \{0, \ldots, N\}^{\mathbb{Z}}$ by

$$\pi_+(x)_n = \begin{cases} i, & x_n = \alpha_i, \\ 0, & x_n \in \{\beta_1, \ldots, \beta_N\}, \end{cases}$$

$$\pi_-(x)_n = \begin{cases} i, & x_n = \beta_i, \\ 0, & x_n \in \{\alpha_1, \ldots, \alpha_N\}. \end{cases}$$

Denote by ν_N the uniform Bernoulli measure on $\{0, \ldots, N\}^{\mathbb{Z}}$, which is uniquely defined by $\nu_N([a]_k) = 1/N^m$ for all $a = (a_1, \ldots, a_m) \in \{0, \ldots, N\}^m$. It was observed in [**13**] that there is a shift-invariant Borel set $X_0 \subset \{0, \ldots, N\}^{\mathbb{Z}}$ of full ν_M-measure so that any $x \in X_0$ has a unique pre-image under π_- and a unique pre-image under π_+ .

A dynamical system is called *intrinsically ergodic* if it admits a unique measure of maximal entropy. One interesting feature of Dyck shifts, which was discovered in [**13**], is that they are not intrinsically ergodic.

THEOREM 5.2. [**13**, Theorem 3] *For any $N \geq 2$ there exists precisely 2 ergodic measures of maximal entropy μ_+ and μ_- for the Dyck shift D_N. The measures μ_+ and μ_- are the pull-back of the uniform Bernoulli measure on $\{0, \ldots, N\}^{\mathbb{Z}}$ via π_+ and π_-, respectively.*

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Recall that a *joining* of two probability preserving $\mathbb{G}$-actions $(X, \mathcal{B}, \mu, T)$ and $(Y, \mathcal{C}, \nu, S)$ is a probability measure λ on $(X\times, \mathcal{B}\otimes\mathcal{C})$ which is $T \times S$-invariant and has $\mu = \lambda \circ \pi_X^{-1}$ and $\nu = \lambda \circ \pi_Y^{-1}$ where $\pi_X : X \times Y \to X$ and $\pi_Y : X \times Y \to Y$ are the obvious projection maps. $(X, \mathcal{B}, \mu, T)$ and $(Y, \mathcal{C}, \nu, S)$ are *disjoint* if the only joining of the systems is the independent joining $\lambda = \mu \times \nu$.

We say that a pair of probability preserving actions of an amenable group $\mathbb{G}$ $(X, \mathcal{B}, \mu, T)$ and $(Y, \mathcal{C}, \nu, S)$ are *intrinsically disjoint* if the independent joining is the only joining which maximizes the entropy.

We use this term to state the following simple lemma.

LEMMA 5.3. (Bernoulli transformations are 'pairwise intrinsically disjoint') *Let $(X, \mathcal{B}, \mu, T)$ and $(Y, \mathcal{C}, \nu, S)$ be Bernoulli transformations with finite entropy, and λ a joining of the two such that $h_\lambda(T \times S) = h_\mu(T) + h_\nu(S)$. Then $\lambda = \mu \times \nu$ is the independent joining.*

Proof. Let $\alpha \subset \mathcal{B}$ and $\beta \subset \mathcal{C}$ be finite partitions which are independent generators for T and S respectively. This is equivalent to the statement that for any $n \in \mathbb{N}$,

$$\frac{1}{n} H_\mu\left(\bigvee_{k=0}^{n-1} T^{-k}\alpha\right) = H_\mu(\alpha) = h_\mu(X, T),$$
$$\frac{1}{n} H_\nu\left(\bigvee_{k=0}^{n-1} T^{-k}\beta\right) = H_\nu(\beta) = h_\mu(Y, S).$$

Since $\alpha \vee \beta$ is a two-sided generator, it follows that

$$h_\lambda(X \times Y, T \times S) = \lim_{n\to\infty} \frac{1}{n} H_\lambda\left(\bigvee_{k=0}^{n} T^{-k}\alpha \vee \bigvee_{k=0}^{n} T^{-k}\beta\right).$$

By subadditivity, for any $m \geq 1$,

$$\frac{1}{m} H_\lambda\left(\bigvee_{k=0}^{m-1} T^{-k}\alpha \vee \bigvee_{k=0}^{m-1} T^{-k}\beta\right) \geq \lim_{n\to\infty} \frac{1}{n} H_\lambda\left(\bigvee_{k=0}^{n-1} T^{-k}\alpha \vee \bigvee_{k=0}^{n-1} T^{-k}\beta\right).$$

Thus, $h_\lambda(X \times Y, T \times S) = H_\nu(\beta) + H_\mu(\alpha)$ if and only if, for every $n \geq 1$,

$$\frac{1}{n} H_\lambda\left(\bigvee_{k=0}^{n} T^{-k}\alpha \vee \bigvee_{k=0}^{n-1} T^{-k}\beta\right) = H_\nu(\beta) + H_\mu(\alpha).$$

This equality can hold if and only if $\{T^{-k}\alpha, T^{-j}\beta\}_{k,j\in\mathbb{Z}}$ are jointly independent. It follows that λ is the independent joining. □

LEMMA 5.4. *Suppose $D_N \cong Y \times Z$ is a direct topological factorization, then precisely one of the direct factors Y and Z is intrinsically ergodic and the other has precisely two ergodic measures of maximal entropy.*

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Proof. Denote by ν_+ and η_+ the projection of μ_+ onto Y and Z respectively. Since the (D_N, σ, μ_+) is isomorphic to a Bernoulli shift, it follows that so are (X, σ, ν_+) and (Y, σ, η_+). We have

$$h_{\mu_+}(D_N, \sigma) \leq h_{\nu_+}(Y, \sigma) + h_{\eta_+}(Z, \sigma) = h_{\nu_+ \times \eta_+}(Y \times Z, \sigma).$$

Since μ_+ is a measure of maximal entropy, the inequality must be an equality, and so by Lemma 5.3 $\mu_+ = \nu_+ \times \eta_+$. Similarly, $\mu_- = \nu_- \times \eta_-$. Since the four combinations $\nu_\pm \times \eta_\pm$ give precisely two ergodic measures on D_N, it follows that either $\nu_+ \neq \nu_-$ and $\eta_+ = \eta_-$ or *vice versa*. □

Remark 5.5. In general, any direct factor of an intrinsically ergodic system is intrinsically ergodic. However, there are intrinsically ergodic homeomorphisms T and S such that $T \times S$ is not intrinsically ergodic (for instance, this is the case if the measures of maximal entropy for T and S have a common zero-entropy factor).

It is obvious that any two systems admitting a non-trivial common zero entropy factor are not intrinsically disjoint.

Question 5.6. Are any two K-systems intrinsically disjoint?

From now on we assume $D_N \cong Y \times Z$ is a direct topological factorization, realized by a shift-equivariant surjective homeomorphism $\Phi : Y \times Z \to D_N$. By the preceding lemma, we also assume, without loss of generality, that Z is intrinsically ergodic. Our goal is to show Z is the trivial one point system. Denote by ν_+ and ν_- the projections of μ_+ and μ_- onto Y and let η denote the unique measure of maximal entropy for Z. It follows from Lemma 5.4 that $\mu_+ \cong \nu_+ \times \eta$ and $\mu_- \cong \nu_- \times \eta$.

For a subshift (X, σ) and $n \in \mathbb{N}$, let $X^{(n)} := \{x \in X : \sigma^n(x) = x\}$ denote the n-periodic points of X.

LEMMA 5.7. *For any $n \in \mathbb{N}$ and $x \in D_N^{(n)}$ the limits*

$$h_\pm(x) := \lim_{k \to \infty} \frac{1}{k} \log \mu_\pm([x_1, \ldots, x_k]) \tag{17}$$

exist and are given by

$$h_\pm(x) = \log(N+1) \pm A(x) \log(N) \tag{18}$$

where

$$A(x) := \min\left(0, \frac{1}{n}(|\{1 \leq j \leq n : \pi_+(x)_j = 0\}| - |\{1 \leq j \leq k : \pi_+(x)_j \neq 0\}|)\right). \tag{19}$$

Proof. Check directly from the definition of μ_+ that

$$\mu_+([a_1, \ldots, a_k]) = \left(\frac{1}{N+1}\right)^k \cdot \frac{1}{N^A},$$

where

$$A := \min(0, \#\{1 \leq t \leq k : a_t \in \{\beta_1, \ldots, \beta_N\}\} - \#\{1 \leq t \leq k : a_t \in \{\alpha_1, \ldots, \alpha_N\}\})$$

is the number of 'unmatched β_j'. The formula for $h_+(x)$ follows directly by setting $a_i = x_i$, taking logarithm, dividing by k and taking the limit $k \to \infty$. The formula for $h_-(x)$ follows by symmetry. □

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LEMMA 5.8. *There exists a sequence of integers* $\{k_j\}_{j=1}^{\infty}$ *with* $\lim_{j\to\infty} k_j = \infty$ *so that, for any* $n \in \mathbb{N}$ *and* $y \in Y^{(n)}$ *and* $z \in Z^{(n)}$, *the limits*

$$h_{\pm}(y) := \lim_{j\to\infty} \frac{1}{k_j} \log \nu_{\pm}([y_1, \ldots, y_{k_j}])$$

and

$$h(z) := \lim_{k\to\infty} \frac{1}{k_j} \log \eta([y_1, \ldots, y_{k_j}])$$

exist and satisfy

$$h_{\pm}(y, z) = h_{\pm}(y) + h(z). \tag{20}$$

Furthermore, $h(z) = h(Z)$ *is independent of* $z \in Z^{(n)}$.

Proof. Fix an integer M big enough so that the 0 coordinate of both Φ and Φ^{-1} are determined by the coordinates $[-M, M]$. Let $x \in D_N^{(n)}$ and write $x = \Phi(y, z)$ with $y \in Y^{(n)}$ and $z \in Z^{(n)}$. Write

$$a_k^{\pm}(x) = -\frac{1}{k} \log \mu_{\pm}([x_1, \ldots, x_k]),$$
$$b_k^{\pm}(y) = -\frac{1}{k} \log \nu_{\pm}([y_1, \ldots, y_k]),$$
$$c_k(z) = -\frac{1}{k} \log \eta([z_1, \ldots, z_k]).$$

Since $\mu_{\pm} = (\nu_{\pm} \times \eta) \circ \Phi^{-1}$, for all $k > 2M$,

$$b_{k-M-n}^{\pm}(y) + c_{k-M-n}(z) \le a_k^{\pm}(x) \le b_{k+M+n}^{\pm}(y) + c_{k+M+n}(z). \tag{21}$$

We know that the sequences $\{a_k^{\pm}(x)\}_{k\ge1}$, $\{b_k^{\pm}(y)\}_{k\ge1}$ and $\{c_k(z)\}_{k\ge1}$ are all non-negative. By Lemma 5.7 the sequence $\{a_k^{\pm}(x)\}_{k\ge1}$ converges for any $x \in \bigcup_{n\in\mathbb{N}} D_N^{(n)}$. It follows that the sequences $\{b_k^{\pm}(y)\}_{k\ge1}$ and $\{c_k(z)\}_{k\ge1}$ are bounded. Thus, for one particular $z \in \bigcup_{n\in\mathbb{N}} Z^{(n)}$ there is a subsequence $\{k_j\}_{j=1}^{\infty}$ along which $\{c_k(z)\}_{k\ge1}$ converges. By (21) $\{b_{k_j}^{\pm}(y)\}_{j\ge1}$ converges along this same subsequence for *any* $y \in \bigcup_{n\in\mathbb{N}} Y^{(n)}$. Again, by (21), it follows that $\{c_{k_j}(z)\}_{j\ge1}$ converges for *any* $z \in \bigcup_{n\in\mathbb{N}} Z^{(n)}$. The formula (20) follows directly from (21) by taking a limit along the sequence $\{k_j\}$.

It remains to show that $h(z) = h(\hat{z})$ for all $z, \hat{z} \in \bigcup_{n\in\mathbb{N}} Z^{(n)}$. Suppose otherwise, $h(z) < h(\hat{z})$. It follows that for all $y \in \bigcup_{n\in\mathbb{N}} Y^{(n)}$

$$h_+(\Phi(y, z)) < h_+(\Phi(y, \hat{z})),$$

and

$$h_-(\Phi(y, z)) < h_-(\Phi(y, \hat{z})).$$

In particular there exist $x, \hat{x} \in \bigcup_{n\in\mathbb{N}} D_N^{(n)}$ with

$$\min\{h_-(x), h_+(x)\} < \min\{h_-(\hat{x}), h_+(\hat{z})\}.$$

By Lemma 5.7 $\min\{h_-(x), h_+(x)\} = \log(N+1)$ for all $x \in \bigcup_{n\in\mathbb{N}} D_N^{(n)}$, which is a contradiction. □

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For $c \in [-1, 1]$ and $n \in \mathbb{N}$ let

$$D_N^{(n,c)} := \{x \in D_N : \sigma_N x = x,\ h_+(x) - h_-(x) = c \log(N)\}.$$

Similarly, set

$$Y^{(n,c)} := \{y \in Y : \sigma_N y = y,\ h_+(y) - h_-(y) = c \log(N)\}.$$

By Lemma 5.7, we have

$$D_N^{(n,c)} := \{x \in D_N : \sigma_N x = x,\ A(x) = c\},$$

where $A(x)$ is given by (19). In other words, $D_N^{(n,c)}$ is the set of legal bi-infinite sequences in $D_N^{(n)}$ in which the number of left brackets minus the number of right brackets is cn. An elementary calculation shows that, for $-n < j < n$ such that $n - j = 0 \bmod 2$,

$$|D_n^{(n,j/n)}| = \binom{n}{\frac{n}{2} + \frac{j}{2}} N^{n/2+|j/2|}. \tag{22}$$

The term $\binom{n}{n/2+j/2}$ in (22) corresponds to selecting the locations of the left brackets within the cycle. The term $N^{n/2+|j/2|}$ corresponds to selecting the 'types' of left brackets independently, or selecting the types of right brackets, according to the sign of j.

In particular,

$$|D_N^{(n,1)}| = |D^{(n,-1)}| = N^n \quad \text{and} \quad |D_N^{(2n,0)}| = \binom{2n}{n} N^n.$$

Our next step in the proof of Theorem 5.1, is to show that for prime N, $|Z^{(N^n)}| = 1$ for all n. A version of the following argument appears in [**17**, Theorem 7] which is the case $d = 1$ of Theorem 3.8.

By Lemma 5.8,

$$|D_N^{(n,c)}| = |Y^{(n,c)}| \cdot |Z^{(n)}|. \tag{23}$$

We now assume N is prime. For $k = 0, 1, 2, \ldots$ we have

$$|Y^{(N^k,1)}| \cdot |Z^{(N^k)}| = |D_N^{(N^k,1)}| = N^{N^k}. \tag{24}$$

Because N is prime, it follows that both $|Y^{(N^k,1)}|$ and $|Z^{(N^k)}|$ are non-negative integer powers of N. In particular, for $k = 0$, we have $N = |Y^{(n,1)}| \cdot |Z^{(1)}|$. Thus, either $|Z^{(1)}| = N$ or $|Z^{(1)}| = 1$.

Suppose first that $|Z^{(1)}| = N$, and so $|Y^{(1,1)}| = 1$. For $n \in \mathbb{N}$, denote by $Y_*^{(n)}$ the set of points whose *least period* is n. It follows that $Y^{(n,1)} = \biguplus_{m|n} Y_*^{(m,1)}$. Also note that m divides $|Y_*^{(m,1)}|$, since $Y_*^{(m,1)}$ is a disjoint union of orbits each of which has cardinality m. Thus, since N is prime the only divisors of N^k are N^j for $j = 0, \ldots, k$ so

$$|Y^{(N^k,1)}| = |Y^{(1,1)}| + \sum_{j=0}^{k} |Y_*^{(N^j,1)}| \equiv |Y^{(1,1)}| \equiv 1 \mod N.$$

On the other hand, $|Y^{(N^k,1)}|$ divides $|D^{(N^k,1)}|$ and $|D^{(N^k,1)}| = N^{N^k}$ by (24). Thus, since we assume N is prime we have that $|Y^{(N^k)}| = N^l$ for some $l = 0, \ldots, N^k$. It follows that

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$N^l \equiv 1 \bmod N$ thus $l = 0$ so $|Y^{(N^k,1)}| = 1$ for all k. It follows that $|Z^{(N^k)}| = |D_N^{(N^k,1)}|$ for all k. On the other hand, we have

$$|D^{(N^k,1)}| \cdot |Y^{(N^k,0)}| = |Z^{(N^k)}| \cdot |Y^{(N^k,0)}| = |D_N^{(N^K,0)}|$$

so we obtain $|D_N^{(N^k,1)}| \leq |D_N^{(N^k,0)}|$, which by (22) is false for sufficiently large k.

We conclude that $|Z^{(1)}| = 1$. Repeating the above argument with Y and Z interchanged, it follows that $|Z^{(N^k)}| = 1$ for all k, and in particular,

$$\liminf_{n\to\infty} \frac{1}{n} |Z^{(n)}| = 0. \tag{25}$$

We will now show this implies $h(Z) = 0$.

LEMMA 5.9. *Suppose the entropy of a subshift X is determined by the growth rate of the periodic points, in the sense that*

$$\lim_{n\to\infty} \frac{1}{n} |X^{(n)}| = h(X).$$

Then for any direct factor Z of X the entropy is determined by the growth rate of its periodic points,

$$\lim_{n\to\infty} \frac{1}{n} |Z^{(n)}| = h(Z),$$

and in particular the limit on the left-hand side exists.

Proof. Note that for any subshift Y and any $n \in \mathbb{N}$,

$$|\{y_{\{1,\dots,n\}} : y \in Y^{(n)}\}| \leq |\{y_{\{1,\dots,n\}} : y \in Y\}|.$$

It is well-known (and easily verified from the definitions) that for any subshift the exponential growth rate of the periodic points is bounded above by the topological entropy

$$\limsup_{n\to\infty} \frac{1}{n} \log |Y^{(n)}| \leq h(Y). \tag{26}$$

Suppose $X \cong Y \times Z$. It follows that $h(X) = h(Y) + h(Z)$. If X satisfies the assumption of the lemma then

$$h(X) = \lim_{n\to\infty} \frac{1}{n} (\log |Y^{(n)}| + \log |Z^{(n)}|).$$

It follows from (26) that

$$\liminf_{n\to\infty} \frac{1}{n} \log |Z^{(n)}| \geq h(Z).$$

Again by (26) applied to Z, we conclude that

$$\lim_{n\to\infty} \frac{1}{n} |Z^{(n)}| = h(Z). \qquad \square$$

LEMMA 5.10. *The topological entropy of any direct factor Z of the N-Dyck shift is determined by the growth rate of its periodic points,*

$$\lim_{n\to\infty} \frac{1}{n} |Z^{(n)}| = h(Z).$$

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Proof. The fact that the topological entropy of D_N is equal to $\lim_{n\to\infty}(1/n)|D_N^{(n)}|$ is a particular case of [**15**, Proposition 3.1], which gives the corresponding result for a bigger family of subshifts. This also follows by a direct computation of the limit $\lim_{n\to\infty}(1/n)|D_N^{(n)}| = \log(N+1) = h(D_N)$, using (22). The last equality holds because $h_{\mu_+}(D_N) = h_{\mu_-}(D_N) = \log(N+1)$ by isomorphism to the Bernoulli $(N+1)$-shift as in [**13**], or by a direct computation. The proof now follows by Lemma 5.9. □

We conclude that $h(Z) = 0$. This implies Z is a trivial 1-point subshift as follows.

Recall that a topological dynamical system has *completely positive entropy* if its only zero-entropy factor is the trivial factor [**1**].

LEMMA 5.11. *The N-Dyck shift D_N has completely positive entropy.*

Proof. The N-Dyck shift D_N is a coded system in the sense of [**2**]. See the remark in [**18**, §2.1]. By [**2**], any coded system has completely positive entropy. The last result follows by observing that a non-trivial factor of a coded system is itself a coded system and thus has positive entropy. □

Remark 5.12. It is possible to complete the proof Theorem 5.1 without using Lemma 5.11 and the notion of 'completely positive entropy'. An alternative argument is to prove the set $\bigcup_{k=1}^{\infty} D_N^{(N^k)}$ of N^k-periodic points is dense in D_N. This property passes to direct factors.

Remark 5.13. 'Entropy like' quantities such as h_+ and h_- defined above can lead to invariants associated to periodic points of certain types of subshifts. Another kind of invariant associated to periodic points are 'multipliers', as in [**10**]. These apply to periodic points of certain types of subshifts, including Dyck shifts. It seems plausible that, using the technology of multipliers and the semi-group invariant introduced in [**14**], it is possible to obtain more general results about direct factorizations of Markov–Dyck shifts and, more generally, about subshifts with Krieger's property (A), introduced in [**14**].

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