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Two-dimensional anisotropic elastic waves emanating from a point source

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Abstract. A two (spatial) dimensional initially strained elastic body is excited by a point impulse. Expressions are found for the various displacement components in a form which is readily evaluated by residues. The solid itself is characterized by three parameters which depend on the material properties and the initial deformation. For the case when two of these parameters are equated, explicit expressions for the displacements are given along the Cartesian axes passing through the origin of the point impulse. Wave front singularities and lacunas are identified and discussed. Some typical numerical results are given.

1. *Introduction.* In the classical infinitesimal theory of elasticity, an isotropic body in a state of zero stress and strain is perturbed by a system of applied forces (or displacements). The assumption that all relevant quantities are small, leads to a linearized theory for the equations of motion describing the dynamic state of stress in the body. A generalization of this theory (due to Cauchy) is achieved by perturbing an elastic body about a state of finite stress and strain. The perturbed equations of motion may still be linearized, but now about a finite deformation. Such a theory will lead to many new and interesting results of both physical and mathematical nature.

In this paper a two (spatial) dimensional elastic body is subjected to a pure homogeneous deformation, upon which is superposed a perturbing body force in the form of a point impulse. The formulation of Green, Rivlin and Shield(1) for the linearized motion of the perturbed body, is used. In earlier papers (2, 3) the author has investigated the shape of the wave fronts emanating from a point source and classified the wave fronts according to certain parameters which appear in the problem. In the present study attention is directed toward explicit expressions for the displacements behind the wave fronts.

In an important paper, concerned with magnetohydrodynamic waves, Weitzner(4) has given a mathematical method for inverting multiple integral transforms connected with (two-dimensional) anisotropic wave problems. Later Burridge(5), while establishing sufficient conditions for the existence of Lacunas in two-dimensional anisotropic wave-motion problems, gave a refinement of Weitzner's method. It is essentially the method of Burridge which is used herein for the transform inversions.

The interesting mathematical question concerning the existence of Lacunas (to

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be defined below) in anisotropic wave motion problems, was evidently first raised by Petrowsky (6) in a now famous paper. This question has received considerable attention since then, e.g. in the recent paper of Bazer and Yen (7).

In section 2 the equations of motion for the various displacement components are set down and a full mathematical statement of the problem is given. These differential equations are solved by means of integral transforms in section 3 wherein formulae for the displacement components are given which involve only simple residue calculations. Certain reciprocal relations among the displacement components are established in section 4. In sections 5, 6 and 7 the singularities of a complex function, which is important in the displacement residue calculations, is discussed. Explicit algebraic expressions for the displacement components along the Cartesian axes centred at the point source are given in section 8. These results apply only to the case in which two of three parameters (α and β) of the problem are set equal to one another. In section 9 the wave-front singularities (along the Cartesian axes) are investigated and some numerical results are presented in section 10.

2. *Equations of motion.* An infinite, isotropic, unstrained, elastic body B_0 is deformed by means of a pure homogeneous deformation into a finitely strained body B. The coordinates of a typical point P_0 of the body B_0 are x_1, x_2 and x_3 measured with respect to a fixed rectangular Cartesian set of axes $Oxyz$. After the homogeneous deformation (which is assumed to be time independent) has been accomplished by the application of a suitable tri-axial force system at infinity, the point P_0 is displaced to the point P with new coordinates x, y and z, again measured with respect to the same (fixed) coordinate system $Oxyz$. The two sets of coordinates are related by

$$x = \lambda_1 x_1, \quad y = \lambda_2 x_2 \quad \text{and} \quad z = \lambda_3 x_3, \tag{2·1}$$

where the positive constants λ_i (stretch parameters) are a measure of the homogeneous deformation. The coordinates of the strained body (x, y, z) will be taken as the active coordinates of the problem since some simplification results by their use.

A time-dependent body force $\epsilon\rho\mathbf{F}'(x, y, z, t)$ is suddenly applied to the finitely strained body B, thus producing a dynamic state of stress in the body B'. ρ is the mass density of the body B and is related to the density ρ_0 of the body B_0 by

$$\rho_0 = \lambda_1\lambda_2\lambda_3\rho. \tag{2·2}$$

is a small number measuring the perturbing body force. In the dynamically disturbed body B', the point P has been displaced to P' with coordinates

$$x + \epsilon u(x, y, z, t), \quad y + \epsilon v(x, y, z, t) \quad \text{and} \quad z + \epsilon w(x, y, z, t).$$

The equations of motion, stress–strain relations, etc., can now be linearized about the known state of finite deformation so that only first-order terms in ϵ are retained. The formulation used here is that of Green *et al.* (1). For the purposes of this paper, it will be assumed that the dynamic strain field is plane, involving only the space coordinates x and y. This condition is met if the perturbing body force is independent of z, has no component in the z-direction and $w \equiv 0$. If the applied body force is

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localized at the origin in the form of a (two-dimensional) point impulse, the displacement functions u and v (which now depend only on x, y and t) satisfy the equations

$$\left.\begin{aligned} L_{11}[u]+L_{12}[v] &= -\rho a_1 \delta(x)\,\delta(y)\,\delta(t),\\ \text{and}\qquad L_{21}[u]+L_{22}[v] &= -\rho a_2 \delta(x)\,\delta(y)\,\delta(t),\end{aligned}\right\} \tag{2·3}$$

where the differential operators L_{ij} are given by

$$\left.\begin{aligned} L_{11} &= (C_{11}+2\tau^{11})\frac{\partial^2}{\partial x^2}+(C_{66}+\tau^{22})\frac{\partial^2}{\partial y^2}-\rho\frac{\partial^2}{\partial t^2},\\ L_{12} &= L_{21} = (C_{12}+\tau^{11}+C_{66})\frac{\partial^2}{\partial x\,\partial y},\\ \text{and}\qquad L_{22} &= (C_{66}+\tau^{11})\frac{\partial^2}{\partial x^2}+(C_{22}+2\tau^{22})\frac{\partial^2}{\partial y^2}-\rho\frac{\partial^2}{\partial t^2}.\end{aligned}\right\} \tag{2·4}$$

Explicit expressions for the constants C_{ij} in equations (2·4) may be found in Green and Zerna (8). The constants depend on the (constant) state of stress τ^{ij} in the finitely strained body, the stretch parameters λ_i and the strain energy density function for the elastic body. The constants a_i simply regulate the impulsive strength of the perturbing body force in each of the two coordinate directions.

The material strain energy density function and stretch parameters will now be restricted so that equations (2·3) are *strictly hyperbolic*. The consequences of this restriction, insofar as the constraints placed on the differential coefficients in (2·4) are concerned, was fully explored in (2). In particular it was there found convenient to replace the independent variables x, y and t by new (dimensional) variables ξ, η and τ where

$$\sqrt{(C_{11}+2\tau^{11})}\,\xi = x, \quad \sqrt{(C_{22}+2\tau^{22})}\,\eta = y \quad \text{and} \quad \sqrt{(\rho)}\,\tau = t. \tag{2·5}$$

The equations of motion then become

$$\left.\begin{aligned} Q_{11}[u]+Q_{12}[v] &= -A_1\delta(\xi)\,\delta(\eta)\,\delta(\tau),\\ \text{and}\qquad Q_{21}[u]+Q_{22}[v] &= -A_2\delta(\xi)\,\delta(\eta)\,\delta(\tau),\end{aligned}\right\} \tag{2·6}$$

where

$$\left.\begin{aligned} Q_{11}\left(\frac{\partial}{\partial\xi},\frac{\partial}{\partial\eta},\frac{\partial}{\partial\tau}\right) &= \frac{\partial}{\partial\xi^2}+\beta\frac{\partial}{\partial\eta^2}-\frac{\partial^2}{\partial\tau^2},\\ Q_{22}\left(\frac{\partial}{\partial\xi},\frac{\partial}{\partial\eta},\frac{\partial}{\partial\tau}\right) &= \alpha\frac{\partial^2}{\partial\xi^2}+\frac{\partial^2}{\partial\eta^2}-\frac{\partial^2}{\partial\tau^2},\\ \text{and}\qquad Q_{12}\left(\frac{\partial}{\partial\xi},\frac{\partial}{\partial\eta}\right) &= Q_{21}\left(\frac{\partial}{\partial\xi},\frac{\partial}{\partial\eta}\right) = \kappa\frac{\partial^2}{\partial\xi\,\partial\eta}.\end{aligned}\right\} \tag{2·7}$$

Here the positive constants α and β are given by

$$\alpha = \frac{C_{66}+\tau^{11}}{C_{11}+2\tau^{11}} \quad \text{and} \quad \beta = \frac{C_{66}+\tau^{22}}{C_{22}+2\tau^{22}},$$

while

$$\kappa^2 = 1+\alpha\beta-\gamma \quad \text{with} \quad \gamma = 1+\alpha\beta-\frac{(C_{12}+\tau^{11}+C_{66})^2}{(C_{11}+2\tau^{11})\,(C_{22}+2\tau^{22})}.$$

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The constants A_1 and A_2 appearing on the right side of equations (2·6) are found to be

$$A_1 = a_1\left[\frac{\rho}{(C_{11}+2\tau^{11})\,(C_{22}+2\tau^{22})}\right]^{\frac{1}{2}}$$

and

$$A_2 = a_2\left[\frac{\rho}{(C_{11}+2\tau^{11})\,(C_{22}+2\tau^{22})}\right]^{\frac{1}{2}}.$$

The basic problem of this paper is to (formally) determine the displacement functions $u(\xi,\eta,\tau)$ and $v(\xi,\eta,\tau)$ which satisfy (in the sense of distributions) equations (2·6) in the domain $-\infty < \xi, \eta < +\infty$, $\tau > 0$, subject to the initial conditions

$$u(\xi,\eta,0) = u_\tau(\xi,\eta,0) = v(\xi,\eta,0) = v_\tau(\xi,\eta,0) = 0.$$

Due to the distributional nature of the loading terms in (2·6), the displacement functions will necessarily be *weak solutions* of the partial differential equations (2·6).

Since the differential operators Q_{ij} have constant coefficients, equations (2·6) may be easily uncoupled to obtain a single (fourth-order) equation for u or v of the form

$$\left.\begin{aligned} Q[u] &= -(A_1Q_{22}-A_2Q_{12})\,[\delta(\xi)\,\delta(\eta)\,\delta(\tau)]\\ \text{and}\quad Q[v] &= -(A_2Q_{11}-A_1Q_{21})\,[\delta(\xi)\,\delta(\eta)\,\delta(\tau)]\end{aligned}\right\} \tag{2·8}$$

where

$$Q\left(\frac{\partial}{\partial\xi},\frac{\partial}{\partial\eta},\frac{\partial}{\partial\tau}\right) = \alpha\frac{\partial^4}{\partial\xi^4}+\gamma\frac{\partial^4}{\partial\xi^2\,\partial\eta^2}+\beta\frac{\partial^4}{\partial\eta^4}-(\alpha+1)\frac{\partial^4}{\partial\xi^2\,\partial\tau^2}-(\beta+1)\frac{\partial^4}{\partial\eta^2\,\partial\tau^2}+\frac{\partial^4}{\partial\tau^4}. \tag{2·9}$$

One final notational change will be helpful. Set

$$u(\xi,\eta,\tau) = A_1u_1(\xi,\eta,\tau)+A_2u_2(\xi,\eta,\tau) \tag{2·10}$$

and

$$v(\xi,\eta,\tau) = A_1v_1(\xi,\eta,\tau)+A_2v_2(\xi,\eta,\tau). \tag{2·11}$$

Then, from equations (2·8) and the fact that the constants A_1 and A_2 are independent,

$$Q[u_1] = -Q_{22}[\delta(\xi)\,\delta(\eta)\,\delta(\tau)], \tag{2·12}$$

$$Q[u_2] = Q_{12}[\delta(\xi)\,\delta(\eta)\,\delta(\tau)], \tag{2·13}$$

$$Q[v_1] = Q_{21}[\delta(\xi)\,\delta(\eta)\,\delta(\tau)], \tag{2·14}$$

and

$$Q[v_2] = -Q_{11}[\delta(\xi)\,\delta(\eta)\,\delta(\tau)]. \tag{2·15}$$

The displacements $u_1,\ldots,v_2$ have a simple physical interpretation. A_1u_1 is the ξ-displacement due to an impulsive source at the origin acting in the ξ-direction, A_2u_2 is the ξ-displacement due to an impulsive source at the origin acting in the η-direction, A_1v_1 is the η-displacement due to an implusive source at the origin acting in the ξ-direction, while A_2v_2 is the η-displacement due to an impulsive source at the origin acting in the η-direction. An alternative matrix formulation of the above problem, in which the displacement components $u_1,\ldots,v_2$ are incorporated into a 2×2 matrix, etc., is preferred by some authors.

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According to John(9) and Gel'fand and Shilov(10), the fourth-order differential operator Q (which is homogeneous of degree four) will be classified as *strictly hyperbolic* if for any (real) values of ω_1 and ω_2 such that $\omega_1^2+\omega_2^2 = +1$, the fourth-degree equation in v

$$Q(\omega_1, \omega_2, v) = 0 \tag{2·16}$$

has four *real* and *distinct* roots. In [2] it was shown that this condition implies that the parameters α, β and γ must fall within the following four cases:

$$\text{case 1} \quad 0 < \alpha < 1, \quad 0 < \beta < 1, \quad -2\sqrt{(\alpha\beta)} < \gamma < 1+\alpha\beta, \tag{2·17}$$

$$\text{case 2} \quad 0 < \alpha < 1, \quad \beta > 1, \quad -2\sqrt{(\alpha\beta)} < \gamma \leqslant 1+\alpha\beta, \tag{2·18}$$

$$\text{case 3} \quad \alpha > 1, \quad 0 < \beta < 1, \quad -2\sqrt{(\alpha\beta)} < \gamma \leqslant 1+\alpha\beta, \tag{2·19}$$

$$\text{case 4} \quad \alpha > 1, \quad \beta > 1, \quad -2\sqrt{(\alpha\beta)} < \gamma < 1+\alpha\beta. \tag{2·20}$$

As mentioned in (2), an important physical consequence of strict hyperbolicity (which necessarily implies that the static version of equations (2·6) are strongly elliptic as defined in Truesdell and Noll(11)) is that the finitely strained body B will be infinitesimally stable in the sense of Hadamard.

3. *Solutions of the equations of motion by integral transforms.* A solution to equation (2·12) for $u_1(\xi, \eta, \tau)$, subject to the zero initial conditions mentioned in section 2, will now be constructed with the help of integral transforms. It is appropriate here to use a double Fourier exponential transform on the (space) variables ξ and η and a Laplace transform on the (time) variable τ. Thus a solution to equation (2·12) is sought in the form

$$u_1(\xi, \eta, \tau) = \frac{1}{(2\pi)^3 i}\int_c\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} \bar{u}_1(\xi_1, \zeta_2, s)\, e^{i(\zeta_1\xi+\zeta_2\eta)+s\tau}\, d\zeta_1\, d\zeta_2\, ds, \tag{3·1}$$

where

$$\bar{u}_1(\zeta_1, \zeta_2, s) = \int_0^{\infty}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} u_1(\xi, \eta, \tau)\, e^{-i(\zeta_1\xi+\zeta_2\eta)-s\tau}\, d\xi\, d\eta\, d\tau. \tag{3·2}$$

Applying the triple transform, indicated in equation (3·2) to equation (2·12), integrating by parts, etc., gives the following algebraic expression for the transformed u_1 displacement

$$\bar{u}_1(\zeta_1, \zeta_2, s) = -\frac{Q_{22}(i\zeta_1, i\zeta_2, s)}{Q(i\zeta_1, i\zeta_2, s)}. \tag{3·3}$$

The evaluation of the triple integral (3·1), with $\bar{u}_1$ given by equation (3·3), will now be carried out by (essentially) following the work of Burridge(5). Some of the inversion details, given below, differ from those used by Burridge but the end result is the same. In particular the plane wave expansion of $\delta(\xi)\,\delta(\tau)$, due to Gel'fand and Shilov(10), is not used.

Introduce polar coordinates (r, θ) by

$$\zeta_1 = r\cos\theta, \quad \zeta_2 = r\sin\theta \quad \text{for} \quad 0 \leqslant r < \infty, \quad 0 \leqslant \theta < 2\pi.$$

Equation (3·1), with the help of (3·3), may then be written as

$$u_1 = \frac{1}{2\pi i}\int_c e^{s\tau}\, ds\, \frac{1}{(2\pi)^2}\int_0^{\infty} r\, dr \int_0^{2\pi} d\theta\, \frac{-Q_{22}(ir\cos\theta, ir\sin\theta, s)}{Q(ir\cos\theta, ir\sin\theta, s)}\, e^{ir(\xi\cos\theta+\eta\sin\theta)}. \tag{3·4}$$

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Setting $s = r\sigma$ and using the facts that Q_{22} is homogeneous of degree 2, while Q is homogeneous of degree 4, gives

$$u_1 = \frac{1}{(2\pi)^2}\int_0^{2\pi} d\theta \int_0^{\infty} e^{ir(\xi\cos\theta+\eta\sin\theta)}\,dr\,\frac{1}{2\pi i}\int_c \frac{-Q_{22}(i\cos\theta,\, i\sin\theta,\, \sigma)}{Q(i\cos\theta,\, i\sin\theta,\, \sigma)}\,e^{\sigma r\tau}\,d\sigma. \quad (3{\cdot}5)$$

Since $Q(i\cos\theta,\, i\sin\theta,\, \sigma) = Q(\cos\theta,\, \sin\theta,\, -i\sigma)$, it follows from equation (2·16) and the conditions of strict hyperbolicity, that there will be four distinct pure imaginary σ roots of $Q(i\cos\theta,\, i\sin\theta,\, \sigma)$. Call these roots (which are simple pole singularities of the Laplace integrand) $\sigma_j = iv_j(\theta)$ where $j = 1, \ldots, 4$. The Laplace inversion is now immediate and gives, for $\tau > 0$,

$$u_1 = \frac{1}{(2\pi)^2}\int_0^{2\pi} d\theta \sum_{j=1}^{4} \frac{-Q_{22}(i\cos\theta,\, i\sin\theta,\, iv_j)}{Q_\sigma(i\cos\theta,\, i\sin\theta,\, iv_j)} \int_0^{\infty} e^{ir(\xi\cos\theta+\eta\sin\theta+v_j\tau)}\,dr. \quad (3{\cdot}6)$$

Unfortunately the second integral in equation (3·6) does not exist in the classical sense. This integral is of the form

$$\mathrm{I}(\lambda) = \int_0^{\infty} e^{ir\lambda}\,dr \quad \text{with} \quad \Im\lambda = 0. \quad (3{\cdot}7)$$

Herein such integrals will be interpreted as

$$I(\lambda) = \lim_{\epsilon\to 0}\int_0^{\infty} e^{ir\lambda-\epsilon r}\,dr \quad (\epsilon > 0). \quad (3{\cdot}8)$$

Then
$$I(\lambda) = \lim_{\epsilon\to 0}\frac{1}{\epsilon - i\lambda} = \lim_{\epsilon\to 0}\frac{\epsilon + i\lambda}{\lambda^2+\epsilon^2} = \frac{i}{\lambda} + \lim_{\epsilon\to 0}\frac{\epsilon}{\lambda^2+\epsilon^2} = \frac{i}{\lambda} + \pi\delta(\lambda), \quad (3{\cdot}9)$$

having used a well known delta-convergent sequence definition of the delta function (10). This interpretation of $I(\lambda)$ gives a result, namely the right-hand side of equation (3·9) which is in agreement with the distributional interpretation of such integrals as given in Gel'fand and Shilov (10).

In view of the above remarks, the second integral in equation (3·6) is now given by

$$u_1 = \lim_{\epsilon\to 0}\frac{1}{(2\pi)^2}\int_0^{2\pi} d\theta \sum_{j=1}^{4} \frac{-Q_{22}(i\cos\theta,\, i\sin\theta,\, iv_j)}{Q_\sigma(i\cos\theta,\, i\sin\theta,\, iv_j)}\,\frac{1}{\epsilon - i(\xi\cos\theta+\eta\sin\theta+v_j\tau)}. \quad (3{\cdot}10)$$

But since $iv_j = \sigma_j$, the integrand of (3·10) becomes

$$\sum_{j=1}^{4} \frac{-Q_{22}(i\cos\theta,\, i\sin\theta,\, \sigma_j)}{Q_\sigma(i\cos\theta,\, i\sin\theta,\, \sigma_j)}\,\frac{1}{\tau[k(\theta)-\sigma_j]} = \frac{-Q_{22}(i\cos\theta,\, i\sin\theta,\, k)}{\tau Q(i\cos\theta,\, i\sin\theta,\, k)}, \quad (3{\cdot}11)$$

where
$$k(\theta) = \frac{\xi\cos\theta + \eta\sin\theta \pm i\epsilon}{i\tau}.$$

Thus the integrand of (3·10) is simply the partial fraction expansion of the right-hand side of equation (3·11). This is one of the key steps in the inversion process and cleverly avoids the individual evaluation of four (complicated) integrals in (3·7) in favour of the much simpler integral

$$u_1 = \lim_{\epsilon\to 0}\frac{1}{4\pi^2\tau}\int_0^{2\pi} \frac{-Q_{22}(i\cos\theta,\, i\sin\theta,\, k)}{Q(i\cos\theta,\, i\sin\theta,\, k)}\,d\theta. \quad (3{\cdot}12)$$

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Since the integrand of equation (3·12) is periodic in θ (with period 2π) the integration limits $(0, 2\pi)$ may be replaced by $(\pi, 3\pi)$. Then making the substitution $\theta = \phi + \pi$, it is easily shown that the new integral so obtained is the same as (3·12) but with ϵ replaced by $-\epsilon$ in the integrand. Thus, using the periodicity of the integrand,

$$u_1 = \lim_{\epsilon \to 0} \frac{1}{4\pi^2 \tau} \int_{-\pi}^{+\pi} \frac{-Q_{22}(i\cos\theta,\, i\sin\theta,\, m)}{Q(i\cos\theta,\, i\sin\theta,\, m)}\, d\theta, \tag{3·13}$$

where

$$m(\theta) = \frac{\xi\cos\theta + \eta\sin\theta \pm i\epsilon}{i\tau}.$$

The integrand of (3·13) can be simplified further still by using the homogeneity property of Q_{22} and Q and noting that both of these functions are even in all their members. Hence

$$u_1 = \lim_{\epsilon \to 0} \frac{1}{4\pi^2 \tau} \int_{-\pi}^{+\pi} \frac{Q_{22}(\cos\theta,\, \sin\theta,\, g_\epsilon)}{Q(\cos\theta,\, \sin\theta,\, g_\epsilon)}\, d\theta \tag{3·14}$$

where $g_\epsilon(\theta) = \bar{\xi}\cos\theta + \bar{\eta}\sin \pm i\epsilon$, having replaced ϵ/τ by ϵ. Here the (dimensionless) variables $\bar{\xi}$ and $\bar{\eta}$ are given by

$$\bar{\xi} = \xi/\tau \quad \text{and} \quad \bar{\eta} = \eta/\tau \tag{3·15}$$

and it is worth noting that the u_1 displacement has the functional form

$$u_1(\xi, \eta, \tau) = \frac{1}{\tau} f(\bar{\xi}, \bar{\eta}).$$

The non-vanishing of ϵ in the integrand of (3·14) insures that the integrand remains finite on the θ integration path. Quoting Burridge(5), another way of saying this is that, '...the integral [for u_1] regarded as a contour integral in the θ-plane must pass around each pole on the contour but not through it in order to avoid divergent integrals. The same effect is accomplished by integrating with respect to θ along a line L_δ parallel to the real θ-axis from $-\pi - i\delta$ to $\pi - i\delta$, and then allowing δ to tend to zero'. Also, since the integral (3·14) is independent of the sign of ϵ, the integral must be real (as is obvious physically). Hence there is no loss in generality in working with the real part ($\Re$) of this integral. Thus, upon combining these results, the expression for u_1 becomes

$$u_1 = \lim_{\delta \to 0} \frac{1}{4\pi^2 \tau} \Re \int_{-\pi - i\delta}^{\pi - i\delta} \frac{Q_{22}(\cos\theta,\, \sin\theta,\, \bar{\xi}\cos\theta + \bar{\eta}\sin\eta)}{Q(\cos\theta,\, \sin\theta,\, \bar{\xi}\cos\theta + \bar{\eta}\sin\theta}\, d\theta \tag{3·16}$$

where (say) $\delta > 0$. The integral can be split into the form

$$\int_{-\pi - i\delta}^{\pi - i\delta} = \int_{-\pi - i\delta}^{-i\delta} + \int_{-i\delta}^{\pi - i\delta}.$$

In the first integral set $\theta = \phi - \pi$. The first integral then becomes identical with the second, so that

$$u_1 = \lim_{\delta \to 0} \frac{1}{2\pi^2 \tau} \Re \int_{-i\delta}^{\pi - i\delta} \frac{Q_{22}(\cos\theta,\, \sin\theta,\, \bar{\xi}\cos\theta + \bar{\eta}\sin\theta)}{Q(\cos\theta,\, \sin\theta,\, \bar{\xi}\cos\theta + \bar{\eta}\sin\theta)}\, d\theta. \tag{3·17}$$

Consider next the mapping $p = \cot\theta$ (3·18)

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from the complex θ-plane to the complex p-plane. Under this mapping the line 1_δ in the θ-plane (directed from $-i\delta$ to $\pi - i\delta$) maps into the (large) circle $\bar{C}_\delta$:

$$p_1^2 + [p_2 - \tfrac{1}{2}(\coth\delta + \tanh\delta)]^2 = (\coth\delta - \tanh\delta)^2/4$$

in the $p(= p_1 + ip_2)$ plane, which is traversed in the clockwise direction. Hence, reversing the direction of $\bar{C}_\delta$ and calling the resulting path C_δ (counterclockwise loop) gives

$$u_1 = -\lim_{\delta\to 0}\frac{1}{2\pi^2\tau}\Re\int_{C_\delta}\frac{\sin^2\theta\, Q_{22}(p,1,\bar{\xi}p+\bar{\eta})}{\sin^4\theta\, Q(p,1,\bar{\xi}p+\bar{\eta})}\,d\theta.$$

But $dp = d\cot\theta = -d\theta/\sin^2\theta$, so

$$u_1 = \lim_{\delta\to 0}\frac{1}{2\pi^2\tau}\Re\int_{C_\delta}\frac{Q_{22}(p,1,\bar{\xi}p+\bar{\eta})}{Q(p,1,\bar{\xi}p+\bar{\eta})}\,dp.$$

Now as $\delta \to 0$ the circle C_δ encloses, in a positive sense, the upper half p-plane (but definitely excluding the real axis), so that

$$u_1(\xi,\eta,\tau) = -\frac{1}{\pi\tau}\Im\sum_p\left\{\text{residues of }\frac{Q_{22}(p,1,\bar{\xi}p+\bar{\eta})}{Q(p,1,\bar{\xi}p+\bar{\eta})}\text{ in the upper half }p\text{-plane}\right\}. \quad (3{\cdot}19)$$

In a similar way the expressions for the 3 other displacement functions u_2, v_1 and v_2 may be written down

$$u_2(\xi,\eta,\tau) = \frac{1}{\pi\tau}\Im\sum_p\left\{\text{residues of }\frac{Q_{12}(p,1)}{Q(p,1,\bar{\xi}p+\bar{\eta})}\text{ in the upper half }p\text{-plane}\right\}, \quad (3{\cdot}20)$$

$$v_1(\xi,\eta,\tau) = \frac{1}{\pi\tau}\Im\sum_p\left\{\text{residues of }\frac{Q_{21}(p,1)}{Q(p,1,\bar{\xi}p+\bar{\eta})}\text{ in the upper half }p\text{-plane}\right\}, \quad (3{\cdot}21)$$

and

$$v_2(\xi,\eta,\tau) = -\frac{1}{\pi\tau}\Im\sum_p\left\{\text{residues of }\frac{Q_{11}(p,1,\bar{\xi}p+\bar{\eta})}{Q(p,1,\bar{\xi}p+\bar{\eta})}\text{ in the upper half }p\text{-plane}\right\}. \quad (3{\cdot}22)$$

The above results were derived under the assumption that the differential operator Q was strictly hyperbolic. This argument in fact goes through under the less restrictive condition that Q be hyperbolic, e.g. that α and/or $\beta = 1$. However, these cases will not be explored further in this paper, and the condition of strict hyperbolicity will be adhered to throughout.

4. *Relations among the displacement components.* Let the right-hand sides of equations (3·19)–(3·22) define three functions ϕ, χ and ψ such that

$$\left.\begin{aligned} u_1(\xi,\eta,\tau) &\equiv \phi(\xi,\eta,\tau|\alpha,\beta,\gamma),\\ u_2(\xi,\eta,\tau) &\equiv \chi(\xi,\eta,\tau|\alpha,\beta,\gamma),\\ v_1(\xi,\eta,\tau) &\equiv \chi(\xi,\eta,\tau|\alpha,\beta,\gamma) \end{aligned}\right\} \quad (4{\cdot}1)$$

since, from equation (2·7) $Q_{12} = Q_{21}$ and

$$v_2(\xi,\eta,\tau) \equiv \psi(\xi,\eta,\tau|\alpha,\beta,\gamma).$$

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Now from equation (3·1), since $\bar{u}_1(\zeta_1, \zeta_2, s)$ is an even function of ζ_1,

$$\phi(\xi, \eta, \tau|\alpha, \beta, \gamma) = \phi(-\xi, \eta, \tau|\alpha, \beta, \gamma). \tag{4·2}$$

Also
$$\phi(\xi, \eta, \tau|\alpha, \beta, \gamma) = \phi(\xi, -\eta, \tau|\alpha, \beta, \gamma), \tag{4·3}$$

since $\bar{u}_1(\zeta_1, \zeta_2, s)$ is an even function of ζ_2. Similarly

$$\chi(\xi, \eta, \tau|\alpha, \beta, \gamma) = -\chi(-\xi, \eta, \tau|\alpha, \beta, \gamma), \tag{4·4}$$

$$\chi(\xi, \eta, \tau|\alpha, \beta, \gamma) = -\chi(\xi, -\eta, \tau|\alpha, \beta, \gamma), \tag{4·5}$$

$$\psi(\xi, \eta, \tau|\alpha, \beta, \gamma) = \psi(-\xi, \eta, \tau|\alpha, \beta, \gamma), \tag{4·6}$$

and
$$\psi(\xi, \eta, \tau|\alpha, \beta, \gamma) = \psi(\xi, -\eta, \tau|\alpha, \beta, \gamma). \tag{4·7}$$

Thus the displacement components $u_1(\xi, \eta, \tau)$ and $v_2(\xi, \eta, \tau)$ are even functions of ξ and η, while the components $u_2(\xi, \eta, \tau)$ and $v_1(\xi, \eta, \tau)$ are odd functions of ξ and η. Also from equations (4·4) and (4·5)

$$u_2(0, \eta, \tau) = u_2(\xi, 0, \tau) = v_1(0, \eta, \tau) = v_1(\xi, 0, \tau) \equiv 0. \tag{4·8}$$

That is, the displacement parallel to the ξ-axis due to an impulsive source at the origin acting in the η-direction is zero at every point on the η-axis ($\xi = 0$) for all time. The other three terms in (4·8) admit similar interpretations.

In the argument leading to equation (3·16) the θ-plane integration path was moved below the real θ-axis. This is arbitrary, so that if δ is replaced by $-\delta$ this integral is along a line parallel to and above the real θ-axis from $+i\delta$ to $\pi + i\delta$. Furthermore, if the mapping $p = \tan\theta$ (rather than $p = \cot\theta$) is used, then it follows that

$$v_2(\xi, \eta, \tau) = -\frac{1}{\pi\tau}\Im\sum_p \left\{\text{residues of } \frac{Q_{11}(1, p, \bar{\xi} + \bar{\eta}p)}{Q(1, p, \bar{\xi} + \bar{\eta}p)} \text{ in the upper half } p\text{-plane}\right\},$$
$$= \psi(\xi, \eta, \tau|\alpha, \beta, \gamma) \quad \text{from} \quad (4·1). \tag{4·9}$$

In the residue expression (4·9) interchange ξ and η, α and β. Then by inspection, having noted the form of the Q, Q_{11} and Q_{22} operators from equation (2·7),

$$\psi(\eta, \xi, \tau|\beta, \alpha, \gamma) = \phi(\xi, \eta, \tau|\alpha, \beta, \gamma), \tag{4·10}$$

hence
$$v_2(\xi, \eta, \tau) = \phi(\eta, \xi, \tau|\beta, \alpha, \gamma). \tag{4·11}$$

In view of the above remarks, it is now possible to express the four displacement components in terms of just two functions ϕ and χ by

$$u(\xi, \eta, \tau) = A_1\phi(\xi, \eta, \tau|\alpha, \beta, \gamma) + A_2\chi(\xi, \eta, \tau|\alpha, \beta, \gamma), \tag{4·12}$$

and
$$v(\xi, \eta, \tau) = A_1\chi(\xi, \eta, \tau|\alpha, \beta, \gamma) + A_2\phi(\eta, \xi, \tau|\beta, \alpha, \gamma). \tag{4·13}$$

Finally, because of the even-odd properties of the functions ϕ and χ, it is only necessary to consider $\xi \geqslant 0$ and $\eta \geqslant 0$. These results considerably simplify any numerical study of the displacement functions.

5. *The real p zeros of $Q(p, 1, \bar{\xi}p + \bar{\eta})$.* The real p zeros of the quartic $Q(p, 1, \bar{\xi}p + \bar{\eta})$ lie on the real p-plane axis and hence contribute nothing to the residue expressions

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(3·16)–(3·19). They do, however, furnish valuable information about the transient displacements, since if there are four such zeros, the displacements will all vanish at the corresponding values of $\bar{\xi}$ and $\bar{\eta}$.

Now
$$Q(p, 1, \bar{\xi}p+\bar{\eta}) = (\bar{\xi}p+\bar{\eta})^4 Q(r, s, 1) = (\bar{\xi}p+\bar{\eta})^4 Q(r, s, -1), \tag{5·1}$$

where
$$r = \frac{p}{\bar{\xi}p+\bar{\eta}} \quad \text{and} \quad s = \frac{1}{\bar{\xi}p+\bar{\eta}}. \tag{5·2}$$

Assuming $(\bar{\xi}p+\bar{\eta})^4 \neq 0$ (this will be justified below), the real p zeros of $Q(p, 1, \bar{\xi}p+\bar{\eta})$ must satisfy
$$Q(r, s, -1) \equiv F(r, s) = 0, \tag{5·3}$$

where $F(r, s) = 0$ describes the normal curve in the (r, s) plane (2). Equations (5·2) may be thought of as the parametric equation of a curve in the (r, s) plane. At each point where this curve intersects the normal curve $F(r, s) = 0$, a real p zero is found. Eliminating p from equations (5·2) gives

$$\bar{\xi}r + \bar{\eta}s = 1 \tag{5·4}$$

which is a straight line. Thus at any point, say (r_0, s_0) where the line $\bar{\xi}r+\bar{\eta}s = 1$ intersects the normal curve $F(r, s) = 0$, the corresponding p value, namely $p = p_0 = r_0/s_0$, will be a real zero of $Q(p, 1, \bar{\xi}p+\bar{\eta})$, i.e. $Q(p_0, 1, \bar{\xi}p_0+\bar{\eta}) = 0$.

What is the significance (for real p zeros) of $\bar{\xi}p+\bar{\eta} = 0$? If $\bar{\xi}p = \bar{\eta} = 0$, then the quartic equation $Q(p, 1, 0) = 0$ becomes

$$\alpha p^4 + \gamma p^2 + \beta = 0, \tag{5·5}$$

and this equation must have one (and hence because of its form two) real p root. But

$$p^2 = \frac{-\gamma \pm \sqrt{(\gamma^2 - 4\alpha\beta)}}{2\alpha}$$

and for $-2\sqrt{(\alpha\beta)} < \gamma \leqslant 2\sqrt{(\alpha\beta)}$, clearly there is no real p root of equation (5·5) and for $\gamma > 2\sqrt{(\alpha\beta)}$, $(\gamma^2 - 4\alpha\beta) > 0$ with the result that $p^2 < 0$ and again there is no real p root of equation (5·5). Consequently there are no real p values for which (simultaneously) $(\bar{\xi}p+\bar{\eta}) = 0$ and $Q(p, 1, \bar{\xi}p+\bar{\eta}) = 0$.

Of special importance are the values of $\bar{\xi}$ and $\bar{\eta}$ (and α, β, γ) for which $Q(p, 1, \bar{\xi}p+\bar{\eta})$ has four real zeros. This can come about in two ways. First, if the line $\bar{\xi}r+\bar{\eta}s = 1$ intersects both branches of the normal curve (from the condition of strict hyperbolicity, the normal curve consists of two ovals, one inside the other), then there will be four real p zeros. This corresponds physically to values of (ξ, η, τ) ahead of the leading wave front where all the (perturbed) displacements are zero until the arrival of the leading front.

Secondly, if the line $\bar{\xi}r+\bar{\eta}s = 1$ intersects the outer branch of the normal curve at four points, then there will also be four real p zeros. This can only happen for certain (α, β, γ) combinations since otherwise the outer branch of the normal curve will be convex [2]. The corresponding $(\bar{\xi}, \bar{\eta})$ values will lie in a cuspoidal triangular domain inside of the leading wave front. Following Petrowsky (6), such domains are called *Lacunas*. In a lacuna the (perturbed) displacements vanish identically. Since the

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lacunas occur for certain values of ξ/τ and η/τ, then in the plane of the spatial coordinates (ξ, η) *the lacunas will move with time.*

Petrowsky(6) has shown that the Cauchy problem for the equation $Q[u] = 0$ leads to a solution with lacunas if and only if the line $\bar{\xi}r + \bar{\eta}s = 1$ intersects the normal curv $F(r, s) = 0$ at four real points.

6. *Some special cases where the p zeros of* $Q(p, 1, \bar{\xi}p + \bar{\eta})$ *are easily found.* In order to perform the residue calculations indicated in equations (3·19)–(3·22), it is necessary to locate the p-plane poles of Q_{22}/Q, etc. Since $Q_{22}(p, 1, \bar{\xi}p + \bar{\eta})$ and $Q(p, 1, \bar{\xi}p + \bar{\eta})$ are polynominals in p, the location and order of the poles will correspond to the location and order of the p zeros of Q. Since Q is a quartic in p, the four zeros of Q may be found exactly by using standard formulae. However, the expressions for the zeros of a quartic tend to become cumbersome when the quartic contains all powers of p with arbitrary coefficients. For this reason it seems advisable to list some special cases in which the zeros of the quartic may be found with comparative ease.

I. $\bar{\eta} = 0$. Displacements may be evaluated at points on the $\bar{\xi}$-axis, without restrictions placed on the parameters α, β and γ.

II. $\bar{\xi} = 0$. Displacements may be evaluated at points on the η-axis, without restrictions placed on the parameters α, β and γ.

III. $\bar{\xi}^2 = \frac{1}{2}(\alpha + 1)$ and $\bar{\eta}^2 = \frac{1}{2}(\beta + 1)$. Displacements may be evaluated at four (moving) points in the (ξ, η) plane, without restrictions placed on the parameters α, β and γ.

IV. $(\alpha - \bar{\xi}^2)(1 - \bar{\xi}^2) = (\beta - \bar{\eta}^2)(1 - \bar{\eta}^2)$ and $(\alpha + 1 - 2\bar{\xi}^2) = (\beta + 1 - 2\bar{\eta}^2)$. There are two possibilities here, $\alpha = \beta$ with $\bar{\xi}^2 = \bar{\eta}^2$ and $\alpha + \beta = 2$ with $\bar{\xi}^2 = \bar{\eta}^2 + \alpha - 1$. Here restrictions are placed on the $(\bar{\xi}, \bar{\eta})$ values as well as on the parameters α, β and γ.

V. $\gamma = 1 + \alpha\beta$. Displacements maybe evaluated at all points in the (ξ, η) plane, but with restrictions placed on the parameters α, β and γ.

VI. $\gamma = \alpha + \beta$. Displacements may be evaluated at all points in the (ξ, η) plane, but with restrictions placed on the parameters α, β and γ.

VII. $(\alpha - \bar{\xi}^2)(1 - \bar{\xi}^2) = 0$ and $(\alpha + 1 - 2\bar{\xi}^2) = 0$. This implies that $\bar{\xi}^2 = \alpha = 1$, in which case the Q operator is no longer strictly hyperbolic.

VIII. $(\alpha - \bar{\xi}^2)(1 - \bar{\xi}^2) = 0$ and $(\beta - \bar{\eta}^2)(1 - \bar{\eta}^2) = 0$. There are four possibilities here in which $(\bar{\xi}^2, \bar{\eta}^2)$ may take on the values (α, β), $(\alpha, 1)$, $(1, \beta)$ and $(1, 1)$. Thus the displacements may be evaluated along certain lines in the (ξ, η) plane, without restrictions placed on the parameters α, β and γ.

IX. $(\beta - \bar{\eta}^2)(1 - \bar{\eta}^2) = 0$ and $(\beta + 1 - 2\bar{\eta}^2) = 0$. This implies that $\bar{\eta}^2 = \beta = 1$, in which case the Q operator is no longer strictly hyperbolic.

It is worth noting that case VI above includes the possibility $\alpha = \beta = \frac{1}{2}\gamma$. When the parameters α, β and γ are so related, and the stretch parameters are unity (no initial deformation), the displacements obtained will correspond to those of the classical (isotropic) infinitesimal theory of elasticity.

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7. *Analysis of the upper half p = plane singularities of Q^{-1} when $0 < \alpha = \beta < 1$ and $\bar{\eta} = 0$.* Henceforth the discussion will be restricted to parameter values for which $\alpha = \beta$ with $0 < \alpha < 1$. The third parameter is free, subject only to the condition that the Q operator remain strictly hyperbolic. In the notation of [2], the parameter values considered here will fall under

case 1·6 where $\alpha = \beta$ and $-2\alpha < \gamma < \alpha(\alpha+1)$,

case 1·7 where $\alpha = \beta$ and $\gamma = \alpha(\alpha+1)$

and case 1·8 where $\alpha = \beta$ and $\alpha(\alpha+1) < \gamma < (\alpha^2+1)$.

Lacunas will always occur under case 1·6 and may occur under case 1·8 (for further details see [3]). However if attention is restricted to displacements along the ξ and η axes, the only (finite in extent) lacunas encountered will come under case 1·6.

When $\alpha = \beta$ and $\bar{\eta} = 0$,

$$Q(p, 1, \bar{\xi}p + \bar{\eta}) \equiv f(p) = (\alpha - \bar{\xi}^2)(1 - \bar{\xi}^2)p^4 + [\gamma - (\alpha+1)\bar{\xi}^2]p^2 + \alpha. \tag{7·1}$$

If the discriminant of this quartic (quadratic in p^2) is denoted by $D(\bar{\xi})$, then

$$D(\bar{\xi}) = (1-\alpha)^2\bar{\xi}^4 - 2(1+\alpha)(\gamma - 2\alpha)\bar{\xi}^2 + (\gamma^2 - 4\alpha^2). \tag{7·2}$$

The zeros of $D(\bar{\xi})$ are important in the analysis of the zeros of $f(p)$. It is not difficult to show that

$-2\alpha < \gamma < 2\alpha$, then for $0 \leqslant \bar{\xi} < \bar{\xi}_1$, $D(\bar{\xi}) < 0$; $D(\bar{\xi}_1) = 0$; $\bar{\xi}_1 < \bar{\xi}$, $D(\bar{\xi}) > 0$,

$\gamma = 2\alpha$, then for $0 = \bar{\xi}$, $D(\bar{\xi}) = 0$; $0 < \bar{\xi}$, $D(\bar{\xi}) > 0$,

$2\alpha < \gamma < (\alpha^2+1)$, then for $0 \leqslant \xi$, $D(\xi) > 0$.

The value of $\bar{\xi}_1$ where the discriminant changes sign is given by

$$\bar{\xi}_1 = \left[\frac{(1+\alpha)(\gamma - 2\alpha) + \sqrt{[4\alpha\{\gamma - (\alpha^2+1)\}\{\gamma - 2\alpha\}]}}{(1-\alpha)^2}\right]^{\frac{1}{2}}. \tag{7·3}$$

Of course there is a corresponding negative value $-\bar{\xi}_1$, but in view of the remarks at the end of section 4, only non-negative $\bar{\xi}$ values need be considered.

When analysing the displacements along $\bar{\eta} = 0$, the point $\bar{\xi} = \bar{\xi}_1$ has a special significance. If the α, β and γ parameters are such that the motion falls under case 1·6, then the vertical line $r = 1/\bar{\xi}_1$ is a bitangent of the normal curve $F(r, s) = 0$. Physically this means that the trailing tip of the (moving) triangular-shaped lacuna is located at $\xi = \bar{\xi}_1\tau$ and $\eta = 0$.

Further analysis of the zeros of the quartic (7·1) follow in a straightforward manner and only the results will be summarized here. Define four functions $\phi(\bar{\xi})$, $\psi(\bar{\xi})$, $h(\bar{\xi})$ and $g(\bar{\xi})$ by

$$\phi(\bar{\xi}) = \frac{1}{2\sqrt{[(\alpha - \bar{\xi}^2)(1-\bar{\xi}^2)]}}[(\alpha+1)\bar{\xi}^2 - \gamma + 2\sqrt{[\alpha(\alpha - \bar{\xi}^2)(1-\bar{\xi}^2)]}]^{\frac{1}{2}}, \tag{7·4}$$

$$\psi(\bar{\xi}) = \frac{1}{2\sqrt{[(\alpha - \bar{\xi}^2)(1-\bar{\xi}^2)]}}[\gamma - (\alpha+1)\bar{\xi}^2 + 2\sqrt{[\alpha(\alpha - \bar{\xi}^2)(1-\bar{\xi}^2)]}]^{\frac{1}{2}}, \tag{7·5}$$

$$h(\bar{\xi}) = \left[\frac{\{\gamma - (\alpha+1)\bar{\xi}^2\} + \sqrt{[D(\bar{\xi})]}}{2(\alpha - \bar{\xi}^2)(1-\bar{\xi}^2)}\right]^{\frac{1}{2}}, \tag{7·6}$$

$$g(\bar{\xi}) = \left[\frac{\{\gamma - (\alpha+1)\bar{\xi}^2\} - \sqrt{[D(\bar{\xi})]}}{2(\alpha - \bar{\xi}^2)(1-\bar{\xi}^2)}\right]^{\frac{1}{2}}. \tag{7·7}$$

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The upper half p-plane poles of $Q^{-1}(p, 1, \bar{\xi}p)$ are now given by:

$1 \leqslant \bar{\xi} < \infty$, (cases 1·6, 1·7, 1·8), no poles.

$\sqrt{\alpha} < \bar{\xi} < 1$, (cases 1·6, 1·7, 1·8), one simple pole at

$$p = ig(\bar{\xi}).$$

$\bar{\xi} = \sqrt{\alpha}$, (case 1·6), no poles; (case 1·7), no poles; (case 1·8), one simple pole at

$$p = i\left[\frac{\alpha}{\gamma - \alpha(\alpha+1)}\right]^{\frac{1}{2}}.$$

$0 < \bar{\xi} < \sqrt{\alpha}$ and $2\alpha \leqslant \gamma < (\alpha^2+1)$, (under case 1·8), two simple poles at

$$p = ih(\bar{\xi}) \quad \text{and} \quad p = ig(\bar{\xi}).$$

$\bar{\xi}_1 < \bar{\xi} < \sqrt{\alpha}$ and $\alpha(\alpha+1) < \gamma < 2\alpha$, (under case 1·8), two simple poles at

$$p = ih(\bar{\xi}) \quad \text{and} \quad p = ig(\bar{\xi}).$$

$\bar{\xi} = \bar{\xi}_1$ and $\alpha(\alpha+1) < \gamma < 2\alpha$, (under case 1·8), one double pole at

$$p = i\left[\frac{2\alpha}{\gamma - (\alpha+1)\bar{\xi}_1^2}\right]^{\frac{1}{2}}.$$

$0 < \bar{\xi} < \bar{\xi}_1$ and $\alpha(\alpha+1) < \gamma < 2\alpha$, (under case 1·8), two simple poles at

$$p = \phi(\bar{\xi}) + i\psi(\bar{\xi}) \quad \text{and} \quad p = -\phi(\bar{\xi}) + i\psi(\bar{\xi}).$$

$0 < \bar{\xi} < \sqrt{\alpha}$ and $\gamma = \alpha(\alpha+1)$, (case 1·7), two simple poles at

$$p = \phi(\bar{\xi}) + i\psi(\bar{\xi}) \quad \text{and} \quad p = -\phi(\bar{\xi}) + i\psi(\bar{\xi}).$$

$\bar{\xi}_1 \leqslant \bar{\xi} < \sqrt{\alpha}$ and $-2\alpha < \gamma < \alpha(\alpha+1)$, (under case 1·6), no poles.

$0 < \bar{\xi} < \bar{\xi}_1$ and $-2\alpha < \gamma < \alpha(\alpha+1)$, (under case 1·6), two simple poles at

$$p = \phi(\bar{\xi}) + i\psi(\bar{\xi}) \quad \text{and} \quad p = -\phi(\bar{\xi}) + i\psi(\bar{\xi}).$$

$\bar{\xi} = 0$ and $2\alpha < \gamma < (\alpha^2+1)$, (under case 1·8), two simple poles at

$$p = ih(0) \quad \text{and} \quad p = ig(0).$$

$\bar{\xi} = 0$ and $\gamma = 2\alpha$, (under case 1·8), one double pole at $p = i$.

$\bar{\xi} = 0$ and $\alpha(\alpha+1) < \gamma < 2\alpha$, (under case 1·8), two simple poles at

$$p = \phi(0) + i\psi(0) \quad \text{and} \quad p = -\phi(0) + i\psi(0).$$

$\bar{\xi} = 0$ and $\alpha(\alpha+1) = \gamma$, (case 1·7), two simple poles at

$$p = \phi(0) + i\psi(0) \quad \text{and} \quad p = -\phi(0) + i\psi(0).$$

$\bar{\xi} = 0$ and $-2\alpha < \gamma < \alpha(\alpha+1)$, (case 1·6), two simple poles at

$$p = \phi(0) + i\psi(0) \quad \text{and} \quad p = -\phi(0) + i\psi(0).$$

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8. *Displacement components along the line* $\bar{\eta} = 0$ *for* $0 < \alpha = \beta < 1$. With the information about the poles of $Q^{-1}(p, 1, \bar{\xi}p)$ from section 7, it is now an easy matter to perform the various residue calculations called for in equations (3·19)–(3·22). The displacement components along the positive $\bar{\xi}$ axis may be expressed in terms of the six functions

$$F_1(\bar{\xi};\alpha,\gamma) = \frac{1}{2\sqrt{2}\sqrt{[D(\bar{\xi})]}}\left[\frac{\bar{\xi}^2-\alpha}{1-\bar{\xi}^2}\right]^{\frac{1}{2}} \frac{-\{\gamma-(\alpha+1)\,\bar{\xi}^2\}+\sqrt{[D(\bar{\xi})]}+2(1-\bar{\xi}^2)}{[-\{\gamma-(\alpha+1)\,\bar{\xi}^2\}+\sqrt{[D(\bar{\xi})]}]^{\frac{1}{2}}}, \tag{8·1}$$

$$F_2(\bar{\xi};\alpha,\gamma) = \frac{1}{2\sqrt{2}}\left[\frac{1}{\sqrt{\alpha}}+\sqrt{\left(\frac{\alpha-\bar{\xi}^2}{1-\bar{\xi}^2}\right)}\right] \times \frac{[\{\gamma-(\alpha+1)\,\bar{\xi}^2\}+\sqrt{[D(\bar{\xi})]}]^{\frac{1}{2}}-[\{\gamma-(\alpha+1)\,\bar{\xi}^2\}-\sqrt{[D(\bar{\xi})]}]^{\frac{1}{2}}}{\sqrt{D(\bar{\xi})}}, \tag{8·2}$$

$$F_3(\bar{\xi};\alpha,\gamma) = \frac{1}{2}\left[\frac{1}{\sqrt{\alpha}}+\sqrt{\left(\frac{\alpha-\bar{\xi}^2}{1-\bar{\xi}^2}\right)}\right] \frac{1}{[\gamma-(\alpha+1)\,\bar{\xi}^2+2\sqrt{[\alpha(\alpha-\bar{\xi}^2)\,(1-\bar{\xi}^2)]}]^{\frac{1}{2}}}, \tag{8·3}$$

$$G_1(\bar{\xi};\alpha,\gamma) = -\frac{1}{2\sqrt{2}\sqrt{[D(\bar{\xi})]}}\left[\frac{1-\bar{\xi}^2}{\bar{\xi}^2-\alpha}\right]^{\frac{1}{2}} \frac{-\{\gamma-(\alpha+1)\,\bar{\xi}^2\}+\sqrt{[D(\bar{\xi})]}+2\alpha(\alpha-\bar{\xi}^2)}{[-\{\gamma-(\alpha+1)\,\bar{\xi}^2\}+\sqrt{[D(\bar{\xi})]}]^{\frac{1}{2}}}, \tag{8·4}$$

$$G_2(\bar{\xi};\alpha,\gamma) = \frac{1}{2\sqrt{2}}\left[\sqrt{\alpha}+\sqrt{\left(\frac{1-\bar{\xi}^2}{\alpha-\bar{\xi}^2}\right)}\right] \times \frac{[\{\gamma-(\alpha+1)\,\bar{\xi}^2\}+\sqrt{[D(\bar{\xi})]}]^{\frac{1}{2}}-[\{\gamma-(\alpha+1)\,\bar{\xi}^2\}-\sqrt{[D(\bar{\xi})]}]^{\frac{1}{2}}}{\sqrt{[D(\bar{\xi})]}}, \tag{8·5}$$

and

$$G_3(\bar{\xi};\alpha,\gamma) = \frac{1}{2}\left[\sqrt{\alpha}+\sqrt{\left(\frac{1-\bar{\xi}^2}{\alpha-\bar{\xi}^2}\right)}\right] \frac{1}{[\gamma-(\alpha+1)\,\bar{\xi}^2+2\sqrt{[\alpha(\alpha-\bar{\xi}^2)\,(1-\bar{\xi}^2)]}]^{\frac{1}{2}}}. \tag{8·6}$$

Explicit expressions for the displacement components $u_1(\xi, 0, \tau)$ and $v_2(\xi, 0, \tau)$ are as follows:

Case 1·6	$1 \leqslant \bar{\xi} < \infty$	$\pi\tau u_1 = 0$	$\pi\tau v_2 = 0,$
$-2\alpha < \gamma < \alpha(\alpha+1)$	$\sqrt{\alpha} \leqslant \bar{\xi} < 1$	$\pi\tau u_1 = F_1(\bar{\xi};\alpha,\gamma)$	$\pi\tau v_2 = G_1(\bar{\xi};\alpha,\gamma),$
	$\bar{\xi}_1 \leqslant \bar{\xi} \leqslant \sqrt{\alpha}$	$\pi\tau u_1 = 0$ (lacuna)	$\pi\tau v_2 = 0$ (lacuna),
	$0 \leqslant \bar{\xi} < \bar{\xi}_1$	$\pi\tau u_1 = F_3(\bar{\xi};\alpha,\gamma)$	$\pi\tau v_2 = G_3(\xi;\alpha,\gamma).$

(8·7)

Case 1·7	$1 \leqslant \bar{\xi} < \infty$	$\pi\tau u_1 = 0$	$\pi\tau v_2 = 0,$
$\gamma = \alpha(\alpha+1)$	$\sqrt{\alpha} < \bar{\xi} < 1$	$\pi\tau u_1 = F_1(\bar{\xi};\alpha,\alpha^2+\alpha)$	$\pi\tau v_2 = G_1(\bar{\xi};\alpha,\alpha^2+\alpha),$
	$\sqrt{\alpha} = \bar{\xi}$	$\pi\tau u_1 = 0$ (point lacuna)	$\pi\tau v_2 = 0$ (point lacuna),
	$0 \leqslant \bar{\xi} < \alpha$	$\pi\tau u_1 = F_3(\bar{\xi};\alpha,\alpha^2+\alpha)$	$\pi\tau v_2 = G_3(\bar{\xi};\alpha,\alpha^2+\alpha).$

(8·8)

Case 1·8	$1 \leqslant \bar{\xi} < \infty$	$\pi\tau u_1 = 0$	$\pi\tau v_2 = 0,$
$\alpha(\alpha+1) < \gamma < 2\alpha$	$\sqrt{\alpha} < \bar{\xi} < 1$	$\pi\tau u_1 = F_1(\bar{\xi};\alpha,\gamma)$	$\pi\tau v_2 = G_1(\bar{\xi};\alpha,\gamma),$
	$\bar{\xi} = \sqrt{\alpha}$	$\pi\tau u_1 = F_2(\sqrt{\alpha};\alpha,\gamma)$	$\pi\tau v_2 = G_1(\sqrt{\alpha};\alpha,\gamma),$
	$\bar{\xi}_1 < \bar{\xi} < \sqrt{\alpha}$	$\pi\tau u_1 = F_2(\bar{\xi};\alpha,\gamma)$	$\pi\tau v_2 = G_2(\bar{\xi};\alpha,\gamma),$
	$0 \leqslant \bar{\xi} < \xi_1$	$\pi\tau u_1 = F_3(\bar{\xi};\alpha,\gamma)$	$\pi\tau v_2 = G_3(\bar{\xi};\alpha,\gamma).$

(8·9)

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Case 1·8		$1 \leqslant \bar{\xi} < \infty$	$\pi\tau u_1 = 0$	$\pi\tau v_2 = 0,$
	$\gamma = 2\alpha$	$\sqrt{\alpha} < \bar{\xi} < 1$	$\pi\tau u_1 = F_1(\bar{\xi}; \alpha, 2\alpha)$	$\pi\tau v_2 = G_1(\bar{\xi}; \alpha, 2\alpha),$
		$\bar{\xi} = \sqrt{\alpha}$	$\pi\tau u_1 = F_2(\sqrt{\alpha}; \alpha, 2\alpha)$	$\pi\tau v_2 = G_1(\sqrt{\alpha}; \alpha, 2\alpha),$
		$0 < \bar{\xi} < \sqrt{\alpha}$	$\pi\tau u_1 = F_2(\bar{\xi}; \alpha, 2\alpha)$	$\pi\tau v_2 = G_2(\bar{\xi}; \alpha, 2\alpha),$
		$\bar{\xi} = 0$	$\pi\tau u_1 = F_3(0; \alpha, 2\alpha)$	$\pi\tau v_2 = G_3(0; \alpha, 2\alpha).$

(8·10)

Case 1·8		$1 \leqslant \bar{\xi} < \infty$	$\pi\tau u_1 = 0$	$\pi\tau v_2 = 0,$
$2\alpha < \gamma < (\alpha^2+1)$		$\sqrt{\alpha} < \bar{\xi} < 1$	$\pi\tau u_1 = F_1(\bar{\xi}; \alpha, \gamma)$	$\pi\tau v_2 = G_1(\bar{\xi}; \alpha, \gamma),$
		$\bar{\xi} = \sqrt{\alpha}$	$\pi\tau u_1 = F_2(\sqrt{\alpha}; \alpha, \gamma)$	$\pi\tau v_2 = G_1(\sqrt{\alpha}; \alpha, \gamma),$
		$0 \leqslant \bar{\xi} < \sqrt{\alpha}$	$\pi\tau u_1 = F_2(\bar{\xi}; \alpha, \gamma)$	$\pi\tau v_2 = G_2(\bar{\xi}; \alpha, \gamma).$

(8·11)

In view of equations (4·4)–(4·7) the above displacement expressions are also valid along the negative $\bar{\xi}$ axis (since u_1 and v_2 are even functions of $\bar{\xi}$), and

$$u_2(\xi, 0, \tau) = v_1(\xi, 0, \tau) \equiv 0, \tag{8·12}$$

(since u_2 and v_1 are odd functions of $\bar{\eta}$). Furthermore, from equations (4·12) and (4·13), the above results also give $u_1(0, \eta, \tau)$ and $v_2(0, \eta, \tau)$. In particular if $\bar{\xi}$ is replaced by $\bar{\eta}$, then the right-hand column in equations (8·7)–(8·11) involving the F functions give $v_2(0, \eta, \tau)$ while the right-hand column involving the G functions give $u_1(0, \eta, \tau)$. Since $u_2(\xi, \eta, \tau)$ and $v_1(\xi, \eta, \tau)$ are odd functions in ξ, then

$$u_2(0, \eta, \tau) = v_1(0, \eta, \tau) \equiv 0. \tag{8·13}$$

Thus explicit algebraic formulae for all the displacement components, valid along the ξ and η-axes, have now been determined subject to the conditions that $\alpha = \beta$ and $0 < \alpha < 1$.

9. *Behaviour of the displacement components $u_1(\xi, 0, \tau)$ and $v_2(\xi, 0, \tau)$ near the wave fronts.* Some of the displacement components of section 8 may tend to become singular (unbounded) at or near certain values of $\bar{\xi}$. From the general theory of linear partial differential equations (**12**), it is known that such singularities may occur only at the wave fronts and that they must be interpreted within the framework of weak solutions to the differential equations.

Since equations (8·7)–(8·11) give explicit expressions for the displacement components along the line $\eta = 0$ (and by implication along $\xi = 0$), it is an easy exercise to examine these formulae for singularities. Their behaviour for ($\bar{\xi} > 0$) near the wave fronts are as follows:

Cases 1·6–1·8 as $\bar{\xi} \to 1 - 0$,

$$\pi\tau u_1 \sim \frac{1}{2}\left[\frac{1-\alpha}{1+\alpha-\gamma}\right]^{\frac{1}{2}} (1-\bar{\xi}^2)^{-\frac{1}{2}} \quad \text{(singular)} \tag{9·1}$$

and

$$\pi\tau v_2 \sim -\frac{\alpha^2+1-\gamma}{2(1+\alpha-\gamma)^{\frac{3}{2}}(1-\alpha)^{\frac{1}{2}}} (1-\bar{\xi}^2)^{\frac{1}{2}}. \tag{9·2}$$

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Case 1·6 as $\bar{\xi} \to \sqrt{\alpha} + 0$

$$\pi\tau u_1 \sim \frac{\alpha(\alpha+1)-\gamma+(1-\alpha)}{2[\alpha(\alpha+1)-\gamma]^{\frac{3}{2}}(1-\alpha)^{\frac{1}{2}}}(\bar{\xi}^2-\alpha)^{\frac{1}{2}}, \tag{9·3}$$

$$\pi\tau v_2 \sim -\frac{(1-\alpha)^{\frac{1}{2}}}{2[\alpha(\alpha+1)-\gamma]^{\frac{1}{2}}}(\xi^2-\alpha)^{-\frac{1}{2}} \quad \text{(singular)}, \tag{9·4}$$

as $\bar{\xi} \to \bar{\xi}_1 - 0$

$$\pi\tau u_1 \sim \frac{1}{2}\left[\frac{1}{\sqrt{\alpha}}+\sqrt{\left(\frac{\alpha-\bar{\xi}_1^2}{1-\bar{\xi}_1^2}\right)}\right]\left[\frac{(\alpha+1)\bar{\xi}_1^2-\gamma}{(2\alpha-\gamma)(\alpha+1)+(\alpha-1)^2\bar{\xi}_1^2}\right]^{\frac{1}{2}}(\bar{\xi}_1^2-\bar{\xi}^2)^{-\frac{1}{2}} \quad \text{(singular)} \tag{9·5}$$

and

$$\pi\tau v_2 \sim \frac{1}{2}\left[\sqrt{\alpha}+\sqrt{\left(\frac{1-\bar{\xi}_1^2}{\alpha-\bar{\xi}_1^2}\right)}\right]\left[\frac{(\alpha+1)\bar{\xi}_1^2-\gamma}{(2\alpha-\gamma)(\alpha+1)+(\alpha-1)^2\bar{\xi}_1^2}\right]^{\frac{1}{2}}(\bar{\xi}_1^2-\bar{\xi}^2)^{-\frac{1}{2}} \quad \text{(singular)}. \tag{9·6}$$

Case 1·7 as $\bar{\xi} \to \sqrt{\alpha} + 0$

$$\pi\tau u_1 \sim \frac{1}{4[\alpha^3(1-\alpha)]^{\frac{1}{4}}}(\bar{\xi}^2-\alpha)^{-\frac{1}{4}}, \quad \text{(singular)}, \tag{9·7}$$

$$\pi\tau v_2 \sim -\frac{1}{4}\left[\frac{1-\alpha}{\alpha}\right]^{\frac{1}{4}}(\bar{\xi}^2-\alpha)^{-\frac{3}{4}} \quad \text{(singular)}, \tag{9·8}$$

as $\bar{\xi} \to \sqrt{\alpha} - 0$

$$\pi\tau u_1 \sim \frac{1}{2\sqrt{2}[\alpha^3(1-\alpha)]^{\frac{1}{4}}}(\alpha-\bar{\xi}^2)^{-\frac{1}{4}} \quad \text{(singular)}, \tag{9·9}$$

and

$$\pi\tau v_2 \sim \frac{1}{2\sqrt{2}}\left[\frac{1-\alpha}{\alpha}\right]^{\frac{1}{4}}(\alpha-\bar{\xi}^2)^{-\frac{3}{4}} \quad \text{(singular)}. \tag{9·10}$$

Case 1·8 as $\bar{\xi} \to \sqrt{\alpha} + 0$

$$\pi\tau u_1 \sim \frac{2}{2\sqrt{\alpha}[\gamma-\alpha(\alpha+1)]^{\frac{1}{2}}} \tag{9·11}$$

and

$$\pi\tau v_2 \sim -\frac{\sqrt{\alpha}(1+\alpha^2-\gamma)}{2[\gamma-\alpha(\alpha+1)]^{\frac{3}{2}}}. \tag{9·12}$$

Case 1·8 as $\bar{\xi} \to \sqrt{\alpha} - 0$

$$\pi\tau u_1 \sim \frac{1}{2\sqrt{\alpha}[\gamma-\alpha(\alpha+1)]^{\frac{1}{2}}}, \tag{9·13}$$

$$\pi\tau v_2 \sim \frac{1}{2}\left[\frac{1-\alpha}{\gamma-\alpha(\alpha+1)}\right]^{\frac{1}{2}}(\alpha-\bar{\xi}^2)^{-\frac{1}{2}} \quad \text{(singular)}. \tag{9·14}$$

10. *Some numerical results.* Some numerical results† are presented in Figs. 1–5. In Fig. 3 the u_1 and v_2 displacement components are plotted as a function of $\bar{\xi} = \xi/\tau$ for $\alpha = \beta = 0{\cdot}64$ and $\gamma = -1$. The corresponding wave front is shown at the top of Fig. 2 and normal curve at the top of Fig. 1. Here a lacuna is clearly in evidence as was predicted for case 1·6. The u_1 displacement (solid curve) and v_2 displacement (dashed curve) are not easily distinguished when $0 \leqslant \bar{\xi} < \bar{\xi}_1$, but for $0{\cdot}8 < \bar{\xi} < 1$ their behaviour is altogether different.

† The author was aided in these calculations by Mr Katholi.

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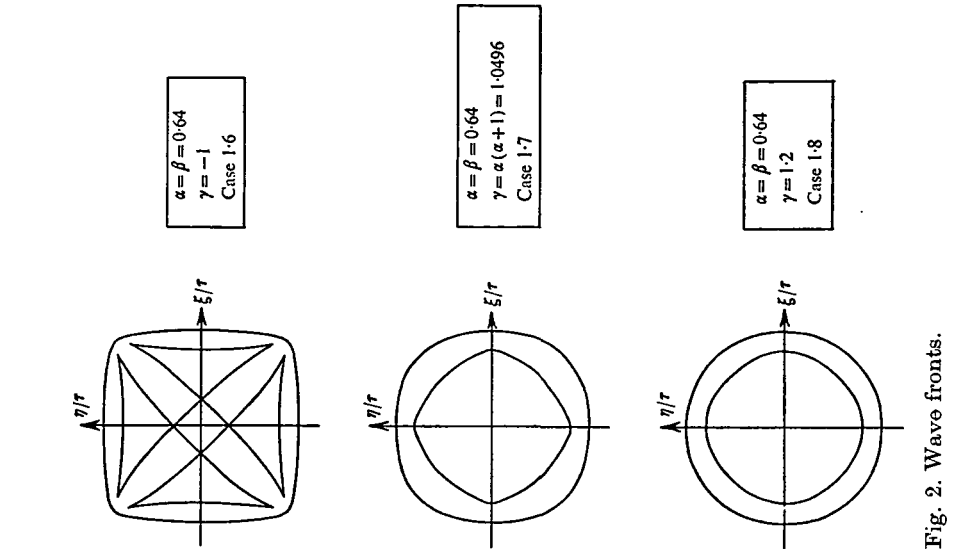


Fig. 2. Wave fronts.

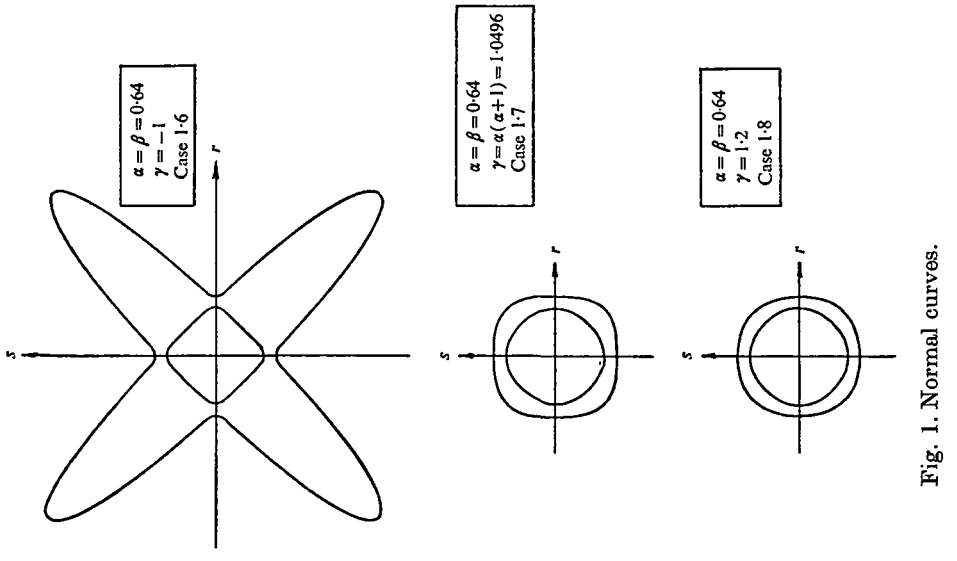


Fig. 1. Normal curves.

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In Fig. 4 the u_1 and v_2 displacement components are plotted for $\alpha = \beta = 0{\cdot}64$ and $\gamma = \alpha(\alpha+1) = 1{\cdot}0496$. The corresponding wave front is shown in the middle of Fig. 2 and the normal curve in the middle of Fig. 1. The parameter values here fall under case 1·7 which is a special case separating solutions with lacunas from solutions without lacunas. By adjusting γ to the value $\alpha(\alpha+1)$, the lacuna crossing the positive $\bar{\xi}$

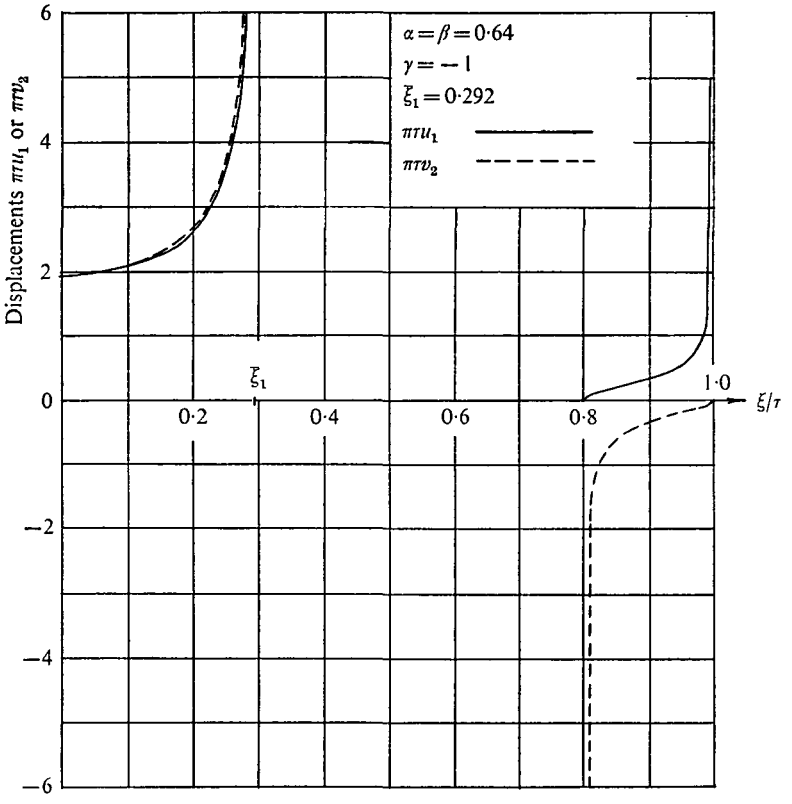


Fig. 3. Displacement components u_1 and v_2 versus ξ/τ along the line $\eta = 0$.

axis, has shrunk to a point at $\bar{\eta} = 0$, $\bar{\xi} = 0{\cdot}8$. By symmetry there are four other such point lacunas. Also the order of the displacement singularity, as $\bar{\xi} \to \sqrt{\alpha}$ has changed from case 1·6 to 1·7 as the formulae of section 9 indicate.

Fig. 5 shows the displacement component curves for $\alpha = \beta = 0{\cdot}64$ and $\gamma = 1{\cdot}2$. These numbers are appropriate to case 1·8 since γ is in the range $\alpha(\alpha+1) < \gamma < 2\alpha$. The wave front is shown at the bottom of Fig. 2 and the wave normal curve at the bottom of Fig. 1. The plots which correspond to equations (8·9)–(8·11) all look essentially alike, and for this reason only a single figure is shown for case 1·8.

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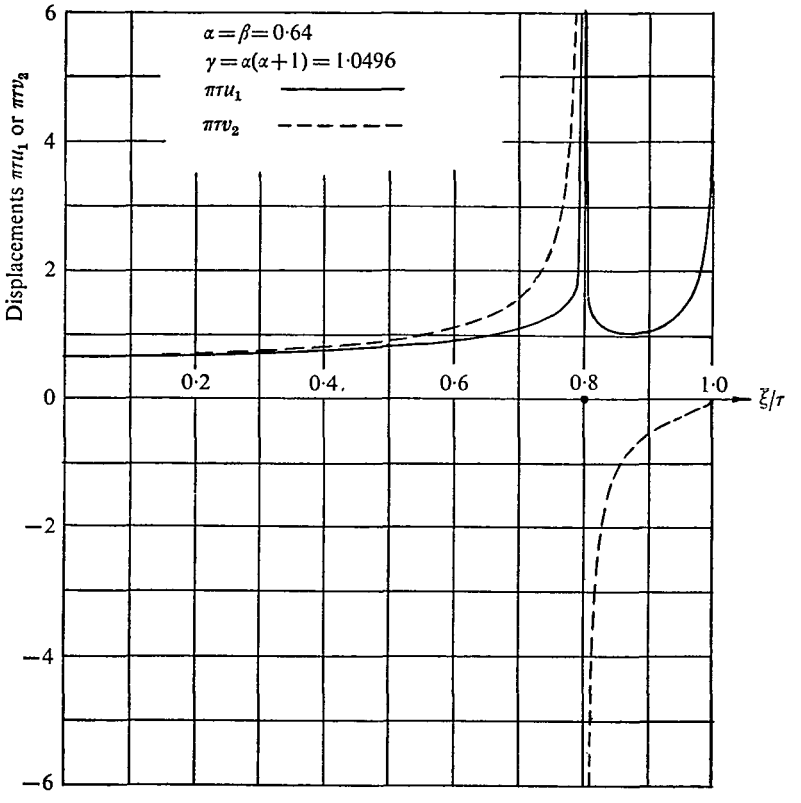


Fig. 4. Displacement components u_1 and v_2 versus ξ/τ along the line $\eta = 0$.

Another interesting feature of the u_1 displacement component curve shown in Fig. 5, is the kink at $\bar{\xi} = \sqrt{\alpha}$. In fact

$$\lim_{\bar{\xi}\to\sqrt{\alpha}-0} \frac{dF_2}{d\bar{\xi}} = +\infty, \tag{10·1}$$

while

$$\lim_{\bar{\xi}\to\sqrt{\alpha}+0} \frac{dF_1}{d\bar{\xi}} = \frac{(3\alpha+1)(\gamma-\bar{\gamma})}{2[\gamma-\alpha(\alpha+1)]^{\frac{3}{2}}} \tag{10·2}$$

where $\bar{\gamma} = \alpha^2+4\alpha/(3\alpha+1)$. And since

$$(\gamma-\bar{\gamma}) = \begin{cases} <0 & \text{if} \quad \alpha(\alpha+1) < \gamma < \bar{\gamma}, \\ 0 & \text{if} \quad \gamma = \bar{\gamma}, \\ >0 & \text{if} \quad \bar{\gamma} < \gamma < (\alpha^2+1), \end{cases}$$

it follows that the u_1 curve may have a variety of shapes near $\bar{\xi} = \sqrt{\alpha}$ depending upon the values of α and γ. Thus $u_1(\xi, 0, \tau)$ is continuous at $\bar{\xi} = \sqrt{\alpha}$ but its $\bar{\xi}$ derivative is not.

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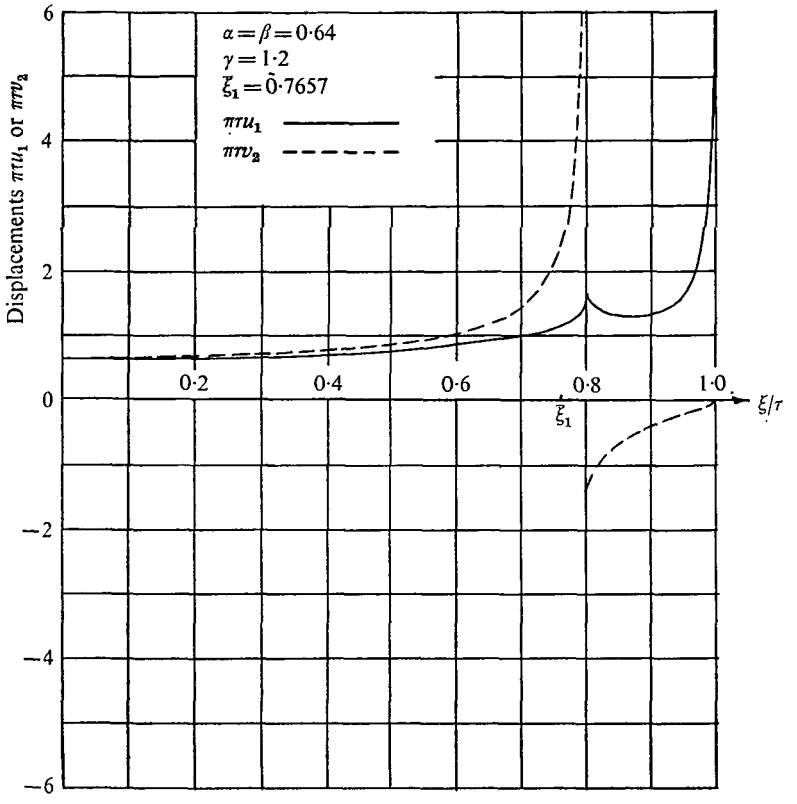


Fig. 5. Displacement components u_1 and v_2 versus ξ/τ along the line $\eta = 0$.

Similarly the right-hand $\bar{\xi}$ derivative of v_2 at $\bar{\xi} = \sqrt{\alpha}$ may change sign since,

$$\lim_{\xi\to\sqrt{\alpha}+0} \frac{dG_1}{d\bar{\xi}} = -\frac{\alpha(\alpha+3)(\alpha^2+1-\gamma)}{2[\gamma-\alpha(\alpha+1)]^{\frac{3}{2}}}(\gamma-\bar{\bar{\gamma}}) \quad \text{where} \quad \bar{\bar{\gamma}} = \frac{\alpha(\alpha^2-\alpha+8)}{(\alpha+3)}. \tag{10·3}$$

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REFERENCES

(1) GREEN, A. E., RIVLIN, R. S. and SHIELD, R. T. *Proc. Roy. Soc.* A **211**, (1952), 128.
(2) PAYTON, R. G. *Arch. Rational Mech. Anal.* **32** (1969), 311.
(3) PAYTON, R. G. *Arch. Rational Mech. Anal.* **35** (1969), 402.
(4) WEITZNER, H. *Phys. Fluids* **4** (1961), 1238.
(5) BURRIDGE, R. *Proc. Cambridge Philos. Soc.* **63** (1967), 819.
(6) PETROWSKY, I. *Rec. Math* [*Mat. Sbornik*] **17** (1945), 289.
(7) BAZER, J. and YEN, D. H. Y. *Comm. Pure Appl. Math.* **22** (1969), 279.
(8) GREEN, A. E. and ZERNA, W. *Theoretical elasticity* (London: Oxford University Press, 1954).
(9) JOHN, F. *Plane waves and Spherical means applied to partial differential equations* (NewYork: John Wiley and Sons, 1965).
(10) GEL'FAND, I. M. and SHILOV, G. E. *Generalized functions*, vol. I. (New York: Academic Press, 1964).
(11) TRUESDELL, C. and NOLL, W. *The non-linear field theories of mechanics* (Berlin-Heidelberg-New York: Springer, 1965).
(12) COURANT, R. and HILBERT, D. *Methods of mathematical physics*, vol. II (New York: John Wiley and Sons, 1962).

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