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An inversion formula for the kernel $K_\nu(x)$

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Abstract. The problem discussed is that of solving the integral equation

$$\int_0^\infty (xu)^\nu K_\nu(xu) f(u)\, du = g(x),$$

where $g(x)$ is given, $K_\nu(z)$ is associated with Bessel functions of purely imaginary argument and $f(x)$ is to be determined.

I prove that, by means of fractional integration, it is possible to reduce this equation to the form of a Laplace transform which can be solved by known methods.

1. *Introduction.* The object of this paper is to solve the integral equation

$$\int_0^\infty (xu)^\nu K_\nu(xu) f(u)\, du = g(x), \tag{1}$$

where $g(x)$ is given, $f(u)$ is an unknown function to be determined and $K_\nu(z)$ is related to the Bessel function $J_\nu(z)$ as follows:

$$K_\nu(z) = \pi\{I_{-\nu}(z) - I_\nu(z)\}/\{2\sin(\nu\pi)\}, \tag{2}$$

where

$$I_\nu(z) = \sum_{m=0}^{\infty} (\tfrac{1}{2}z)^{\nu+2m}/\{m!\,\Gamma(\nu+m+1)\}, \tag{3}$$

and $J_\nu(z)$ is equal to the series on the right of (3) with the even terms negative.

I shall show that (1) can be reduced to the Laplace transform by means of fractional integration and so, since the Laplace transform can be inverted in two ways, (1) can be solved for $f(u)$.

We need Mellin transforms as well as fractional integration and so the next two sections will deal with these two subjects. The reduction of (1) is first discussed in section 4.

2. *The Mellin transform.* The Mellin transform of $f(z)$ is denoted by $\mathfrak{M}\{f(z)\}$. If $\mathfrak{M}\{f(z)\} = F(s)$ we shall also write $f(z) = \mathfrak{M}^{-1}\{F(s)\}$ and $\mathfrak{M}^{-1}$ will denote the inverse Mellin transform. Formally we have

$$F(s) = \mathfrak{M}\{f(z)\} = \int_0^\infty f(z)\, z^{s-1}\, dz \tag{4}$$

and

$$f(z) = \mathfrak{M}^{-1}\{F(s)\} = \frac{1}{2\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty} F(s)\, x^{-s}\, ds. \tag{5}$$

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The simplest conditions are given in L_2 space, which we shall use here. If

$$f(u) \in L_2(0, \infty)$$

and the l.i.m. is with index 2 then

$$F(s) = \mathfrak{M}\{f(z)\} = \underset{N\to\infty}{\text{l.i.m.}} \int_{1/N}^{N} f(z)\, z^{s-1}\, dz \tag{6}$$

and also $$F(s) \in L_2(\tfrac{1}{2} - i\infty, \tfrac{1}{2} + i\infty).$$

If $$F(s) \in L_2(\tfrac{1}{2} - i\infty, \tfrac{1}{2} + i\infty)$$

then $$f(z) = \mathfrak{M}^{-1}\{F(s)\} = \frac{1}{2\pi i} \underset{N\to\infty}{\text{l.i.m.}} \int_{\frac{1}{2}-iN}^{\frac{1}{2}+iN} F(s)\, z^{-s}\, ds \tag{7}$$

and also $f(z) \in L_2(0, \infty)$, ((4), Theorem 71, page 94).

If the integral of (4) and the l.i.m. of (6) both exist then they are equal. The same is true for (5) and (7), a remark which will be of use later.

If $x > 0$, from (6) or (7) it follows easily that if $f(zx)$ is considered as a function of z with x as a parameter then

$$\mathfrak{M}\{f(xz)\} = x^{-s}\, \mathfrak{M}\{f(z)\}. \tag{8}$$

The following lemma is useful.

LEMMA 1. *If (i) $x > 0$, (ii) $f(z)$ and $g(z)$ both belong to $L_2(0, \infty)$, (iii) $\mathfrak{M}\{f(z)\} = F(s)$, $\mathfrak{M}\{g(z)\} = G(s)$ and $G(s)$ is bounded on the line $s = \frac{1}{2} + it$, $-\infty < t < \infty$, then*

$$\int_0^\infty g(xz) f(z)\, dz \in L_2(0, \infty) \tag{9}$$

and

$$\mathfrak{M}\left\{\int_0^\infty g(xz) f(z)\, dz\right\} = G(s)\, F(1-s), \tag{10}$$

where the integrals of (9) *and* (10) *are taken as functions of x.*

Proof. Since $f(z) \in L_2(0, \infty)$ from (6) we have $F(s) \in L_2(\frac{1}{2} - i\infty, \frac{1}{2} + i\infty)$ and so, on writing $1 - s$ for s, we also have $F(1-s) \in L_2(\frac{1}{2} - i\infty, \frac{1}{2} + i\infty)$. From condition (*iii*) $G(s)$ is bounded on the line $s = \frac{1}{2} + it$, $-\infty < t < \infty$, i.e. on this line we have

$$|G(s)\, F(1-s)| < B|F(1-s)|,$$

where B is a finite positive constant. Hence $G(s)\, F(1-s) \in L_2(\frac{1}{2} - i\infty, \frac{1}{2} + i\infty)$ and so, from (7), we have

$$\mathfrak{M}^{-1}\{G(s)\, F(1-s)\} = \frac{1}{2\pi i} \underset{N\to\infty}{\text{l.i.m.}} \int_{\frac{1}{2}-iN}^{\frac{1}{2}+iN} G(s)\, F(1-s)\, x^{-s}\, ds. \tag{11}$$

We now proceed to deduce from (11) the following result:

$$\mathfrak{M}^{-1}\{G(s)\, F(1-s)\} = \frac{1}{2\pi i} \int_{\frac{1}{2}-i\infty}^{\frac{1}{2}+i\infty} G(s)\, F(1-s)\, x^{-s}\, ds. \tag{12}$$

Denote the integral in (12) by I. From (*ii*) above and the statements accompanying (6) we see that $F(s)$, $G(s)$, and so $G(1-s)$, all belong to $L_2(\frac{1}{2} - i\infty, \frac{1}{2} + i\infty)$. Hence, on applying Schwartz's inequality to I ((5), page 381), it follows that I converges absolutely. Consequently the value of I must be the same as the l.i.m. of (11); hence (12) is established.

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One more step is required before we can establish the lemma. We may apply the Parseval theorem for Mellin transforms ((4), Theorem 72, page 95) to I, an application justified by condition (*ii*). From (12) we then obtain

$$\mathfrak{M}^{-1}\{G(s)\,F(1-s)\} = \int_0^\infty g(xz)f(z)\,dz. \tag{13}$$

To establish (9), we have just proved that $G(s)\,F(1-s) \in L_2(\frac{1}{2}-i\infty, \frac{1}{2}+i\infty)$ and therefore, from the statement following (7), we have $\mathfrak{M}^{-1}\{G(s)\,F(1-s)\} \in L_2(0,\infty)$. Hence the right-hand side of (13), as a function of x, belongs to $L_2(0,\infty)$ and (9) is established.

To establish (10) we note that both sides of (13) belong to $L_2(0,\infty)$ and so by (6) we may apply the operator $\mathfrak{M}$ to both sides. This application transforms (13) to (10) and so establishes (10).

In order to apply Lemma 1 we need the following properties of $K_\nu(z)$.

$$\mathfrak{M}\{z^\nu K_\nu(z)\} = 2^{s+\nu-2}\,\Gamma(\tfrac{1}{2}s)\,\Gamma(\tfrac{1}{2}s+\nu), \tag{14}$$

where $s = \sigma + it$, σ and t real, and $\sigma > \max(0, -2\nu)$ ((4), 7.9.12, page 197).

From the asymptotic expansion of $\Gamma(s+a)$ ((7), page 272), for large

$$|s| \quad \text{and} \quad |\arg s| < \pi - \delta \quad (\delta > 0),$$

we have
$$2^{s+\nu-2}\,\Gamma(\tfrac{1}{2}s)\,\Gamma(\tfrac{1}{2}s+\nu) = \pi s^{\nu-1}\exp[s(\log s) - s]\{1+O(s^{-1})\}, \tag{15}$$

where O is the usual Landau order symbol.

We now write $s = \frac{1}{2}+it$ in (15) and note the restriction just stated, $|\arg s| < \pi - \delta$. For large positive t we then have $\arg(\frac{1}{2}+it) = \frac{1}{2}\pi + O(t^{-1})$ and for large negative t we have $\arg(4+it) = -\frac{1}{2}\pi + O(t^{-1})$. Hence on the line $s = \frac{1}{2}+it$ we have for large positive or negative t,

$$|2^{s+\nu-2}\,\Gamma(\tfrac{1}{2}s)\,\Gamma(\tfrac{1}{2}s+\nu)| = \pi e^{-\frac{1}{2}}t^{\nu-\frac{1}{2}}\exp(-\tfrac{1}{2}\pi|t|)\{1+O(|t|^{-1})\}. \tag{16}$$

If $\nu \geqslant 0$ the singularities of the s function on the left of (15) occurs only when s is real and $s \leqslant 0$. Hence, from (16), it follows that $\mathfrak{M}\{z^\nu K_\nu(z)\}$ is bounded on the line $s = \frac{1}{2}+it$.

We may now apply Lemma 1 to the function $K_\nu(z)$.

LEMMA 2. *If* (*i*) $x > 0$, (*ii*) $\nu \geqslant 0$ *and* (*iii*) $f(z) \in L_2(0,\infty)$ *then*

$$\int_0^\infty (xu)^\nu K_\nu(xu) f(u)\,du \in L_2(0,\infty) \tag{17}$$

and
$$\mathfrak{M}\left\{\int_0^\infty (xu)^\nu K_\nu(xu) f(u)\,du\right\} = 2^{s+\nu-2}\,\Gamma(\tfrac{1}{2}s)\,\Gamma(\tfrac{1}{2}s+\nu)\,F(1-s), \tag{18}$$

where the left-hand side of (17) *is considered to be a function of* x.

Proof. In order to use Lemma 1 we must establish the following two results: $\mathfrak{M}\{z^\nu K_\nu(z)\}$ is bounded on the line $s = \frac{1}{2}+it$, condition (*iii*), and $z^\nu K_\nu(z) \in L_2(0,\infty)$, condition (*ii*). The first of these requirements has already been established just before the statement of this lemma. The second is readily established by noting that, from

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(2) and (3), $z^\nu K_\nu(z)$ is an entire function which is finite at $z = 0$ and which has the asymptotic expansion

$$z^\nu K_\nu(z) = \pi^{\frac{1}{2}} 2^{-\frac{1}{2}} z^{\nu-\frac{1}{2}} e^{-z}\{1 + O(z^{-1})\}, \tag{19}$$

for large $|z|$ and $|\arg z| < \frac{3}{2}\pi$ ((6), page 202). From the exponential factor with z real and positive it follows that $z^\nu K_\nu(z) \in L_2(0, \infty)$.

We are now justified in writing $g(z) = z^\nu K_\nu(z)$ in Lemma 1 and then, after changing the variable of integration from z to u, (9) becomes (17) and (10) becomes (18) (from (14)).

This completes the proof of Lemma 2.

3. *Fractional integration.* There are several types of fractional integration; we shall need only one type here. The operation will be denoted by $\mathfrak{R}(\alpha, \beta: m)$ and when this acts on the function $f(x)$ it will be written as on the left of (20) below and defined by the expression on the right of (20).

$$\mathfrak{R}(\alpha, \beta: m) f(x) = \frac{m}{\Gamma(\alpha)} x^\beta \int_x^\infty (v^m - x^m)^{\alpha-1} v^{-\beta-m\alpha+m-1} f(v)\, dv. \tag{20}$$

The case $m = 1$ has been studied in detail by Kober ((3)) and the formula of (20) with $m > 0$ has been proposed by Erdelyi ((1)). There is no essential difference between the two cases $m = 1$ and $m > 0$ for on writing $v = u^m$, $x = z^m$ and $f(z^m) = F(z)$ it is easy to prove that

$$\mathfrak{R}(\alpha, \beta/m: 1) f(x) = \mathfrak{R}(\alpha, \beta: m) F(z). \tag{21}$$

Kober's proofs for the case $m = 1$ then apply, with perhaps some minor modifications, to the case $m > 0$. Hence, from (3), a set of sufficient conditions for the validity of (20) are

$$\alpha > 0, \quad \beta > -\tfrac{1}{2}, \quad m > 0 \quad \text{and} \quad f(x) \in L_2(0, \infty). \tag{22}$$

If these conditions are satisfied then we have $\mathfrak{R}(\alpha, \beta: m) f(x) \in L_2(0, \infty)$ also.

We require the following important formula. As in section 2, $\mathfrak{M}$ denotes the operation of forming the Mellin transform and then, if the conditions (22) hold, we have

$$\mathfrak{M}\{\mathfrak{R}(\alpha, \beta: m) f(x)\} = \frac{\Gamma[(s+\beta)/m]}{\Gamma[\alpha + (s+\beta)/m]} \mathfrak{M}\{f(x)\} \tag{23}$$

((3), page 203, (1), page 222).

When $\alpha = 0$ equation (23) becomes, after application of $\mathfrak{M}^{-1}$,

$$\mathfrak{R}(0, \beta: m) f(x) = f(x) \tag{24}$$

or symbolically $\mathfrak{R}(0, \beta: m) = 1$. (24) is a standard convention in fractional integration theory and is one which we shall use here.

We can now prove

LEMMA 3. *If (i) $\nu \geqslant \mu > 0$, (ii) $x > 0$ and (iii) $f(x) \in L_2(0, \infty)$ then*

$$\mathfrak{R}(\nu - \mu, 2\mu: 2) \int_0^\infty (xu)^\nu K_\nu(xu) f(u)\, du = 2^{\nu-\mu} \int_0^\infty (xu)^\mu K_\mu(xu) f(u)\, du, \tag{25}$$

where the integrals are considered as functions of x.

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Proof. From Lemma 2, section 2, the first integral of (25) belongs to $L_2(0,\infty)$ and so we may apply the operator $\mathfrak{K}$ to it, by (22). It also follows that the left-hand side of (25) belongs to $L_2(0,\infty)$ so that we may apply the operator $\mathfrak{M}$, by (6). Hence from (18), of Lemma 2, and (23) we have

$$\mathfrak{M}\left\{\mathfrak{K}(\nu-\mu, 2\mu, 2)\int_0^\infty (xu)^\nu K_\nu(xu) f(u)\,du\right\}$$

$$= \frac{\Gamma[\frac{1}{2}(s+2\mu)]}{\Gamma[\nu-\mu+\frac{1}{2}(s+2\mu)]}\, 2^{s+\nu-2}\,\Gamma(\tfrac{1}{2}s)\,\Gamma(\tfrac{1}{2}s+\nu)\,F(1-s) \quad (26)$$

$$= 2^{\nu-\mu}\,\mathfrak{M}\left\{\int_0^\infty (xu)^\mu K_\mu(xu) f(u)\,du\right\}, \quad (27)$$

after first cancelling out the factor $\Gamma(\frac{1}{2}s+\nu)$ from the right-hand side of (26) and then using (18), with ν replaced by μ.

From (17) and the remark following (22) the functions in (27) operated upon by $\mathfrak{M}$ both belong to $L_2(0,\infty)$. Hence, from the remark following (6), each side of (27) belongs to $L_2(\frac{1}{2}-i\infty, \frac{1}{2}+i\infty)$ and so, from (7) we may apply the operator $\mathfrak{M}^{-1}$ to (27). This application finally proves (25) and completes the proof of Lemma 3.

In the case when $\nu = \mu$ the convention expressed by (24) shows immediately that both sides of (25) are the same.

4. *The inversion of* (1). For convenience we use the following language. If

$$\int_0^\infty (xu)^\nu K_\nu(xu) f(u)\,du = g(x) \quad (28)$$

we say that $g(x)$ is the K_ν transform of $f(x)$. If

$$\int_0^\infty e^{-xu} f(u)\,du = h(x) \quad (29)$$

we shall say that $h(x)$ is the Laplace transform of $f(x)$, as is customary, and write $h(x) = \mathfrak{L}\{f(x)\}$ or, in some cases $f(x) = \mathfrak{L}^{-1}\{h(x)\}$, as equivalent to (29).

The Laplace transform has been much studied ((8)); for example, if $h(x)$ is given and $f(x)$ is to be determined then a solution of (29) is given by

$$f(x) = \mathfrak{L}^{-1}\{h(x)\} = \text{l.i.m.}_{T\to\infty} \frac{1}{2\pi i}\int_{-iT}^{iT} h(s)\,e^{xs}\,ds \quad (x \geqslant 0), \quad (30)$$

where the l.i.m. is with index 2. If $x < 0$ then the integral on the right of (30) vanishes. For conditions and proof of this theorem see (8), page 81. Since (29) can be solved for $f(x)$ a reduction of (28) to (29), which is our aim, will also enable us to solve (28).

THEOREM 1. *If* (*i*) $\nu \geqslant \frac{1}{2}$, (*ii*) $x > 0$ *and* (*iii*) $f(x)$ *is a solution of*

$$\int_0^\infty (xu)^\nu K_\nu(xu) f(u)\,du = g(x) \quad (31)$$

which belongs to $L_2(0,\infty)$ *then*

$$f(x) = 2^{1-\nu}\pi^{-\frac{1}{2}}\,\mathfrak{L}^{-1}\{\mathfrak{K}(\nu-\tfrac{1}{2}, 1:2)\,g(x)\}. \quad (32)$$

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Proof. From condition (*iii*) and from (17) we see that both sides of (31) belong to $L_2(0,\infty)$. Hence, by (22), we may apply the operator $\mathfrak{R}(\nu-\frac{1}{2}, 1:2)$ to (31) whence, from (25) with $\mu = \frac{1}{2}$, we obtain

$$2^{\nu-\frac{1}{2}}\int_0^\infty (xu)^{\frac{1}{2}} K_{\frac{1}{2}}(xu) f(u)\,du = \mathfrak{R}(\nu-\tfrac{1}{2}, 1:2)\, g(x). \tag{33}$$

But ((6), (13), page 80) $$K_{\frac{1}{2}}(z) = \pi^{\frac{1}{2}} 2^{-\frac{1}{2}} z^{-\frac{1}{2}} e^{-z}, \tag{34}$$

so that (33) may be rewritten in the form

$$\int_0^\infty e^{-xu} f(u)\,du = 2^{1-\nu}\pi^{-\frac{1}{2}}\,\mathfrak{R}(\nu-\tfrac{1}{2}, 1:2)\, g(x), \tag{35}$$

which is the same as (32) and so the theorem is established.

When $\nu = \frac{1}{2}$ equations (31) and (32) are identical, by (24). When $\nu > \frac{1}{2}$ the theorem enables us to find those solutions of (31), regarded as an integral equation for $f(x)$, which belong to $L_2(0,\infty)$.

THEOREM 2. *If* (*i*) $0 < \nu \leqslant \frac{1}{2}$, (*ii*) $x > 0$, (*iii*) $h(x) = \mathfrak{L}\{f(x)\}$ *and* (*iv*) $f(x) \in L_2(0,\infty)$ *then* $2^{\nu-1}\pi^{\frac{1}{2}}\,\mathfrak{R}(\frac{1}{2}-\nu, 2\nu:2)\, h(x)$ *is the* K_ν *transform of* $f(x)$.

Proof. From condition (*iii*) we have

$$\int_0^\infty e^{-xu} f(u)\,du = h(x). \tag{36}$$

From (34) we can rewrite (36) in the form

$$\int_0^\infty (xu)^{\frac{1}{2}} K_{\frac{1}{2}}(xu) f(u)\,du = 2^{-\frac{1}{2}}\pi^{\frac{1}{2}} h(x). \tag{37}$$

From (*iv*) and (17) both sides of (37) belong to $L_2(0,\infty)$. Hence, by (22), we may apply the operator $2^{\nu-1}\,\mathfrak{R}(\frac{1}{2}-\nu, 2\nu:2)$ to (37) and then use (25), Lemma 3. Some slight changes in (25), such as writing $\nu = \frac{1}{2}$ and then substituting ν for μ, give us finally

$$\int_0^\infty (xu)^\nu K_\nu(xu) f(u)\,du = 2^{\nu-1}\pi^{\frac{1}{2}}\,\mathfrak{R}(\tfrac{1}{2}-\nu, 2\nu:2)\, h(x), \tag{38}$$

which completes the proof of the theorem.

When $0 < \nu \leqslant \frac{1}{2}$ this theorem enables us to construct a table of K_ν transforms from a table of Laplace transforms. The reason why the cases $\nu \geqslant \frac{1}{2}$ and $0 < \nu \leqslant \frac{1}{2}$ must be treated differently is because in the operator $\mathfrak{R}(\alpha, b:m)$ we must have $\alpha \geqslant 0$.

5. *The inverse of the operator* $\mathfrak{R}(\alpha,\beta:m)$. In order to complete the theory of the K_ν transform we must find the inverse of the operator $\mathfrak{R}(\alpha,\beta:m)$. We shall denote this inverse operation by $\mathfrak{R}(\alpha,\beta:m)^{-1}$.

From (23) we may reasonably expect that

$$\mathfrak{M}\{\mathfrak{R}(\alpha,\beta:m)^{-1} f(x)\} = \frac{\Gamma[\alpha+(s+\beta)/m)]}{\Gamma[(s+\beta)/m]}\,\mathfrak{M}\{f(x)\}. \tag{39}$$

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If we use the formula of (23) again the ratio of the Gamma functions on the right of (39) can be accounted for by the operator $\mathfrak{R}(-\alpha, m\alpha+\beta: m)$ but, owing to the presence of $-\alpha$ with $\alpha > 0$, the definition of (20) would then be invalid. We must therefore obtain a meaning for $\mathfrak{R}(\alpha, \beta: m)^{-1}$ by other methods, when we shall have the extra advantage of obtaining a meaning not only for $\mathfrak{R}(\alpha, \beta: m)^{-1}$ but for $\mathfrak{R}(-\alpha, m\alpha+\beta: m)$ at the same time.

In order to find an interpretation for $\mathfrak{R}(\alpha, \beta: m)^{-1}$ we rewrite (39) in the following form. First, let n be a positive integer satisfying $n \geqslant \alpha > 0$, then (39) can be written as

$$\mathfrak{M}\{\mathfrak{R}(\alpha, \beta: m)^{-1} f(x)\} = \frac{\Gamma[\alpha+(s+\beta)/m]}{\Gamma[n+(s+\beta)/m]} \frac{\Gamma[n+(s+\beta)/m]}{\Gamma[(s+\beta)/m]} \mathfrak{M}\{f(x)\}. \tag{40}$$

By (23), the first Gamma-function ratio on the right of (40) can be accounted for by the operator $\mathfrak{R}(n-\alpha, m\alpha+\beta: m)$ which, since $n \geqslant \alpha$, comes within the definition (20). For the second Gamma function ratio in (40) let $\mathfrak{M}\{f(x)\} = F(s)$. Then proceeding formally we have, by (5),

$$f(x) = \frac{1}{2\pi i} \int_{\frac{1}{2}-i\infty}^{\frac{1}{2}+i\infty} F(s)\, x^{-s}\, ds. \tag{41}$$

If we can differentiate through the integral sign in (41) we then have

$$\begin{aligned} x^{\beta+mn} \left(\frac{1}{x^{m-1}} \frac{d}{dx}\right)^n \{x^{-\beta} f(x)\} \\ &= \frac{1}{2\pi i} \int_{\frac{1}{2}-i\infty}^{\frac{1}{2}+i\infty} [-s-\beta-m(n-1)] \dots [-s-\beta-m][-s-\beta] F(s)\, x^{-s}\, ds \\ &= (-1)^n m^n \frac{1}{2\pi i} \int_{\frac{1}{2}-i\infty}^{\frac{1}{2}+i\infty} \frac{\Gamma[n+(s+\beta)/m]}{\Gamma[(s+\beta)/m]} F(s)\, x^{-s}\, ds. \end{aligned} \tag{42}$$

Hence (42) can be written in the form

$$\mathfrak{M}\left\{\frac{(-1)^n}{m^n} x^{\beta+mn} \left(\frac{1}{x^{m-1}} \frac{d}{dx}\right)^n [x^{-\beta} f(x)]\right\} = \frac{\Gamma[n+(s+\beta)/m]}{\Gamma[(s+\beta)/m]} \mathfrak{M}\{f(x)\}. \tag{43}$$

On combining (40) with the $\mathfrak{R}$ operator which accounts for the first Gamma function ratio of (40), namely $\mathfrak{R}(n-\alpha, m\alpha+\beta: m)$ and in addition using (43) we may then write

$$\mathfrak{R}(\alpha, \beta: m)^{-1} f(x) = \mathfrak{R}(n-\alpha, m\alpha+\beta: m) \left[\frac{(-1)^n}{m^n} x^{\beta+mn} \left(\frac{1}{x^{m-1}} \frac{d}{dx}\right)^n \{x^{-\beta} f(x)\}\right], \tag{44}$$

where n is a positive integer and $n \geqslant \alpha > 0$. We shall take (44) to be the definition of the operator inverse to $\mathfrak{R}(\alpha, \beta: m)$. If we wish to use (39) we must first show that the integral in (41) can be differentiated, with respect to x, through the integral sign.

In the special case $\alpha = n$ (a positive integer), $\beta = 0$ and $m = 1$ it is easy to verify that (44) gives the operation inverse to $\mathfrak{R}(n, 0: 1)$ as defined by (20).

As stated above, (44) gives us a meaning for operators $\mathfrak{R}(\alpha_1, \beta_1: m_1)$ when α_1 is negative. Thus we may take the right-hand side of (44) to be equal to $\mathfrak{R}(-\alpha, m\alpha+\beta: m)$ where n is a positive integer and $n > \alpha > 0$.

For a discussion of the problem above for the case $m = 1$ see (2), page 688.

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6. *Some preliminary lemmas.*

LEMMA 4. *If n is a positive integer then*

$$\frac{(-1)^n}{2^n} x^{2\mu+2n} \left(\frac{1}{x}\frac{d}{dx}\right)^n \{x^{-2\mu}(xu)^\mu K_\mu(xu)\} = 2^{-n}(xu)^{\mu+n} K_{\mu+n}(xu). \tag{45}$$

Proof. From (6), (6), page 79, we have

$$\left(\frac{1}{z}\frac{d}{dz}\right)^n \left\{\frac{K_\mu(z)}{z^\mu}\right\} = (-1)^n \frac{K_{\mu+n}(z)}{z^{\mu+n}}. \tag{46}$$

(45) follows from (46) on writing $z = xu$ and regarding x as the variable for the differentiations.

LEMMA 5. *If (i) n is a positive integer, (ii) $\mu > 0$, (iii) $x \geqslant \delta > 0$ and (iv) $f(x) \in L_2(0, \infty)$ then*

$$(-2)^{-n} x^{2\mu+2n} \left(\frac{1}{x}\frac{d}{dx}\right)^n \left\{x^{-2\mu}\int_0^\infty (xu)^\mu K_\mu(xu) f(u)\, du\right\} = 2^{-n}\int_0^\infty (xu)^{\mu+n} K_{\mu+n}(xu) f(u)\, du. \tag{47}$$

Proof. If the order of differentiation and integration on the left of (47) can be interchanged then (47) would follow immediately from (45). Now at each stage of this differentiation we have to differentiate expressions of the type $x^{-a}I$ where $a > 0$, I is given by

$$I = \int_0^\infty (xu)^{\mu+r} K_{\mu+r}(xu) f(u)\, du \tag{48}$$

and r is a positive integer. Consequently the validity of the desired interchange of order will be established if we can prove that I is uniformly convergent with respect to $x > \delta > 0$.

Let J denote the integrand of (48) then, from Schwartz's inequality ((5), page 381), we have

$$\left|\int_N^\infty J\, du\right|^2 \leqslant \int_N^\infty |(xu)^{\mu+r} K_{\mu+r}(ux)|^2\, du \int_N^\infty |f(u)|^2\, du \tag{49}$$

$$\leqslant x^{-1}\int_N^\infty |v^{\mu+r} K_{\mu+r}(v)|^2\, dv \int_N^\infty |f(u)|^2\, du. \tag{50}$$

From (2) and (3) $z^\nu K_\nu(z)$ is an entire function finite at the origin and so, from the asymptotic expansion (19), it follows that the first integral on the right of (50) converges and is bounded for all positive (and zero) values of N and x. Let B denote the upper bound of this integral. Since $f(x) \in L_2(0, \infty)$, given $\epsilon\,(> 0)$ we can choose N_0 independent of x $(\geqslant \delta > 0)$ so that

$$\int_N^\infty |f(u)|^2\, du < \delta\epsilon^2/B \quad (N > N_0). \tag{51}$$

Hence

$$\left|\int_N^\infty J\, du\right| < \epsilon \quad (N > N_0), \tag{52}$$

where N_0 is independent of x $(\geqslant \delta > 0)$. This establishes the uniform convergence of (48) with respect to x and so justifies the change in the order of differentiation and

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integration in (47). Equation (47) now follows immediately from (45) and the proof of the lemma is complete.

LEMMA 6. *If* (*i*) $\nu \geqslant \mu > 0$, (*ii*) $x > \delta > 0$ *and* (*iii*) $f(x) \in L_2(0, \infty)$ *then*

$$\mathfrak{R}(\nu-\mu, 2\mu: 2)^{-1}\int_0^\infty (xu)^\mu K_\mu(xu) f(u)\, du = 2^{\mu-\nu}\int_0^\infty (xu)^\nu K_\nu(xu) f(u)\, du. \tag{53}$$

Proof. We first apply the operator $\mathfrak{R}(n-\nu+\mu, 2\nu: 2)$ to (47) and then use the definition of $\mathfrak{R}(\nu-\mu, 2\mu: 2)^{-1}$ given by (44).

From condition (*iii*) and (17) both sides of (47) belong to $L_2(0, \infty)$ and so we may apply the operator $\mathfrak{R}(\alpha, \beta: m)$ to (47) if $\alpha \geqslant 0$. In (44) write $\alpha = \nu-\mu$, $\beta = 2\mu$, $m = 2$ and therefore $m\alpha+\beta = 2\nu$. Also choose n to be any positive integer satisfying $n \geqslant \nu-\mu$ and then apply the operator $\mathfrak{R}(n-\nu+\mu, 2\nu: 2)$ to (47). By (44) the result is

$$\mathfrak{R}(\nu-\mu, 2\mu: 2)^{-1}\int_0^\infty (xu)^\mu K_\mu(xu) f(u)\, du$$
$$= 2^{-n}\mathfrak{R}(n-\nu+\mu, 2\nu: 2)\int_0^\infty (xu)^{\mu+n} K_{\mu+n}(xu) f(u)\, du. \tag{54}$$

Since all the conditions of Lemma 3, section 3, are satisfied we may apply (25) to the right-hand side of (54) and this application shows that the right-hand side of (54) is equal to the right-hand side of (53), so completing the proof of the lemma.

7. *The inversion of* (1) *continued.* Using the language of section 4 we can now prove

THEOREM 3. *If* (*i*) $\nu \geqslant \frac{1}{2}$, (*ii*) $x > 0$, (*iii*) $h(x) = \mathfrak{Q}\{f(x)\}$ *and* $\mathfrak{R}(\nu-\frac{1}{2}, 1: 2)^{-1}h(x)$ *exists and* (*iv*) $f(x) \in L_2(0, \infty)$ *then* $2^{\nu-1}\pi^{\frac{1}{2}}\mathfrak{R}(\nu-\frac{1}{2}, 1: 2)^{-1}h(x)$ *is the* K_ν *transform of* $f(x)$.

Condition (*iii*) assumes that $h(x)$ is differentiable and can be operated upon by the formula on the right of (44).

Proof. Condition (*iii*) is equivalent to the statement that

$$\int_0^\infty e^{-xu} f(u)\, du = h(x) \tag{55}$$

and from (34) this may be rewritten in the form

$$\int_0^\infty (xu)^{\frac{1}{2}} K_{\frac{1}{2}}(xu) f(u)\, du = 2^{-\frac{1}{2}}\pi^{\frac{1}{2}}h(x). \tag{56}$$

From (iv) and (17) both sides of (56) belong to $L_2(0, \infty)$.

We now apply the operator $\mathfrak{R}(\nu-\frac{1}{2}, 1: 2)^{-1}$ to (56) and then use (53), of Lemma 6, with $\mu = \frac{1}{2}$ and $\nu \geqslant \frac{1}{2}$. The result is

$$2^{\frac{1}{2}-\nu}\int_0^\infty (xu)^\nu K_\nu(xu) f(u)\, du = 2^{-\frac{1}{2}}\pi^{\frac{1}{2}}\mathfrak{R}(\nu-\tfrac{1}{2}, 1: 2)^{-1}h(x). \tag{57}$$

To complete the proof of the theorem we divide (57) by $2^{\frac{1}{2}-\nu}$.

This theorem, which is the converse of Theorem 1, enables us to construct tables of K_ν transforms from tables of Laplace transforms when $\nu > \frac{1}{2}$.

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THEOREM 4. *If* (*i*) $0 < \nu \leqslant \frac{1}{2}$, (*ii*) $x > 0$, (*iii*) $\mathfrak{Q}^{-1}\{\mathfrak{R}(\frac{1}{2}-\nu, 2\nu: 2)^{-1} g(x)\}$ *exists and* (*iv*) $f(x)$ *is a solution of*

$$\int_0^\infty (xu)^\nu K_\nu(xu) f(u)\, du = g(x) \tag{58}$$

which belongs to $L_2(0,\infty)$ *then*

$$f(x) = 2^{1-\nu}\pi^{-\frac{1}{2}}\mathfrak{Q}^{-1}\{\mathfrak{R}(\tfrac{1}{2}-\nu, 2\nu: 2)^{-1} g(x)\}. \tag{59}$$

Proof. The conditions allow us to apply the operator $\mathfrak{R}(\frac{1}{2}-\nu, 2\nu: 2)^{-1}$ to (58) and then to apply (53), Lemma 6, with $\nu = \frac{1}{2}$ inserted in equation (53). The result is

$$2^{\nu-\frac{1}{2}}\int_0^\infty (xu)^{\frac{1}{2}} K_{\frac{1}{2}}(xu) f(u)\, du = \mathfrak{R}(\tfrac{1}{2}-\nu, 2\nu: 2)^{-1} g(x). \tag{60}$$

On using (34) it is evident that (60) is in the form of a Laplace transform and, in fact, that it is the Laplace transform equivalent to (59).

This completes the proof of Theorem 4 and (59) gives us that solution of (58) which belongs to $L_2(0,\infty)$.

8. *A generalization of the* K_ν *kernel.* In this section we consider very briefly the integral equation

$$\int_0^\infty H(xu) f(u)\, du = g(x), \tag{61}$$

where $f(x)$ is to be determined, $g(x)$ is given and

$$H(z) = \frac{1}{2\pi i}\int_{\frac{1}{2}-i\infty}^{\frac{1}{2}+i\infty} \prod_{i=1}^{n} \Gamma[(s+\nu_i)/n]\, z^{-s}\, ds. \tag{62}$$

For simplicity we assume that $\nu_i \neq \nu_j$, $i \neq j$ and that $\nu_i \geqslant 0\,(i = 1, 2, \ldots, n)$. When $n = 2$, $\nu_1 = 0$ and $\nu_2 = 2\nu$ we have $H(z) = 4z^\nu K_\nu(2z)$.

We shall show briefly that a formal solution of (61) is given by

$$f(x) = n^{-\frac{1}{2}}(2\pi)^{\frac{1}{2}(1-n)}\mathfrak{Q}^{-1}\left\{\prod_{i=1}^{n} \mathfrak{R}((\nu_i+1-i)/n, i-1: n)\, g(x/n)\right\}. \tag{63}$$

From (4), (5) and (62) we have

$$\mathfrak{M}\{H(z)\} = \prod_{i=1}^{n} \Gamma[(s+\nu_i)/n]. \tag{64}$$

By (23) if the operator $\mathfrak{R}((\nu_i+1-i)/n, i-1: n)$ acts on $H(z)$ the result is to substitute the factor $\Gamma[(s+i-1)/n]$ for the factor $\Gamma[(s+\nu_i)/n]$ in $\mathfrak{M}\{H(z)\}$. If $\nu_i \geqslant i-1$ we use the definition (20) for the operator $\mathfrak{R}$ but if $\nu_i < i-1$ then we must use the definition of (44).

On applying a succession of such operators we have

$$\begin{aligned}\mathfrak{M}\left\{\prod_{i=1}^{n} \mathfrak{R}((\nu_i+1-i)/n, i-1: n)\, H(z)\right\} &= \prod_{i=1}^{n} \Gamma[(s+i-1)/n] \\ &= n^{\frac{1}{2}-s}(2\pi)^{\frac{1}{2}(n-1)}\Gamma(s) \\ &= n^{\frac{1}{2}}(2\pi)^{\frac{1}{2}(n-1)}\mathfrak{M}\{e^{-nx}\},\end{aligned} \tag{65}$$

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by (7), page 234 and (4), page 190. Hence

$$\prod_{i=1}^{n} \mathfrak{R}((\nu_i + 1 - i)/n, i - 1 : n)\, H(z) = n^{\frac{1}{2}}(2\pi)^{\frac{1}{2}(n-1)} e^{-nz}. \tag{66}$$

The operators $\mathfrak{R}$ are commutative and we assume that they are also commutative with integration. On applying the succession of $\mathfrak{R}$ operators of (66) to (61) we find that

$$n^{\frac{1}{2}}(2\pi)^{\frac{1}{2}(n-1)} \int_0^\infty e^{-nxu} f(u)\, du = \prod_{i=1}^{n} \mathfrak{R}((\nu_i + 1 - i)/n,\ i - 1 : n)\, g(x). \tag{67}$$

From this it follows that (63) is a formal solution of (61).

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