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OPTIMAL BANDWIDTH SELECTION FOR ROBUST GENERALIZED METHOD OF MOMENTS ESTIMATION

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A two-step generalized method of moments estimation procedure can be made robust to heteroskedasticity and autocorrelation in the data by using a nonparametric estimator of the optimal weighting matrix. This paper addresses the issue of choosing the corresponding smoothing parameter (or bandwidth) so that the resulting point estimate is optimal in a certain sense. We derive an asymptotically optimal bandwidth that minimizes a higher-order approximation to the asymptotic mean-squared error of the estimator of interest. We show that the optimal bandwidth is of the same order as the one minimizing the mean-squared error of the nonparametric plugin estimator, but the constants of proportionality are significantly different. Finally, we develop a data-driven bandwidth selection rule and show, in a simulation experiment, that it may substantially reduce the estimator's mean-squared error relative to existing bandwidth choices, especially when the number of moment conditions is large.

1. INTRODUCTION

Since the seminal paper by Hansen (1982) the generalized method of moments (GMM) has become a popular method for the estimation of partially specified models based on moment conditions. It is well known (Sargan, 1958, 1959; Hansen, 1982) that a careful choice of the weighting matrix in the GMM criterion function leads to an efficient estimator. A particular version of optimal GMM, the two-step GMM estimator, enjoys efficiency even in the presence of heteroskedasticity and autocorrelation (HAC) in the data-generating process by employing a nonparametric plugin estimator of the optimal weighting matrix. This paper develops a data-driven selection rule for the corresponding smoothing parameter of the nonparametric estimator such that the resulting point estimator minimizes a suitably defined mean-squared error (MSE) criterion.

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Many instances of poor finite sample performance of GMM estimators have been reported in the literature. See for example, Hansen, Heaton, and Yaron (1996) and references therein. As an attempt to improve the properties different extensions and new estimators have been proposed, e.g., the empirical likelihood estimator introduced by Owen (1988) and Qin and Lawless (1994), the exponential tilting estimator of Kitamura and Stutzer (1997) and Imbens, Spady, and Johnson (1998), and the continuous updating estimator by Hansen, Heaton, and Yaron (1996). Newey and Smith (2004) show that all these estimators are members of a larger class of generalized empirical likelihood (GEL) estimators. A different approach termed "fixed-b" asymptotics is based on deriving more accurate approximations of estimators and test statistics based on an asymptotic sequence in which the HAC smoothing parameter tends to infinity at the same rate as the sample size (see for example, Kiefer and Vogelsang, 2002a, 2002b, 2005). Instead of treating the smoothing parameter as proportional to the sample size, Sun and Phillips (2008) and Sun, Phillips, and Jin (2008) develop a higher-order asymptotic theory based on which they find the optimal rate at which the smoothing parameter (here a bandwidth) minimizes the coverage probability error or length of confidence intervals.

Similar in spirit, the present paper derives the optimal growth rate of the bandwidth to minimize an asymptotic mean-squared error (AMSE) criterion. We approximate the MSE of the second-step GMM estimator by the MSE of the first few terms in a stochastic expansion. Since the proposed semiparametric estimator is first-order equivalent to ordinary GMM estimators in the iid (independent and identically distributed) case, the optimal bandwidth derived in this paper will minimize its second-order effects on the estimator and lead to second-order efficiency gains. In an unpublished dissertation, Jun (2007) independently develops a similar expansion and arrives at the same MSE-optimal bandwidth as derived in this paper, however under a slightly different set of assumptions.[1]

Other bandwidth choices for HAC-robust estimation have been suggested by Andrews (1991), Newey and West (1994), and Andrews and Monahan (1992), for example, and are very popular in applied research. In this paper, we show that these are suboptimal choices if MSE-optimal point estimation is of main interest. In finite samples, the existing methods can select bandwidths that are significantly different from the MSE-optimal bandwidth even though they share the same asymptotic order relative to the sample size. The difference is due to the other methods minimizing the AMSE of the weighing matrix estimator instead of minimizing the AMSE of the GMM estimator itself; they guarantee accurate estimates of the optimal weighting matrix, but not necessarily of the parameter of interest.

In the linear regression framework with potential autocorrelation and heteroskedasticity, there are several papers (e.g., Robinson, 1991; Xiao and Phillips, 1998; Tamaki, 2007) that derive higher-order expansions of the MSE of semiparametric frequency domain estimators to determine an optimal bandwidth that minimizes higher-order terms of such expansions. In the present paper, however,

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we allow for nonlinear models and overidentification which significantly complicate the problem and require a different set of tools to derive such expansions.

To approximate higher-order moments of the GMM estimator we develop a stochastic expansion of the estimator similar to the approach in Nagar (1959). See Rothenberg (1984) for an introduction to Nagar-type expansions and for further references. Several other authors have analyzed higher-order properties of GMM and GEL estimators using similar tools. Rilstone, Srivastava, and Ullah (1996) and Newey and Smith (2004) provide expressions for the higher-order bias and variance of GMM and GEL estimators when the data are iid. Anatolyev (2005) derives the higher-order bias in the presence of serial correlation.

Finally, Goldstein and Messer (1992) present general conditions under which functionals of nonparametric plugin estimators achieve the optimal rate of convergence. Depending on the functional undersmoothing the plugin estimator relative to the smoothing parameter used to optimally estimate the nonparametric quantity itself may be necessary.

The paper is organized as follows. The first section introduces the econometric setup and derives a higher-order expansion of the two-step GMM estimator and the optimal bandwidth that minimizes an approximate MSE based on that expansion. The third section describes an approach to estimate the infeasible optimal bandwidths, followed by a simulation experiment that demonstrates the procedure's performance in finite samples. The paper concludes with an appendix containing all mathematical proofs.

Let $\text{vec}(\cdot)$ denote the column-by-column stacking operation and $\text{vech}(\cdot)$ the column-by-column stacking operation of entries on and above the diagonal of a symmetric matrix. By $K_{m,n}$ denote the $mn \times mn$ commutation matrix so that, for any $m \times n$ matrix M, $K_{m,n}\text{vec}(M) = \text{vec}(M')$. Let $\otimes$ be the Kronecker product and $\nabla^r F(\beta)$, with $r \in \mathbb{Z}$ and $F(\beta)$ a matrix being r times differentiable in β, denote the matrix of r-th order partial derivatives with respect to β, recursively defined as in Rilstone, Srivastava, and Ullah (1996). $\nabla^0 F(\beta) := F(\beta)$. This notation for derivatives will sometimes be used to save space and simplify notation. $\|\cdot\|$ denotes the euclidean (matrix) norm and M' the transpose of a matrix M, "with probability approaching one" is abbreviated "w.p.a. 1" and "with probability one" by "w.p. 1". The notation $x_T = O_p(1)$ means that the sequence $\{x_T\}_{T=1}^{\infty}$ is uniformly tight.

2. OPTIMAL BANDWIDTH

In this section, we introduce the basic framework, define an appropriate MSE criterion, and find the optimal bandwidth that minimizes it. The idea is to derive a higher-order approximation of the second-step estimator and to compute the MSE from the moments of this approximation. A higher-order analysis is required in this setup because first-order asymptotics does not depend on the smoothing parameter.

Consider estimation of a parameter $\beta_0 \in \mathcal{B}$ from the moment equation $Eg(X_t, \beta_0) = 0$ given a data sample $\{x_t\}_{t=1}^{T}$. If the dimension of the range of

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g is at least as large as the dimension of the parameter β_0, then a popular estimator of β_0 is the two-step GMM estimator defined as follows. First, estimate β_0 by some $\sqrt{T}$-consistent estimator, say $\tilde{\beta}$, that is then used to construct a consistent estimator $\hat{\Omega}(\tilde{\beta})$ of the long-run variance $\Omega_0 := \sum_{s=-\infty}^{\infty} \Gamma(s)$, $\Gamma(s) := E[g(X_{t+s}, \beta_0)g(X_t, \beta_0)']$. In a second step, compute the GMM estimator $\hat{\beta}$ of β_0 with weighting matrix $\hat{\Omega}(\tilde{\beta})^{-1}$, viz.

$$\hat{\beta} := \arg\min_{\beta \in \mathcal{B}} \hat{g}(\beta)' \hat{\Omega}_T(\tilde{\beta})^{-1} \hat{g}(\beta), \tag{1}$$

where $\hat{g}(\beta) := T^{-1} \sum_{t=1}^{T} g(x_t, \beta)$. The second step improves the first-step estimator in terms of efficiency. In fact, $\hat{\beta}$ is optimal in the sense that it achieves the lowest asymptotic variance among all estimators of the form $\hat{\beta}_W := \arg\min_{\beta \in \mathcal{B}} \hat{g}(\beta)' W \hat{g}(\beta)$ for some positive definite weighting matrix W (see Hansen, 1982).

In the special case of an iid process $\{X_t\}$, Ω_0 collapses to $\Omega_0 = E[g(X_t, \beta_0) g(X_t, \beta_0)']$ and can simply be estimated by its sample analog $\hat{\Omega}_T(\tilde{\beta}) := T^{-1} \sum_{t=1}^{T} g(x_t, \tilde{\beta}) g(x_t, \tilde{\beta})'$. When the iid assumption is not justified, one can perform inference robust to autocorrelated and/or heteroskedastic X_t processes. Robustness here means that potential dependence and heteroskedasticity are treated nonparametrically and one does not have to be explicit about the data-generating process of the X_t's. To that end, one needs to "smooth" the observations $g(x_{t+s}, \tilde{\beta}) g(x_t, \tilde{\beta})'$ over s to ensure that $\hat{\Omega}_T(\tilde{\beta})$ is consistent. In this paper, we use a nonparametric kernel estimator of the form

$$\hat{\Omega}_T(\tilde{\beta}) := \frac{1}{T} \sum_{s=1-T}^{T-1} \sum_{t=\max\{1,1-s\}}^{\min\{T,T-s\}} k\left(\frac{s}{S_T}\right) g_{t+s}(\tilde{\beta}) g_t(\tilde{\beta})'$$

with $g_t(\beta) := g(x_t, \beta)$ and k a kernel function. S_T, with $S_T \to \infty$ as $T \to \infty$, is a so-called bandwidth parameter that governs the degree of smoothing. Andrews (1991) derives a range of rates (in terms of T) at which S_T is allowed to diverge in order to guarantee consistency of $\hat{\Omega}_T(\tilde{\beta})$. These conditions, however, do not suggest rules for choosing S_T for a fixed sample size T. Small values of S_T imply averaging over only a few observations which decreases the variability of the estimator $\hat{\Omega}_T(\tilde{\beta})$, but increases its bias. On the other hand, large bandwidths yield inclusion of more distant lags in the above sum, thereby increasing the variance, but decreasing the bias of the estimator. Below we show that the choice of S_T affects the bias and variance of the second-step estimator $\hat{\beta}$ in a similar way. This trade-off can be used to derive decision rules on how to pick S_T in finite samples. For example, Andrews (1991) derives the optimal bandwidth minimizing a truncated AMSE criterion that balances bias and variance of $\hat{\Omega}_T(\tilde{\beta})$, thereby guaranteeing "good" properties of the estimator of the optimal weighting matrix. However, in the GMM estimation framework, the second-step estimator $\hat{\beta}$ is the quantity of interest and, thus, the bandwidth should be chosen so as to take into

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account the bias and variance trade-off of $\hat{\beta}$, rather than that of $\hat{\Omega}_T(\tilde{\beta})$. To that end the subsequent analysis develops a higher-order expansion of the MSE of the second-step estimator and then minimizes the leading terms with respect to the bandwidth.

Assumption 1. (a) The process $\{X_t\}_{t=-\infty}^{\infty}$ taking values in $\mathcal{X} \subset \mathbb{R}^m$ is fourth-order stationary and α-mixing with mixing coefficients $\alpha(j)$ satisfying $\sum_{j=1}^{\infty} j^2 \alpha(j)^{(\nu-1)/\nu} < \infty$ for some $\nu > 1$. **(b)** $\{x_t\}_{t=1}^{T}$ is an observed sample of $\{X_t\}_{t=-\infty}^{\infty}$. **(c)** $h(\cdot,\beta) := (g(\cdot,\beta)', \text{vec}(\nabla g(\cdot,\beta) - E\nabla g(\cdot,\beta))', \text{vec}(\nabla^2 g(\cdot,\beta) - E\nabla^2 g(\cdot,\beta))')'$ is a measurable function for every $\beta \in \mathcal{B}$. **(d)** $\sup_{t\geq 1} E[\|h(x_t,\beta_0)\|^{4\nu}] < \infty$. **(e)** $\sup_{t\geq 1} E[\sup_{\beta\in\mathcal{B}} \|\nabla^k g(x_t,\beta)\|^2] < \infty$ for $k = 1, 2, 3$. **(f)** There is a first-step estimator $\tilde{\beta}$ satisfying $\tilde{\beta} - \beta_0 = O_p(T^{-1/2})$.

As a slight abuse of notation, in the remainder, x_t represents the random variable X_t as well as the observation x_t, but the distinction should be clear from the context. Furthermore, dropping β as an argument of a function means that the function is evaluated at β_0, e.g., $\hat{\Omega}_T := \hat{\Omega}_T(\beta_0)$ or $g_t := g_t(\beta_0)$. Let $G_0 := EG(x_t)$, $G(x_t,\beta) := \partial g_t(\beta)/\partial\beta'$, $G_t(\beta) := G(x_t,\beta)$, and the sample counterpart $G_T(\beta) := T^{-1}\sum_{t=1}^{T} \partial g_t(\beta)/\partial\beta'$.

Following Parzen (1957), let q be the characteristic exponent that characterizes the smoothness of the kernel k at zero: $q := \max\{\alpha \in [0,\infty) : g_\alpha \text{ exists and } 0 < |g_\alpha| < \infty\}$ with $g_\alpha := \lim_{z\to 0} \frac{1-k(z)}{|z|^\alpha}$. For example, for the Bartlett, Parzen, and Tukey–Hanning kernel the values of q are 1, 2, and 2, respectively.

Assumption 2. Let the kernel k satisfy the following conditions: **(a)** $k : \mathbb{R} \to [-1,1]$ satisfies $k(0) = 1$, $k(x) = k(-x)\ \forall x \in \mathbb{R}$, $\int_{-\infty}^{\infty} k^2(x)dx < \infty$, $\int_{-\infty}^{\infty} |k(x)|dx < \infty$, $k(\cdot)$ is continuous at 0 and at all but a finite number of other points, and $S_T \to \infty$, $S_T^2/T \to 0$, $S_T^q/T \to 0$ for some $q \in [0,\infty)$ for which $g_q, \|f^{(q)}\| \in [0,\infty)$ where $f^{(q)} := \frac{1}{2\pi}\sum_{j=-\infty}^{\infty} |j|^q \Gamma(j)$. **(b)** $\int_{-\infty}^{\infty} \bar{k}(x)dx < \infty$ with

$$\bar{k}(x) := \begin{cases} \sup_{y\geq x} |k(y)|, & x \geq 0 \\ \sup_{y\leq x} |k(y)|, & x < 0 \end{cases}.$$

Assumptions 1 and 2(a) imply Assumptions A, B, and C in Andrews (1991) applied to $\{g_t\}$ and $\{G_t\}$, allowing us to use his consistency and rate of convergence results for HAC estimators. The necessity of Assumption 2(b) is explained in Jansson (2002).

Assumption 3. (a) $g : \mathcal{X}\times\mathcal{B} \to \mathbb{R}^l$, $l \geq p$, and $\beta_0 \in int(\mathcal{B})$ is the unique solution to $Eg(x_t,\beta_0) = 0$, $\mathcal{B} \subset \mathbb{R}^p$ is compact. **(b)** $rank(G_0) = p$. **(c)** For any $x \in \mathcal{X}$, $g(x,\cdot)$ is twice continuously differentiable in a neighborhood $\mathcal{N}$ of β_0. **(d)** There exists a function $d : \mathcal{X}\times\mathcal{X} \to \mathbb{R}$ with $\sum_{s=-\infty}^{\infty} Ed(x_{t+s},x_t) < \infty$ so that g satisfies the condition $\|\nabla(g_{t+s}(\beta)g_t(\beta)') - \nabla(g_{t+s}(\beta_0)g_t(\beta_0)')\| \leq d(x_{t+s},x_t)$ w.p. 1 for $\beta \in \mathcal{N}$. **(e)** There exists a function $b : \mathcal{X} \to \mathbb{R}$ with $Eb(x_t) < \infty$ such that $\|\nabla^k g(x,\beta) - \nabla^k g(x,\beta_0)\| \leq b(x)\|\beta - \beta_0\|$ for $k = 2, 3$. **(f)** Ω_0 is positive definite.

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The following proposition is the first main result of the paper, presenting an expansion of the second-step GMM estimator $\hat{\beta}$ up to the lowest orders involving the bandwidth S_T. This approximation constitutes a crucial ingredient for the computation of the optimal bandwidth.

PROPOSITION 1. *Under Assumptions 1–3, $\hat{\beta}$ satisfies the stochastic expansion*

$$\hat{\beta} = \beta_0 + \kappa_{1,T} T^{-1/2} + \kappa_{2,T} S_T^{1/2} T^{-1} + \kappa_{3,T} S_T^{-q} T^{-1/2} + o_p(\eta_T T^{-1/2}) \tag{2}$$

with $\eta_T := S_T^{1/2} T^{-1/2} + S_T^{-q}$, $\kappa_{i,T} = O_p(1)$ *for* $i = 1, 2, 3$,

$$\kappa_{1,T} := -H_0 F_T$$

$$\kappa_{2,T} := H_0 \sqrt{T/S_T} \left(\hat{\Omega}_T - \overline{\Omega}_T \right) P_0 F_T$$

$$\kappa_{3,T} := H_0 S_T^q \left(\overline{\Omega}_T - \Omega_0 \right) P_0 F_T$$

and $F_T(\beta) := \sqrt{T}\hat{g}(\beta)$, $\quad F_T := F_T(\beta_0)$, $\overline{\Omega}_T := E\hat{\Omega}_T$, $\Sigma_0 := (G_0' \Omega_0^{-1} G_0)^{-1}$, $H_0 := \Sigma_0 G_0' \Omega_0^{-1}$, *and* $P_0 := \Omega_0^{-1} - \Omega_0^{-1} G_0 H_0$.

Since the lowest-order term, $-H_0 F_T T^{-1/2}$, does not depend on the bandwidth, expansion (2) illustrates the well-known fact that nonparametric estimation of the GMM weighting matrix does not affect first-order asymptotics as long as that nonparametric estimator is consistent. The other two terms in the expansion involve the bandwidth and arise from the bias ($S_T^q(\overline{\Omega}_T - \Omega_0)$) and variance ($\sqrt{T/S_T}(\hat{\Omega}_T - \overline{\Omega}_T)$) of the nonparametric estimator of the weighting matrix. In a similar expansion to (2) for the iid case, these two components do not appear and the next higher-order term after $\kappa_{1,T} T^{-1/2}$ is of order T^{-1} (see Newey and Smith, 2004), which here is part of the remainder and plays no role in determining the optimal bandwidth derived below.

Anatolyev (2005) does not explicitly present a stochastic expansion such as (2), but computes the higher-order bias B_T of $\hat{\beta}$ which turns out to be of order T^{-1}, i.e., $E[\hat{\beta} - \beta_0] = B_T T^{-1} + o(T^{-1})$, and therefore does not depend on the bandwidth. Interestingly, one can show that the two higher-order terms in (2) do not contribute to that bias (Jun, 2007).

In the class of GMM estimators defined by (1) and indexed by the bandwidth S_T, we now characterize the most efficient one under quadratic loss. Specifically, we rank estimators according to the MSE of the approximation $\zeta_T := \beta_0 + \kappa_{1,T} T^{-1/2} + \kappa_{2,T} S_T^{1/2} T^{-1} + \kappa_{3,T} S_T^{-q} T^{-1/2}$.

THEOREM 1. *Suppose Assumptions 1–3 hold. Let $\mathcal{W} \in \mathbb{R}^{p \times p}$ be a weighting matrix. Define the weighted MSE $MSE_T := E[(\zeta_T - \beta_0)' \mathcal{W} (\zeta_T - \beta_0)]$. Then*

$$MSE_T = \nu_1 T^{-1} + \nu_2 S_T T^{-2} + \nu_3 S_T^{-2q} T^{-1} + o(\eta_T^2 T^{-1})$$

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with

$$\nu_1 := \sum_{s=-\infty}^{\infty} E\left[g_t' H_0' \mathcal{W} H_0 g_{t+s}\right]$$

$$\nu_2 := \lim_{T\to\infty} \frac{T}{S_T} E\left[F_T' P_0' (\hat{\Omega}_T - \overline{\Omega}_T) H_0' \mathcal{W} H_0 (\hat{\Omega}_T - \overline{\Omega}_T) P_0 F_T\right]$$

$$-2 \lim_{T\to\infty} \frac{T}{S_T} E\left[F_T' P_0' (\hat{\Omega}_T - \overline{\Omega}_T) H_0' \mathcal{W} H_0 F_T\right]$$

$$\nu_3 := \lim_{T\to\infty} S_T^{2q} E\left[F_T' P_0' (\overline{\Omega}_T - \Omega_0) H_0' \mathcal{W} H_0 (\overline{\Omega}_T - \Omega_0) P_0 F_T\right],$$

where all the limits exist and are finite.

As explained above, the bias of the approximation ζ_T is zero so that MSE expansion (1) represents only the variance of ζ_T. Despite the lack of a bias component of ζ_T, the expansion displays the first-order tradeoff that is relevant for choosing a bandwidth: $\nu_2 S_T T^{-2}$ increases in S_T and $\nu_3 S_T^{-2q} T^{-1}$ decreases in S_T. The terms have the standard order of squared bias and variance of HAC estimators as derived in Andrews (1991).

Under additional conditions, the moments of the approximation ζ_T correspond to the moments of a formal Edgeworth expansion of the cumulative distribution function (cdf) of the second-step estimator $\hat{\beta}$. The (finite) moments of such an Edgeworth expansion can be used to approximate the distribution of $\hat{\beta}$ up to the specified order even when the corresponding moments of $\hat{\beta}$ do not exist (Götze and Hipp, 1978; Rothenberg, 1984; Magdalinos, 1992). In this sense, we can regard (1) as an approximation of the MSE of $\hat{\beta}$ when $\hat{\beta}$ possesses second moments and as the MSE of an approximate estimator that shares the same Edgeworth expansion up to a specified order.

Remark 1. For linear instrumental variable models with iid variables and normal errors, Kinal (1980) shows that $\hat{\beta}$ has finite moments up to order $l - p$. Similar results have been conjectured for GMM and generalized empirical likelihood estimators (e.g., Kunitomo and Matsushita, 2003; Guggenberger, 2008). Therefore, one should be careful in interpreting the MSE approximation in cases when the degree of overidentification is less than two. Other loss functions may then be more appropriate (see Zaman, 1981, for example).

Having established Theorem 1, the calculation of an MSE-optimal bandwidth, S_T^*, becomes straightforward: for the second-order term to attain its fastest possible rate of convergence, the terms of order $S_T T^{-2}$ and $S_T^{-2q} T^{-1}$ have to be balanced which is the case for $S_T^* = c^*(q) T^{1/(1+2q)}$ and some constant $c^*(q)$. We refer to this bandwidth as the MSE($\hat{\beta}$)-optimal bandwidth. The bandwidth derived in Andrews (1991), which minimizes the MSE of $\hat{\Omega}_T(\tilde{\beta})$, we call the MSE($\hat{\Omega}$)-optimal bandwidth.

COROLLARY 1. *Under the assumptions of Theorem 1 and if $l > p$, minimizing the lowest order of MSE_T involving the bandwidth yields the optimal*

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bandwidth growth rate $T^{1/(1+2q)}$. *Moreover,* $S_T^* = c^*(q)T^{1/(1+2q)}$ *minimizes the higher-order AMSE defined as the limit of* $HMSE_T := T^{(1+4q)/(1+2q)}\{MSE_T - \nu_1 T^{-1}\}$ *with*

$$c^*(q) := \left(\frac{c_0\nu_3}{\nu_2}\right)^{1/(1+2q)}, \qquad c_0 := \begin{cases} 2q, & sign(\nu_2) = sign(\nu_3) \\ -1, & sign(\nu_2) \neq sign(\nu_3) \end{cases}. \tag{3}$$

The expressions for ν_2 and ν_3 show that the optimal bandwidth growth rate is governed by the convergence rate of the covariances between the moment functions and the HAC estimator. The bias and variance of the HAC estimator itself only play an indirect role; in particular, the AMSE of $\hat{\beta}$ is not an increasing function of the AMSE of the HAC estimator. In consequence, none of the existing procedures minimizing the AMSE of the HAC estimator (Andrews, 1991; Andrews and Monahan, 1992; Newey and West, 1994 among others) are optimal in the above sense.

The convergence rate of the MSE of $\hat{\beta}$ is not affected by the bandwidth choice because it is of order $O(T^{-1})$, only the second-order terms of the MSE, converging at an optimal rate of $O(T^{-(1+4q)/(1+2q)})$, are. By choosing kernels of very high order q, this rate can be made arbitrarily close to $O(T^{-2})$. Nevertheless, kernels of order smaller than or equal to 2 are popular because, unlike kernels of higher order, they can produce positive definite covariance matrix estimates.

Remark 2. Interestingly, the semiparametric estimator $\hat{\beta}$ converges at rate $T^{-1/2}$, but the optimal bandwidth minimizing MSE($\hat{\beta}$) is of the same order as the optimal bandwidth minimizing MSE($\hat{\Omega}$). This result contrasts the findings in other semiparametric settings such as those studied by Powell and Stoker (1996) and Goldstein and Messer (1992), for example, in which undersmoothing the nonparametric plugin estimator leads to $T^{-1/2}$-convergence rates of smooth functionals of that nonparametric plugin estimator.

To gain more insight into which features of the data-generating process determine the value of the optimal bandwidth and to be able to directly estimate the quantities involved, the following proposition derives more explicit expressions for the constants ν_i.

PROPOSITION 2. *Assume that* $\{g_t\}$ *follows a linear Gaussian process, viz.*

$$g_t = \sum_{s=0}^{\infty} \Psi_s e_{t-s}$$

for $t = 1, \ldots, T$, $e_t \sim N(0, \Sigma_e)$ *iid, and* Ψ_s *satisfies* $\sum_{s=0}^{\infty} s^4 \|\Psi_s\| < \infty$. *Define* $\mu_i := \int_{-\infty}^{\infty} k^i(x)dx$ *for* $i = 1, 2$ *and* $\Omega_0^{(q)} := 2\pi f^{(q)}$. *Then*

$$\nu_1 = tr(\Omega_0 H_0' \mathcal{W} H_0),$$
$$\nu_2 = (2\mu_1 + \mu_2)(l-p)tr(\Sigma_0 \mathcal{W}),$$
$$\nu_3 = g_q^2 tr(\Omega_0^{(q)} H_0' \mathcal{W} H_0 \Omega_0^{(q)} P_0).$$

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2.1. Linear IV Model

In this subsection, we specialize the expressions in Proposition 2 to a stylized instrumental variable model that allows us to analyze the difference between the MSE($\hat{\beta}$)-optimal and the MSE($\hat{\Omega}$)-optimal bandwidths and subsequently serves as the data-generating process for the Monte Carlo simulations. Let y_t and w_t be random variables satisfying

$$y_t = \beta_0 w_t + \varepsilon_t$$

and z_t an l-dimensional random vector of instruments such that

$$w_t = \gamma\, \iota' z_t + v_t,$$

where $\iota := (1, \ldots, 1)' \in \mathbb{R}^l$ and $\gamma \in \mathbb{R}$. Define $x_t = (y_t, w_t, z_t')'$ so that $g(x_t, \beta) = (y_t - \beta w_t) z_t$. Further let $\{\varepsilon_t\}$ and $\{v_t\}$ be AR(1) processes with autocorrelation coefficient $|\rho| \in (0, 1)$, viz. $\varepsilon_t = \rho \varepsilon_{t-1} + \eta_t$ and $v_t = \rho v_{t-1} + u_t$, where

$$\begin{pmatrix} \eta_t \\ u_t \end{pmatrix} \sim iid\ N\left(0, \begin{pmatrix} 1 & \sigma_{12} \\ \sigma_{12} & 1 \end{pmatrix}\right), \qquad \sigma_{12} \neq 0.$$

The instruments follow a VAR(1) process, $z_t = \rho_z z_{t-1} + \epsilon_t$ with $\rho_z := diag(0, \bar{\rho}, \ldots, \bar{\rho})$, $|\bar{\rho}| \in (0, 1)$, and $\epsilon_t \sim iid\ N(0, I_l)$ independent of $\{(\eta_t, u_t)\}$. Then, one can show that

$$c^*(1) = \left(\frac{4 c_0 g_1^2 \rho^2 \bar{\rho}^2 (1 - \bar{\rho}^2)}{(2\mu_1 + \mu_2)(1 - \rho\bar{\rho})(1 + \rho\bar{\rho})(-l + (l-2)\rho\bar{\rho} + \bar{\rho}^2 + \rho\bar{\rho}^3)^2} \right)^{1/3}$$

$$c^*(2) = \left(\frac{4 c_0 g_2^2 \rho^2 \bar{\rho}^2 (1 + \rho\bar{\rho})(1 - \bar{\rho}^2)}{(2\mu_1 + \mu_2)(1 - \rho\bar{\rho})^3(-l + (l-2)\rho\bar{\rho} + \bar{\rho}^2 + \rho\bar{\rho}^3)^2} \right)^{1/5}.$$

In this specific example, we can easily compare the MSE($\hat{\beta}$)-optimal to the MSE($\hat{\Omega}$)-optimal bandwidth derived in Andrews (1991), $S_T = (q g_q^2 / \mu_2\, \alpha(q) T)^{1/(1+2q)}$ with $\alpha(q) = 2\text{vec}(\Omega_0^{(q)})' \mathcal{W}_A \text{vec}(\Omega_0^{(q)}) / tr(\mathcal{W}_A (I_{l^2} + K_{ll})(\Omega_0 \otimes \Omega_0))$. For $\mathcal{W}_A = I_{l^2}$, the constants of proportionality become $\alpha(1) = 8(l-1)\rho\rho_2^2 / f(\rho, \rho_2, l)$ and $\alpha(2) = 8(l-1)\rho\rho_2^2(1 + \rho\rho_2)^2 / ((1 - \rho\rho_2)^2 f(\rho, \rho_2, l))$ with $f(\rho, \rho_2, l) := (1 - \rho\rho_2)^2 [l(l+1) + 2(l^2 - l - 2)\rho\rho_2 + (l^2\rho^2 - l(2 + 3\rho^2) + 4\rho^2 - 2)\rho_2^2 + 8\rho\rho_2^3 + 2(1 + (l-3)\rho^2)\rho_2^4 - 4\rho\rho_2^5 + 2\rho^2\rho_2^6]$.

Notice that the MSE($\hat{\beta}$)-optimal and the MSE($\hat{\Omega}$)-optimal bandwidth both adapt to the persistence of the error processes (ρ), the persistence of the instruments ($\bar{\rho}$), and the number of instruments (l), but through very different functional forms. Therefore, we expect there to be scenarios in which the MSE($\hat{\Omega}$)-optimal bandwidth is clearly not MSE($\hat{\beta}$)-optimal and the two bandwidths may differ significantly. The simulation evidence in Section 4 confirms these findings.

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3. DATA-DRIVEN BANDWIDTH CHOICE

The optimal bandwidth S_T^* is infeasible because it depends on several unknown quantities. In the case in which $\{g_t\}$ is a linear Gaussian process, we require knowledge of Ω_0, $\Omega_0^{(q)}$, and G_0. This section describes a data-driven approach to select the optimal bandwidth by estimating the required quantities based on parametric approximating models for $\{g_t\}$, similarly as proposed in Andrews (1991). The idea is to first construct the first-step estimator $\tilde{\beta}$, then fit a parsimonious autoregressive (AR) model to $\{g_t(\tilde{\beta})\}_{t=1}^T$ and, finally, to substitute its parameter estimates into analytical formulas for Ω_0 and $\Omega_0^{(q)}$ assuming that the AR is the true model. Together with the usual sample average estimator for G_0, these estimates are then substituted into the expressions of ν_2 and ν_3 in Proposition 2 to yield estimates of the optimal bandwidths.

We focus on estimating a univariate AR(1) model for each component of $\{g_t(\tilde{\beta})\}$, although other approximating models like vector autoregressions or models with more lags could be considered. Let $\hat{\rho}_i$ and $\hat{\sigma}_i$ be the estimated coefficient and the residual variance of the i-th estimated AR(1) process. We can construct estimators of Ω_0 and $\Omega_0^{(q)}$ as $\hat{\Omega}_0 := diag(\hat{\omega}_1, \ldots, \hat{\omega}_l)$ and $\hat{\Omega}_0^{(q)} := diag(\hat{\omega}_1^{(q)}, \ldots, \hat{\omega}_l^{(q)})$ with $\hat{\omega}_i := \hat{\sigma}_i^2/(1-\hat{\rho}_i)^2$, $\hat{\omega}_i^{(1)} := 2\hat{\sigma}_i^2\hat{\rho}_i/[(1-\hat{\rho}_i)^3(1+\hat{\rho}_i)]$, and $\hat{\omega}_i^{(2)} := 2\hat{\sigma}_i^2\hat{\rho}_i/[(1-\hat{\rho}_i)^4]$. Then, estimate H_0, Σ_0, and P_0 by $\hat{H}_0 := \hat{\Sigma}_0 G_T(\tilde{\beta})'\hat{\Omega}_0^{-1}$, $\hat{\Sigma}_0 := (G_T(\tilde{\beta})'\hat{\Omega}_0^{-1}G_T(\tilde{\beta}))^{-1}$, and $\hat{P}_0 := \hat{\Omega}_0^{-1} - \hat{\Omega}_0^{-1}G_T(\tilde{\beta})\hat{H}_0$. Finally, substitute all these expressions into the formulas of Proposition 2 to get estimates $\hat{\nu}_2$ and $\hat{\nu}_3$ of ν_2 and ν_3, and the estimator of the optimal bandwidth,

$$\hat{S}_T := \left(\frac{c_0\hat{\nu}_3}{\hat{\nu}_2}\right)^{1/(1+2q)} T^{1/(1+2q)}.$$

The difference in performance one incurs by using $\hat{S}_T$ instead of the infeasible bandwidth minimizing the finite-sample MSE of $\hat{\beta}$ has four sources: the error made by replacing the MSE of $\hat{\beta}$ by the MSE of the first terms in the higher-order expansion (ζ_T), the error due to the large sample approximation of the MSE, the estimation error in $\hat{\Omega}_0$ and $\hat{\Omega}_0^{(q)}$, and the error made by potential misspecification of the approximating parametric model for $\{g_t\}$. In practice, one hopes that these errors are small. As mentioned in the discussion after Theorem 1, the first type of error vanishes with the sample size under additional assumptions. The second and third type also disappear as $T \to \infty$. The fourth type of error can typically be conjectured to be negligible because the MSE of $\hat{\beta}$ tends to be relatively flat around its minimum (as is the case in the Monte Carlo simulations, for example), so that misspecification in the approximating model is not expected to have a large impact on the properties of the resulting GMM estimator. Nevertheless, the applied researcher should bear in mind that the plugin procedure is not "automatic" and some thought has to go into selecting an appropriate approximating model and the potential impact of the aforementioned types of errors has to be considered.

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Remark 3. As in Andrews and Monahan (1992) one may want to consider prewhitening the series $\{g_t(\tilde{\beta})\}_{t=1}^{T}$ before fitting the AR process. The reported increases in accuracy of test statistics in Andrews and Monahan (1992) and Newey and West (1994) are expected to occur with the procedure presented here as well.

4. SIMULATIONS

In this section, we discuss a small simulation experiment that illustrates the theoretical findings from the previous sections and, in particular, shows that the MSE($\hat{\beta}$)-optimal bandwidth may lead to a substantially lower finite-sample MSE of $\hat{\beta}$ relative to choosing the MSE($\hat{\Omega}$)-optimal bandwidth.

We simulate the model from Section 2.1, denoted by "AR(1)-HOM", for different degrees of serial correlation ($\rho \in \{0.01, 0.1, 0.5, 0.9, 0.99\}$), weak

TABLE 1. Bandwidths ("bw"), bias, standard deviation ("SD"), and MSE of $\hat{\beta}$ when computed based on the MSE($\hat{\Omega}$)-optimal ("optimal"), the MSE($\hat{\Omega}$)-optimal ("Andrews"), $S_T = T^{1/(1+2q)}$ ("naive"), or the simulated MSE-minimizing ("sim") bandwidth

		AR(1)-HOM, $T = 64, l = 10$							
		$\gamma = 0.1$				$\gamma = 2$			
ρ		Optimal	Andrews	Naive	Sim	Optimal	Andrews	Naive	Sim
0.01	bw	0.651	1.150	4.000	0.000	0.645	1.098	4.000	0.000
	Bias	0.242	0.244	0.251	0.242	0.001	0.001	0.001	0.001
	sd	0.185	0.188	0.211	0.185	0.011	0.012	0.013	0.011
	MSE	0.093	0.095	0.107	0.093	0.000	0.000	0.000	0.000
0.1	bw	0.659	1.319	4.000	0.000	0.650	1.241	4.000	0.000
	Bias	0.263	0.266	0.274	0.262	0.001	0.001	0.001	0.001
	sd	0.193	0.197	0.219	0.193	0.012	0.013	0.014	0.012
	MSE	0.106	0.109	0.123	0.106	0.000	0.000	0.000	0.000
0.5	bw	0.954	3.225	4.000	0.000	0.918	2.948	4.000	0.000
	Bias	0.427	0.442	0.444	0.425	0.003	0.003	0.003	0.003
	sd	0.232	0.242	0.248	0.232	0.021	0.022	0.023	0.021
	MSE	0.236	0.254	0.259	0.235	0.000	0.001	0.001	0.000
0.9	bw	1.891	8.382	4.000	2.300	2.015	8.525	4.000	3.400
	Bias	0.788	0.786	0.785	0.786	0.012	0.012	0.013	0.012
	sd	0.324	0.364	0.326	0.322	0.077	0.083	0.076	0.075
	MSE	0.725	0.750	0.723	0.721	0.006	0.007	0.006	0.006
0.99	bw	2.488	18.469	4.000	1.200	2.566	15.929	4.000	4.500
	Bias	0.844	0.846	0.838	0.842	0.063	0.060	0.061	0.060
	sd	0.541	0.597	0.532	0.519	0.228	0.228	0.221	0.221
	MSE	1.005	1.071	0.986	0.978	0.056	0.055	0.052	0.052

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TABLE 2. Ratios of MSE ("MSE ratio") and higher-order MSE ("HMSE ratio") based on the MSE($\hat{\beta}$)-optimal bandwidth divided by those based on the MSE($\hat{\Omega}$)-optimal bandwidth. μ^2/l is the standardized concentration parameter measuring the strength of the instruments

		AR(1)-HOM, $T = 64$					
		$\gamma = 0.1$			$\gamma = 2$		
ρ	l	MSE ratio	HMSE ratio	μ^2/l	MSE ratio	HMSE ratio	μ^2/l
0.01	2	0.945	0.251	1.951	0.994	0.763	760.534
	3	1.009	0.879	2.458	0.995	0.799	946.225
	4	0.958	0.686	2.607	0.982	0.782	1052.094
	5	0.984	0.759	2.798	0.984	0.787	1118.489
	10	0.977	0.735	3.056	0.970	0.747	1186.655
	15	0.989	0.735	3.083	0.971	0.747	1223.093
	25	0.992	0.714	3.148	0.990	0.733	1227.450
0.1	2	0.985	0.749	2.017	0.993	0.779	754.774
	3	0.983	0.754	2.582	0.990	0.777	943.791
	4	0.987	0.600	2.727	0.967	0.721	1050.000
	5	0.972	0.710	2.941	0.975	0.729	1120.367
	10	0.968	0.671	3.253	0.951	0.671	1203.529
	15	0.977	0.660	3.304	0.957	0.668	1252.836
	25	0.985	0.638	3.422	0.973	0.649	1279.956
0.5	2	1.015	0.453	2.296	1.011	0.797	595.776
	3	0.976	0.613	3.214	1.005	0.669	770.276
	4	0.980	0.631	3.331	0.931	0.587	864.166
	5	0.963	0.522	3.662	0.942	0.557	945.503
	10	0.930	0.441	4.295	0.867	0.459	1105.717
	15	0.925	0.415	4.517	0.824	0.433	1236.639
	25	0.911	0.361	4.907	0.811	0.387	1395.522
0.9	2	1.001	0.675	5.083	1.133	0.763	217.609
	3	1.019	0.535	7.358	1.072	0.597	308.766
	4	1.010	0.508	8.283	1.078	0.443	373.754
	5	0.976	0.471	9.171	1.070	0.476	439.970
	10	0.967	0.375	11.156	0.865	0.339	665.258
	15	0.957	0.279	12.162	0.910	0.276	898.255
	25	0.974	0.193	13.180	0.926	0.188	1248.500
0.99	2	1.228	0.426	8.982	1.082	0.745	99.308
	3	1.036	0.608	12.269	1.254	0.496	144.471
	4	0.980	0.406	15.458	1.187	0.386	185.307
	5	0.984	0.323	16.999	1.095	0.356	230.003
	10	0.938	0.096	23.119	1.010	0.231	409.860
	15	0.956	0.171	27.098	1.033	0.000	601.000
	25	0.924	0.079	32.242	0.952	0.127	948.061

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and strong instruments ($\gamma \in \{0.1, 2\}$) and increasing number of instruments ($l \in \{2, 3, 4, 5, 10, 15, 25\}$). We also consider two variants of the model, one in which the outcome equation is replaced by a model with heteroskedastic errors, $y_t = \beta_0 w_t + |w_t|\varepsilon_t$ (referred to as "AR(1)-HET"), and one in which the AR(1) error process is replaced by an MA(1), i.e., $\varepsilon_t = \rho\eta_{t-1} + \eta_t$ and $v_t = \rho u_{t-1} + u_t$. We simulate 1,000 samples and, to save space, present only results for sample size $T = 64$, $\bar{\rho} = 0.9$, $\sigma_{12} = 0.9$, $\beta_0 = 1$ and the Bartlett kernel. Other parameter combinations yield similar results. Since the model is linear, the GMM estimator of β_0 simplifies to $(Z'W\hat{\Omega}_T(\tilde{\beta})^{-1}W'Z)^{-1}Z'W\hat{\Omega}_T(\tilde{\beta})^{-1}W'Y$ where Y, W, and Z, are the vector/matrices of stacked observations of the outcome variable, the regressors, and the instruments, respectively. All results below are based on this formula.

TABLE 3. Bandwidths ("bw"), bias, standard deviation ("SD"), and MSE of $\hat{\beta}$ when computed based on the MSE($\hat{\Omega}$)-optimal ("optimal"), the MSE($\hat{\Omega}$)-optimal ("Andrews"), $S_T = T^{1/(1+2q)}$ ("naive"), or the simulated MSE-minimizing ("sim") bandwidth

		AR(1)-HET, $T = 64$, $l = 10$							
		$\gamma = 0.1$				$\gamma = 2$			
ρ		Optimal	Andrews	Naive	Sim	Optimal	Andrews	Naive	Sim
0.01	bw	0.651	1.150	4.000	0.000	0.645	1.098	4.000	0.000
	Bias	0.188	0.190	0.197	0.188	0.001	0.001	0.001	0.001
	sd	0.235	0.240	0.263	0.235	0.012	0.012	0.014	0.012
	MSE	0.091	0.094	0.108	0.090	0.000	0.000	0.000	0.000
0.1	bw	0.659	1.319	4.000	0.000	0.650	1.241	4.000	0.000
	Bias	0.204	0.207	0.215	0.204	0.001	0.001	0.002	0.001
	sd	0.248	0.256	0.279	0.248	0.013	0.013	0.015	0.013
	MSE	0.103	0.108	0.124	0.103	0.000	0.000	0.000	0.000
0.5	bw	0.954	3.225	4.000	0.000	0.918	2.948	4.000	0.000
	Bias	0.331	0.342	0.344	0.330	0.002	0.003	0.003	0.002
	sd	0.337	0.349	0.355	0.337	0.022	0.023	0.024	0.022
	MSE	0.223	0.239	0.244	0.222	0.000	0.001	0.001	0.000
0.9	bw	1.891	8.382	4.000	2.300	2.015	8.525	4.000	2.300
	Bias	0.619	0.620	0.619	0.618	0.011	0.012	0.011	0.011
	sd	0.588	0.612	0.591	0.586	0.076	0.084	0.076	0.076
	MSE	0.729	0.758	0.732	0.725	0.006	0.007	0.006	0.006
0.99	bw	2.488	18.469	4.000	0.000	2.566	15.929	4.000	3.400
	Bias	0.675	0.676	0.670	0.671	0.056	0.052	0.053	0.054
	sd	0.781	0.818	0.759	0.745	0.235	0.237	0.226	0.226
	MSE	1.064	1.125	1.024	1.005	0.058	0.059	0.054	0.054

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TABLE 4. Ratios of MSE ("MSE ratio") and higher-order MSE ("HMSE ratio") based on the MSE($\hat{\beta}$)-optimal bandwidth divided by those based on the MSE($\hat{\Omega}$)-optimal bandwidth. μ^2/l is the standardized concentration parameter measuring the strength of the instruments

		AR(1)-HET, $T = 64$					
		$\gamma = 0.1$			$\gamma = 2$		
ρ	l	MSE ratio	HMSE ratio	μ^2/l	MSE ratio	HMSE ratio	μ^2/l
0.01	2	0.905	0.220	1.951	0.992	0.753	760.534
	3	1.026	0.931	2.458	0.999	0.787	946.225
	4	0.956	0.719	2.607	0.975	0.788	1052.094
	5	0.995	0.772	2.798	0.979	0.789	1118.489
	10	0.964	0.729	3.056	0.972	0.766	1186.655
	15	0.990	0.738	3.083	0.972	0.758	1223.093
	25	0.991	0.740	3.148	0.995	0.753	1227.450
0.1	2	0.977	0.789	2.017	0.992	0.761	754.774
	3	0.962	0.718	2.582	0.985	0.754	943.791
	4	0.985	0.621	2.727	0.956	0.733	1050.000
	5	0.977	0.722	2.941	0.971	0.736	1120.367
	10	0.949	0.666	3.253	0.962	0.689	1203.529
	15	0.985	0.665	3.304	0.958	0.680	1252.836
	25	0.986	0.662	3.422	0.982	0.662	1279.956
0.5	2	1.034	0.600	2.296	1.002	0.784	595.776
	3	0.995	0.642	3.214	0.984	0.676	770.276
	4	0.984	0.686	3.331	0.902	0.589	864.166
	5	0.971	0.529	3.662	0.967	0.562	945.503
	10	0.934	0.447	4.295	0.877	0.466	1105.717
	15	0.917	0.427	4.517	0.785	0.434	1236.639
	25	0.925	0.367	4.907	0.779	0.418	1395.522
0.9	2	1.011	0.709	5.083	1.152	0.751	217.609
	3	1.039	0.575	7.358	1.093	0.601	308.766
	4	1.006	0.487	8.283	1.059	0.472	373.754
	5	0.982	0.464	9.171	1.134	0.495	439.970
	10	0.961	0.359	11.156	0.831	0.351	665.258
	15	0.937	0.283	12.162	0.939	0.300	898.255
	25	0.982	0.178	13.180	0.944	0.197	1248.500
0.99	2	1.134	0.278	8.982	1.190	0.780	99.308
	3	1.045	0.559	12.269	1.181	0.473	144.471
	4	0.955	0.441	15.458	1.133	0.455	185.307
	5	0.953	0.306	16.999	1.080	0.391	230.003
	10	0.945	0.078	23.119	0.986	0.229	409.860
	15	0.976	0.144	27.098	1.001	0.000	601.000
	25	0.895	0.095	32.242	0.846	0.153	948.061

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For each of the three different data-generating processes, Tables 1, 3, and 5 report four different bandwidths ("bw") averaged over the simulation samples: "optimal", "Andrews", "naive", and "sim", referring to the MSE($\hat{\beta}$)-optimal, the MSE($\hat{\Omega}$)-optimal, the naive choice $S_T = T^{1/(1+2q)}$, and to the (infeasible) bandwidth that minimizes the simulated MSE($\hat{\beta}$) over a grid of bandwidths, respectively. The optimal bandwidth is estimated based on the procedure in Section 3. The table also shows the bias, standard deviation ("SD"), and MSE of $\hat{\beta}$. In almost all cases considered here, the MSE($\hat{\beta}$)-optimal bandwidth is closer to the one minimizing the simulated MSE than the MSE($\hat{\Omega}$)-optimal bandwidth. In all scenarios, the MSE($\hat{\beta}$)-optimal bandwidth is smaller than the MSE($\hat{\Omega}$)-optimal bandwidth one, in some cases substantially smaller.

Tables 2, 4, and 6 show the ratios of MSE ("MSE ratio") and higher-order MSE ("HMSE ratio"), as defined in Corollary 1, based on the MSE($\hat{\beta}$)-optimal

TABLE 5. Bandwidths ("bw"), bias, standard deviation ("SD"), and MSE of $\hat{\beta}$ when computed based on the MSE($\hat{\Omega}$)-optimal ("optimal"), the MSE($\hat{\Omega}$)-optimal ("Andrews"), $S_T = T^{1/(1+2q)}$ ("naive"), or the simulated MSE-minimizing ("sim") bandwidth

		MA(1), $T = 64$, $l = 10$							
		$\gamma = 0.1$				$\gamma = 2$			
ρ		Optimal	Andrews	Naive	Sim	Optimal	Andrews	Naive	Sim
0.01	bw	0.651	1.150	4.000	0.000	0.645	1.098	4.000	0.000
	Bias	0.242	0.244	0.251	0.242	0.001	0.001	0.001	0.001
	sd	0.185	0.188	0.211	0.185	0.011	0.012	0.013	0.011
	MSE	0.093	0.095	0.107	0.093	0.000	0.000	0.000	0.000
0.1	bw	0.655	1.308	4.000	0.000	0.647	1.232	4.000	0.000
	Bias	0.261	0.264	0.273	0.261	0.001	0.001	0.001	0.001
	sd	0.192	0.196	0.218	0.192	0.012	0.013	0.014	0.012
	MSE	0.105	0.108	0.122	0.105	0.000	0.000	0.000	0.000
0.5	bw	0.796	2.486	4.000	0.000	0.769	2.328	4.000	0.000
	Bias	0.359	0.373	0.376	0.358	0.002	0.002	0.003	0.002
	sd	0.215	0.226	0.240	0.214	0.017	0.018	0.019	0.017
	MSE	0.175	0.190	0.199	0.174	0.000	0.000	0.000	0.000
0.9	bw	0.915	3.068	4.000	0.000	0.875	2.875	4.000	0.000
	Bias	0.458	0.473	0.474	0.456	0.003	0.004	0.004	0.003
	sd	0.225	0.238	0.247	0.225	0.021	0.023	0.024	0.021
	MSE	0.260	0.280	0.286	0.259	0.000	0.001	0.001	0.000
0.99	bw	0.922	3.098	4.000	0.000	0.880	2.903	4.000	0.000
	Bias	0.478	0.494	0.495	0.477	0.004	0.004	0.004	0.004
	sd	0.226	0.239	0.247	0.226	0.022	0.024	0.025	0.022
	MSE	0.280	0.300	0.306	0.278	0.001	0.001	0.001	0.001

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TABLE 6. Ratios of MSE ("MSE ratio") and higher-order MSE ("HMSE ratio") based on the MSE($\hat{\beta}$)-optimal bandwidth divided by those based on the MSE($\hat{\Omega}$)-optimal bandwidth. μ^2/l is the standardized concentration parameter measuring the strength of the instruments

		MA(1), $T = 64$					
		$\gamma = 0.1$			$\gamma = 2$		
ρ	l	MSE ratio	HMSE ratio	μ^2/l	MSE ratio	HMSE ratio	μ^2/l
0.01	2	0.944	0.246	1.951	0.994	0.763	760.522
	3	1.009	0.880	2.458	0.995	0.799	946.220
	4	0.958	0.686	2.607	0.982	0.782	1052.082
	5	0.984	0.759	2.798	0.984	0.787	1118.457
	10	0.977	0.735	3.056	0.970	0.747	1186.630
	15	0.989	0.735	3.083	0.971	0.747	1223.086
	25	0.992	0.714	3.148	0.990	0.733	1227.355
0.1	2	0.984	0.766	2.008	0.993	0.779	754.471
	3	0.995	0.743	2.567	0.990	0.778	943.247
	4	0.982	0.591	2.711	0.967	0.723	1049.296
	5	0.972	0.711	2.922	0.974	0.729	1118.885
	10	0.968	0.672	3.233	0.951	0.673	1201.270
	15	0.978	0.662	3.285	0.958	0.670	1249.736
	25	0.984	0.639	3.402	0.977	0.651	1275.644
0.5	2	1.043	0.459	1.925	0.999	0.790	618.982
	3	0.988	0.642	2.591	0.991	0.680	787.576
	4	0.976	0.599	2.696	0.931	0.609	878.617
	5	0.955	0.554	2.939	0.955	0.578	945.049
	10	0.923	0.474	3.414	0.871	0.482	1061.880
	15	0.911	0.460	3.570	0.846	0.465	1143.806
	25	0.915	0.437	3.918	0.867	0.441	1252.310
0.9	2	1.025	0.307	1.548	1.000	0.794	431.764
	3	0.968	0.631	2.149	0.994	0.676	553.344
	4	0.963	0.554	2.207	0.918	0.601	618.183
	5	0.970	0.542	2.416	0.934	0.562	666.128
	10	0.928	0.450	2.856	0.851	0.458	761.458
	15	0.919	0.400	3.013	0.829	0.424	830.616
	25	0.914	0.354	3.378	0.811	0.407	935.298
0.99	2	1.002	0.442	1.459	1.000	0.794	395.023
	3	0.988	0.770	2.036	0.993	0.676	506.511
	4	0.964	0.570	2.086	0.916	0.601	565.875
	5	0.976	0.554	2.284	0.931	0.560	609.690
	10	0.931	0.449	2.701	0.850	0.457	697.341
	15	0.924	0.426	2.848	0.834	0.432	760.882
	25	0.917	0.383	3.192	0.807	0.407	857.193

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bandwidth divided by those based on the MSE($\hat{\Omega}$)-optimal bandwidth. The number of instruments is fixed at $l = 10$, but other numbers yield qualitatively the same results. μ^2/l denotes the standardized concentration parameter measuring the strength of the instruments (Stock, Wright, and Yogo, 2002). The MSE ratios demonstrate that the MSE($\hat{\beta}$)-optimal bandwidth may lead to substantial MSE gains relative to the MSE($\hat{\Omega}$)-optimal bandwidth. The gains are particularly large, up to more than 20%, when the number of instruments is large and the estimator of the optimal weighting matrix becomes less precise. As predicted by the theoretical results in the previous sections, the MSE($\hat{\beta}$)-optimal bandwidth may also lead to dramatic higher-order MSE gains relative to the MSE($\hat{\Omega}$)-optimal bandwidth of up to more than 90%.

Unlike the MSE($\hat{\Omega}$)-optimal bandwidth defined by Andrews (1991), the MSE($\hat{\beta}$)-optimal bandwidth is formally not defined for the case $l = p$ in which the estimator $\hat{\beta}$ is independent of the weighting matrix. To study scenarios in which l/p is close to this boundary, we conclude this section by considering a robustness check in which l/p approaches one. Table 7 reports the

TABLE 7. Robustness check: ratios of MSE ("MSE ratio") and higher-order MSE ("HMSE ratio") based on the MSE($\hat{\beta}$)-optimal bandwidth divided by those based on the MSE($\hat{\Omega}$)-optimal bandwidth. $T = 128$

Model	l/p	MSE ratio	HMSE ratio
AR(1)-HOM	5	0.967	0.564
	2	0.990	0.654
	1.50	0.986	0.388
	1.33	1.005	0.754
	1.25	1.012	0.647
	1.14	0.997	0.707
	1.11	0.998	0.690
AR(1)-HET	5	0.964	0.568
	2	0.983	0.662
	1.50	0.982	0.333
	1.33	1.005	0.769
	1.25	1.023	0.529
	1.14	1.003	0.758
	1.11	0.992	0.713
MA(1)	5	0.962	0.570
	2	0.993	0.685
	1.50	0.997	0.482
	1.33	1.007	0.741
	1.25	1.015	0.830
	1.14	1.005	0.689
	1.11	0.997	0.676

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same MSE and HMSE ratio as Tables 2, 4, and 6, but for a sequence $l/p \in \{5/1, 4/2, 3/2, 4/3, 5/4, 8/7, 10/9\}$ that approaches one. T, ρ, $\bar{\rho}$, and γ are fixed at values 128, 0.5, 0.9, and 2, respectively, but other values yield similar results. The MSE based on the MSE($\hat{\beta}$)-optimal bandwidth stays close to or slightly smaller than the one based on the MSE($\hat{\Omega}$)-optimal bandwidth for all values of l/p. Similarly, the HMSE is significantly smaller for the optimal bandwidth. In the case of the AR(1)-HET model, the HMSE gains are even up to 67%.

5. CONCLUSION

This paper develops a selection procedure for the bandwidth of a HAC estimator of the optimal GMM weighting matrix which minimizes the asymptotic MSE of the resulting two-step GMM estimator. We show that it is of the same order as the bandwidth minimizing the MSE of the nonparametric plugin estimator, but the constants of proportionality differ significantly. The simulation study suggests that the data-driven version of the selection procedure works well in finite samples and may substantially reduce the first- and second-order MSE of the GMM estimator relative to existing, suboptimal choices, especially when the number of moment conditions is large.

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APPENDIX

A. Proofs

LEMMA 1. *Under Assumptions 1 and 2 and 3(a)–(c),* $\hat{\beta} - \beta_0 = \psi T^{-1/2} + o_p(T^{-1/2})$ *with* $\psi = -G_0^{-1}\sqrt{T}\hat{g} = O_p(1)$.

Proof. We need to check the assumptions of Newey and McFadden (1994, Thm. 3.2) with $\hat{W}$ replaced by $\hat{\Omega}_T(\tilde{\beta})$. First of all, by Assumptions 1 and 2 and Andrews (1991, Thm. 1(b)) $\hat{\Omega}_T(\tilde{\beta}) \xrightarrow{p} \Omega_0$. (i), (ii), and (v) hold by assumption. (iii) and (iv) hold by Assumption 1 and White and Domowitz (1984, Thm. 2.3 and 2.4). ■

Proof of Proposition 1. Step I: Expansion of the optimal weighting matrix. A Taylor expansion of $\hat{\Omega}_T(\tilde{\beta})$ around β_0 yields

$$\begin{aligned}\operatorname{vec}\left(\hat{\Omega}_T(\tilde{\beta})\right) &= \operatorname{vec}\left(\hat{\Omega}_T\right) + \nabla\hat{\Omega}_T(\bar{\beta})(\tilde{\beta}-\beta_0)\\ &= \operatorname{vec}(\Omega_0) + \operatorname{vec}\left(\hat{\Omega}_T - \Omega_0\right) + \nabla\Omega_0(\tilde{\beta}-\beta_0) + \left(\nabla\hat{\Omega}_T - \nabla\Omega_0\right)(\tilde{\beta}-\beta_0)\\ &\quad + \left(\nabla\hat{\Omega}_T(\bar{\beta}) - \nabla\hat{\Omega}_T\right)(\tilde{\beta}-\beta_0), \qquad \textbf{(A.1)}\end{aligned}$$

where $\bar{\beta}$ lies on the line segment joining $\tilde{\beta}$ and β_0. By Assumptions 1 and 2 and Andrews (1991, Prop. 1(a) and (b)), $\hat{\Omega}_T - \Omega_0 = \omega_{1,T} S_T^{1/2} T^{-1/2} + \omega_{2,T} S_T^{-q}$ with $\omega_{i,T} = O_p(1)$, $i = 1, 2$, and $\nabla\hat{\Omega}_T - \nabla\Omega_0 = O_p(\eta_T)$. Next, we show that $\nabla\hat{\Omega}_T(\bar{\beta}) - \nabla\hat{\Omega}_T = O_p(\|\tilde{\beta} - \beta_0\|)$. To this end, let $\bar{g}_t := g_t(\bar{\beta})$. Notice that, by Assumption 1(f), $\bar{\beta} \in \mathcal{N}$ w.p.a. 1 and, thus, $\bar{\beta} \in \mathcal{N}$ w.p.a. 1. From Assumption 3(d), we get

$$\begin{aligned}\left\|\nabla\hat{\Omega}_T(\bar{\beta}) - \nabla\hat{\Omega}_T\right\| &\le \frac{1}{T}\sum_{s=1-T}^{T-1}\sum_{t=\max\{1,1-s\}}^{\min\{T,T-s\}}\left\|k\left(\frac{s}{S_T}\right)\left[\nabla\bar{g}_{t+s}\bar{g}_t' - \nabla g_{t+s}g_t'\right]\right\|\\ &\le \frac{C}{T}\sum_{s=1-T}^{T-1}\sum_{t=\max\{1,1-s\}}^{\min\{T,T-s\}} d(x_{t+s}, x_t)\,\|\bar{\beta}-\beta_0\|\\ &= C\left(\sum_{s=-\infty}^{\infty} Ed(x_{t+s}, x_t) + o_p(1)\right)\|\bar{\beta}-\beta_0\|\\ &= O_p(\|\tilde{\beta}-\beta_0\|) \qquad \textbf{(A.2)}\end{aligned}$$

which holds w.p.a. 1 and for some constant C. (A.1) together with (A.2) and the first-order asymptotics in Assumption 1(f) then imply

$$\hat{\Omega}_T(\tilde{\beta}) = \Omega_0 + \omega_{1,T} S_T^{1/2} T^{-1/2} + \omega_{2,T} S_T^{-q} + R_T \qquad \textbf{(A.3)}$$

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with $R_T = O_p(\eta_T T^{-1/2})$, $\omega_{1,T} := \sqrt{T/S_T}(\hat{\Omega}_T - \overline{\Omega}_T) = O_p(1)$, and $\omega_{2,T} := S_T^q (\overline{\Omega}_T - \Omega_0) = O(1)$.

Step II: Expansion of the second-step GMM estimator. Write the second-stage estimator $\hat{\theta} := (\hat{\beta}', \hat{\lambda}')'$ of $\theta_0 := (\beta_0', 0)' \in \mathcal{B} \times \Lambda$, $\Lambda := [0, \infty)^l$, as the solution to

$$\begin{pmatrix} G_T(\hat{\beta})'\hat{\lambda} \\ \hat{g}(\hat{\beta}) + \hat{\Omega}_T(\tilde{\beta})\hat{\lambda} \end{pmatrix} = 0. \tag{A.4}$$

Further, define $\hat{m}(\theta) := \frac{1}{T}\sum_{t=1}^T m_t(\theta)$ with

$$m_t(\theta) := -\begin{pmatrix} G(x_t, \beta)'\lambda \\ g_t(\beta) + \Omega_0 \lambda \end{pmatrix}$$

for some $\theta := (\beta', \lambda')' \in \mathcal{B} \times \Lambda$. Then use the expansion in (A.3) to rewrite (A.4) as

$$0 = \hat{m}(\hat{\theta}) - \begin{pmatrix} 0 \\ (\omega_{1,T} S_T^{1/2} T^{-1/2} + \omega_{2,T} S_T^{-q} + R_T)'\hat{\lambda} \end{pmatrix}. \tag{A.5}$$

Next, consider $\hat{\lambda} = -\hat{\Omega}_T(\tilde{\beta})^{-1}\hat{g}(\hat{\beta})$. By Assumptions 1 and 2 and Andrews (1991, Prop. 1(a) and (b), Thm. 1(b)), $\hat{\Omega}_T(\tilde{\beta}) - \Omega_0 = O_p(\eta_T)$. Also, by an expansion of $\hat{g}(\hat{\beta})$ around β_0, Lemma 1, Assumption 1, and the Central Limit Theorem (CLT), $\hat{g}(\hat{\beta}) = (I_l - G_0 H_0)\hat{g} + o_p(T^{-1/2})$, and thus

$$\begin{aligned} \hat{\lambda} &= -\left[\Omega_0 + O_p(\eta_T)\right]^{-1}\left((I_l - G_0 H_0) F_T T^{-1/2} + o_p(T^{-1/2})\right) \\ &= -P_0 F_T T^{-1/2} + O_p(\eta_T T^{-1/2}) + o_p(T^{-1/2}). \end{aligned} \tag{A.6}$$

Consider the following expansion of $\hat{m}(\hat{\theta})$ around θ_0:

$$\hat{m}(\hat{\theta}) = \hat{m}(\theta_0) + \nabla\hat{m}(\theta_0)(\hat{\theta} - \theta_0) + \frac{1}{2}\nabla^2\hat{m}(\bar{\theta})\left[(\hat{\theta} - \theta_0) \otimes (\hat{\theta} - \theta_0)\right],$$

where $\bar{\theta}$ lies on the line segment joining $\hat{\theta}$ and θ_0. By Assumptions 1(e) and 3(c) and (d), $\nabla^2\hat{m}(\bar{\theta})$ is $O_p(1)$. Lemma 1, Assumption 1, and the CLT for mixing sequences then imply

$$\hat{m}(\hat{\theta}) = \hat{m}(\theta_0) + M_1(\hat{\theta} - \theta_0) + O_p(T^{-1}). \tag{A.7}$$

Substituting (A.6) and (A.7) into (A.5) and solving for $\hat{\theta} - \theta_0$ yields

$$\begin{aligned} \hat{\theta} - \theta_0 = &-M_1^{-1}\hat{m}(\theta_0) - M_1^{-1}\begin{pmatrix} 0 \\ (\omega_{1,T}' S_T^{1/2} T^{-1/2} + \omega_{2,T}' S_T^{-q}) P_0 F_T T^{-1/2} \end{pmatrix} \\ &+ O_p(T^{-1}) + O_p(\eta_T^2 T^{-1/2}) + o_p(\eta_T T^{-1/2}), \end{aligned}$$

where

$$M_1 = -\begin{pmatrix} 0 & G_0' \\ G_0 & \Omega_0 \end{pmatrix}, \qquad M_1^{-1} = -\begin{pmatrix} -\Sigma_0 & H_0 \\ H_0' & P_0 \end{pmatrix}.$$

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Therefore,

$$\sqrt{T}(\hat{\beta}-\beta_0) = -H_0 F_T + H_0\omega'_{1,T} P_0 F_T S_T^{1/2} T^{-1/2} + H_0\omega'_{2,T} P_0 F_T S_T^{-q} + o_p(\eta_T).$$

Since F_T, $\omega_{1,T}$, and $\omega_{2,T}$ are $O_p(1)$, we also have that $\kappa_{i,T} = O_p(1)$ for $i = 1, 2, 3$. ■

Proof of Theorem 1. We need to derive the order of $E[\kappa'_{i,T}\mathcal{W}\kappa_{j,T}]$, $i, j \in \{1, 2, 3\}$. Consider the case $i = j = 1$:

$$E[\kappa'_{1,T}\mathcal{W}\kappa_{1,T}] = E[F'_T H'_0 \mathcal{W} H_0 F_T] = \frac{1}{T}\sum_{s,t=1}^{T} E[g'_t H'_0 \mathcal{W} H_0 g_s]$$
$$= \sum_{s=-(T-1)}^{T-1}\left(1-\frac{s}{T}\right) E[g'_t H'_0 \mathcal{W} H_0 g_{t-s}] \to \sum_{s=-\infty}^{\infty} E[g'_t H'_0 \mathcal{W} H_0 g_{t-s}].$$

The limiting sum can be shown to be finite using Assumption 1, the Hölder Inequality, and the mixing inequality of Hall and Heyde (1980, Cor. A.2). Similarly, we can show that $E[\kappa'_{2,T}\mathcal{W}\kappa_{2,T}]$ and $E[\kappa'_{3,T}\mathcal{W}\kappa_{3,T}]$ are $O(1)$, $E[\kappa'_{1,T}\mathcal{W}\kappa_{2,T}] = O(S_T^{1/2}T^{-1/2})$, but $E[\kappa'_{1,T}\mathcal{W}\kappa_{3,T}] = o(\eta_T)$ and $E[\kappa'_{2,T}\mathcal{W}\kappa_{3,T}] = o(1)$. ■

Proof of Proposition 2. From the proof of Theorem 1,

$$E[\kappa'_{1,T}\mathcal{W}\kappa_{1,T}] \to \sum_{s=-\infty}^{\infty} E[g'_t H'_0 \mathcal{W} H_0 g_{t-s}] = \mathrm{vec}(\Omega_0)'\mathrm{vec}\left(H'_0 \mathcal{W} H_0\right).$$

By Andrews (1991, Prop. 1(b)) we have $S_T^q(\overline{\Omega}_T - \Omega_0) \to -g_q\Omega_0^{(q)}$ and thus

$$E[\kappa'_{3,T}\mathcal{W}\kappa_{3,T}] = E[F'_T P'_0 S_T^q(\overline{\Omega}_T - \Omega_0) H'_0 \mathcal{W} H_0 S_T^q(\overline{\Omega}_T - \Omega_0) P_0 F_T]$$
$$\to g_q^2 \sum_{s=-\infty}^{\infty} E\left[g'_t P'_0 \Omega_0^{(q)} H'_0 \mathcal{W} H_0 \Omega_0^{(q)} P_0 g_{t-s}\right]$$
$$= g_q^2 \mathrm{vec}(\Omega_0)'\mathrm{vec}\left(P'_0\Omega_0^{(q)} H'_0 \mathcal{W} H_0 \Omega_0^{(q)} P_0\right)$$
$$= g_q^2 tr\left(\Omega_0^{(q)} H'_0 \mathcal{W} H_0 \Omega_0^{(q)} P_0\right)$$

because, for conformable matrices A, B, $tr(AB) = tr(BA)$, and $P_0 G_0 H_0 = 0$. Next, consider the terms

$$E\left[\kappa'_{1,T}\mathcal{W}\kappa_{2,T}\right] = E\left[F'_T H'_0 \mathcal{W} H_0 \sqrt{T/S_T}(\hat{\Omega}_T - \overline{\Omega}_T) P_0 F_T\right]$$
$$= E\left[(F_T \otimes F_T)'\mathrm{vec}\left(H'_0 \mathcal{W} H_0 \sqrt{T/S_T}(\hat{\Omega}_T - \overline{\Omega}_T) P_0\right)\right]$$
$$= E\left[(F_T \otimes F_T)'\left(P'_0 \otimes H'_0 \mathcal{W} H_0\right)\mathrm{vec}\left(\sqrt{T/S_T}(\hat{\Omega}_T - \overline{\Omega}_T)\right)\right]$$
$$= E\left[\left(F_T \otimes F_T \otimes \mathrm{vec}\left(\sqrt{T/S_T}(\hat{\Omega}_T - \overline{\Omega}_T)\right)\right)'\right]\mathrm{vec}\left(P'_0 \otimes H'_0 \mathcal{W} H_0\right)$$

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and, similarly,

$$
\begin{aligned}
&E[\kappa_{2,T}'\mathcal{W}\kappa_{2,T}] \\
&\quad = E\left[F_T' P_0'\sqrt{T/S_T}(\hat{\Omega}_T - \overline{\Omega}_T)H_0'\mathcal{W}H_0\sqrt{T/S_T}(\hat{\Omega}_T - \overline{\Omega}_T)P_0 F_T\right] \\
&\quad = \frac{T}{S_T}\mathrm{vec}(P_0' \otimes P_0')' E\left[F_T \otimes F_T \otimes (\hat{\Omega}_T - \overline{\Omega}_T) \otimes (\hat{\Omega}_T - \overline{\Omega}_T)\right]\mathrm{vec}(H_0'\mathcal{W}H_0).
\end{aligned}
$$

In order to find expressions for these two cross-products involving $\sqrt{T/S_T}(\hat{\Omega}_T - \overline{\Omega}_T)$, we make use of the BN-decomposition of the linear process $\{g_t\}$, viz.

$$g_t = \Psi e_t + \tilde{e}_{t-1} - \tilde{e}_t,$$

where $\Psi := \sum_{j\geq 0}\Psi_j$, $\tilde{e}_t := \sum_{j\geq 0}\tilde{\Psi}_j e_{t-j}$, and the tail sums $\tilde{\Psi}_j := \sum_{k\geq j+1}\Psi_k$. With this representation of $\{g_t\}$ we can calculate limiting variances and covariances of g_t based only on Ψe_t and disregard the transient part of the process, $\tilde{e}_{t-1} - \tilde{e}_t$. Since $e_t \sim N(0, \Sigma_e)$, third and fourth moments are zero. Therefore,

$$
\begin{aligned}
&S_T^{1/2}T^{3/2}E\left[F_T \otimes F_T \otimes \mathrm{vec}\left(\sqrt{T/S_T}(\hat{\Omega}_T - \overline{\Omega}_T)\right)\right] \\
&\quad = E\left[\sum_{s,t=1}^{T} g_t \otimes g_s \otimes \left(\sum_{\upsilon=1-T}^{T-1}\sum_{u=\max\{1,1-\upsilon\}}^{\min\{T,T-\upsilon\}} k\left(\frac{\upsilon}{S_T}\right)\mathrm{vec}\left(g_{u+\upsilon}g_u' - E[g_{u+\upsilon}g_u']\right)\right)\right] \\
&\quad = E\left[\sum_{t=1}^{T}\Psi e_t \otimes \Psi e_t \otimes \left(\sum_{\substack{u=1\\u\neq t}}^{T}\mathrm{vec}\left(\Psi e_u \Psi e_u' - E[\Psi e_u \Psi e_u']\right)\right)\right] + o(1) \\
&\qquad + E\left[\sum_{s,t=1}^{T}\Psi e_t \otimes \Psi e_s \otimes \left(\sum_{\substack{\upsilon=1-T\\\upsilon\neq 0}}^{T-1}\sum_{u=\max\{1,1-\upsilon\}}^{\min\{T,T-\upsilon\}} k\left(\frac{\upsilon}{S_T}\right)\mathrm{vec}\left(\Psi e_{u+\upsilon}\Psi e_u'\right)\right)\right] \\
&\quad = \sum_{s,t=1}^{T}\sum_{\substack{\upsilon=1-T\\\upsilon\neq 0}}^{T-1}\sum_{u=\max\{1,1-\upsilon\}}^{\min\{T,T-\upsilon\}} k\left(\frac{\upsilon}{S_T}\right)E\left[\Psi e_t \otimes \Psi e_s \otimes \Psi e_u \otimes \Psi e_{u+\upsilon}\right] + o(1) \\
&\quad = \sum_{s,t=1}^{T} k\left(\frac{t-s}{S_T}\right)E\left[\Psi e_t \otimes \Psi e_s \otimes \Psi e_s \otimes \Psi e_t\right] \\
&\qquad + \sum_{s,t=1}^{T} k\left(\frac{t-s}{S_T}\right)E\left[\Psi e_t \otimes \Psi e_s \otimes \Psi e_t \otimes \Psi e_s\right] + o(1).
\end{aligned}
$$

One can show that $E[\Psi e_t \otimes \Psi e_s \otimes \Psi e_s \otimes \Psi e_t] = E[\Psi e_t \otimes \Psi e_s \otimes \Psi e_t \otimes \Psi e_s] = \mathrm{vec}(\Omega_0 \otimes \Omega_0)$ when $s \neq t$ and 0 otherwise. Noticing $\frac{1}{S_T T}\sum_{s,t=1}^{T} k((t-s)/S_T) \to \mu_1$, we then have

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$$\sqrt{\frac{T}{S_T}}E\left[\kappa_{1,T}'\mathcal{W}\kappa_{2,T}\right] \to 2\mu_1 \text{vec}(\Omega_0 \otimes \Omega_0)' \text{vec}\left(P_0 \otimes H_0'\mathcal{W}H_0\right)$$
$$= 2\mu_1 tr((\Omega_0 \otimes \Omega_0)(P_0 \otimes H_0'\mathcal{W}H_0))$$
$$= 2\mu_1 tr(\Omega_0 P_0 \otimes \Omega_0 H_0'\mathcal{W}H_0)$$
$$= 2\mu_1 tr(I_l - G_0H_0)tr(\Omega_0\Omega_0^{-1}G_0\Sigma_0\mathcal{W}\Sigma_0G_0'\Omega_0^{-1})$$
$$= 2\mu_1(l - tr(H_0G_0))tr(\Sigma_0\mathcal{W}\Sigma_0G_0'\Omega_0^{-1}G_0)$$
$$= 2\mu_1(l-p)tr(\Sigma_0\mathcal{W})$$

which uses the fact that $H_0G_0 = I_p$. By a similar derivation, $E[\kappa_{2,T}'\mathcal{W}\kappa_{2,T}] \to \mu_2(l-p)tr(\Sigma_0\mathcal{W})$ so that $\nu_2 = (2\mu_1 + \mu_2)(l-p)tr(\Sigma_0\mathcal{W})$. ■

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