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# Center foliation: absolute continuity, disintegration and rigidity

RÉGIS VARÃO

*Instituto de Ciências Matemáticas e de Computação, Universidade de São Paulo, Avenida Trabalhador São-Carlense, 400, 13566-590, São Carlos, SP, Brasil*
*(e-mail: jregis@impa.br)*

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*Abstract*. In this paper we address the issues of absolute continuity for the center foliation, as well as the disintegration on the non-absolute continuous case and rigidity of volume-preserving partially hyperbolic diffeomorphisms isotopic to a linear Anosov automorphism on $\mathbb{T}^3$. It is shown that the disintegration of volume on center leaves for these diffeomorphisms may be neither atomic nor Lebesgue, in contrast to the dichotomy (Lebesgue or atomic) obtained by Avila, Viana and Wilkinson [Absolute continuity, Lyapunov exponents and rigidity I: Geodesic flows. *Preprint*, 2012, arXiv:1110.2365v2] for perturbations of time-one of geodesic flow. In the case of atomic disintegration of volume on the center leaves of an Anosov diffeomorphism on $\mathbb{T}^3$, we show that it has to be one atom per leaf. Moreover, we show that not even a $C^1$ center foliation implies a rigidity result. However, for a volume-preserving partially hyperbolic diffeomorphism isotopic to a linear Anosov automorphism, assuming the center foliation is $C^1$ and transversely absolutely continuous with bounded Jacobians, we obtain smooth conjugacy to its linearization.

## 1. *Introduction and statements*

We study the measure-theoretical properties of the center foliation of partially hyperbolic diffeomorphisms for which the center leaves are non-compact. Two main issues are those of absolute continuity and rigidity:

- When is the center foliation absolutely continuous? Concerning the disintegration, what can be said otherwise?
- Concerning rigidity: does absolute continuity imply greater regularity?

To better illustrate these questions we mention a recent result of Avila, Viana and Wilkinson [**1**, Theorem A], which provides answers to these issues for a certain class of partially hyperbolic diffeomorphisms. They show that perturbations (preserving the

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Liouville probability measure) of the time-one for the geodesic flow on a closed negatively curved surface satisfy the following dichotomy.

- *Atomic disintegration*: If the center foliation is non-absolutely continuous, then there exist $k \in \mathbb{N}$ and a full volume subset that intersects each center leaf on exactly $k$ orbits.
- *Rigidity*: If the center foliation is absolutely continuous then the diffeomorphism is the time one of a smooth flow. In particular, the center foliation is a smooth foliation.

In fact the arguments used in [**1**] can be applied to some other types of partially hyperbolic diffeomorphisms. For instance, the proof of Avila, Viana and Wilkinson's result reduces the problem to a cocycle with compact fibers. Hence, similar arguments can be used to work on some large class of volume-preserving skew-products on a three-dimensional manifold (plus some hypothesis, such as dynamical coherence and ergodicity). In private communications with Viana and Wilkinson the author has been informed that Part II of [**1**] will contain all these results as well as some other generalities.

On three-dimensional manifolds the only known examples of transitive partially hyperbolic diffeomorphism are of skew-product type, perturbation of time-one of Anosov flows and diffeomorphisms derived from a linear Anosov automorphism (DA). In fact, it has been conjectured by E. Pujals that these are the only possibilities (see [**4**] for precise statements).

In this paper, we focus on the third type, DA diffeomorphisms, since for the first two types of partially hyperbolic diffeomorphisms the questions concerning the absolute continuity of the center foliation and rigidity are fairly well understood, as mentioned above.

We say that $f$ is a DA diffeomorphism if it is partially hyperbolic and lies in the isotopy class of some hyperbolic linear automorphism $A$. We refer to $A$ as the linearization of $f$. Every partially hyperbolic diffeomorphism has the split $TM = E^s \oplus E^c \oplus E^u$ on the tangent space (see §2 for definitions), where $E^s$ is a contracting direction, $E^u$ is an expanding direction and the center direction $E^c$ has an intermediate behavior. We consider Anosov diffeomorphisms (see §2 for definition) which are partially hyperbolic diffeomorphisms. That means that they have the splitting $TM = E^s \oplus E^c \oplus E^u$ and the center direction is uniformly contracting or expanding. It then makes sense to talk about the center foliation of an Anosov (partially hyperbolic) diffeomorphism, as we do in Theorems 1.1 and 1.3. A center foliation (see §2) is an invariant foliation by $f$ tangent to the $E^c$ direction. We also mention that all diffeomorphisms treated in this work are assumed to be at least $C^{1+\alpha}$, for some $\alpha > 0$. This implies, in particular, that a volume-preserving Anosov diffeomorphism on the three-dimensional torus $\mathbb{T}^3$ is ergodic.

We now state our results. For a non-absolutely continuous center foliation, we show that it is possible to have a disintegration (see §2.2 for its definition) which is non-Lebesgue and non-atomic. Our conclusion exhibits a new example of behavior for the center foliation of a volume-preserving partially hyperbolic diffeomorphism in a three-dimensional manifold:

THEOREM 1.1. *For volume-preserving partially hyperbolic Anosov diffeomorphisms on* $\mathbb{T}^3$*, for which the center foliation is non-absolutely continuous:*

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(i) *there exist Anosov diffeomorphisms for which the disintegration of volume on the center leaves is neither Lebesgue nor atomic—in fact, such diffeomorphisms fill a dense subset of an infinite-dimensional manifold in the neighborhood of any hyperbolic linear automorphisms in the space of volume-preserving maps;*
(ii) *the conditional measures are singular measures with respect to the volume on the center leaf;*
(iii) *if the disintegration is atomic, then there is exactly one atom per leaf—that is, there exists a set of full volume that intersects each center leaf in one point.*

In the setting of DA diffeomorphisms there have been some results concerning the non-absolute continuity of the center foliation (for instance, [**7**, **14**]), but no further description. Theorem 1.1 gives more information than merely non-absolute continuity. Ruelle and Wilkinson [**13**] exhibited a $C^1$ open set inside the set of volume-preserving partially hyperbolic (skew-product) diffeomorphisms that has atomic disintegration; their method is also used in [**1**] to analyze the perturbation of a time-one of a geodesic flow. More recently, Homburg [**11**] proved that Ruelle and Wilkinson's example has in fact one atom per leaf. Item (iii) of Theorem 1.1 also states that atomic disintegration implies one atom per leaf. But the methods are different, for instance to prove item (iii) we only need to know the behavior of the center foliation, whereas [**11**] used the absolute continuity of the stable and unstable foliations. Item (iii) has recently been generalized, using the ideas from this paper, for any conservative DA diffeomorphism on $\mathbb{T}^3$ by the author in joint work with Ponce and Tahzibi [**12**].

THEOREM 1.2. *Let $f : \mathbb{T}^3 \to \mathbb{T}^3$ be a volume-preserving partially hyperbolic Anosov diffeomorphism. Then the disintegration of volume on the center leaves is atomic if and only if the partition by center leaves is a measurable partition.*

The next result shows, in contrast to the dichotomy from [**1**] (for another class of partially hyperbolic diffeomorphisms), that absolute continuity has no rigidity implications in our case.

THEOREM 1.3. *There exist volume-preserving Anosov diffeomorphisms $f$ on $\mathbb{T}^3$ for which the center foliation is a $C^1$ foliation (in particular, absolutely continuous) but $f$ is not $C^1$-conjugate to its linearization.*

*In fact, such diffeomorphisms fill a dense subset of an infinite-dimensional manifold in the neighborhood of any hyperbolic linear automorphism in the space of volume-preserving maps.*

The next result provides a better understanding of the conjugacy, and besides being an important result in its own right, will help us in the proof of Theorems 1.3 and 1.4.

PROPOSITION 1.1. *Let $f$ be a volume-preserving partially hyperbolic Anosov diffeomorphism on $\mathbb{T}^3$. The Lyapunov exponents of any two periodic points on each of the directions (stable, center, unstable) are the same if and only if $f$ is $C^1$ conjugate to its linearization.*

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Note that Theorem 1.3 implies that to obtain a rigidity result we must impose some stronger conditions on the center foliation besides being a $C^1$ foliation. And we do so by asking the center foliation to be transversely absolute continuous with bounded Jacobian (see §2.2.1).

THEOREM 1.4. *Let $f$ be a volume-preserving DA diffeomorphism on $\mathbb{T}^3$, with linearization $A$. If the center foliation is a $C^1$ foliation and transversely absolutely continuous with bounded Jacobians, then $f$ is $C^1$ conjugate to $A$ and, hence, is an Anosov diffeomorphism.*

The hypothesis of $C^1$ center foliation already gives a local bound for the Jacobians of the center holonomy. Hence, the additional hypothesis to the $C^1$ condition needed to obtain the rigidity result is a natural one, in the sense that some local property is not enough and we ask it to be a uniform one.

*Organization of the paper.* In §2 we give some basic definitions such as the notion of disintegration of a measure, and absolute continuity. In §3 we study the behavior of non-absolute continuous center foliation, and prove Theorems 1.1 and 1.2. We begin §4 with a discussion of how the Lyapunov exponents vary with respect to their linearization, and then prove Proposition 1.1 and Theorem 1.3. In §5 we construct some conditional measures, not probabilities, on each center leaf with some dynamical meaning; we use these measures to prove Theorem 1.4.

## 2. *Preliminaries*

2.1. *Partially hyperbolic diffeomorphism.* A diffeomorphism $f$ of a compact Riemannian manifold $M$ is called a *partially hyperbolic diffeomorphism* if there are positive constants $\lambda < \hat{\gamma} < 1 < \gamma < \mu$ and $C > 1$ and a $Df$-invariant splitting of $TM = E^u \oplus E^c \oplus E^s$ where

$$\frac{1}{C}\mu^n\|v\| < \|Df^n v\|, \quad v \in E^u_x \setminus \{0\};$$

$$\frac{1}{C}\hat{\gamma}^n\|v\| < \|Df^n v\| < C\gamma^n\|v\|, \quad v \in E^c_x \setminus \{0\};$$

$$\|Df^n v\| < C\lambda^n\|v\|, \quad v \in E^s_x \setminus \{0\}.$$

We say that a partially hyperbolic diffeomorphism is *dynamically coherent* if the subbundles $E^s \oplus E^c$ and $E^c \oplus E^u$ integrate into invariant foliations, $\mathcal{F}^{cs}$ and $\mathcal{F}^{cu}$, respectively. This implies, in particular, that there is a center foliation $\mathcal{F}^c$ which is obtained by an intersection of the other two: $\mathcal{F}^c = \mathcal{F}^{cs} \cap \mathcal{F}^{cu}$. Brin, Buragov and Ivanov [**6**] proved the following theorem.

THEOREM 2.1. *Every partially hyperbolic diffeomorphism on $\mathbb{T}^3$ is dynamically coherent.*

2.2. *Disintegration of measure.* Let $(M, \mu, \mathcal{B})$ be a probability space, where $M$ is a compact metric space, $\mu$ a probability and $\mathcal{B}$ a Borel $\sigma$-algebra. Given a partition $\mathcal{P}$ of $M$ by measurable sets, we associate the probability space $(\mathcal{P}, \widetilde{\mu}, \widetilde{\mathcal{B}})$, where $\widetilde{\mu} := \pi_*\mu$, $\widetilde{\mathcal{B}} := \pi_*\mathcal{B}$, and $\pi : M \to \mathcal{P}$ is the canonical projection that associates to a point of $M$ the partition element that contains it.

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For a given partition $\mathcal{P}$, a family $\{\mu_P\}_{P\in\mathcal{P}}$ is a *system of conditional measures* for $\mu$ (with respect to $\mathcal{P}$) if:

(i) given $\phi \in C^0(M)$, then $P \mapsto \int \phi\mu_P$ is measurable;
(ii) $\mu_P(P) = 1$, $\widetilde{\mu}$-almost everywhere;
(iii) if $\phi \in C^0(M)$, then $\int_M \phi\, d\mu = \int_{\mathcal{P}}(\int_P \phi\, d\mu_P)\, d\widetilde{\mu}(P)$.

We call $\mathcal{P}$ a *measurable partition* (with respect to $\mu$) if there exist a family $\{A_i\}_{i\in\mathbb{N}}$ of Borel sets and a set $F$ of full $\mu$-measure such that for every $P \in \mathcal{P}$ there exists a sequence $\{B_i\}_{i\in\mathbb{N}}$, where $B_i \in \{A_i, A_i^c\}$ such that $P \cap F = \bigcap_{i\in\mathbb{N}} B_i \cap F$.

The following result is known as Rokhlin's disintegration theorem.

THEOREM 2.2. *Let $\mathcal{P}$ be a measurable partition of a compact metric space $M$ and $\mu$ a Borel probability. Then there exists a disintegration by conditional measures for $\mu$.*

*Remark.* In Theorem 1.1 the meaning of 'disintegration of volume on the center leaves is neither Lebesgue nor atomic' is that, on a foliated box, since the center foliation forms a measurable partition, we can apply Rokhlin's disintegration theorem and the conditional measures are neither Lebesgue nor atomic. This notion is independent of the choice of foliated box (see Lemma 5.1), which explains why we do not need to say instead that the disintegration is locally neither Lebesgue nor atomic.

2.2.1. *Absolute continuity.* We say that the foliation $\mathcal{F}$ is *transversely absolutely continuous with bounded Jacobians* if, given an angle $\theta \in (0, \pi/2]$, there exists $K \geq 1$ such that, for any two transversals $\tau_1$, $\tau_2$ to $\mathcal{F}$ of angle at least $\theta$ with the leaves of $\mathcal{F}$, and any measurable set $A \subset \tau_1$, then

$$K^{-1}\lambda_{\tau_1}(A) \leq \lambda_{\tau_2}(h_{1,2}^{\mathcal{F}}(A)) \leq K\lambda_{\tau_1}(A),$$

where $\lambda_{\tau_i}$ is the Lebesgue measure on the transversal $\tau_i$ and $h_{1,2}^{\mathcal{F}}$ is the $\mathcal{F}$-holonomy from $\tau_1$ to $\tau_2$.

Let $\mathcal{F}$ be a foliation; we disintegrate the volume inside a foliated box. If the conditional measure $m_L$ on the leaf $L$ satisfies $m_L \ll Leb_L$ for almost every leaf, then $\mathcal{F}$ is said to be an *absolutely continuous foliation*, where $Leb_L$ is the Lebesgue measure on the leaf $L$.

To understand absolute continuity for partially hyperbolic diffeomorphisms with non-compact center leaves, our starting point is a result due to Gogolev [**7**]. He works with Anosov diffeomorphism with splitting of the form $E^s \oplus E^{wu} \oplus E^{uu}$, where $E^{wu}$ is the weak unstable direction, $E^{uu}$ is the strong unstable direction and $E^s$ is the stable, that is, contracting direction. That means the Anosov diffeomorphism can also be seen as a partially hyperbolic diffeomorphism, where $E^{wu}$ is the center direction. Recall that $\lambda^{uu}(x)$ is the Lyapunov exponent on the strong unstable direction at $x \in \mathbb{T}^3$. That is, $\lambda^{uu}(x) = \lim_{n\to\infty}(1/n)\log \|Df^n v^{uu}\|$, where $v^{uu} \in E^{uu}(x)$ has norm one.

THEOREM 2.3. *Let $f : \mathbb{T}^3 \to \mathbb{T}^3$ be an Anosov diffeomorphism with splitting of the form $E^s \oplus E^{wu} \oplus E^{uu}$. Then $\mathcal{F}^c_f$ is absolutely continuous if and only if $\lambda^{uu}(p) = \lambda^{uu}(q)$ for all periodic points $p$ and $q$.*

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2.3. *Geometric property.* By a Derived from Anosov (DA) diffeomorphism $f : \mathbb{T}^3 \to \mathbb{T}^3$ we mean a partially hyperbolic diffeomorphism homotopic to a linear Anosov diffeomorphism $A$. We call this linear Anosov diffeomorphism the linearization of $f$. In fact, $f$ is semi-conjugated to its linearization. In the theorem below, the proof of which can be found in Sambarino ([**15**]; see also [**16**]), we list the main properties of the semi-conjugacy.

THEOREM 2.4. *Let $B : \mathbb{R}^3 \to \mathbb{R}^3$ be a linear hyperbolic isomorphism. Then there exists $C > 0$ such that if $G : \mathbb{R}^3 \to \mathbb{R}^3$ is a homeomorphism such that $\sup\{\|G(x) - B(x)\| \mid x \in \mathbb{R}^3\} = K < \infty$ then there exists $H : \mathbb{R}^3 \to \mathbb{R}^3$ continuous and surjective satisfying:*

- *$B \circ H = H \circ G$;*
- *$\|H(x) - x\| \leq CK$ for all $x \in \mathbb{R}^3$;*
- *$H(x)$ is characterized as the unique point $y$ such that*

$$\|B^n(y) - G^n(x)\| \leq CK \quad \text{for all } n \in \mathbb{Z};$$

- *$H(x) = H(y)$ if and only if $\|G^n(x) - G^n(y)\| \leq 2CK$, for all $n \in \mathbb{Z}$, and if and only if $\sup_{n\in\mathbb{Z}}\{\|G^n(x) - G^n(y)\|\} < \infty$;*
- *if $B \in SL(3, \mathbb{Z})$ and $G$ is the lift of $g : \mathbb{T}^3 \to \mathbb{T}^3$ then $H$ induces $h : \mathbb{T}^3 \to \mathbb{T}^3$ continuous and onto such that $B \circ h = h \circ g$ and $\text{dist}_{C^0}(h, id) \leq C\text{dist}_{C^0}(B, g)$.*

The geometrical property we shall need later is given by Hammerlindl [**9**]:

PROPOSITION 2.1. *Let $f$ be a DA diffeomorphism and $A$ be its linearization. Denote by $\tilde{f}$ and $\tilde{A}$ the lift to $\mathbb{R}^n$ of $f$ and $A$, respectively. Then for each $k \in \mathbb{Z}$ and $C > 1$ there is $M > 0$ such that for $x, y \in \mathbb{R}^n$,*

$$\|x - y\| > M \Rightarrow \frac{1}{C} < \frac{\|\tilde{f}^k(x) - \tilde{f}^k(y)\|}{\|\tilde{A}^k(x) - \tilde{A}^k(y)\|} < C.$$

*More generally, for each $k \in \mathbb{Z}$, $C > 1$, and a linear map $\pi : \mathbb{R}^n \to \mathbb{R}^n$ there is an $M > 0$ such that for all $x, y \in \mathbb{R}^n$*

$$\|\pi(x - y)\| > M \Rightarrow \frac{1}{C} < \frac{\|\pi(\tilde{f}^k(x) - \tilde{f}^k(y))\|}{\|\pi(\tilde{A}^k(x) - \tilde{A}^k(y))\|} < C.$$

## 3. *Non-absolute continuity*

We devote this section this the proof of Theorems 1.1 and 1.2. We prove items (i)–(iii) of Theorem 1.1 in §§3.1, 3.2 and 3.3, respectively. We then prove Theorem 1.2 in §3.4.

3.1. *Proof of Theorem 1.1(i).* Consider a volume-preserving linear Anosov automorphism $A$ with the splitting $TM = E^{ss} \oplus E^{ws} \oplus E^u$. Let $\phi$ be a volume-preserving diffeomorphism different from the identity which preserves the stable manifold $\mathcal{F}^s_A$ of $A$ as well as being volume-preserving restricted to the stable manifold of $A$. Recall that $\mathcal{F}^s_A$ is a foliation by two-dimensional planes tangent to $E^{ss} \oplus E^{ws}$. We can take $\phi$ as $\phi(x, y, z) = (R_z(x, y), z)$ on some small ball and as the identity outside this ball, where $R_z$ is a volume-preserving map. By Baraviera and Bonnatti [**2**] one has $\int \lambda^{ws}_A \, d\text{Vol} \neq \int \lambda^{ws}_{A\circ\phi} \, d\text{Vol}$. Let $h$ be the conjugacy between $A$ and $f$, i.e. $f \circ h = h \circ A$. We claim that $h$ is volume-preserving and sends center leaves of $A$ to center leaves of $f$.

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To see that $h$ is volume-preserving note that $f$ and $A$ have the same topological entropy $\lambda_A^u$. Hence, $h_*\text{Vol}$ is a measure of maximal entropy. Observe that the perturbation $A \circ \phi$ of $A$ preserves the $E^u$ exponent, which means that by the equilibrium state theory (see Bowen [**5**]) the potentials 0 and $-\log \|Df_{|E^u}\|$ are cohomological and therefore give the same equilibrium states. That is, $h_*\text{Vol} = \text{Vol}$. And the fact that $h(\mathcal{F}^c) = \mathcal{F}^c$ comes from [**8**, Lemma 2].

CLAIM. *We claim that $\mathcal{F}^c_{A\circ\phi}$ is not absolutely continuous.*

*Proof.* Suppose that $\mathcal{F}^c_{A\circ\phi}$ is absolutely continuous. Then Theorem 2.3 implies that $\lambda^{ss}_f(p) = \text{const}$ for all periodic points $p \in \mathbb{T}^3$. By construction we have $\lambda^u_f(p) = \lambda^u_A$. Since we are in the volume-preserving case, $\lambda^{ws}(p)$ is also constant at periodic points. Therefore, by Proposition 1.1, $f$ is $C^1$-conjugate to $A$, but this would imply that $\int \lambda^{ws}_f \, d\text{Vol} = \int \lambda^{ws}_A \, d\text{Vol}$ which is absurd by [**2**, Theorem 3]. □

CLAIM. *The disintegration of volume on center leaves of $A \circ \phi$ is neither Lebesgue nor atomic.*

*Proof.* It is not Lebesgue because it is not absolutely continuous. And to see that it is not atomic, note that since $h$ is volume-preserving and sends center leaves onto center leaves we can induce (by push-forward) the disintegration on the center leaves of $A$ to the center leaves of $A \circ \phi$. And since the disintegration for $A$ is Lebesgue, this implies that the disintegration for $A \circ \phi$ is not atomic. □

3.2. *Proof of Theorem 1.1(ii).* One can suppose that the Anosov diffeomorphism $f$ has a two-dimensional unstable leaf, otherwise work with $f^{-1}$. That is, the center foliation is expanded by $f$. Working with $f$ or $f^{-1}$ (both volume-preserving) does not affect the disintegration of volume.

By ergodicity we know that the Birkhoff set

$$B = \left\{ x \in \mathbb{T}^3 \;\middle|\; \frac{1}{n}\sum_{i=0}^{n-1} \delta_{f^i(x)} \to \text{Vol as } n \to \infty \right\}$$

has full measure.

CLAIM. *If there is a center leaf such that $\mathcal{F}^c \cap B$ has positive Lebesgue measure, then the center foliation is absolutely continuous.*

*Proof.* Let $D$ be any disc on the center foliation and consider the construction

$$\mu_n = \frac{1}{n}\sum_{j=0}^{n-1} f^j_*\left(\frac{m_D}{m_D(D)}\right),$$

where $m_D$ means the Lebesgue measure on the center leaf. If a subsequence of $\{\mu_n\}$ converges to a measure $\mu$, then the disintegration of $\mu$ on the center leaves is absolutely continuous with respect to the Lebesgue measure. This is a well-known construction of measures, studied by Pesin and Sinai in the 1980s. For more references see [**3**, Ch. 11] and the references therein. Although Pesin and Sinai studied these measures for the case

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of the disc $D$ in the unstable foliation, for the center foliation, in our case the construction is the same. Gogolev and Guysinsky [**8**] worked explicitly on this case and we refer the reader to that paper.

We make a slightly different construction: instead of the disc $D$, as above, we take $D \cap B$ which has positive Lebesgue measure on the center leaf. By hypothesis there exists such a disc. And any limit of a convergent subsequence of such measures still has conditional measures absolutely continuous to the Lebesgue measure on the center leaf [**3**, Lemma 11.12].

Denote $D_0 := D \cap B$. Let us see that $\mu_n = (1/n) \sum_{j=0}^{n-1} f_*^j(m_{D_0}/m_{D_0}(D_0))$ actually converges to the volume measure. In order to simplify notation we may suppose that $m_{D_0}(D_0) = 1$, otherwise just carry it in the calculation.

Let $\phi : \mathbb{T}^3 \to \mathbb{R}$ be a continuous function. To obtain the desired convergence we shall see that

$$\lim_{n\to\infty} \int \phi \, d\mu_n = \int \phi \ d\text{Vol}.$$

Let $\phi^+ := \int \phi \ d\text{Vol}$. In particular, $\phi^+$ is the time average for a Birkhoff point, that is, $\phi^+ = \lim_{n\to\infty}(1/n) \sum_{k=0}^{n} \phi(f^k(x))$, for all $x \in B$. Then

$$\begin{aligned}\int_{\mathbb{T}^3} \phi \, d\mu_n &= \frac{1}{n} \sum_{j=0}^{n} \int_{\mathbb{T}^3} \phi \, df_*^j(m_{D_0}) = \frac{1}{n} \sum_{j=0}^{n} \int_{f^j(D_0)} \phi \, df_*^j(m_{D_0}) \\ &= \int_{f^j(D_0)} \frac{1}{n} \sum_{j=0}^{n} \phi \, df_*^j(m_{D_0}) = \int_{D_0} \frac{1}{n} \sum_{j=0}^{n} \phi \circ f^j \, dm_{D_0}.\end{aligned}$$

By the dominated convergence theorem we can take the limit inside the integral, obtaining

$$\begin{aligned}\lim_{n\to\infty} \int_{\mathbb{T}^3} \phi \, d\mu_n &= \int_{D_0} \lim_{n\to\infty} \frac{1}{n} \sum_{j=0}^{n} \phi \circ f^j \, dm_{D_0} \\ &= \int_{D_0} \phi^+ \, dm_{D_0} = \phi^+,\end{aligned}$$

proving that $\mu_n$ converges to the volume measure. Hence, the volume measure has Lebesgue disintegration on the center leaves, which proves the claim. □

From the claim, since we are in the case where the center foliation is non-absolutely continuous, we must have that the center foliation intersects $B$ on a set of zero Lebesgue measure. But the conditional measures give full measure to $B$, since $B$ has full volume measure. Therefore the conditional measures are singular with respect to the Lebesgue measure. And item (ii) is proved.

3.3. *Proof of Theorem 1.1(iii).* As well as in the above item, we suppose the Anosov diffeomorphism $f$ has a two-dimensional unstable leaf (otherwise work with $f^{-1}$). That is, the center foliation is expanded by $f$.

In what follows let $\{R_i\}_{i=1}^{l}$ be the rectangles of a fixed finite Markov partition. On each Markov rectangle, we may apply Rokhlin's disintegration theorem on the segments of center leaves. A segment of a center leaf, which contains $x$, on the rectangle $R_i$ is the

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connected component of $R_i \cap \mathcal{F}^c(x)$ that contains $x$. Let $R(x)$ denote the Markov partition that contains $x$. Therefore, $m_x$ means the conditional measure for the disintegration of volume on $R(x)$. By hypothesis volume has atomic disintegration on the center leaves. Hence, the conditional measures $m_x$ are sums of atomic measures, where atomic measure is a multiple of a Dirac measure. These atomic measures are supported on the atoms of the disintegration. Let

$$A := \{\text{atoms}\} = \left\{ \bigcup_{x \in \mathbb{T}^3} \operatorname{supp} m_x \right\}.$$

Recall that since $f$ is volume-preserving the set $A$ is invariant by $f$. We start by proving that the atoms have the same weight.

LEMMA 3.1. *There exists* $\gamma_0 \in [0, 1]$ *such that* $\{x \in A \mid m_x(x) = \gamma_0\}$ *has full volume measure.*

*Proof.* Consider the set $A_\gamma := \{x \in A \mid m_x(x) \leq \gamma\}$, $\gamma \in [0, 1]$. The Markov property $f(\mathcal{F}^c_{R(x)}(x)) \supset \mathcal{F}^c_{R(f(x))}(f(x))$ and Lemma 5.1 together imply that $f_* m_x(I) \leq m_{f(x)}(I)$ where $I$ is any subset of the connected component of $\mathcal{F}^c_{f(x)} \cap R(f(x))$ that contains $f(x)$, in particular, $f_* m_x(f(x)) \leq m_{f(x)}(f(x))$. If $f(x) \in A_\gamma$, then

$$m_x(x) = m_x(f^{-1} \circ f(x)) = f_* m_x(f(x)) \leq m_{f(x)}(f(x)) \leq \gamma.$$

Hence, $f^{-1}(A_\gamma) \subset A_\gamma$.

By ergodicity, since our Anosov diffeomorphism is volume-preserving on $\mathbb{T}^3$, $A_\gamma$ has full volume measure or zero volume measure. Let $\gamma_0$ be the discontinuity point of the function $\gamma \in [0, 1] \mapsto \operatorname{Vol}(A_\gamma)$. This implies that almost every atom has weight $\gamma_0$. □

LEMMA 3.2. *On every Markov partition* $R_i$ *the conditional measures have the same number of atoms.*

*Proof.* The conditional measures $m_x$ are probabilities. We also know that $m_x$ are the sum of atomic measures, but the above lemma says that all these atomic measures have the same weight. Since all the atoms have the same weight $\gamma_0$ the conditional measures must have $1/\gamma_0$ atoms. □

In fact, we have the following lemma.

LEMMA 3.3. $\gamma_0 = 1$.

*Proof.* Let $R$ be some fixed rectangle from our initial Markov partition. Recall that $A_{\gamma_0} := \{x \in A \mid m_x(x) = \gamma_0\}$. Suppose that $\gamma_0 \neq 1$, which means that the set $A_{\gamma_0} \cap R$ has more than one atom on each center segment. From Rokhlin's disintegration theorem we have that $A_{\gamma_0}$ is a measurable set. Consider the set $\hat{R}$ to be the set of pairs $(x, y)$, such that $x, y \in A_{\gamma_0}$, that are in the same center segment. That is,

$$\hat{R} = \{(x, y) \in R \times R \mid x, y \in \mathcal{F}^c_R(y)\} \cap A_{\gamma_0} \times A_{\gamma_0},$$

where $\mathcal{F}^c_R$ is the segment of the center leaf restricted to $R$. Then $\hat{R}$ is a measurable set of $R \times R$. Since the function distance $d$ on $\hat{R}$ is measurable, we obtain the measurable

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function

$$h : \hat{R} \to \mathbb{R}$$
$$(x, y) \mapsto d(x, y).$$

Let $B_i = h^{-1}([1/i, \infty))$. Since $B_i \nearrow \hat{R}$, and $\hat{R}$ has positive volume (in this case the volume is Vol $\times$ Vol), there exists $i_0$ such that $\text{Vol} \times \text{Vol}(B_{i_0}) > 0$. Since $h(x, y) = h(y, x)$, the set $B_{i_0}$ is of the form $B_0 \times B_0$. Hence, $\text{Vol}(B_0) > 0$. Since $d_{\mathcal{F}^c}(x, y) \geq d(x, y)$ and the center foliation expands uniformly, consider some very large iteration $n$ of $f$. Then the points on $B_0$ will be uniformly very far away, hence points that were in the same local center leaf are very far away, and therefore are no longer in the same local center leaf. That means that we decrease the number of atoms on each Markov rectangle. This is absurd. Therefore $\gamma_0 = 1$. □

We conclude with the following lemma.

LEMMA 3.4. *There is one atom per center leaf.*

*Proof.* Now we have an invariant set $A_1$ of full measure that has only one atom on each local center leaf. But suppose there are two atoms $x$, $y$ in the same center leaf. Iterating these atoms backward, they come together, hence it would contradict that on a local center leaf there is only one atom. □

3.4. *Proof of Theorem 1.2.* For this proof we assume the center foliation to be contracting (otherwise consider $f^{-1}$).

Suppose that $\{\mathcal{F}^c(x)\}_{x \in \mathbb{T}^3}$ is a measurable partition. Then we can apply Rokhlin's theorem and decompose the volume on probabilities $m_x$ on center leaves. Let

$$S_L = \{x \in \mathbb{T}^3 \mid m_x(\mathcal{F}^c(x, L)) \geq 0.6\},$$

where $\mathcal{F}^c(x, L)$ is the segment of $\mathcal{F}^c(x)$ of length $L$ on the induced metric and centered at $x$.

Since $m_x$ are probabilities, $\bigcup_{n=1}^{\infty} S_n = \mathbb{T}^3$, therefore there exists $L \in \mathbb{R}$ such that $\text{Vol}(S_L) > 0$. Since $f$ contracts the center leaf, then $f^{-1}(\mathcal{F}^c(f(x), L)) \supset \mathcal{F}^c(x, L)$. Since $f_* m_x = m_{f(x)}$, for $x \in S_L$,

$$m_{f(x)}(\mathcal{F}^c(f(x), L)) = m_x(f^{-1}(\mathcal{F}^c(f(x), L))) \geq m_x(\mathcal{F}^c(x, L)) \geq 0.6.$$

So $f(x) \in S_L$, and by ergodicity $f(S_L) \subset S_L$ implies $\text{Vol}(S_L) = 1$.

CLAIM. $\text{diam}^c S_L \cap \mathcal{F}^c(x) \leq 2L$, *where* $\text{diam}^c$ *means the diameter of the set inside the center leaf.*

*Proof.* Suppose that there exist $y_1, y_2 \in A_L \cap \mathcal{F}^c(x)$ with $d^c(y_1, y_2) > 2L$, where $d^c$ is the distance inside the center leaf. Then

$$\mathcal{F}^c(y_1, L) \cap \mathcal{F}^c(y_2, L) = \emptyset \quad \text{and} \quad m_x(\mathcal{F}^c(y_i, L)) \geq 0.6, \ i = 1, 2.$$

Then

$$1 \geq m_x(\mathcal{F}^c(y_1, L) \cup \mathcal{F}^c(y_2, L)) = m_x(\mathcal{F}^c(y_1, L)) + m_x(\mathcal{F}^c(y_2, L)) \geq 0.6 + 0.6 = 1.2.$$

This is absurd and concludes the proof of the claim. □

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CLAIM. *The disintegration is atomic.*

*Proof.* Define

$$L_0 = \inf\{L \in [0, \infty) \mid \mathrm{Vol}(S_L) = 1\}.$$

We want to prove that $L_0 = 0$, which would imply atomicity. So let us suppose that $L_0 > 0$.

Note that $\mathrm{Vol}(S_{L_0}) = 1$. To see that, take a sequence $L_n \to L_0$ and observe that $S_{L_0} = \bigcap_n S_{L_n}$. Let $\lambda = \inf \|Df^{-1}|E^c\|$ and let $\varepsilon < 1$ be such that $\varepsilon\lambda > 1$. For $x \in S_{\lambda L_0}$,

$$m_{f(x)}(\mathcal{F}^c(f(x), \varepsilon L_0)) = m_x(f^{-1}(\mathcal{F}^c(f(x), \varepsilon L_0))) \geq m_x(\mathcal{F}^c(x, L_0)) \geq 0.6.$$

Therefore $f(x) \in S_{\varepsilon L_0}$. By ergodicity we may suppose that $S_{L_0}$ is $f$-invariant, hence $\mathrm{Vol}(S_{\varepsilon L_0}) = 1$. This is absurd by the choice of $L_0$ and since $\varepsilon L_0 < L_0$. Therefore $L_0 = 0$. □

Let us prove the converse. Suppose that we have atomic decomposition; we want to see that the partition through center leaves is a measurable partition.

Lifting $f$ to $\mathbb{R}^3$, by Hammerlindl [**9**] we may find a disk $\tilde{D}^2$ transverse to the center foliation, and by quasi-isometry of the center foliation we may take this disk as big as we want. So take a disk such that its projection $D^2 = \pi(\tilde{D}^2)$ has the property

$$\mathcal{F}^c_x \cap D^2 \neq \emptyset, \quad \text{for all } x \in \mathbb{T}^3.$$

Since the decomposition is atomic, we already know that it has one atom per leaf. Let us define the following set of full measure:

$$\hat{M} = \bigcup_{p \in A} \mathcal{F}^c_{\mathrm{loc}}(p),$$

where $A$ is the set of atoms, $\mathcal{F}^c_{\mathrm{loc}}(p)$ is the segment of center leaf such that the right extreme point is $p$ and the left extreme point is on $D^2$ and $\#\mathcal{F}^c_{\mathrm{loc}}(p) \cap D^2 = 1$.

Since $D^2$ is a separable metric space, $\{\mathcal{F}^c_{\mathrm{loc}}(p)\}_{p \in A}$ is a measurable partition for $\hat{M}$. Therefore, we have a family of subsets $\{A_i\}_{i \in \mathbb{N}}$ of $\hat{M}$ such that for all $p \in A$,

$$\mathcal{F}^c_{\mathrm{loc}}(p) = \bigcap_{i \in \mathbb{N}} B_i \quad \text{where } B_i \in \{A_i, A_i^c\}.$$

## 4. *Conjugacy*

We begin by discussing how Lyapunov exponents of a DA diffeomorphism on $\mathbb{T}^3$ vary with respect to its linearization.

PROPOSITION 4.1. *Let $f : \mathbb{T}^3 \to \mathbb{T}^3$ be a DA diffeomorphism, not necessarily ergodic nor volume-preserving, and let $A$ be its linearization. Then $\int \lambda^u_f(x)\ d\mathrm{Vol}(x) \leq \lambda^u_A$.*

*Proof.* Suppose that $\int \lambda^u_f(x)\ d\mathrm{Vol}(x) > \lambda^u_A$. Then there exists a set $B$ of positive volume and a constant $\alpha$ such that $\lambda^u_f(x) > \alpha > \lambda^u_A$, for all $x \in B$. Define

$$B_N = \{x \in B \mid \|Df^n|E^u_x\| \geq e^{n\alpha}, \forall n \geq N\}.$$

Note that

$$B = \bigcup_{N=1}^{\infty} B_N;$$

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this means that there is $N_0$ such that $\mathrm{Vol}(B_{N_0}) > 0$. Since $\mathcal{F}^u_f$ is absolutely continuous, there is $x \in B_{N_0}$ such that $\mathcal{F}^u_f(x) \cap B_{N_0}$ has positive volume on the unstable leaf.

We now want to use the quasi-isometry of the unstable foliation, so we lift everything, and use $\sim$ for the lift under the projection $\pi : \mathbb{R}^3 \to \mathbb{T}^3$. Since $\pi$ is locally an isometry, $\widetilde{B}_{N_0} = \pi^{-1}(B_{N_0}) = \{x \in \widetilde{B} \mid \|D\widetilde{f}^n|E^u_x\| \geq e^{n\alpha}, \forall n \geq N\}$. We also recall the notation that $\mathrm{Vol}^u$ means the volume coming from the Riemannian metric restricted to the unstable leaf, and $d^u$ the distance inside the unstable leaf.

We now fix $k, C > 1$ and $M$ as given in Proposition 2.1.

Let $\widetilde{I} \subset \mathcal{F}^u_{\widetilde{f}}(x)$ be a compact segment with $\mathrm{Vol}^u(\widetilde{I} \cap \widetilde{B}_{N_0}) > 0$ such that $\mathrm{length}(\widetilde{I}) =: l(\widetilde{I}) > M$. Then

$$\begin{aligned} l(\widetilde{f}^n(\widetilde{I})) &= \int_{\widetilde{f}^n(\widetilde{I})} d\mathrm{Vol}^u = \int_{\widetilde{I}} (\widetilde{f}^n)^*\ d\mathrm{Vol}^u \geq \int_{\widetilde{I}\cap B_{N_0}} (\widetilde{f}^n)^*\ d\mathrm{Vol}^u \\ &\geq \int_{\widetilde{I}\cap\widetilde{B}_{N_0}} \|D\widetilde{f}^n|E^u_x\|\ d\mathrm{Vol}^u(x) \geq e^{n\alpha}\mathrm{Vol}^u(\widetilde{I}\cap B_{N_0}). \end{aligned}$$

Let $x_0, y_0 \in \widetilde{I}$ be its extreme points. Then $d^u(\widetilde{f}^n(x_0), \widetilde{f}^n(y_0)) = l(\widetilde{f}^n(I))$. Since the unstable foliation is quasi-isometric, there exists $C_0 > 1$ such that $(1/C_0)d^u(x, y) \leq \|x - y\| \leq C_0 d^u(x, y)$, for all $x \in \mathcal{F}^u_{\widetilde{f}}(y)$. We may suppose also that $\|x_0 - y_0\| > M$ (by assuming $l(\widetilde{I})$ sufficiently greater that $M$). Also since the unstable manifold expands uniformly we get $\|\widetilde{f}^n(x_0) - \widetilde{f}^n(y_0)\| > M$ for all $n$. Therefore we may apply Proposition 2.1 with $\widetilde{f}^n(x)$ and $\widetilde{f}^n(x)$ in the role of $x$ and $y$, respectively, as in Proposition 2.1. That is,

$$\frac{1}{C} \leq \frac{\|\widetilde{f}^k(\widetilde{f}^n(x_0)) - \widetilde{f}^k(\widetilde{f}^n(y_0))\|}{\|A^k(\widetilde{f}^n(x_0)) - A^k(\widetilde{f}^n(y_0))\|} < C \quad \text{for all } n \in \mathbb{N}.$$

But

$$\begin{aligned} \frac{\|\widetilde{f}^k(\widetilde{f}^n(x_0)) - \widetilde{f}^k(\widetilde{f}^n(y_0))\|}{\|A^k(\widetilde{f}^n(x_0)) - A^k(\widetilde{f}^n(y_0))\|} &\geq \frac{(1/C_0)l(\widetilde{f}^{n+k}(\widetilde{I}))}{\|A^k(\widetilde{f}^n(x_0) - \widetilde{f}^n(y_0))\|} \\ &\geq \frac{e^{(n+k)\alpha}}{C_0 e^{k\lambda^u_A}} \frac{\mathrm{Vol}^u(\widetilde{I}\cap A_{N_0})}{\|\widetilde{f}^n(x_0) - \widetilde{f}^n(y_0)\|} \\ &\geq \frac{e^{(n+k)\alpha}}{C_0 e^{k\lambda^u_A}} \frac{\mathrm{Vol}^u(\widetilde{I}\cap A_{N_0})}{\|\widetilde{f}^n(x_0) - \widetilde{f}^n(y_0)\|} \frac{\|A^n(x_0) - A^n(y_0)\|}{\|A^n(x_0) - A^n(y_0)\|} \\ &\geq \frac{e^{(n+k)\alpha}}{C_0^2 C e^{k\lambda^u_A}} \frac{\mathrm{Vol}^u(\widetilde{I}\cap A_{N_0})}{\|A^n(x_0) - A^n(y_0)\|} \\ &\geq \frac{e^{(n+k)\alpha}}{C_0^2 C e^{(n+k)\lambda^u_A}} \mathrm{Vol}^u(\widetilde{I}\cap A_{N_0}). \end{aligned}$$

We used Proposition 2.1 in the fourth inequality above. Notice that by assumption

$$\frac{e^{(n+k)\alpha}}{e^{(n+k)\lambda^u_A}} \to \infty \quad \text{as } n \to \infty,$$

which is a contradiction. □

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The same type of argument as above yields the following corollary.

COROLLARY 4.1.
$$\int \lambda_f^s(x)\ d\mathrm{Vol}(x) \geq \lambda^s(A).$$

The following corollary for the case of Anosov systems will be used later.

COROLLARY 4.2. *Let $f$ be an Anosov diffeomorphism with the split $TM = E^{ss} \oplus E^{ws} \oplus E^u$ on the tangent space, and the center foliation $\mathcal{F}^{ws}$ is absolutely continuous. Then $\lambda_f^{ws} \geq \lambda_A^{ws}$.*

*Proof.* The proof goes as before, with a minor change. We apply Proposition 2.1 with the linear map $\pi : \mathbb{R}^n \to \mathbb{R}^n$ which is the projection onto a center foliation of the linearization. The projection is with respect to the system of coordinates given by the foliations of the linearization $(x_{ss}, x_{ws}, x_u) \in \mathbb{R}^n$. □

4.1. *Proof of Proposition 1.1.* We only have to prove one of the implications, as the converse is a direct consequence of the $C^1$-conjugacy.

Let us suppose that $f$ is a partially hyperbolic diffeomorphism with the split $TM = E^{ss} \oplus E^{ws} \oplus E^u$ on the tangent space. The next three lemmas concern this case; the other case ($E^s \oplus E^{wu} \oplus E^{uu}$) is reduced to this one by applying the inverse. $\mathrm{Per}(f)$ is the set of periodic points of $f$ and $\lambda_f^*(\mathrm{Vol})$, $* \in \{ss, ws, u\}$, is the Lyapunov exponent in the $*$ direction which is constant volume almost everywhere.

LEMMA 4.1.
$$\lambda_f^u(\mathrm{Vol}) = \lambda_f^u(p) \quad \textit{for all } p \in \mathrm{Per}(f).$$

*Proof.* By ergodicity the set of transitive points $\mathcal{T}$ has total volume. We may assume that all points of $\mathcal{T}$ have well-defined Lyapunov exponents. Let us prove that $\lambda_f^u(x)$ equals the Lyapunov exponent of the periodic points.

For $x \in \mathcal{T}$, given $\varepsilon > 0$, let $\delta > 0$ be such that by uniform continuity,
$$|\log \|Df|E^u_{y_1}\| - \log \|Df|E^u_{y_2}\|| < \varepsilon \quad \text{if } d(y_1, y_2) < \delta.$$

From the shadowing lemma there is $\alpha$ such that for every $\alpha$-pseudo-orbit is $\delta$ shadowed by a real orbit. Given $N_0 \in \mathbb{N}$, there is $n_0 \in \mathbb{N}$ and $n_0 > N_0$ such that $\{\ldots, f^{n_0-1}(x), x, f(x), \ldots, f^{n_0-1}(x), \ldots\}$ is an $\alpha$-pseudo-orbit. Since it is a pseudo-periodic orbit it is $\delta$ shadowed by a periodic point with period $n_0$; call this point $q$. Using the fact that $E^u$ is one-dimensional, then
$$\left| \frac{1}{n_0} \log \|Df^{n_0}|E^u_{y_1}\| - \frac{1}{n_0} \log \|Df^{n_0}|E^u_{y_2}\| \right| < \varepsilon.$$
Since we already know that $\lambda_f^u(x)$ exists, this implies that $\lambda_f^u(x)$ is $\varepsilon$-close to $\lambda_f^u(q)$. □

LEMMA 4.2.
$$\lambda_f^u(\mathrm{Vol}) = \lambda_A^u.$$

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*Proof.* We know that the topological entropy of $A$ is $\lambda^u_A$, and the conjugacy gives $h_{\text{top}}(f) = h_{\text{top}}(A)$. From the theory of equilibrium states [**5**] the measure of maximal entropy is given by the potential $\psi = 0$ and the equilibrium state for the potential $\psi = -\log \lambda^u$ gives the SRB measure, which is Vol in our case. And to see that both equilibrium states are the same we just need to see that both potentials are cohomologous [**5**]. This means that both measures coincide if and only if

$$\frac{1}{n}\sum_{i=1}^{n}(-\log \|Df_{f^i(x)}|E^u\|) = \text{const} \quad \text{for all } x \text{ such that } f^n(x) = x,$$

which is true by hypothesis.

Finally Pesin's formula gives that $h_f(\text{Vol}) = \int \lambda^u_f \, d\text{Vol} = \lambda^u_f$. Gathering all this together, we have the equalities

$$\lambda^u_A = h_{\text{top}}(A) = h_{\text{top}}(f) = h_f(\text{Vol}) = \int \lambda^u_f \, d\text{Vol} = \lambda^u_f(p) = \lambda^u_f(\text{Vol}),$$

and the lemma is proved. □

LEMMA 4.3.

$$\lambda^{ws}_f(p) = \lambda^{ws}_A.$$

*Proof.* By the above lemma we already know that $\lambda^u_f(p) = \lambda^u_A$; and $\lambda^{ss}_f(p) \geq \lambda^{ss}_A$ by Corollary 4.1. Hence, since we are in the volume-preserving case $\lambda^{ss}_f + \lambda^{ws}_f + \lambda^u_f = \lambda^{ss}_A + \lambda^{ws}_A + \lambda^u_A$, we just need to see that $\lambda^{ws}_f(p) \geq \lambda^{ws}_A$, which is Corollary 4.2. □

The above lemmas imply that

$$\lambda^*_f(p) = \lambda^*_A(h(p)) \quad \text{for all } p \in \text{Per}(f).$$

The above equality gives what is known as periodic data, hence by Gogolev and Guysinsky [**8**], $f$ is $C^1$ conjugate to its linearization. □

4.2. *Proof of Theorem 1.3.* Consider a linear Anosov automorphism $A$ with splitting $TM = E^{ss}_A \oplus E^{ws}_A \oplus E^u_A$. Let $\phi$ be a volume-preserving diffeomorphism which leaves invariant the center unstable foliation $\mathcal{F}^{cu}_A$ (tangent to $E^{ws} \oplus E^u$), such that $\phi \neq \text{Id}$ in a small ball $\phi(x, y, z) = (x, h_x(y, z))$, and $\phi = \text{Id}$ outside it, where $x, y, z$ are the coordinates relative to $E^{ss}_A$, $E^{ws}_A$, and $E^u_A$, respectively. We may also suppose that it is centered at a fixed point. Define $f := A \circ \phi$; then $\mathcal{F}^{cu}_f = \mathcal{F}^{cu}_A$. Therefore $\mathcal{F}^{cu}_f$ is a $C^1$ foliation. And because $\mathcal{F}^{cs}_f$ (foliation tangent to $E^{ss}_f \oplus E^{ws}_f$) is a codimension-one stable foliation of an Anosov diffeomorphism, it is a $C^1$ foliation [**10**, Corollary 2.3.4].

Since we suppose that $\phi$ is not the identity it has changed the center Lyapunov exponent of the periodic point for which $\phi$ is 'centered', and not on some other periodic point outside this ball [**2**]. Hence by Proposition 1.1 $f$ is not $C^1$ conjugated to its linearization and has a $C^1$ center foliation. □

## 5. *Rigidity*

In this section, $f$ will be as in the hypothesis of Theorem 1.4. We also mention that since we will be working on the lift throughout this section, we allow ourselves an abuse of notation and use $f$ instead of $\widetilde{f}$ for the lift.

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The goal of this section is to prove Theorem 1.4. But first we construct some *conditional measures with dynamical meaning*. We shall associate to each center leaf a class of measures differing from each other by a multiplication of a positive real number in such a way that on each foliated box the normalized element of this class will give the Rokhlin disintegration of the measure. When the foliation satisfies the hypothesis of Theorem 1.4 we shall be able to construct on each leaf a representative with some dynamical meaning, and thus will help us to obtain some information on the center Lyapunov exponent of $f$.

The following lemma enables us to understand how two different disintegrations (on different foliated boxes) relate to one another in the intersection. For our construction later on we want to know exactly how this happens, given by the number $a(\xi)$, which appears in the proof the Lemma 5.1. Because the lemma has a short proof we include it, as in [**1**], for the sake of completeness.

In what follows we use the notation $m_x^{\mathcal{B}}$ for the conditional measure, given by Rokhlin's disintegration, on the leaves of the foliated box $\mathcal{B}$.

LEMMA 5.1. (Avila, Viana and Wilkinson [**1**]) *Let $m$ be any Borel measure. For any foliated boxes $\mathcal{B}$, $\mathcal{B}'$ and $m$-almost every $x \in \mathcal{B} \cap \mathcal{B}'$ the restrictions of $m_x^{\mathcal{B}}$ and $m_x^{\mathcal{B}'}$ to $\mathcal{B} \cap \mathcal{B}'$ coincide up to a constant factor.*

*Proof.* Let $\Sigma$ be a transversal to the foliated box $\mathcal{B}$ and $\mu_{\mathcal{B}}$ the measure on $\Sigma$ obtained as the projection of $m|\mathcal{B}$ along local leaves. Consider any measurable set $\mathcal{C} \subset \mathcal{B}$ and let $\mu_{\mathcal{C}}$ be the projection of $m|\mathcal{C}$ on $\Sigma$,

$$\frac{d\mu_{\mathcal{C}}}{d\mu_{\mathcal{B}}} \in (0, 1], \quad \mu_{\mathcal{C}} \text{ almost every point.}$$

For any measurable set $E \subset \mathcal{C}$

$$m(E) = \int_{\Sigma} m_{\xi}^{\mathcal{B}}(E)\, d\mu_{\mathcal{B}}(\xi) = \int_{\Sigma} m_{\xi}^{\mathcal{B}}(E) \frac{d\mu_{\mathcal{B}}}{d\mu_{\mathcal{C}}}(\xi)\, d\mu_{\mathcal{C}}(\xi).$$

By essential uniqueness, this proves that the disintegration of $m|\mathcal{C}$ is given by

$$m_{\xi}^{\mathcal{C}} = \frac{d\mu_{\mathcal{B}}}{d\mu_{\mathcal{C}}}(\xi) m_{\xi}^{\mathcal{B}}, \quad \mu_{\mathcal{C}} \text{ almost every point } \xi.$$

Take $\mathcal{C} = \mathcal{B} \cap \mathcal{B}'$. Therefore $(d\mu_{\mathcal{B}}/d\mu_{\mathcal{C}})(\xi) m_{\xi}^{\mathcal{B}}|\mathcal{C} = m_{\xi}^{\mathcal{C}} = (d\mu_{\mathcal{B}'}/d\mu'_{\mathcal{C}})(\xi) m_{\xi}^{\mathcal{B}'}|\mathcal{C}$, where $\mu'_{\mathcal{C}}$ is the projection of measure $\mu$ on the transversal $\Sigma'$ relative to the $\mathcal{B}'$ box. Hence

$$m_{\xi}^{\mathcal{B}}|\mathcal{C} = a(\xi) m_{\xi}^{\mathcal{B}'}|\mathcal{C},$$

where $a(\xi) = (d\mu_{\mathcal{B}'}/d\mu'_{\mathcal{C}})(\xi)((d\mu_{\mathcal{B}}/d\mu_{\mathcal{C}})(\xi))^{-1}$. □

The above lemma implies the existence of a family $\{[m_x] \mid x \in M\}$ of measures defined up to scaling and satisfying $m_x(M \backslash \mathcal{F}_x) = 0$. The map $x \mapsto [m_x]$ is constant on leaves of $\mathcal{F}$ and the conditional probabilities $m_x^{\mathcal{B}}$ coincide almost everywhere with the normalized restrictions of $[m_x]$.

We observe that disintegration of a measure is an almost everywhere concept, but in our case, since we shall be considering a $C^1$ center foliation, the conditional measures, of volume, are defined everywhere. In this case the number $a(\xi) = (d\mu_{\mathcal{B}'}/d\mu'_{\mathcal{C}})(\xi)$ $((d\mu_{\mathcal{B}}/d\mu_{\mathcal{C}})(\xi))^{-1}$ is indeed defined everywhere.

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Recall that in this section we are working on the lift, and in order not to carry unnecessary notation we shall use $f$ for the lift, as well as not using $\sim$ when considering the invariant foliations. Let $B := \mathcal{W}^{su}(0)$, which is the saturation by unstable leaves of the stable manifold of $0 \in \mathbb{R}^3$. By the semi-conjugacy we know that every segment of center leaf which has size large enough keeps increasing by forward iteration. Let $\gamma_0$ be a length with this property. Let $B_0$ be the two-dimensional topological surface such that each center leaf intersects $B$ and $B_0$ at two points that are on the same center leaf and at a distance $\gamma_0$ inside the center leaf. Let $B_k := f^k(B_0)$. Therefore, for each point $\xi \in B$ there is a unique point $q_k(\xi) \in B_k$ that is on the same center leaf as $\xi$. Since it will be clear to which point $\xi$ we associate $q_k(\xi)$, we use $q_k$ instead to simplify notation.

We are now ready for the construction that will eventually give the *conditional measures with dynamical meaning*. At each step $k$ we pick a conditional measure $m_{x,k}$ that satisfies a 'dynamical property' (Lemma 5.2). In the end we will be able to get a conditional measure $m_x$ with bounded densities and this will allow us to calculate the Lyapunov exponent on the center direction and with a little more effort finish the proof by applying Proposition 1.1.

From the discussion of Lemma 5.1 we know that on each center leaf $\mathcal{F}^c_f(\xi)$, $\xi \in B$, there is a family of conditional measures which differ from each other up to multiplication by a positive number.

We now start the construction of the conditional measures $m_{x,k}$. Because on each center leaf there is a family of conditional measures equal up to scaling, we can uniquely determine an element of this family to be the one which gives a certain value for a given set. Therefore, we uniquely define $m_{x,k}$ by the measure which satisfies

$$m_{\xi,k}([0, q_k]) = \lambda^k,$$

where $\lambda$ is the center eigenvalue of $A$, the linearization of $f$, and $[0, q_k]$ means the segment $[\xi, q_k(\xi)]$ inside the center leaf of $\xi$.

LEMMA 5.2.

$$f_* m_{x,k} = \lambda^{-1} m_{f(x),k+1}.$$

*Proof.* Because these measures are equal up to rescaling, we only need to check that they give the same measure for the same set. Recall that $B_k = f^k(B_0)$, and therefore $f^{-1}([0, q_{k+1}]) = [0, q_{k+1}]$.

$$\begin{aligned} f_* m_{x,k}([0, q_{k+1}]) &= m_{x,k}(f^{-1}([0, q_{k+1}])) \\ &= m_{x,k}([0, q_k]) \\ &= \lambda^k = \lambda^{-1}\lambda^{k+1} = \lambda^{-1} m_{f(x),k+1}([0, q_{k+1}]). \end{aligned}$$ □

Therefore if the sequence $m_{x,k}$ converges we would get

$$f_* m_x = \lambda^{-1} m_{f(x)}.$$

But we have to understand more clearly how these measures $m_{x,k}$ are behaving.

By Lemma 5.1, for two foliated boxes $\mathcal{B}$ and $\mathcal{B}'$, we have

$$m^{\mathcal{B}}_x \frac{d\mu_{\mathcal{B}}}{d\mu_{\mathcal{C}}} = m^{\mathcal{B}'}_x \frac{d\mu_{\mathcal{B}'}}{d\mu'_{\mathcal{C}}}.$$

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We apply this formula to the boxes $\mathcal{B}$ and $\mathcal{B}_k$, where $\mathcal{B}$ includes the segments of center leaves between $B$ and $B_0$, and similarly $\mathcal{B}_k$ is formed by the segments of center leaves bounded by $B$ and $B_k$. Also $\mathcal{C} = \mathcal{B}$, $\mathcal{B}' = \mathcal{B}_k$ and $m = \text{Vol}$. Recall that $\lambda^k m_{x,k} = m_x^{\mathcal{B}_k}$ by the definition of the disintegration. Then

$$m_x^{\mathcal{B}}.1 = m_x^{\mathcal{B}_k} \frac{d\mu_{\mathcal{B}_k}}{d\mu_{\mathcal{B}}} = \lambda^{-k} m_{x,k} \frac{d\mu_{\mathcal{B}_k}}{d\mu_{\mathcal{B}}}.$$

We have thus proved the following lemma.

LEMMA 5.3. *On $\mathcal{B}$,*

$$m_{x,k} = \left(\frac{d\mu_{\mathcal{B}_k}}{d\mu_{\mathcal{B}}}\right)^{-1} \lambda^k m_x^{\mathcal{B}}.$$

To establish the convergence of the measures we shall need our next result.

LEMMA 5.4. *If $\mathcal{F}_f^c$ satisfies the hypothesis of Theorem 1.4, then there is a uniform constant $\alpha$ such that*

$$\frac{1}{\alpha} \frac{l(\mathcal{F}_f^c(x) \cap \mathcal{B}_k)}{l(\mathcal{F}_f^c(x) \cap \mathcal{B})} \leq \frac{d\mu_{B_k}}{d\mu_{\mathcal{B}}}(x) \leq \alpha \frac{l(\mathcal{F}_f^c(x) \cap \mathcal{B}_k)}{l(\mathcal{F}_f^c(x) \cap \mathcal{B})},$$

*where $l$ is the length.*

*Proof.* By the Lebesgue differentiation theorem we calculate the Radon–Nikodym derivative $(d\mu_{\mathcal{B}_k}/d\mu_{\mathcal{B}})(x)$ as

$$\frac{d\mu_{\mathcal{B}_k}}{d\mu_{\mathcal{B}}}(x) = \lim_{\epsilon \to 0} \frac{\mu_{\mathcal{B}_k}(B(x, \epsilon))}{\mu_{\mathcal{B}}(B(x, \epsilon))}.$$

Recall that the sets $\mu_{\mathcal{B}_k}(B(x, \epsilon))$ and $\mu_{\mathcal{B}}(B(x, \epsilon))$ have measure $\text{Vol}(\pi_{\mathcal{B}_k}^{-1}(B(x, \epsilon)))$ and $\text{Vol}(\pi_{\mathcal{B}}^{-1}(B(x, \epsilon)))$ respectively, where $\pi_{\mathcal{B}_k} : \mathcal{B}_k \to B$ and $\pi_{\mathcal{B}} : \mathcal{B} \to B$ are the projections along the center leaves. That is, we calculate the volume of the sets which are saturated by the center leaves from $B(x, \epsilon)$ up to the top lid of $\mathcal{B}_k$ and $\mathcal{B}$, respectively. But the transversely absolute continuity with bounded Jacobians allows us to estimate the volume of this set as base times height, as each slice has uniform volume compared to the base. And because those two saturated sets have the same base, the fraction $(d\mu_{\mathcal{B}_k}/d\mu_{\mathcal{B}})(x)$ is simply the height of $\mathcal{B}_k$ divided by the height of $\mathcal{B}$. □

Hence,

$$\frac{d\mu_{\mathcal{B}_k}}{d\mu_{\mathcal{B}}}(x) = \alpha_{x,k} \frac{l(\mathcal{F}_f^c(x) \cap \mathcal{B}_k)}{l(\mathcal{F}_f^c(x) \cap \mathcal{B})},$$

where $\alpha_{x,k} \in [1/\alpha, \alpha]$, for all $x \in \mathbb{R}^3$ and $k \in \mathbb{N}$.

Therefore on $\mathcal{B}$, using Lemma 5.3, we get

$$m_{x,k} = \left(\alpha_{x,k} \frac{l(\mathcal{F}_f^c(x) \cap \mathcal{B}_k)}{l(\mathcal{F}_f^c(x) \cap \mathcal{B})}\right)^{-1} \lambda^k m_x^{\mathcal{B}}.$$

For each $x$ there is a subsequence $\alpha_{x,k_{i(x)}}$ that converges to some $\tilde{\alpha}_x$ as $i(x) \to \infty$.

LEMMA 5.5. *There is $\beta > 0$ such that, for all $x \in \mathbb{R}^3$,*

$$\lambda^k / l(\mathcal{F}_f^c(x) \cap \mathcal{B}_k) \in [1/\beta, \beta].$$

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*Proof.* Let $x \in \mathcal{F}^c_f(y)$. Since the semi-conjugacy maps center leaves to center leaves, $\|f^n(x) - f^n(y)\|$ measures the growth of the center foliation and $\|A^n(H(x)) - A^n(H(y))\|$ grows as $\lambda^n$. Hence we need to estimate the fraction

$$\frac{\|A^n(H(x)) - A^n(H(y))\|}{\|f^n(x) - f^n(y)\|} = \frac{\|H \circ f^n(x) - H \circ f^n(y)\|}{\|f^n(x) - f^n(y)\|}.$$

By the triangle inequality,

$$\frac{\|H \circ f^n(x) - H \circ f^n(y)\|}{\|f^n(x) - f^n(y)\|} \leq \frac{\|H(f^n(x)) - f^n(x)\|}{\|f^n(x) - f^n(y)\|} + \frac{\|f^n(x) - f^n(y)\|}{\|f^n(x) - f^n(y)\|} + \frac{\|H(f^n(y)) - f^n(y)\|}{\|f^n(x) - f^n(y)\|}$$

and

$$\frac{\|H \circ f^n(x) - H \circ f^n(y)\|}{\|f^n(x) - f^n(y)\|} \geq -\frac{\|H(f^n(x)) - f^n(x)\|}{\|f^n(x) - f^n(y)\|} + \frac{\|f^n(x) - f^n(y)\|}{\|f^n(x) - f^n(y)\|} - \frac{\|H(f^n(y)) - f^n(y)\|}{\|f^n(x) - f^n(y)\|}.$$

We know that $H$ is at a bounded distance of the identity and $\|f^n(x) - f^n(y)\|$ is large. □

By the above lemma we may assume that $\lambda^k / l(\mathcal{F}^c_x \cap \mathcal{B}_k)$ goes to one as $k$ increases; otherwise incorporate it in the constant $\alpha_{x,k}$. Then sending $k_{i(x)}$ to infinity,

$$m_x := \lim_{k_{i(x)} \to \infty} m_{x,k_{i(x)}} = (l(\mathcal{F}^c_x \cap \mathcal{B})/\tilde{\alpha}_x) m^{\mathcal{B}}_x.$$

By going to a subsequence we obtain a convergent measure, but we want it to have the specific property of $f_* m_x = \lambda m_{f(x)}$. Therefore we have to be more careful about how to define them. We have seen above that $f_* m_{x,k} = \lambda^{-1} m_{f(x),k+1}$, hence for fixed $x$ there is $k_{i(x)}$ defined as above, but if we define $k_{i(f(x))} = k_{i(x)} + 1$ we obtain the limit satisfying $f_* m_x = \lambda^{-1} m_{f(x)}$. This means that for fixed $x$ we can define on the orbit of $x$ measures satisfying the mentioned dynamical property.

The measures are in fact indexed by $B = W^{su}$. Hence, to get a proper definition on the whole space we define the measures on the orbits of fundamental domains of $B$.

From the above we conclude that we did get measures on each center leaf with the property that $f_* m_x = \lambda^{-1} m_{f(x)}$. The construction of such measures will help us to get information of the center Lyapunov exponent, since we may recover $\lambda$ by the equality

$$\frac{df_* m_x}{dm_{f(x)}} = \lambda^{-1}.$$

Let us explore the above relation in more depth.

LEMMA 5.6. *By the above, the center Lyapunov exponent of $f$ exists everywhere and is equal to* $\log \lambda$.

*Proof.* Note that

$$\frac{df^n_* m_x}{dm_{f^n(x)}}(f^n(x)) = \lambda^{-n}.$$

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Let us calculate the Radon–Nikodym derivative in another way. Let $I_\delta^n \subset \mathcal{F}^c_{f^n(x)}$ be a segment of length $\delta$ around $f^n(x)$. Then

$$\frac{df^n_* m_x}{dm_{f^n(x)}}(f^n(x)) = \lim_{\delta\to 0} \frac{f^n_* m_x(I_\delta^n)}{m_{f^n(x)}(I_\delta^n)}.$$

Let us denote by $\rho_x$ the density of $m_x$ with respect to the Lebesgue measure on the center leaf $\mathcal{F}^c_f(x)$. From the construction of $m_x$ there exists $C > 1$ such that $1/C < \rho_x < C$. Then

$$\begin{aligned}\frac{df^n_* m_x}{dm_{f^n(x)}}(f^n(x)) &= \lim_{\delta\to 0} \frac{m_x(f^{-n}(I_\delta^n))}{m_{f^n(x)}(I_\delta^n)} = \lim_{\delta\to 0} \frac{\int_{f^{-n}(I_\delta^n)} \rho_x \, d\lambda_x}{\int_{I_\delta^n} \rho_{f^n(x)} \, d\lambda_{f(x)}} \\ &\le C^2 \lim_{\delta\to 0} \frac{\int_{f^{-n}(I_\delta^n)} d\lambda_x}{\int_{I_\delta^n} d\lambda_{f(x)}} \le \lim_{\delta\to 0} C^2 \frac{\int_{I_\delta^n} \|Df^{-n}\| \, d\lambda_x}{\int_{I_\delta^n} d\lambda_{f(x)}} \\ &\le C^2 \|Df^{-n}(x)\|.\end{aligned}$$

Then

$$\lim_{\delta\to 0} \frac{df^n_* m_x}{dm_{f^n(x)}}(I_\delta^n) \le C^2 \|Df^{-n}(x)\|.$$

Analogously,

$$\lim_{\delta\to 0} \frac{df^n_* m_x}{dm_{f^n(x)}}(I_\delta^n) \ge C^{-2} \|Df^{-n}(x)\|.$$

Adding all the relations, we get

$$C^{-2} \|Df^{-n}(x)\| \le \lambda^{-n} \le C^2 \|Df^{-n}(x)\|.$$

By applying '$\lim_{n\to\infty}(1/n)\log$' to the above inequalities we get

$$\lambda^c(x) = \log \lambda. \qquad \square$$

We are now in a position to give our final proof.

*Proof of Theorem 1.4:* First, let us prove that $f$ is an Anosov diffeomorphism. We just need to analyze the behavior of $Df$ on the center direction. Let $\varepsilon > 0$ be such that $\lambda_\varepsilon := \log \lambda - \varepsilon > 0$. We assume the center direction of the linearization $A$ to be expanding. Recall that $\lambda$ is the eigenvalue of the center direction for $A$. Since the center exponent exists for every $x$ and equals $\log \lambda$, given $x \in \mathbb{T}^3$, there are $n_x \in \mathbb{N}$ and a neighborhood $\mathcal{U}_x$ of $x$ such that, for all $y \in \mathcal{U}_x$, $|Df^{n_x}|E^c_y| \ge e^{n_x \lambda_\varepsilon}$. Since $\mathbb{T}^3$ is a compact manifold, take a finite cover $\mathcal{U}_{x_1} \ldots \mathcal{U}_{x_l}$. Let $C_i < 1$ small enough so that, for $y \in \mathcal{U}_{x_i}$, $|Df^n|E^c_y| \ge C_{x_i} e^{n\lambda_\varepsilon}$ for all $n \in \{0, 1, \ldots, n_{x_i}\}$. Let $C := \min_i C_{x_i}$. We then have $|Df^n|E^c_y| \ge C e^{n\lambda_\varepsilon}$ for all $y \in \mathbb{T}^3$ and $n \in \mathbb{N}$.

Since, in particular, the center foliation is absolutely continuous, from Gogolev [**7**], one of the extremal exponents is constant at periodic points. On the other hand, the above gives in particular that at the periodic points the central exponent is also constant. Since we are in the conservative case and the sum of Lyapunov exponents is equal to one, all of the exponents are constant at periodic points. Then Proposition 1.1 gives that $f$ is $C^1$-conjugate to its linearization. $\square$

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