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ASYMPTOTIC EXPANSIONS OF GENERALIZED QUANTILES AND EXPECTILES FOR EXTREME RISKS

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Generalized quantiles of a random variable were defined as the minimizers of a general asymmetric loss function, which include quantiles, expectiles and M-quantiles as their special cases. Expectiles have been suggested as potentially better alternatives to both Value-at-Risk and expected shortfall risk measures. In this paper, we first establish the first-order expansions of generalized quantiles for extreme risks as the confidence level $\alpha \uparrow 1$, and then investigate the first-order and/or second-order expansions of expectiles of an extreme risk as $\alpha \uparrow 1$ according to the underlying distribution belonging to the max-domain of attraction of the Fréchet, Weibull, and Gumbel distributions, respectively. Examples are also presented to examine whether and how much the first-order expansions have been improved by the second-order expansions.

1. INTRODUCTION

Let $\Phi_i : \Re_+ \to \Re_+$ be a convex and strictly increasing function satisfying that $\Phi_i(0) = 0$ and $\Phi_i(1) = 1$ for $i = 1, 2$. Bellini and Bignozzi [2] defined the generalized quantiles $x^*_\alpha = x^*_\alpha(X)$ of a random variable X as the minimizers of the following general asymmetric loss function:

$$\rho_\alpha(u) = \alpha\Phi_1(u_+) + (1-\alpha)\Phi_2(u_-), \quad u \in \Re,$$

where $u_+ = \max\{u, 0\}$ and $u_- = \max\{-u, 0\}$ for any $u \in \Re$, and $\alpha \in (0, 1)$ represents the confidence level. That is, the generalized quantile x^*_α satisfies that

$$x^*_\alpha = \mathsf{argmin}_{x\in\Re}\mathsf{E}\rho_\alpha(X - x). \tag{1.1}$$

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Choosing the functions Φ_1 and Φ_2 appropriately, several other notions of quantiles in the literature are seen to be particular cases of the generalized quantiles:

- Quantiles (see, e.g., Koenker [17] or Denuit et al. [10]): $\Phi_1(x) = \Phi_2(x) = x$;
- Expectiles (Newey and Powell [22]): $\Phi_1(x) = \Phi_2(x) = x^2$;
- L_β-quantiles (Chen [6]): $\Phi_1(x) = \Phi_2(x) = x^\beta$ with $\beta \geq 1$;
- M-quantiles (Breckling and Chambers [5]): $\Phi_1(x) = \Phi_2(x) = \Phi(x)$ with $\Phi(x)$ differentiable and $\Phi'_+(0) = 0$.

The properties of the generalized quantiles (especially, expectiles) were investigated by Bellini and Bignozzi [2] and Delbaen [9] from the point of view of the axiomatic theory of risk measures. It was proved by Bellini and Bignozzi [2] that expectiles $e_\alpha(X)$ are the only generalized quantiles that are coherent risk measures in the case of $\alpha \geq 1/2$. For the definition of coherent risk measures, see Artzner et al. [1]. Expectiles are becoming increasingly popular in the econometric literature; see, for example, De Rossi and Harvey [8], Kuan Yeh, and Hsu [18] and Embrechts et al. [12]. A risk measure is said to be elicitable if it can be defined as the minimizer of a suitable expected loss function (see Gneiting [15]; Ziegel [24]). Bellini and Bignozzi [2] showed that, among all coherent risk measures, the only elicitable risk measures are the expectiles. It is known from Ziegel [24] that elicitability is a key property of a risk measure because it provides a natural methodology to perform backtesting. Since Value-at-Risk (VaR) does not possess subadditivity (and, hence, not coherent), and Expected Shortfall (ES) is not elicitable (and, hence, the backtesting of ES is not straightforward), expectiles have been suggested (see Emmer, Kratz, and Tasche [13]) as potentially better alternatives to both VaR and ES.

The purposes of this paper are to establish the first-order expansions of generalized quantiles for extreme risks as the confidence level $\alpha \uparrow 1$, and to investigate the first-order and/or second-order expansions of expectiles of an extreme risk as $\alpha \uparrow 1$ according to the underlying distribution belonging to the max-domain of attraction of the Fréchet, Weibull and Gumbel distributions, respectively. The main results on generalized quantiles and expectiles are given in Sections 2 and 3, respectively. Some examples are presented in Section 4 to illustrate that the second-order expansions are more accurate than the first-order expansions.

Throughout, the notation "$\sim$" means asymptotic equivalence; that is, for functions g and h, $g(x) \sim h(x)$ as $x \to x_0$ means that $\lim_{x\to x_0} g(x)/h(x) = 1$. For any distribution function F of a random variable X, denote be $q_\alpha = q_\alpha(X)$ the (ordinary) quantile of F at confidence level $\alpha \in (0, 1)$.

2. FIRST-ORDER EXPANSIONS OF GENERALIZED QUANTILES

In this section, we derive the first-order expansions of generalized quantiles when the underlying survival function is of regular variation (RV). Recall that a measurable function $h : \Re_+ \to \Re$ that is eventually positive is said to be of RV at infinity with index $\gamma \in \Re$, denoted by $h \in \mathrm{RV}_\gamma$, if for any $x > 0$,

$$\lim_{t\to\infty} \frac{h(tx)}{h(t)} = x^\gamma.$$

Standard references on RV and 2RV (whose definition is given in Section 3.1) are Bingham, Goldie, and Teugels [4], de Haan and Ferreira [7], and Resnick [23].

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PROPOSITION 2.1: *Let Φ_1 and Φ_2 be two increasing, differentiable and convex functions satisfying that $\Phi_i'(\infty) = \lim_{x\to\infty}\Phi_i'(x) < \infty$ for $i = 1, 2$. If the distribution function F of X is continuous such that $\overline{F} \in \mathrm{RV}_{-\gamma}$ with $\gamma > 0$, then the generalized α-quantile x_α^* of F satisfies that*

$$x_\alpha^* \sim \left(\frac{\Phi_1'(\infty)}{\Phi_2'(\infty)}\right)^{1/\gamma} q_\alpha, \quad \alpha \uparrow 1. \tag{2.1}$$

In particular, if $\Phi_1'(\infty) = \Phi_2'(\infty) < \infty$, then $x_\alpha^* \sim q_\alpha$ as $\alpha \uparrow 1$.

PROOF: First, note that the generalized quantile x_α^* satisfies the following equation:

$$\alpha \mathsf{E}\left[1_{\{X>x_\alpha^*\}}\Phi_1'\left((X-x_\alpha^*)_+\right)\right] = (1-\alpha)\mathsf{E}\left[1_{\{X<x_\alpha^*\}}\Phi_2'\left((X-x_\alpha^*)_-\right)\right]. \tag{2.2}$$

Here, it should be pointed out that Eq. (6) in Corollary 3 of Bellini et al. [3] is not correct only under the assumption that F is continuous. It is easy to see that

$$\begin{aligned}\mathsf{E}\left[1_{\{X>x\}}\Phi_1'\left((X-x)_+\right)\right] &= \int_x^\infty \Phi_1'(y-x)dF(y)\\ &= -\Phi_1'(y-x)\overline{F}(y)\Big|_x^\infty + \int_x^\infty \overline{F}(y)d\Phi_1'(y-x)\\ &= \Phi_1'(0)\overline{F}(x) + \int_0^\infty \overline{F}(y+x)d\Phi_1'(y).\end{aligned}$$

By Potter's theorem (see Proposition B.1.9(5) in de Haan and Ferreira [7]) and the dominated convergence theorem, we have

$$\lim_{x\to\infty}\int_0^\infty \frac{\overline{F}(y+x)}{\overline{F}(x)}d\Phi_1'(y) = \Phi_1'(\infty) - \Phi_1'(0).$$

Hence,

$$\mathsf{E}\left[1_{\{X>x\}}\Phi_1'\left((X-x)_+\right)\right] \sim \Phi_1'(\infty)\overline{F}(x), \quad x\to\infty. \tag{2.3}$$

On the other hand,

$$\mathsf{E}\left[1_{\{X<x\}}\Phi_2'\left((X-x)_-\right)\right] \sim \Phi_2'(\infty), \quad x\to\infty, \tag{2.4}$$

by the dominated convergence theorem. Substituting (2.3) and (2.4) into (2.2) and noting that $x_\alpha^* \to \infty$ as $\alpha \uparrow 1$, we conclude that

$$\Phi_1'(\infty)\overline{F}(x_\alpha^*) \sim (1-\alpha)\Phi_2'(\infty), \quad \alpha\uparrow 1.$$

In view of $\overline{F} \in \mathrm{RV}_{-\gamma}$, $\gamma > 0$, it follows that (2.1) holds. ■

In Proposition 2.1, the assumption that $\Phi_i(\infty) < \infty$ for each i is restrictive. In the sequel, we try to remove this assumption. First, we consider the L_β-quantile for $\beta > 1$.

PROPOSITION 2.2: *Let $\Phi_1(x) = \Phi_2(x) = x^\beta$ for $\beta > 1$, and let X be a random variable with survival function $\overline{F} \in \mathrm{RV}_{-\gamma}$, $\gamma > \beta$, and $\mathsf{E}|X|^{\beta-1} < \infty$. Then*

$$x_\alpha^* \sim J_{\beta,\gamma}^{1/\gamma} q_\alpha, \quad \alpha\uparrow 1,$$

where

$$J_{\beta,\gamma} = (\beta-1)B(\gamma-\beta+1, \beta-1).$$

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PROOF: For $\gamma > \beta > 1$, $J_{\beta,\gamma}$ is well defined. Since $\Phi_i'(0) = 0$ for $i = 1, 2$, from Corollary 3 in Bellini et al. [3], it follows that the generalized α-quantile x_α^* is the unique solution to the following equation:

$$\alpha \mathsf{E}\left[(X-x)_+^{\beta-1}\right] = (1-\alpha)\mathsf{E}\left[(X-x)_-^{\beta-1}\right]. \tag{2.5}$$

For $\beta > 1$,

$$\mathsf{E}[(X-x)_+^{\beta-1}] = (\beta-1)\int_x^\infty \overline{F}(u)(u-x)^{\beta-2}du = (\beta-1)x^{\beta-1}\int_1^\infty \overline{F}(ux)(u-1)^{\beta-2}du.$$

By Potter's theorem and the dominated convergence theorem, we have

$$\lim_{x\to\infty}\int_1^\infty \frac{\overline{F}(ux)}{\overline{F}(x)}(u-1)^{\beta-2}du = \int_1^\infty u^{-\gamma}(u-1)^{\beta-2}du = B(\gamma-\beta+1,\beta-1).$$

Hence,

$$\mathsf{E}\left[(X-x)_+^{\beta-1}\right] \sim J_{\beta,\gamma}x^{\beta-1}\overline{F}(x), \quad x\to\infty. \tag{2.6}$$

On the other hand,

$$\frac{\mathsf{E}\left[(X-x)_-^{\beta-1}\right]}{x^{\beta-1}} = \mathsf{E}\left[\left(1-\frac{X}{x}\right)_+^{\beta-1}\right] \longrightarrow 1, \quad x\to\infty,$$

by the dominated convergence theorem since $(1-X/x)_+^{\beta-1} \le (1\vee 2^{\beta-2})(1+|X|^{\beta-1})$ for any $x > 1$. Then

$$\mathsf{E}\left[(X-x)_-^{\beta-1}\right] \sim x^{\beta-1}, \quad x\to\infty. \tag{2.7}$$

Substituting (2.6) and (2.7) into (2.5) yields that $J_{\beta,\gamma}\overline{F}(x_\alpha^*) \sim 1-\alpha$ as $\alpha\uparrow 1$. Since $\overline{F} \in \mathrm{RV}_{-\gamma}$, we have $x_\alpha^* \sim J_{\beta,\gamma}^{1/\gamma}q_\alpha$ as $\alpha\uparrow 1$. This completes the proof of the proposition. ■

In the next proposition, we will generalize Proposition 2.2 to the more general convex functions Φ_1 and Φ_2 which are regularly varying at infinity.

PROPOSITION 2.3: *Let X be a random variable with a continuous distribution function F satisfying $\overline{F} \in \mathrm{RV}_{-\gamma}$, $\gamma > 0$, and let Φ_1 and Φ_2 be two increasing convex functions such that $\Phi_1'' \in \mathrm{RV}_{\beta_1-1}$ and $\Phi_2 \in \mathrm{RV}_{\beta_2+1}$ with $\beta_2 > 0$ and $\gamma > \beta_1 > 0$. Assume that $\mathsf{E}[(X_-)^{\beta_2+\epsilon}] < \infty$ for some $\epsilon > 0$. Then*

$$x_\alpha^* \sim \left[B(\gamma-\beta_1,\beta_1)\right]^{1/(\beta_2-\beta_1+\gamma)}H^{\leftarrow}(1-\alpha), \quad \alpha\uparrow 1,$$

with $H(x) = \overline{F}(x)\Phi_1''(x)x/\Phi_2'(x)$ and $H^{\leftarrow}(1-\alpha) = \inf\{x : H(x) \le 1-\alpha\}$ for $\alpha \in (0,1)$. Specially, if $\Phi_1(x) = \Phi_2(x)$, then $\beta_1 = \beta_2$ and

$$x_\alpha^* \sim \left[\beta_2 B(\gamma-\beta_2,\beta_2)\right]^{1/\gamma}q_\alpha, \quad \alpha\uparrow 1.$$

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PROOF: First, we show that

$$\lim_{x\to\infty}\frac{1}{\Phi_2'(x)}\mathsf{E}\left[\Phi_2'\left((X-x)_-\right)I_{\{X<x\}}\right]=1. \tag{2.8}$$

To this end, note that for any constant $M>0$,

$$(x-M)_+I_{\{X\le M\}}\le(X-x)_-\le(x+M)I_{\{X>-M\}}+(X-x)_-I_{\{X\le -M\}},\quad x>0.$$

Since Φ_2' is increasing, we have

$$\begin{aligned}\Phi_2'\big((x-M)_+\big)F(M)&\le\mathsf{E}\left[\Phi_2'((X-x)_-)I_{\{X<x\}}\right]\\&\le\Phi_2'(x+M)\overline{F}(-M)+\mathsf{E}\left[\Phi_2'\big((X-x)_-\big)I_{\{X\le -M\}}\right].\end{aligned} \tag{2.9}$$

The second term in the right-hand side of (2.9) can be written as

$$\mathsf{E}\left[\Phi_2'\left((X-x)_-\right)I_{\{X\le -M\}}\right]=\int_{-\infty}^{-M}\Phi_2'(x-y)dF(y).$$

From Proposition B.1.9(11) in de Haan and Ferreira [7], it follows that $\Phi_2'\in\mathrm{RV}_{\beta_2}$. Then, by Potter's theorem (see Proposition B.1.9(5) in de Haan and Ferreira [7]), for any $\delta_1>0$ and $\delta_2\in(0,\epsilon)$, there exists x_0 such that for all $x\ge x_0$ and $y\le -M$,

$$\frac{\Phi_2'(x-y)}{\Phi_2'(x)}\le(1+\delta_1)\left(1-\frac{y}{x}\right)^{\beta_2+\delta_2}.$$

Noting that $\mathsf{E}[(X_-)^{\beta_2+\epsilon}]<\infty$, by C_r-inequality, we have $\mathsf{E}[|1-X/x|^{\beta_2+\delta_2}]$ is bounded uniformly for $x>x_0$. Then, by the dominated convergence theorem,

$$\lim_{x\to\infty}\int_{-\infty}^{-M}\frac{\Phi_2'(x-y)}{\Phi_2'(x)}dF(y)=F(-M).$$

Also, since $\lim_{x\to\infty}\Phi_2'(x+M)/\Phi_2'(x)=1$ for any M, it follows from the second inequality in (2.9) that

$$\limsup_{x\to\infty}\frac{1}{\Phi_2'(x)}\mathsf{E}\left[\Phi_2'\left((X-x)_-\right)I_{\{X<x\}}\right]\le 1. \tag{2.10}$$

In view of the first inequality in (2.9), it is easy to see that

$$\liminf_{x\to\infty}\frac{1}{\Phi_2'(x)}\mathsf{E}\left[\Phi_2'\left((X-x)_-\right)I_{\{X<x\}}\right]\ge F(M) \tag{2.11}$$

holds for all $M>0$. Then (2.8) follows by letting $M\to\infty$ and combining (2.10) and (2.11).

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Next, observe that

$$\begin{aligned}\mathsf{E}\left[\Phi_1'\left((X-x)_+\right)I_{\{X>x\}}\right] &= \int_x^\infty \Phi_1'(y-x)dF(y) = \int_1^\infty \Phi_1'(x(u-1))dF(xu)\\ &= \Phi_1'(0)\overline{F}(x) + \int_1^\infty \overline{F}(xu)\Phi_1''(x(u-1))xdu\\ &= \Phi_1'(0)\overline{F}(x) + \int_0^\infty \overline{F}(x(u+1))\Phi_1''(xu)x\,du.\end{aligned}$$

It is shown in the Appendix that

$$\lim_{x\to\infty}\int_0^\infty \frac{\overline{F}(x(u+1))\Phi_1''(xu)}{\overline{F}(x)\Phi_1''(x)}du = B(\gamma-\beta_1,\beta_1). \tag{2.12}$$

Since $\beta_1 > 0$, $\lim_{x\to\infty} x\Phi_1''(x) = \infty$, which implies $\lim_{x\to\infty}\Phi_1'(0)/(x\Phi_1''(x)) = 0$. Then

$$\lim_{x\to\infty}\frac{1}{\overline{F}(x)\Phi_1''(x)x}\mathsf{E}\left[\Phi_1'\left((X-x)_+\right)I_{\{X>x\}}\right] = B(\gamma-\beta_1,\beta_1). \tag{2.13}$$

In view of the fact that $x_\alpha^* \to \infty$ as $\alpha \uparrow 1$, substituting (2.8) and (2.13) into (2.2) yields that

$$B(\gamma-\beta_1,\beta_1)\overline{F}(x_\alpha^*)\Phi_1''(x_\alpha^*)x_\alpha^* \sim (1-\alpha)\Phi_2'(x_\alpha^*), \quad \alpha\uparrow 1. \tag{2.14}$$

Define $H(x) = \overline{F}(x)\Phi_1''(x)x/\Phi_2'(x)$. Then $H \in \mathrm{RV}_{\beta_1-\beta_2-\gamma}$ with $\beta_1-\beta_2-\gamma < 0$ and, hence,

$$x_\alpha^* \sim [B(\gamma-\beta_1,\beta_1)]^{1/(\beta_2-\beta_1+\gamma)}\, H^{\leftarrow}(1-\alpha), \quad \alpha\uparrow 1.$$

Moreover, if $\Phi_1 = \Phi_2$, then $\beta_1 = \beta_2$. By Karamata's theorem, we have $\lim_{x\to\infty}\Phi_1''(x)x/\Phi_2'(x) = \beta_1$. Then (2.14) reduces to

$$\overline{F}(x_\alpha^*) \sim \frac{1-\alpha}{\beta_1 B(\gamma-\beta_1,\beta_1)}, \quad \alpha\uparrow 1,$$

which implies that $x_\alpha^* \sim [\beta_1 B(\gamma-\beta_1,\beta_1)]^{1/\gamma} q_\alpha$ as $\alpha\uparrow 1$. This completes the proof of the proposition. ■

3. EXPANSIONS OF EXPECTILES

For a random variable X with $\mathsf{E}|X| < \infty$, its α-expectile $e_\alpha = e_\alpha(X)$ is defined as the solution to the following equation:

$$e_\alpha - \mathsf{E}[X] = \frac{2\alpha-1}{1-\alpha}\mathsf{E}[(X-e_\alpha)_+]. \tag{3.1}$$

The Fisher–Tippett theorem gives the only possible limit laws for maxima $M_n = \max\{X_1,\ldots,X_n\}$ when properly normalized and centered, where $\{X_n, n\ge 1\}$ is a sequence of independent and identically distributed random variables with a common distribution function F; see Theorem 3.2.3 in Embrechts, Kluppelberg, and Mikosch [11] or Theorem 2.4.1 in Galambos [14]. The three possible limit laws are the Fréchet, Weibull and Gumbel distributions. Denote by $F \in \mathrm{MDA}(G)$ if F belongs to the maximum domain of attraction of some extreme distribution G.

In this section, we will establish the first-order and/or the second-order expansions in separate ways according to the different MDAs of the Fréchet, Weibull, and Gumbel distributions.

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3.1. The Fréchet Case

Let X be a random variable with distribution function F in the MDA of the Fréchet distribution, that is, $\overline{F} \in \mathrm{RV}_{-\gamma}$ for some $\gamma > 1$. Then, by Theorem 11 in Bellini et al. [3], we have

$$\overline{F}(e_\alpha) \sim \overline{F}(q_\alpha)(\gamma - 1), \quad \alpha \uparrow 1,$$

which implies that

$$e_\alpha \sim (\gamma - 1)^{-1/\gamma} q_\alpha, \quad \alpha \uparrow 1. \tag{3.2}$$

As pointed out by Bellini et al. [3], Eq. (3.2) implies that expectiles e_α are asymptotically larger than quantiles q_α for $\gamma \in (1, 2)$, while quantiles q_α are asymptotically larger than expectiles e_α for $\gamma > 2$. That is, expectiles e_α [quantiles q_α] are closer to the center of the distribution than the corresponding quantiles q_α [expectiles e_α] as α goes to 1 for $\gamma > 2$ [$\gamma < 2$].

The above first-order expansion may be crude. We shall investigate the second-order expansions of expectiles, which determine the convergence rates of the first-order expansions. Recall that a measurable function $h : \Re_+ \to \Re$ that is eventually positive is said to be of second-order regular variation (2RV) with the first-order parameter $\gamma \in \Re$ and the second-order parameter $\rho \le 0$, denoted by $h \in 2\mathrm{RV}_{\gamma,\rho}$, if there exists some ultimately positive or negative function $A(t)$ with $A(t) \to 0$ as $t \to \infty$ such that

$$\lim_{t\to\infty} \frac{h(tx)/h(t) - x^\gamma}{A(t)} = x^\gamma \int_1^x u^{\rho-1} du, \quad \forall\, x > 0.$$

Here, $A(t)$ is referred as an auxiliary function of h. Further properties of 2RV are given by Hua and Joe [16] and Lv Mao, and Hu [19].

Proposition 3.1: *Let X be a random variable having a distribution function F such that $\overline{F} \in 2\mathrm{RV}_{-\gamma,\rho}$ with $\gamma > 1$, $\rho < 0$ and auxiliary function A. Then, as $\alpha \uparrow 1$,*

$$e_\alpha(X) = \begin{cases} \left(\dfrac{2\alpha - 1}{\gamma - 1}\right)^{1/\gamma} q_\alpha \left(1 + \xi_{\gamma,\rho} A(q_\alpha)(1 + o(1))\right), & \mu = 0 \text{ or } \rho > -1, \\ \left(\dfrac{2\alpha - 1}{\gamma - 1}\right)^{1/\gamma} q_\alpha + \frac{\mu}{\gamma}(1 + o(1)), & \mu \neq 0 \text{ and } \rho < -1, \end{cases}$$

where $\mu = EX$ and

$$\xi_{\gamma,\rho} = \frac{1}{\gamma\rho}\left(\frac{(\gamma - 1)^{1-\rho/\gamma}}{\gamma - \rho - 1} - 1\right).$$

Proof: We first prove the desired result for the case that $\mu = 0$. By Remark 1 in Hua and Joe [16], $\overline{F} \in 2\mathrm{RV}_{-\gamma,\rho}$ with $\gamma > 0, \rho < 0$ and auxiliary function A means that there exists a constant $c > 0$, such that

$$\overline{F}(x) = c x^{-\gamma}\left(1 + \frac{A(x)}{\rho}(1 + o(1))\right), \quad x \to \infty. \tag{3.3}$$

Also, from the proof of Proposition 5 in Hua and Joe [16], it follows that

$$\mathsf{E}[(X - x)_+] = x\overline{F}(x)\left(\frac{1}{\gamma - 1} + \frac{1}{(\gamma - 1)(\gamma - 1 - \rho)} A(x)(1 + o(1))\right), \quad x \to \infty.$$

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Thus,

$$\mathsf{E}[(X-x)_+] = cx^{1-\gamma}\left(\frac{1}{\gamma-1}+\frac{A(x)}{\rho(\gamma-\rho-1)}(1+o(1))\right), \quad x\to\infty,$$

Since $\mu = 0$ and $e_\alpha \to \infty$ as $\alpha \uparrow 1$, we have

$$\begin{aligned}\frac{1-\alpha}{2\alpha-1} &= \frac{1}{e_\alpha}\mathsf{E}[(X-e_\alpha)_+]\\ &= ce_\alpha^{-\gamma}\left(\frac{1}{\gamma-1}+\frac{A(e_\alpha)}{\rho(\gamma-\rho-1)}(1+o(1))\right)\\ &= ce_\alpha^{-\gamma}\left(\frac{1}{\gamma-1}+\frac{(\gamma-1)^{-\rho/\gamma}}{\rho(\gamma-\rho-1)}A(q_\alpha)(1+o(1))\right), \quad \alpha\uparrow 1,\end{aligned}$$

where the last equality follows from that (3.2) and $A \in \mathrm{RV}_\rho$, $\rho < 0$. This implies

$$e_\alpha = c^{1/\gamma}\left(\frac{2\alpha-1}{1-\alpha}\right)^{1/\gamma}(\gamma-1)^{-1/\gamma}\left(1+\frac{(\gamma-1)^{1-\rho/\gamma}}{\gamma\rho(\gamma-\rho-1)}A(q_\alpha)(1+o(1))\right), \quad \alpha\uparrow 1. \tag{3.4}$$

On the other hand, by Proposition 2.5 in Mao and Hu [20], (3.3) implies that

$$U(t) = \left(\frac{1}{\overline{F}}\right)^{\leftarrow}(t) = (ct)^{1/\gamma}\left(1+\frac{1}{\gamma\rho}A(U(t))(1+o(1))\right), \quad t\to\infty.$$

Then

$$q_\alpha = U\left(\frac{1}{1-\alpha}\right) = c^{1/\gamma}(1-\alpha)^{-1/\gamma}\left(1+\frac{1}{\gamma\rho}A(q_\alpha)(1+o(1))\right), \quad \alpha\uparrow 1. \tag{3.5}$$

Substituting (3.5) into (3.4) yields that, as $\alpha \uparrow 1$,

$$\begin{aligned}e_\alpha &= q_\alpha(2\alpha-1)^{1/\gamma}(\gamma-1)^{-1/\gamma}\left(1-\frac{1}{\gamma\rho}A(q_\alpha)(1+o(1))\right)\\ &\quad\times\left(1+\frac{(\gamma-1)^{1-\rho/\gamma}}{\gamma\rho(\gamma-\rho-1)}A(q_\alpha)(1+o(1))\right)\\ &= q_\alpha\left(\frac{2\alpha-1}{\gamma-1}\right)^{1/\gamma}(1+\xi_{\gamma,\rho}A(q_\alpha)(1+o(1)))\,.\end{aligned}$$

Next, we consider the case that $\mu \neq 0$. Define $Y = X - \mu$. Then the distribution function of Y is $G(x) = F(x+\mu)$. From (3.3), it follows that

$$\begin{aligned}\overline{G}(x) &= c(x+\mu)^{-\gamma}\left[1+\frac{1}{\rho}A(x+\mu)(1+o(1))\right]\\ &= cx^{-\gamma}\left[1-\gamma\mu x^{-1}+\frac{1}{\rho}A(x)(1+o(1))\right], \quad x\to\infty.\end{aligned} \tag{3.6}$$

Now, we consider two cases.

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(1) If $\rho < -1$, then $\overline{G} \in 2\mathrm{RV}_{-\gamma,-1}$ with auxiliary function $A^*(x) = \gamma\mu/x$. Applying the above arguments to Y yields that

$$e_\alpha(Y) = q_\alpha(Y)\left(\frac{2\alpha-1}{\gamma-1}\right)^{1/\gamma}(1+\xi_{\gamma,-1}A^*(q_\alpha(Y))(1+o(1))). \quad \alpha \uparrow 1.$$

Then, by translation invariance property of $e_\alpha(\cdot)$ and $q_\alpha(\cdot)$, that is, $e_\alpha(X-\mu) = e_\alpha(X) - \mu$ and $q_\alpha(X-\mu) = q_\alpha(X) - \mu$, we have

$$\begin{aligned} e_\alpha(X) &= \mu + (q_\alpha - \mu)\left(\frac{2\alpha-1}{\gamma-1}\right)^{1/\gamma}\left(1+\xi_{\gamma,-1}\frac{\gamma\mu}{q_\alpha}(1+o(1))\right) \\ &= \left(\frac{2\alpha-1}{\gamma-1}\right)^{1/\gamma} q_\alpha + \frac{\mu}{\gamma}(1+o(1)), \quad \alpha \uparrow 1. \end{aligned}$$

(2) If $\rho > -1$, then $\overline{G} \in 2\mathrm{RV}_{-\gamma,\rho}$ with auxiliary function A. Similarly, we have

$$\begin{aligned} e_\alpha(X) &= \mu + (q_\alpha - \mu)\left(\frac{2\alpha-1}{\gamma-1}\right)^{1/\gamma}(1+\xi_{\gamma,\rho}A(q_\alpha)(1+o(1))) \\ &= \left(\frac{2\alpha-1}{\gamma-1}\right)^{1/\gamma} q_\alpha\,(1+\xi_{\gamma,\rho}A(q_\alpha)(1+o(1))), \quad \alpha \uparrow 1, \end{aligned}$$

where the last equality follows from the fact that $\lim_{\alpha\uparrow 1} q_\alpha A(q_\alpha) = \infty$ since $1+\rho > 0$.

Combining the two above cases and the case that $\mu = 0$, we obtain the desired results. This completes the proof of the proposition. ■

Remark 3.2: For $\rho = -1$ and $\mu = \mathsf{E}[X] \neq 0$, we can establish a similar result. Since $\overline{G} \in 2\mathrm{RV}_{-\gamma,-1}$ with auxiliary function $A^*(x) = \gamma\mu/x + A(x)$, then, as $\alpha \uparrow 1$,

$$\begin{aligned} e_\alpha(X) &= \mu + (q_\alpha - \mu)\left(\frac{2\alpha-1}{\gamma-1}\right)^{1/\gamma}(1+\xi_{\gamma,-1}A^*(q_\alpha)(1+o(1))) \\ &= \left(\frac{2\alpha-1}{\gamma-1}\right)^{1/\gamma} q_\alpha + \left[\left(1-\left(\frac{2\alpha-1}{\gamma-1}\right)^{1/\gamma}\right)\mu + \left(\frac{2\alpha-1}{\gamma-1}\right)^{1/\gamma}\xi_{\gamma,-1}q_\alpha A^*(q_\alpha)\right](1+o(1)) \\ &= \left(\frac{2\alpha-1}{\gamma-1}\right)^{1/\gamma} q_\alpha + \left[\frac{1}{\gamma}\mu + \left(\frac{2\alpha-1}{\gamma-1}\right)^{1/\gamma}\xi_{\gamma,-1}q_\alpha A(q_\alpha)\right](1+o(1)). \end{aligned}$$

3.2. The Weibull Case

PROPOSITION 3.3: *Let X be a random variable with distribution function F in the MDA of the Weibull distribution, that is, $\overline{F}(\hat{x} - 1/\cdot) \in \mathrm{RV}_{-\gamma}$ for some $\gamma > 0$, where $\hat{x}$ is the right*

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endpoint of the support of F. Then

$$\hat{x} - q_\alpha = o\left(\hat{x} - e_\alpha\right), \quad \alpha \uparrow 1. \tag{3.7}$$

In particular, if $\overline{F}(\hat{x} - 1/x) \sim cx^{-\gamma}$ as $x \to \infty$ for some $c > 0$, then

$$\hat{x} - e_\alpha \sim \eta_\gamma \left(\hat{x} - q_\alpha\right)^{\gamma/(1+\gamma)}, \quad \alpha \uparrow 1, \tag{3.8}$$

where

$$\eta_\gamma = \left((\gamma + 1)\left(\hat{x} - \mathsf{E}X\right)\right)^{1/(1+\gamma)}. \tag{3.9}$$

Proof: From Lemma 3.2 and Remark 3.3 in Mao and Hu [20], it follows that

$$\mathsf{E}\left[(X - x)_+\right] = \frac{1}{\gamma + 1}\left(\hat{x} - x\right)\overline{F}(x)(1 + o(1)), \quad x \uparrow \hat{x}.$$

Since $e_\alpha \uparrow \hat{x}$ as $\alpha \uparrow 1$, we have

$$\left(\hat{x} - e_\alpha\right)\overline{F}(e_\alpha) \sim (\gamma + 1)\left(\hat{x} - \mathsf{E}X\right)(1 - \alpha), \quad \alpha \uparrow 1. \tag{3.10}$$

Define $\tilde{e}_\alpha = \hat{x} - e_\alpha$ and $\tilde{q}_\alpha = \hat{x} - q_\alpha$. Since $\overline{F}(q_\alpha) \sim 1 - \alpha$ as $\alpha \uparrow 1$, then, for any $\delta > 0$ and α close enough to 1,

$$\overline{F}\left(\hat{x} - \tilde{q}_\alpha\right) < \delta^\gamma \overline{F}\left(\hat{x} - \tilde{e}_\alpha\right) \sim \overline{F}\left(\hat{x} - \delta\tilde{e}_\alpha\right).$$

Then $\tilde{q}_\alpha < \delta\tilde{e}_\alpha$ for any $\delta > 0$ and α close enough to 1. This implies (3.7).

If $\overline{F}(\hat{x} - 1/x) \sim cx^{-\gamma}$ as $x \to \infty$ for some $c > 0$, then, by (3.10), we have

$$\left(\hat{x} - e_\alpha\right)^{1+\gamma} \sim (\gamma + 1)\left(\hat{x} - \mathsf{E}X\right)\left(\hat{x} - q_\alpha\right)^\gamma, \quad \alpha \uparrow 1,$$

implying (3.8). ■

For the Weibull case, Eq. (3.7) implies that the expectiles e_α are closer to the center of the distribution F than the corresponding quantiles q_α as α goes to 1.

In order to establish the second-order expansions of expectiles e_α as $\alpha \uparrow 1$ for the Weibull case, we need impose the second-order condition upon F.

Proposition 3.4: *Let X be a random variable with distribution function F such that $\overline{F}(\hat{x} - 1/\cdot) \in 2\mathrm{RV}_{-\gamma,\rho}$ with auxiliary function $A(x)$, $\gamma > 0$ and $\rho < 0$. Then, as $\alpha \uparrow 1$,*

$$\hat{x} - e_\alpha = \begin{cases} \dfrac{\eta_\gamma}{(2\alpha - 1)^{1/(\gamma+1)}}\left(\hat{x} - q_\alpha\right)^{\gamma/(\gamma+1)}\left(1 - H_{\gamma,\rho}A_0(q_\alpha)(1 + o(1))\right), & \rho > -1, \\ \dfrac{\eta_\gamma}{(2\alpha - 1)^{1/(\gamma+1)}}\left(\hat{x} - q_\alpha\right)^{\gamma/(\gamma+1)}\left(1 - H_{\gamma,\rho}\left(\hat{x} - q_\alpha\right)^{\gamma/(\gamma+1)}(1 + o(1))\right), & \rho < -1, \end{cases}$$

with

$$H_{\gamma,\rho} = \begin{cases} \dfrac{\eta_\gamma^{-\rho}}{\gamma + 1}\left(\dfrac{1}{\rho} + \dfrac{1}{\gamma + 1 - \rho}\right), & \rho > -1, \\ \dfrac{\eta_\gamma}{(\gamma + 1)(\hat{x} - \mathsf{E}X)}, & \rho < -1, \end{cases}$$

$A_0(q_\alpha) = A\left(\left(\hat{x} - q_\alpha\right)^{-\gamma/(\gamma+1)}\right)$ and η_γ given by (3.9).

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PROOF: From Remark 1 in Hua and Joe [16], it follows that, for some $c > 0$,

$$\overline{F}\left(\hat{x} - \frac{1}{t}\right) = ct^{-\gamma}\left(1 + \frac{A(t)}{\rho}(1 + o(1))\right), \quad t \to \infty, \tag{3.11}$$

and, hence,

$$\overline{F}(x) = c(\hat{x} - x)^{\gamma}\left(1 + \frac{A^*(x)}{\rho}(1 + o(1))\right), \quad x \uparrow \hat{x}, \tag{3.12}$$

where $A^*(x) = A\left(1/(\hat{x} - x)\right)$. Also, it is shown in the Appendix that

$$\frac{\mathsf{E}[(X - x)_+]}{(\hat{x} - x)\overline{F}(x)} = \frac{1}{\gamma + 1}\left(1 + \frac{A^*(x)}{\gamma + 1 - \rho}(1 + o(1))\right), \quad x \uparrow \hat{x}, \tag{3.13}$$

Substituting (3.12) into (3.13) and noting that $e_\alpha \uparrow \hat{x}$ as $\alpha \uparrow 1$, we have

$$e_\alpha - \mathsf{E}X = \frac{(2\alpha - 1)\, c}{(1 - \alpha)(\gamma + 1)}\left(\hat{x} - e_\alpha\right)^{\gamma+1}\left(1 + W_{\gamma,\rho}A^*(e_\alpha)(1 + o(1))\right), \quad \alpha \uparrow 1, \tag{3.14}$$

with $W_{\gamma,\rho} = \frac{1}{\rho} + \frac{1}{\gamma+1-\rho}$. By Proposition 3.3, we have

$$A^*(e_\alpha) \sim A\left(\frac{1}{\eta_\gamma}(\hat{x} - q_\alpha)^{-\frac{\gamma}{\gamma+1}}\right) \sim \eta_\gamma^{-\rho} A\left((\hat{x} - q_\alpha)^{-\frac{\gamma}{\gamma+1}}\right), \quad \alpha \uparrow 1,$$

implying that

$$A^*(q_\alpha) = o\left(A^*(e_\alpha)\right), \quad \alpha \uparrow 1. \tag{3.15}$$

Now, two cases arise.

(1) $\rho > -1$. Since $\hat{x} - e_\alpha = o\left(A^*(e_\alpha)\right)$ as $\alpha \uparrow 1$ and $e_\alpha - \mathsf{E}X$ can be written as $\hat{x} - \mathsf{E}X - (\hat{x} - e_\alpha)$, (3.14) can be rewritten as

$$\hat{x} - \mathsf{E}X = \frac{(2\alpha - 1)c}{(1 - \alpha)(\gamma + 1)}(\hat{x} - e_\alpha)^{\gamma+1}\left(1 + W_{\gamma,\rho}A^*(e_\alpha)(1 + o(1))\right). \tag{3.16}$$

On the other hand, from (3.12), we have

$$\frac{1 - \alpha}{c} = (\hat{x} - q_\alpha)^{\gamma}\left(1 + \frac{A^*(q_\alpha)}{\rho}(1 + o(1))\right), \quad \alpha \uparrow 1. \tag{3.17}$$

Substituting (3.17) and (3.15) into (3.16) yields that

$$(\hat{x} - e_\alpha)^{\gamma+1} = \frac{\gamma + 1}{2\alpha - 1}(\hat{x} - \mathsf{E}X)(\hat{x} - q_\alpha)^{\gamma}\left(1 - W_{\gamma,\rho}A^*(e_\alpha)(1 + o(1))\right), \quad \alpha \uparrow 1,$$

and, hence,

$$\hat{x} - e_\alpha = \frac{\eta_\gamma}{(2\alpha - 1)^{1/(\gamma+1)}}(\hat{x} - q_\alpha)^{\frac{\gamma}{\gamma+1}}\left(1 - \frac{\eta_\gamma^{-\rho}W_{\gamma,\rho}}{\gamma + 1}A_0(q_\alpha)(1 + o(1))\right), \quad \alpha \uparrow 1,$$

with $A_0(q_\alpha) = A\left((\hat{x} - q_\alpha)^{-\gamma/(\gamma+1)}\right)$.

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(2) $\rho < -1$. Since $A^*(e_\alpha) = o(\hat{x} - e_\alpha)$ and $A^*(q_\alpha) = o(A^*(e_\alpha))$ as $\alpha \uparrow 1$, we have

$$\hat{x} - \mathsf{E}X - (\hat{x} - e_\alpha)(1 + o(1)) = \frac{(2\alpha - 1)c}{(1-\alpha)(\gamma+1)}(\hat{x} - e_\alpha)^{\gamma+1}, \quad \alpha \uparrow 1. \tag{3.18}$$

Substituting (3.17) into (3.18) yields that

$$(\hat{x} - q_\alpha)^\gamma \left(\hat{x} - \mathsf{E}X - (\hat{x} - e_\alpha)(1 + o(1))\right) = \frac{2\alpha - 1}{\gamma + 1}(\hat{x} - e_\alpha)^{\gamma+1}. \tag{3.19}$$

Replacing the $\hat{x} - e_\alpha$ in the left-hand side of (3.19) by the asymptotic expansion in (3.8), we have

$$\frac{2\alpha - 1}{\gamma + 1}(\hat{x} - e_\alpha)^{\gamma+1} = (\hat{x} - q_\alpha)^\gamma \left((\hat{x} - \mathsf{E}X) - \eta_\gamma (\hat{x} - q_\alpha)^{\frac{\gamma}{\gamma+1}}(1 + o(1))\right), \quad \alpha \uparrow 1,$$

and, hence,

$$\hat{x} - e_\alpha = \frac{\eta_\gamma}{(2\alpha - 1)^{1/(\gamma+1)}}(\hat{x} - q_\alpha)^{\frac{\gamma}{\gamma+1}} \left(1 - \frac{\eta_\gamma (\hat{x} - q_\alpha)^{\gamma/(\gamma+1)}}{(\gamma+1)(\hat{x} - \mathsf{E}X)}(1 + o(1))\right), \quad \alpha \uparrow 1.$$

This completes the proof of the proposition. ■

Remark 3.5: Under the conditions of Proposition 3.4, a refinement of the proof allows us to derive the following second-order expansion of $\hat{x} - e_\alpha$ as $\alpha \uparrow 1$ for the case $\rho = -1$:

$$\hat{x} - e_\alpha = \frac{\eta_\gamma}{(2\alpha - 1)^{1/(\gamma+1)}}(\hat{x} - q_\alpha)^{\gamma/(\gamma+1)} \times \left\{1 - \frac{\eta_\gamma}{\gamma + 1}\left(\frac{(\hat{x} - q_\alpha)^{\gamma/(\gamma+1)}}{\hat{x} - \mathsf{E}X} - \frac{\gamma + 1}{\gamma + 2}A_0(q_\alpha)\right)(1 + o(1))\right\}, \quad \alpha \uparrow 1.$$

3.3. The Gumbel Case

For the Gumbel case, we cannot obtain the explicit first-order expansion of expectiles e_α as $\alpha \uparrow 1$ in terms of quantiles q_α. However, we have the following result, which gives us some intuition on e_α and q_α as $\alpha \uparrow 1$.

PROPOSITION 3.6: *Let X be a random variable with distribution function F in the MDA of the Gumbel distribution, and denote by $\hat{x}$ the right endpoint of the support of F. Then, as $\alpha \uparrow 1$,*

$$\overline{F}(q_\alpha) = \begin{cases} o\big(\overline{F}(e_\alpha)\big), & \hat{x} = \infty; \\ o\big((\hat{x} - e_\alpha)\overline{F}(e_\alpha)\big), & \hat{x} < \infty. \end{cases}$$

PROOF: Denote by $a(t)$ the auxiliary function of the tail quantile function $U(t) = (1/\overline{F})^{\leftarrow}(t)$. Then, by Lemma 3.2 and Remark 3.3 in Mao and Hu [20], we have

$$\begin{aligned} \mathsf{E}\left[(X - x)_+\right] &= a\left(\frac{1}{\overline{F}(x)}\right)\overline{F}(x)(1 + o(1)) \\ &= \begin{cases} o\big(x\overline{F}(x)\big), & \hat{x} = \infty, \\ o\big((\hat{x} - x)\overline{F}(x)\big), & \hat{x} < \infty. \end{cases} \end{aligned} \tag{3.20}$$

Now, we consider two cases.

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(1) For $\hat{x} = \infty$, substituting (3.20) into (3.1), we have

$$(1-\alpha)e_\alpha \sim (1-\alpha)(e_\alpha - \mathsf{E}X) = o\big(e_\alpha \overline{F}(e_\alpha)\big), \quad \alpha \uparrow 1.$$

This means that $\overline{F}(q_\alpha) \sim 1-\alpha = o\big(\overline{F}(e_\alpha)\big)$ as $\alpha \uparrow 1$.

(2) For $\hat{x} < \infty$, substituting (3.20) into (3.1), we have

$$(1-\alpha)(\hat{x} - \mathsf{E}X) = o\big((\hat{x}-e_\alpha)\overline{F}(e_\alpha)\big), \quad \alpha \uparrow 1.$$

This means that $\overline{F}(q_\alpha) \sim o\big((\hat{x}-e_\alpha)\overline{F}(e_\alpha)\big)$ as $\alpha \uparrow 1$.

This completes the proof of the proposition. ■

Also from Proposition 3.6, it is seen that, for any distribution in the MDA of the Gumbel distribution, its expectiles e_α are closer to the center of the distribution than the corresponding quantiles q_α when α is close to 1.

4. EXAMPLES

In this section, three examples are presented to illustrate the power of the second-order expansions of expectiles in Propositions 3.1 and 3.4. In the first two examples, we consider Pareto and Student distributions which belong to the MDA of the Fréchet distribution. The third example deals with Beta distribution which belongs to the MDA of the Weibull distribution.

Example 4.1 (Pareto Distribution): Let X be a random variable with a Pareto distribution function F given by

$$F(x) = 1 - \left(\frac{\theta}{x+\theta}\right)^\gamma, \quad x > 0,$$

with parameters $\gamma > 1$ and $\theta > 0$. It is easy to verify that $\overline{F} \in 2\mathrm{RV}_{-\gamma,-1}$ with auxiliary function $A(x) = \theta\gamma/x$ and $\mathsf{E}[X] = \theta/(\gamma-1)$. By Remark 3.2, we have, as $\alpha \uparrow 1$,

$$e_\alpha(X) = \left(\frac{2\alpha-1}{\gamma-1}\right)^{1/\gamma} q_\alpha + \theta\left[(\gamma-1)^{-1/\gamma} - \frac{\gamma-2}{\gamma-1}\right](1+o(1)).$$

In Figure 1, we compare the first-order and the second-order expansions with the exact true value of $e_\alpha(X)$ when $\gamma = 1.3$ and $\theta = 5$. Similarly, in Figure 2, we set $\gamma = 3$ and $\theta = 6$. We find that the second-order expansion is more accurate than the first-order one.

Example 4.2 (Student t_ν Distribution): Let X be a random variable having a standard Student t_ν distribution function F with density function given by

$$f(x) = \frac{\Gamma((\nu+1)/2)}{\sqrt{\nu\pi}\,\Gamma(\nu/2)}\left(1+\frac{x^2}{\nu}\right)^{-(\nu+1)/2}, \quad \nu > 1.$$

It is known from Example 3 and Proposition 6 in Hua and Joe [16] that $\overline{F} \in 2\mathrm{RV}_{-\nu,-2}$ with auxiliary function $A(x) = \nu^2(\nu+1)x^{-2}/(\nu+2)$ and $\mathsf{E}[X] = 0$. By Proposition 3.1, we have

$$e_\alpha(X) = \left(\frac{2\alpha-1}{\nu-1}\right)^{1/\nu} q_\alpha \left(1 + \xi_{\nu,-2}\frac{\nu^2(\nu+1)}{\nu+2} q_\alpha^{-2}(1+o(1))\right), \quad \alpha \uparrow 1,$$

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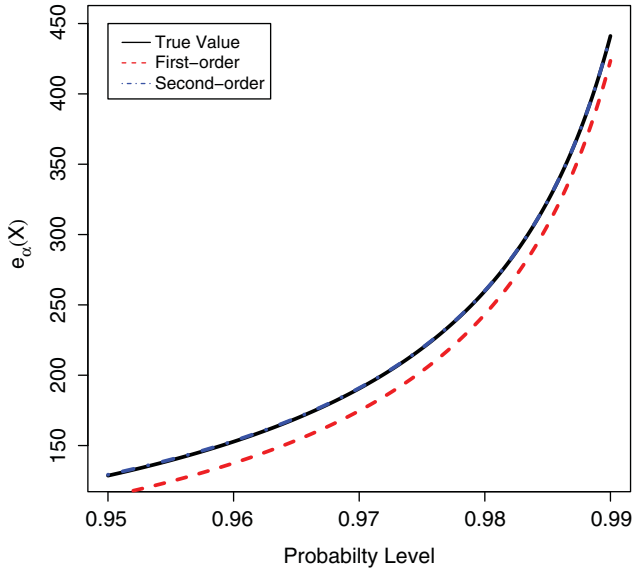


FIGURE 1. Pareto distribution with $\gamma = 1.3$ and $\theta = 5$.

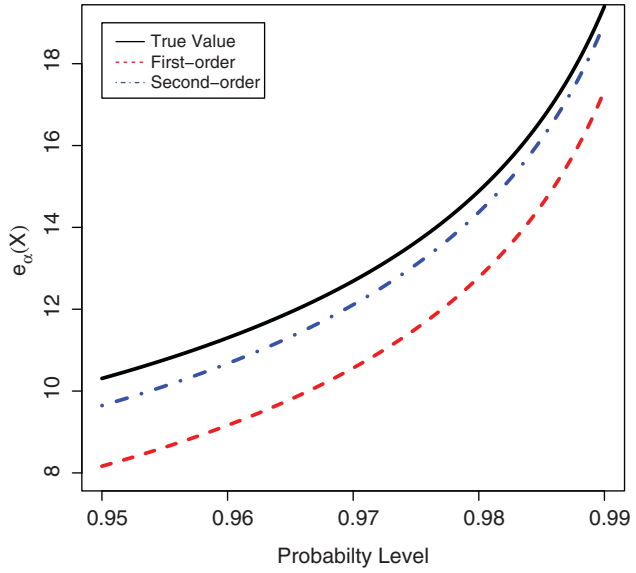


FIGURE 2. Pareto distribution with $\gamma = 3$ and $\theta = 6$.

with $\xi_{\nu,-2} = (1/2\nu)(1 - ((\nu-1)^{1+2/\nu})/(\nu+1))$. In Figures 3 and 4, we compare the first-order and the second-order expansions with the exact true values of $e_\alpha(X)$ when the degrees of freedom $\nu = 1.4$ and $\nu = 2.5$.

Example 4.3 (Beta Distribution): Let X be a random variable having Beta (d, b) distribution with density function given by

$$f(x) = \frac{1}{B(d,b)} x^{d-1}(1-x)^{b-1}, \quad 0 < x < 1, \quad d, b > 0.$$

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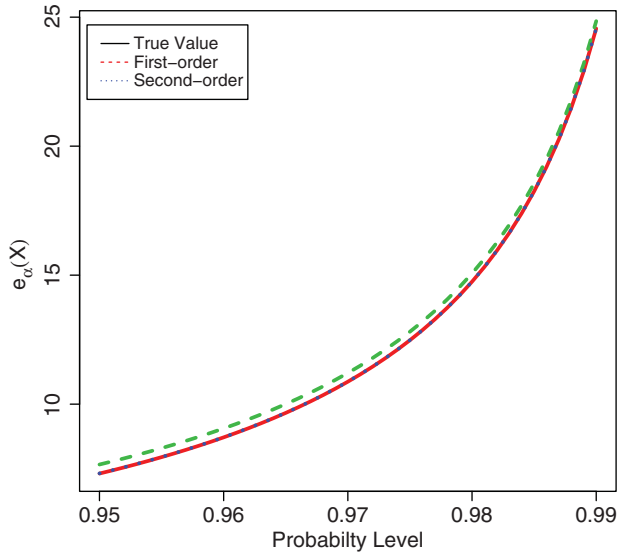


FIGURE 3. Student distribution with $\nu = 1.4$.

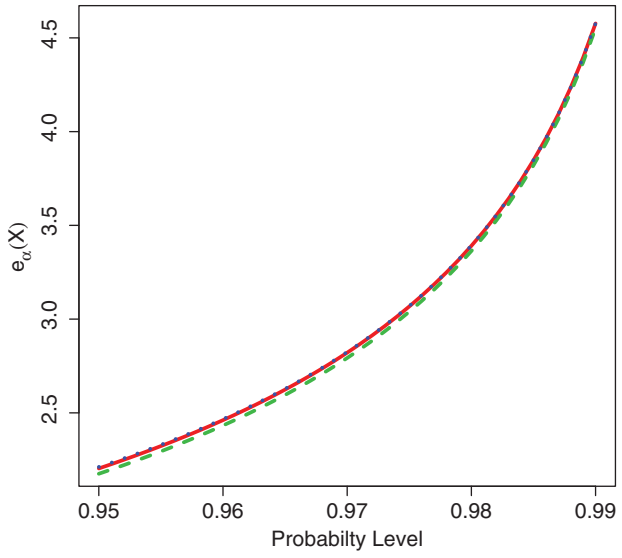


FIGURE 4. Student distribution with $\nu = 2.5$.

It is proved in Mao and Hu [20] that $\overline{F}(1-1/\cdot) \in 2\mathrm{RV}_{-b,-1}$ with auxiliary function $A(t) = b(d-1)t^{-1}/(b+1)$. Note that $\mathsf{E}[X] = d/(d+b)$ and $\hat{x} = 1$. By Remark 3.5, we have

$$1 - e_\alpha = \frac{\eta_b}{(2\alpha-1)^{1/(b+1)}}(1-q_\alpha)^{\frac{b}{b+1}} \times \left\{1 - \frac{\eta_b}{b+1}(1-q_\alpha)^{\frac{b}{b+1}}\left(\frac{d+b}{b} - \frac{b(d-1)}{b+2}\right)(1+o(1))\right\}$$

as $\alpha \uparrow 1$, where $\eta_b = \left(\frac{b(b+1)}{d+b}\right)^{1/(1+b)}$. In Figure 5, we compare the first-order and the second-order expansions with the exact true value of $1 - e_\alpha(X)$ as $\alpha \uparrow 1$ when $(d, b) = (2, 0.5)$.

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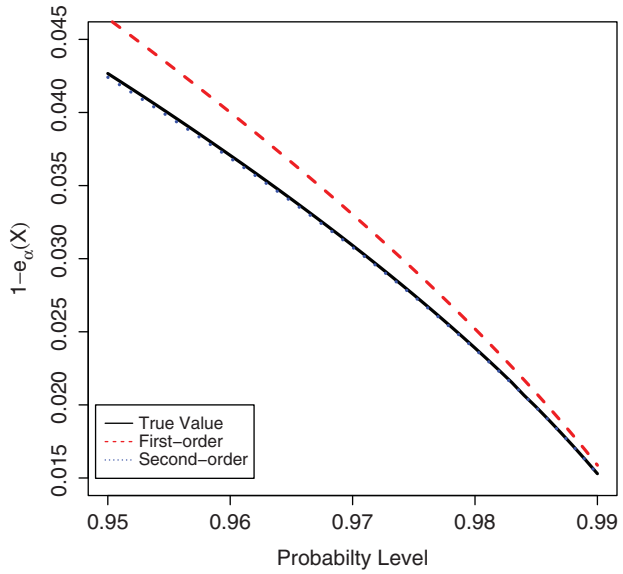


FIGURE 5. Beta distribution with $d = 2$ and $b = 0.5$.

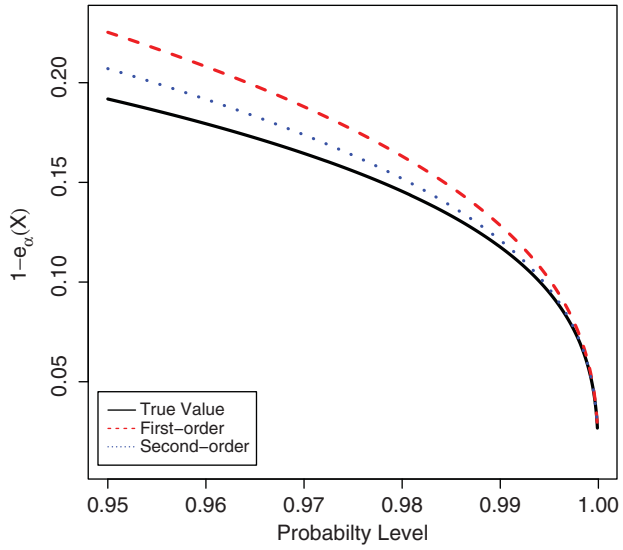


FIGURE 6. Beta distribution with $d = 3$ and $b = 2$.

Similarly, in Figure 6, we set $(d, b) = (3, 2)$. From numerical computation, it is seen that the second-order expansion is more accurate than the first-order one when the parameter b is small.

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APPENDIX

The Proof of (2.12): First, note that $\overline{F} \in \mathrm{RV}_{-\gamma}$. Then, by Potter's theorem, for any $\delta_1 > 0$, $(\gamma - \beta_1) \wedge (2\beta_1) > 2\delta_2 > 0$, there exists x_0 such that for $x \geq x_0$ and $u > 0$,

$$\frac{\overline{F}(x(u+1))}{\overline{F}(x)} \leq (1+\delta_1)(u+1)^{-\gamma+\delta_2}. \tag{A.1}$$

We consider the following two cases.

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(1) *Case 1:* $0 < \beta_1 \le 1$. Since $\Phi_1'' \in \mathrm{RV}_{\beta_1-1}$, then, by Proposition B.1.9 (7) in de Haan and Ferreira [7] and Potter's theorem, there exist $c > 0$ and $x_1 > x_0$ such that for $x \ge x_1$,

$$\frac{\Phi_1''(xu)}{\Phi_1''(x)} \le cu^{\beta_1-1-\delta_2}, \quad 0 < u < 1,$$

and

$$\frac{\Phi_1''(xu)}{\Phi_1''(x)} \le (1+\delta_1)u^{\beta_1-1+\delta_2}, \quad u \ge 1. \tag{B.2}$$

Combining this with (A.1), we have

$$\frac{\overline{F}(x(u+1))\Phi_1''(xu)}{\overline{F}(x)\Phi_1''(x)} \le \begin{cases} c(1+\delta_1)(u+1)^{-\gamma+\delta_2}u^{\beta_1-1-\delta_2}, & 0 < u < 1, \\ (1+\delta_1)^2(u+1)^{-\gamma+\delta_2}u^{\beta_1-1+\delta_2}, & u \ge 1. \end{cases}$$

It is easy to see that the right-hand side of the inequality is integrable over $0 < u < \infty$ for $2\delta_2 < (\gamma-\beta_1) \wedge (2\beta_1)$. Then, by the dominated convergence theorem, we have

$$\begin{aligned} \lim_{x\to\infty} \int_0^\infty \frac{\overline{F}(x(u+1))\Phi_1''(xu)}{\overline{F}(x)\Phi_1''(x)} du &= \int_0^\infty (u+1)^{-\gamma}u^{\beta_1-1}du \\ &= \int_1^\infty u^{-\gamma}(u-1)^{\beta_1-1}du \\ &= \int_0^1 u^{\gamma}(1-u)^{\beta_1-1}u^{-\beta_1-1}du \\ &= B(\gamma-\beta_1,\beta_1). \end{aligned} \tag{C.3}$$

This completes the proof of (2.12) for $0 < \beta_1 \le 1$.

(2) *Case 2:* $\beta_1 > 1$. Since $\Phi_1'' \in \mathrm{RV}_{\beta_1-1}$, by Proposition B.1.9 (6) in de Haan and Ferreira (2006) and Potter's theorem, there exist $c > 0$ and $x_2 > x_0$ such that for $x \ge x_2$,

$$\frac{\Phi_1''(xu)}{\Phi_1''(x)} \le c, \quad 0 < u < 1,$$

and (B.2) holds for $u \ge 1$. Combining this with (A.1), we have

$$\frac{\overline{F}(x(u+1))\Phi_1''(xu)}{\overline{F}(x)\Phi_1''(x)} \le \begin{cases} c(1+\delta_1)(u+1)^{-\gamma+\delta_2}, & 0 < u < 1, \\ (1+\delta_1)^2(u+1)^{-\gamma+\delta_2}u^{\beta_1-1+\delta_2}, & u \ge 1. \end{cases}$$

Note that the right-hand side of the inequality is integrable over $0 < u < \infty$ for $2\delta_2 < \gamma-\beta_1$. Then, by the dominated convergence theorem, (C.3) holds for $\beta_1 > 1$.

So, we complete the proof of (3.12). ■

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THE PROOF OF EQ. (3.13):: Let $U(t) = \left(1/\overline{F}\right)^{\leftarrow}(t)$ denote the tail quantile function of F. From (3.11), it follows that

$$\frac{1}{\overline{F}(\hat{x} - 1/t)} = c^{-1}t^{\gamma}\left(1 - \frac{A(t)}{\rho}(1 + o(1))\right), \quad t \to \infty.$$

By Proposition 2.6 in Lv et al. [19], we have

$$\hat{x} - U(t) = c^{-1/\gamma}t^{-1/\gamma}\left(1 - \frac{1}{\gamma\rho}A\left(\frac{1}{\hat{x} - U(t)}\right)(1 + o(1))\right), \quad t \to \infty,$$

implying that

$$\lim_{t\to\infty} \frac{\frac{U(tx)-U(t)}{\hat{x}-U(t)} - (1 - x^{-1/\gamma})}{A(1/(\hat{x} - U(t)))} = x^{-1/\gamma}\frac{x^{\rho/\gamma} - 1}{\rho\gamma}, \quad x > 0,$$

since $A(1/(\hat{x} - U(\cdot))) \in \mathrm{RV}_{\rho/\gamma}$. Then, by similar arguments to that in the proof of Lemma 4.2 in Mao and Hu [21], we have

$$\begin{aligned}
\frac{\mathsf{E}[(X - U(t))_+]}{t^{-1}(\hat{x} - U(t))} &= \int_0^1 \frac{U(t/v) - U(t)}{\hat{x} - U(t)}dv \\
&= \int_0^1 \left(1 - v^{1/\gamma}\right)dv + \int_0^1 v^{1/\gamma}\frac{v^{-\rho/\gamma} - 1}{\rho\gamma}dv \cdot A\left(\frac{1}{\hat{x} - U(t)}\right)(1 + o(1)) \\
&= \frac{1}{\gamma + 1} - A\left(\frac{1}{\hat{x} - U(t)}\right)\frac{1}{\gamma\rho}\left(\frac{\gamma}{\gamma + 1} - \frac{\gamma}{\gamma + 1 - \rho}\right)(1 + o(1)) \\
&= \frac{1}{\gamma + 1} + A\left(\frac{1}{\hat{x} - U(t)}\right)\frac{1}{(\gamma + 1)(\gamma + 1 - \rho)}(1 + o(1)) \\
&= \frac{1}{\gamma + 1}\left(1 + A\left(\frac{1}{\hat{x} - U(t)}\right)\frac{1}{\gamma + 1 - \rho}(1 + o(1))\right), \quad t \to \infty.
\end{aligned}$$

Replacing $U(t)$ by x yields (3.13) since $U(t) \to \hat{x}$ as $t \to \infty$.

■

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