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Theory of particle diffusion in electrostatic turbulent plasma using extended direct-interaction approximation

M. TAGUCHI

College of Industrial Technology, Nihon University, 2-11-1 Shin-ei,
Narashino, Chiba 275-0005, Japan
(taguchi.masayoshi@nihon-u.ac.jp)

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Abstract. The theory of particle diffusion in an electrostatic turbulent plasma is formulated by applying the direct-interaction approximation (DIA) to subensemble-averaged functions instead of conventional ensemble-averaged ones. This theory approximately incorporates the Lagrangian description into the DIA through decorrelation trajectories. The running diffusion coefficient is shown to be calculated by solving a nonlinear ordinary differential equation together with an equation for decorrelation trajectories and by averaging initial conditions at the starting point of trajectories.

1. Introduction

Considerable number of articles have been devoted to systematic formulations of statistical plasma turbulence theory [1, 2]. However, the rigorous mathematical treatment of the systematic formulation is far from our reach and so various approximate approaches are presented here. Representative approach to the plasma turbulent theory beyond the quasi-linear regime is the renormalized perturbation theory by the direct-interaction approximation (DIA) [2–4].

The running diffusion coefficient in the electrostatic turbulence is described by the time integral of the two-point Lagrangian correlation of the stochastic velocity. This Lagrangian correlation and the resulting running diffusion coefficient are calculated using the second cumulant and the Corrsin approximations [2, 5–7]. These results can also be obtained in the framework of kinetic formulation by employing a closed set of equations for an ensemble-averaged distribution function and a response function [8, 9]. In this paper we refer the approximation leading to this closed set of equations as the DIA. However, these approximations, which relied on the Eulerian description, failed to reproduce the results of numerical simulation and the qualitative theoretical estimation in the extreme turbulent regime [2]. This failure is mainly due to the lack of correct treatment of particle trapping in a fluctuating electrostatic potential. The formulation based on the Lagrangian description is favorable in order to take the trapping effect into account. The Lagrangian formulation is, however, not an easy work because we need the information of particle trajectories in the stochastic electrostatic potential.

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Vlad et al. [2, 10–12] presented a powerful tool for estimating the Lagrangian correlation in the whole range from the quasi-linear to the nonlinear regimes. They have introduced a set of deterministic decorrelation trajectories which are determined by the Eulerian correlation of the potential, and they have intuitively approximated the Lagrangian velocity correlation using these trajectories instead of stochastic real trajectories. The Lagrangian correlation and the resulting running diffusion coefficient obtained by the DCT method in the framework of a test-particle approach are shown to successfully reproduce those of numerical simulation.

In the DCT method the fluctuations of trajectories in the subensemble are neglected. Vlad and Spineanu [13] recently extended and improved the DCT method by taking into account the fluctuations of decorrelation trajectories in considering the nested subensembles. In this paper we choose another approach for extending the DCT method and also extending the formulation by the DIA. We incorporate the idea of the decorrelation trajectory into the renormalized perturbation theory by the DIA, and present a new statistical formulation for the theory of particle diffusion in an electrostatic plasma turbulence. In Sec. 2, starting from a kinetic equation in the presence of electrostatic fluctuations and using the functional integral representation, we will present a closure theory for an ensemble-averaged distribution function and its response function using the DIA. The running diffusion coefficient in the DIA is shown to be calculated by solving a nonlinear ordinary differential equation. In Sec. 3, we will repeat the formulation of Sec. 2 using subensemble-averaged functions instead of conventional ensemble-averaged functions, and incorporate the Lagrangian description into the DIA through the decorrelation trajectories. The running diffusion coefficients in this extended DIA is found to be obtained by solving a nonlinear ordinary differential equation together with an equation for the decorrelation trajectories. In Sec. 4, we show that the concluding results obtained in Sec. 3 can also be derived in the framework of a test-particle approach.

2. Formulation by DIA

Let us consider a fluctuating electrostatic potential $\phi(\boldsymbol{r}, t)$ in a strong constant magnetic field $\boldsymbol{B} = B\mathbf{e}_z$, where $\boldsymbol{r} = x\mathbf{e}_x + y\mathbf{e}_y + z\mathbf{e}_z$; $\mathbf{e}_i$ is the unit vector in the i-direction and the subscripts $\parallel$ and $\perp$ refer parallel and perpendicular to the magnetic field. Then we start with the following kinetic equation for a distribution function $f(\boldsymbol{r}, t)$ in the presence of electrostatic fluctuations:

$$\partial_t f + v_\parallel \partial_\parallel f + \sqrt{\epsilon}\boldsymbol{v}(\boldsymbol{r}, t) \cdot \nabla_\perp f - C(f) = 0, \tag{2.1}$$

with

$$C(f) = -\sqrt{\epsilon}\eta_\parallel \partial_\parallel f - \sqrt{\epsilon}\boldsymbol{\eta}_\perp(t)\nabla_\perp f, \tag{2.2}$$

where $\partial_\parallel = \partial/\partial z$; $\nabla_\perp = \mathbf{e}_x\partial/\partial x + \mathbf{e}_y\partial/\partial y$; the function $f(\boldsymbol{r}, t)$ is normalized as $\int f(\boldsymbol{r}, t)\, d\boldsymbol{r} = 1$; the effect of collisions is modeled by the Gaussian random variation $\boldsymbol{\eta}(t)$ of velocity [2]; $\boldsymbol{v}(\boldsymbol{r}, t)$ represents the drift velocity given by $\boldsymbol{v}(\boldsymbol{r}, t) = -[\nabla\phi(\boldsymbol{r}, t) \times \mathbf{e}_z]/B$ and the stochastic parallel velocity due to the electrostatic acceleration [12] is neglected. In (2.1) we have introduced a formal ordering parameter ϵ, which is finally set to be unity. The correlation of the collisional variation is assumed

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to be

$$\langle \eta_\parallel(t)\eta_\parallel(t')\rangle_\parallel \;=\; v\chi_\parallel R[v(t-t')], \tag{2.3}$$

$$\langle \boldsymbol{\eta}_\perp(t)\boldsymbol{\eta}_\perp(t')\rangle_\perp \;=\; v\chi_\perp R[v(t-t')]\mathbf{I}, \tag{2.4}$$

where $\mathbf{I} = \mathbf{e}_x\mathbf{e}_x + \mathbf{e}_y\mathbf{e}_y$; v is the collision frequency; $\chi_\parallel$ and $\chi_\perp$ are the parallel and perpendicular cross-field classical diffusion coefficients and $\langle\cdot\rangle_\parallel$ and $\langle\cdot\rangle_\perp$ denote the ensemble average over the random variables $\eta_\parallel(t)$ and $\boldsymbol{\eta}_\perp(t)$. We assume Gaussian, stationary and homogeneous for the fluctuating electrostatic potential $\phi(\boldsymbol{r},t)$, and write the Eulerian correlation functions as

$$\boldsymbol{E}_\phi(\boldsymbol{r}-\boldsymbol{r}',t-t') = \langle\phi(\boldsymbol{r},t)\boldsymbol{v}(\boldsymbol{r}',t')\rangle_\phi = \frac{1}{B}\left(\boldsymbol{e}_z\times\nabla' E\right) \tag{2.5}$$

and

$$\boldsymbol{E}_v(\boldsymbol{r}-\boldsymbol{r}',t-t') = \langle\boldsymbol{v}(\boldsymbol{r},t)\boldsymbol{v}(\boldsymbol{r}',t')\rangle_\phi = -\frac{1}{B^2}\left(\boldsymbol{e}_z\times\nabla'\right)\left(\boldsymbol{e}_z\times\nabla'\right)E \tag{2.6}$$

with

$$E(\boldsymbol{r}-\boldsymbol{r}',t-t') = \langle\phi(\boldsymbol{r},t)\phi(\boldsymbol{r}',t')\rangle_\phi, \tag{2.7}$$

where $\langle\cdot\rangle_\phi$ means the ensemble average over the fluctuating potential $\phi(\boldsymbol{r},t)$. The correlation functions $R(t)$ and $E(\boldsymbol{r},t)$ are further assumed to be even functions with respect to t and $\boldsymbol{r}$. For later use we note $\nabla\cdot\boldsymbol{E}_\phi(\boldsymbol{r}-\boldsymbol{r}',t-t') = -\nabla'\cdot\boldsymbol{E}_\phi(\boldsymbol{r}-\boldsymbol{r}',t-t') = 0$ and $\nabla\cdot\boldsymbol{E}_v(\boldsymbol{r}-\boldsymbol{r}',t-t') = -\nabla'\cdot\boldsymbol{E}_v(\boldsymbol{r}-\boldsymbol{r}',t-t') = \mathbf{0}$.

We now introduce a functional representation for a functional $F[f]$ of the distribution function [14, 15]:

$$F[f] = \mathscr{N}\int \mathscr{D}[f']\mathscr{D}[\hat{f}]\,F[f']\,e^{-\mathscr{L}/\epsilon} \tag{2.8}$$

with

$$\mathscr{L} = \int dtd\boldsymbol{r}\hat{f}(\boldsymbol{r},t)\left\{\partial_t + [v_\parallel + \sqrt{\epsilon}\eta_\parallel(t)]\partial_\parallel + \sqrt{\epsilon}[\boldsymbol{v}(\boldsymbol{r},t) + \boldsymbol{\eta}_\perp(t)]\cdot\nabla_\perp\right\}f'(\boldsymbol{r},t), \tag{2.9}$$

where the measures $\mathscr{D}[f']$ and $\mathscr{D}[\hat{f}]$ mean integrations over all possible functions $f'(\boldsymbol{r},t)$ and $\hat{f}(\boldsymbol{r},t)$, and $\mathscr{N}$ is the normalization constant irrelevant to the following calculation. The ensemble average of a functional $F[f]$ over the random variables $\phi(\boldsymbol{r},t)$, $\eta_\parallel(t)$ and $\boldsymbol{\eta}_\perp$, which is denoted as $\langle\,\cdot\,\rangle \equiv \langle\,\cdot\,\rangle_{\phi,\parallel,\perp}$, yields the ensemble-averaged functional

$$\langle F[f]\rangle = \mathscr{N}\int \mathscr{D}[f']\mathscr{D}[\hat{f}]\,F[f']\,\langle e^{-\mathscr{L}/\epsilon}\rangle, \tag{2.10}$$

where we note that f' and $\hat{f}$ are merely integral variables in the functional representation. We next introduce the following generating functional with the infinitesimal external sources $\zeta(\boldsymbol{r},t)$ and $\xi(\boldsymbol{r},t)$:

$$Z[\zeta,\xi] = \mathscr{N}\int \mathscr{D}[f']\mathscr{D}[\hat{f}]\;\; e^{-S/\epsilon} \equiv e^{W/\epsilon} \tag{2.11}$$

with

$$S = -\epsilon\log\langle e^{-\mathscr{L}/\epsilon}\rangle - \int dt\,d\boldsymbol{r}\left[\zeta(\boldsymbol{r},t)f'(\boldsymbol{r},t) + \xi(\boldsymbol{r},t)\hat{f}(\boldsymbol{r},t)\right], \tag{2.12}$$

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and using this functional we define

$$g(\boldsymbol{r},t)\equiv\frac{\delta W}{\delta\zeta(\boldsymbol{r},t)},\qquad G(\boldsymbol{r}-\boldsymbol{r}',t-t')\equiv\epsilon\frac{\delta^2 W}{\delta\xi(\boldsymbol{r}',t')\delta\zeta(\boldsymbol{r},t)}. \tag{2.13}$$

In the limit of $\xi=\zeta=0$, the one-point function $g(\boldsymbol{r},t)$ becomes the ensemble-averaged distribution function, *i.e.* $g(\boldsymbol{r},t)=\langle f(\boldsymbol{r},t)\rangle$, and the causality leads to the property $\langle\hat{f}(\boldsymbol{r}_1,t_1)\cdots\rangle=0$, if the time t_1 is the latest time in the ensemble average [14, 15].

Performing the ensemble average in (2.12) over the random variables $\phi(\boldsymbol{r},t)$, $\eta_\parallel(t)$ and $\boldsymbol{\eta}_\perp(t)$, and using the Wentzel–Kramers–Brillouin (WKB) approximation for the identity

$$\int\mathscr{D}[f']\mathscr{D}[\hat{f}]\frac{\delta}{\delta\hat{f}}e^{-S/\epsilon}=0, \tag{2.14}$$

we can obtain the equation for the ensemble-averaged distribution function $g(\boldsymbol{r},t)$:

$$\begin{aligned}&\partial_t g(\boldsymbol{r},t)+v_\parallel\partial_\parallel g(\boldsymbol{r},t)-v\chi_\parallel\partial_\parallel\int dt'd\boldsymbol{r}'R[v(t-t')]G(\boldsymbol{r}-\boldsymbol{r}',t-t')\partial'_\parallel g(\boldsymbol{r}',t')\\&-v\chi_\perp\nabla_\perp\cdot\int dt'd\boldsymbol{r}'R[v(t-t')]G(\boldsymbol{r}-\boldsymbol{r}',t-t')\cdot\nabla'_\perp g(\boldsymbol{r}',t')\\&-\nabla_\perp\cdot\int dt'd\boldsymbol{r}'\boldsymbol{E}_v(\boldsymbol{r}-\boldsymbol{r}',t-t')G(\boldsymbol{r}-\boldsymbol{r}',t-t')\cdot\nabla'_\perp g(\boldsymbol{r}',t')=\xi(\boldsymbol{r},t).\end{aligned} \tag{2.15}$$

The running diffusion tensor $\boldsymbol{D}_\perp(t)$ is expressed in terms of the ensemble-averaged distribution function as

$$\boldsymbol{D}_\perp(t)=\frac{1}{2}\frac{d}{dt}\int d\boldsymbol{r}\,\boldsymbol{r}_\perp\boldsymbol{r}_\perp g(\boldsymbol{r},t)=\frac{1}{2}\int d\boldsymbol{r}\,\boldsymbol{r}_\perp\boldsymbol{r}_\perp\frac{\partial g(\boldsymbol{r},t)}{\partial t}. \tag{2.16}$$

Substituting the (2.15) into (2.16) and integrating by parts, we have $\boldsymbol{D}_\perp(t)=D_\perp^{\mathrm{cl}}(t)\mathbf{I}+\boldsymbol{D}_\perp^{\mathrm{an}}(t)$ with $D_\perp^{\mathrm{cl}}(t)=v\chi_\perp\int_0^t d\tau R(v\tau)$ and

$$\boldsymbol{D}_\perp^{\mathrm{an}}(t)=\int_0^t d\tau\int d\boldsymbol{r}'\boldsymbol{E}_v(\boldsymbol{r}',\tau)G_(\boldsymbol{r}',\tau), \tag{2.17}$$

where ϵ is set to be unity. The evolution equation for the response function $G(\boldsymbol{r}-\boldsymbol{r}',t-t')$ is obtained by taking the functional differentiation of (2.15) with respect to $\xi(\boldsymbol{r}',t')$ in the form

$$\begin{aligned}&\left(\partial_t+v_\parallel\partial_\parallel\right)G(\boldsymbol{r}-\boldsymbol{r}',t-t')-\nabla_\perp\cdot\boldsymbol{D}^{\mathrm{an}}(t-t')\cdot\nabla_\perp G(\boldsymbol{r}-\boldsymbol{r}',t-t')\\&-\left[D_\parallel^{\mathrm{cl}}(t-t')\partial_\parallel^2+D_\perp^{\mathrm{cl}}(t-t')\nabla_\perp^2\right]G(\boldsymbol{r}-\boldsymbol{r}',t-t')=\delta(\boldsymbol{r}-\boldsymbol{r}')\delta(t-t'),\end{aligned} \tag{2.18}$$

where $D_\parallel^{\mathrm{cl}}(t)=v\chi_\parallel\int_0^t d\tau R(v\tau)$; and we have neglected the three-point function $\delta G(\boldsymbol{r}-\boldsymbol{r}'',t-t'')/\delta\xi(\boldsymbol{r}',t')$ and used the temporal and spatial Markovian approximation. Equations (2.15) and (2.18) are closed for the ensemble-averaged distribution function $g(\boldsymbol{r},t)$ and the response function $G(\boldsymbol{r}-\boldsymbol{r}',t-t')$ in setting the external sources ξ and ζ to be zero [8, 9].

The Fourier transform of the response function $G(\boldsymbol{r}-\boldsymbol{r}',\tau)$ is obtained from (2.18) in the form

$$G(\boldsymbol{k},\tau)=H(\tau)\exp\left[-ik_\parallel v_\parallel\tau-\frac{\chi_\parallel}{v}\psi(v\tau)k_\parallel^2-\frac{\chi_\perp}{v}\psi(v\tau)k_\perp^2-\boldsymbol{k}_\perp\boldsymbol{k}_\perp:\boldsymbol{\mu}(\tau)\right], \tag{2.19}$$

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where $\boldsymbol{\mu}(t) = \int_0^t \boldsymbol{D}_\perp^{\mathrm{an}}(\tau)d\tau$, $\psi(\tau) = \int_0^\tau d\tau_1 \int_0^{\tau_1} d\tau_2 R(\tau_2)$ and H is the Heaviside step function. Substituting the expression (2.19) into (2.17), we have

$$\frac{d^2\boldsymbol{\mu}}{dt^2} = \int \frac{d\boldsymbol{k}}{(2\pi)^3}\boldsymbol{E}_v(\boldsymbol{k},t) \times \exp\left[-ik_\parallel v_\parallel t - \frac{\chi_\parallel}{v}\psi(vt)k_\parallel^2 - \frac{\chi_\perp}{v}\psi(vt)k_\perp^2 - \boldsymbol{k}_\perp\boldsymbol{k}_\perp : \boldsymbol{\mu}(t)\right], \quad (2.20)$$

so that the running diffusion tensor is calculated by solving this differential equation with the initial conditions $\boldsymbol{\mu}(0) = \boldsymbol{0}$ and $d\boldsymbol{\mu}(0)/dt = \boldsymbol{0}$. The differential equations similar to (2.20) were obtained for the diffusion coefficient of fluctuating magnetic field lines [5, 6], and for particle diffusion coefficient in a fluctuating magnetic field [5, 9] and fluctuating electrostatic potential [2].

The Kubo number, defined by $K = V\tau_c/\lambda_\perp$, where V is the amplitude of stochastic velocity, τ_c is the correlation time and $\lambda_\perp$ is the perpendicular correlation length, is a measure of the Lagrangian nonlinearity due to the position dependence of the fluctuating electrostatic potential [2, 10]. In the extreme turbulent regime of high K, the subdiffusive behavior of the running diffusion coefficient is obtained in numerical simulations and theoretical estimations when we neglect particle collisions and the average parallel velocity for the static potential. The vanishing of the asymptotic running diffusion coefficient in this regime cannot be derived from the formulation in this section because the solution to (2.20) leads us to the finite diffusion coefficient (Bohm diffusion coefficient), i.e. $\boldsymbol{D}_\perp^{\mathrm{an}}(\infty) = \mathrm{d}\boldsymbol{\mu}/\mathrm{dt}\,|_{t=\infty} = \mathrm{const}$. This failure may be caused by neglecting the effect of particle trapping in the fluctuating potential. In the next section, we try to include the trapping effect in the formulation by the DIA.

3. Extended DIA

3.1. Subensemble-averaged response function

In this section, we extend the formulation of the previous section by incorporating the idea of the DCT method into the DIA. Let us now define the subensemble of realization of the fluctuating potential determined by the condition

$$\phi(\boldsymbol{r},t) = \phi^0, \quad \boldsymbol{v}(\boldsymbol{r},t) = \boldsymbol{v}^0, \quad (3.1)$$

and represent the subensemble average by $\langle\cdot\rangle_0 \equiv \langle\cdot\rangle_{\phi,\parallel,\perp;0}$. The subensemble average for the functional $F[f]$ is written as [2, 10, 11]

$$\langle F[f]\rangle_0 = \frac{\langle F[f]\delta(\phi^0 - \phi(\boldsymbol{r},t))\delta(\boldsymbol{v}^0 - \boldsymbol{v}(\boldsymbol{r},t))\rangle}{\langle\delta(\phi^0 - \phi(\boldsymbol{r},t))\delta(\boldsymbol{v}^0 - \boldsymbol{v}(\boldsymbol{r},t))\rangle}, \quad (3.2)$$

so that the functional integral representation for this subensemble average becomes

$$\langle F[f]\rangle_0 = \mathscr{N}\int \mathscr{D}[f']\mathscr{D}[\hat{f}]F[f']e^{-L_0/\epsilon} \quad (3.3)$$

with

$$e^{-L_0/\epsilon} = \langle e^{-\mathscr{L}/\epsilon}\delta(\phi^0 - \phi(\boldsymbol{r},t))\delta(\boldsymbol{v}^0 - \boldsymbol{v}(\boldsymbol{r},t))\rangle = \frac{1}{(2\pi)^3}\int dq d\boldsymbol{k}_\perp e^{iq\phi^0 + i\boldsymbol{k}_\perp\cdot\boldsymbol{v}^0}\langle e^{-\mathscr{L}/\epsilon - iq\phi(\boldsymbol{r},t) - i\boldsymbol{k}_\perp\cdot\boldsymbol{v}(\boldsymbol{r},t)}\rangle. \quad (3.4)$$

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We repeat the procedure leading to (2.20) using the subensemble average instead of the conventional ensemble average. Similar to (2.11), we introduce a generating functional

$$Z_0[\zeta,\xi] = \mathcal{N}\int \mathcal{D}[f']\mathcal{D}[\hat{f}]e^{-S_0/\epsilon} \equiv e^{W_0/\epsilon} \tag{3.5}$$

with

$$S_0 = L_0 - \int dt\, d\boldsymbol{r}\left[\zeta(\boldsymbol{r},t)f'(\boldsymbol{r},t) + \xi(\boldsymbol{r},t)\hat{f}(\boldsymbol{r},t)\right], \tag{3.6}$$

and define the following functions:

$$g_0(\boldsymbol{r},t) = \frac{\delta W_0}{\delta\zeta(\boldsymbol{r},t)}, \quad \hat{g}_0(\boldsymbol{r},t) = \frac{\delta W_0}{\delta\xi(\boldsymbol{r},t)}, \tag{3.7}$$

$$G_0(\boldsymbol{r}-\boldsymbol{r}',t-t') = \epsilon\frac{\delta^2 W_0}{\delta\xi(\boldsymbol{r}',t')\delta\zeta(\boldsymbol{r},t)} = \epsilon\frac{\delta\hat{g}_0(\boldsymbol{r}',t')}{\delta\zeta(\boldsymbol{r},t)}. \tag{3.8}$$

In the $\zeta = \xi = 0$ limit, we have $g_0(\boldsymbol{r},t)|_{\zeta=\xi=0} = \langle f(\boldsymbol{r},t)\rangle_0$, and $\hat{g}_0(\boldsymbol{r},t)|_{\zeta=\xi=0} = \langle\hat{f}(\boldsymbol{r},t)\rangle_0 = 0$ due to the causality. Performing the subensemble average in L_0, we have

$$\begin{aligned}
L_0 = {} & \int dt'd\boldsymbol{r}'\hat{f}(\boldsymbol{r}',t')\left[\partial_{t'} + v_\parallel\partial'_\parallel\right]f'(\boldsymbol{r}',t') - \frac{1}{2}v\chi_\parallel\int dt'dt''d\boldsymbol{r}'d\boldsymbol{r}''R[v(t'-t'')] \\
& \times\hat{f}(\boldsymbol{r}',t')\hat{f}(\boldsymbol{r}'',t'')\partial'_\parallel f'(\boldsymbol{r}',t')\partial''_\parallel f'(\boldsymbol{r}'',t'') - \frac{1}{2}v\chi_\perp\int dt'dt''d\boldsymbol{r}'d\boldsymbol{r}''R[v(t'-t'')] \\
& \times\hat{f}(\boldsymbol{r}',t')\hat{f}(\boldsymbol{r}'',t'')\nabla'_\perp f'(\boldsymbol{r}',t')\cdot\nabla''_\perp f'(\boldsymbol{r}'',t'') - \frac{1}{2}\int dt'dt''d\boldsymbol{r}'d\boldsymbol{r}''\boldsymbol{E}_v(\boldsymbol{r}'-\boldsymbol{r}'',t'-t'') \\
& :\hat{f}(\boldsymbol{r}',t')\hat{f}(\boldsymbol{r}'',t'')\nabla'_\perp f'(\boldsymbol{r}',t')\nabla''_\perp f'(\boldsymbol{r}'',t'') + \frac{1}{2}\frac{1}{\langle\phi(\boldsymbol{r},t)^2\rangle} \\
& \times\left[\sqrt{\epsilon}\phi^0 + \int dt'd\boldsymbol{r}'\hat{f}(\boldsymbol{r}',t')\boldsymbol{E}_\phi(\boldsymbol{r}-\boldsymbol{r}',t-t')\cdot\nabla'_\perp f'(\boldsymbol{r}',t')\right]^2 \\
& + \frac{1}{\langle v(\boldsymbol{r},t)^2\rangle}\left[\sqrt{\epsilon}\boldsymbol{v}^0 + \int dt'd\boldsymbol{r}'\hat{f}(\boldsymbol{r}',t')\boldsymbol{E}_v(\boldsymbol{r}-\boldsymbol{r}',t-t')\cdot\nabla'_\perp f'(\boldsymbol{r}',t')\right]^2,
\end{aligned} \tag{3.9}$$

where $\langle v(\boldsymbol{r},t)^2\rangle = 2\langle v_x(\boldsymbol{r},t)^2\rangle$ is used with the assumption $\langle v_x(\boldsymbol{r},t)^2\rangle = \langle v_y(\boldsymbol{r},t)^2\rangle$.

The evolution equation for $\hat{g}_0(\boldsymbol{r}',t')$ is derived in the WKB approximation from the identity $\int\mathcal{D}[f']\mathcal{D}[\hat{f}]\delta/\delta f'e^{-S_0/\epsilon} = 0$:

$$\begin{aligned}
& \partial_{t'}\hat{g}_0(\boldsymbol{r}',t') + v_\parallel\partial'_\parallel\hat{g}_0(\boldsymbol{r}',t') \\
& \quad + \int dt''d\boldsymbol{r}''\{\boldsymbol{E}_v(\boldsymbol{r}'-\boldsymbol{r}'',t'-t'') : g_0(\boldsymbol{r}'',t'')\nabla''_\perp\hat{g}_0(\boldsymbol{r}'',t'')\nabla'_\perp\hat{g}_0(\boldsymbol{r}',t') \\
& \quad + v\chi_\parallel R[v(t'-t'')]g_0(\boldsymbol{r}'',t'')\partial''_\parallel\hat{g}_0(\boldsymbol{r}'',t'')\partial'_\parallel\hat{g}_0(\boldsymbol{r}',t') \\
& \quad + v\chi_\perp R[v(t'-t'')]g_0(\boldsymbol{r}'',t'')\nabla''_\perp\hat{g}_0(\boldsymbol{r}'',t'')\cdot\nabla'_\perp\hat{g}_0(\boldsymbol{r}',t') \\
& \quad + v\chi_\parallel R[v(t'-t'')]\partial'_\parallel G_0(\boldsymbol{r}''-\boldsymbol{r}',t''-t')\partial''_\parallel\hat{g}_0(\boldsymbol{r}'',t'')
\end{aligned}$$

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$$+ v\chi_\perp R[v(t'-t'')]\nabla'_\perp G_0(\boldsymbol{r}''-\boldsymbol{r}', t''-t')\cdot\nabla''_\perp \hat{g}_0(\boldsymbol{r}'', t'')$$

$$+ \boldsymbol{E}_v(\boldsymbol{r}'-\boldsymbol{r}'', t'-t'') : \nabla'_\perp G_0(\boldsymbol{r}''-\boldsymbol{r}', t''-t')\nabla''_\perp \hat{g}_0(\boldsymbol{r}'', t'')\}$$

$$+ \frac{1}{\langle\phi(\boldsymbol{r},t)^2\rangle}\left[\sqrt{\epsilon}\phi^0 - \int dt''d\boldsymbol{r}'' g_0(\boldsymbol{r}'', t'')\boldsymbol{E}_\phi(\boldsymbol{r}-\boldsymbol{r}'', t-t'')\right.$$

$$\left.\cdot\nabla''_\perp \hat{g}_0(\boldsymbol{r}'', t'')\right]\boldsymbol{E}_\phi(\boldsymbol{r}-\boldsymbol{r}', t-t')\cdot\nabla'_\perp \hat{g}_0(\boldsymbol{r}', t')$$

$$+ \frac{2}{\langle\boldsymbol{v}(\boldsymbol{r},t)^2\rangle}\left[\sqrt{\epsilon}\boldsymbol{v}^0 - \int dt''d\boldsymbol{r}'' g_0(\boldsymbol{r}'', t'')\boldsymbol{E}_v(\boldsymbol{r}-\boldsymbol{r}'', t-t'')\right.$$

$$\left.\cdot\nabla''_\perp \hat{g}_0(\boldsymbol{r}'', t'')\right]\cdot\boldsymbol{E}_v(\boldsymbol{r}-\boldsymbol{r}', t-t')\cdot\nabla'_\perp \hat{g}_0(\boldsymbol{r}', t')$$

$$= -\zeta(\boldsymbol{r}', t'). \tag{3.10}$$

Taking the functional differentiation of this equation with respect to $\zeta(\boldsymbol{r}, t)$, setting the external sources ζ and ξ to be zero and setting $\epsilon = 1$, we find the equation for the subensemble-averaged response function $G_0(\boldsymbol{r}-\boldsymbol{r}', t-t')$:

$$\partial_{t'} G_0(\boldsymbol{r}-\boldsymbol{r}', t-t') + v_\parallel \partial'_\parallel G_0(\boldsymbol{r}-\boldsymbol{r}', t-t')$$

$$+ \frac{\phi^0}{\langle\phi(\boldsymbol{r},t)^2\rangle}\boldsymbol{E}_\phi(\boldsymbol{r}-\boldsymbol{r}', t-t')\cdot\nabla'_\perp G_0(\boldsymbol{r}-\boldsymbol{r}', t-t')$$

$$+ \frac{2\boldsymbol{v}^0}{\langle v(\boldsymbol{r},t)^2\rangle}\cdot\boldsymbol{E}_v(\boldsymbol{r}-\boldsymbol{r}', t-t')\cdot\nabla'_\perp G_0(\boldsymbol{r}-\boldsymbol{r}', t-t')$$

$$+ \nabla'_\perp\cdot\boldsymbol{D}_0(t-t')\cdot\nabla'_\perp G_0(\boldsymbol{r}-\boldsymbol{r}', t-t')$$

$$+ D^{\mathrm{cl}}_\parallel(t-t')\partial'^2_\parallel G_0(\boldsymbol{r}-\boldsymbol{r}', t-t')$$

$$+ D^{\mathrm{cl}}_\perp(t-t')\nabla'^2_\perp G_0(\boldsymbol{r}-\boldsymbol{r}', t-t')$$

$$= -\delta(\boldsymbol{r}-\boldsymbol{r}')\delta(t-t'), \tag{3.11}$$

where

$$\boldsymbol{D}_0(t-t') = \int_0^{t-t'} d\tau \int d\boldsymbol{r}'\boldsymbol{E}_v(\boldsymbol{r}', \tau)G_0(\boldsymbol{r}', \tau). \tag{3.12}$$

We now write the Fourier transform of the response function $G_0(\boldsymbol{r}-\boldsymbol{r}', \tau)$ in the following form:

$$G_0(\boldsymbol{k}, \tau) = H(\tau)\hat{G}_0(\boldsymbol{k}_\perp, \tau)\exp\left[-ik_\parallel v_\parallel\tau - \frac{\chi_\parallel}{v}\psi(v\tau)k_\parallel^2 - \frac{\chi_\perp}{v}\psi(v\tau)k_\perp^2 - \boldsymbol{k}_\perp\boldsymbol{k}_\perp : \boldsymbol{\mu}_0(\tau)\right], \tag{3.13}$$

with $\hat{G}_0(\boldsymbol{k}_\perp, 0) = 1$ and $\boldsymbol{\mu}_0(t) = \int_0^t d\tau\boldsymbol{D}_0(\tau)$. Then the function $\hat{G}_0(\boldsymbol{k}_\perp, \tau)$ is determined from the equation

$$\partial_\tau\hat{G}_0(\boldsymbol{k}_\perp, \tau) + \frac{i}{(2\pi)^3}\int d\boldsymbol{k}' A_{\boldsymbol{k}\boldsymbol{k}'}\left[\frac{\phi^0}{\langle\phi(\boldsymbol{r},t)^2\rangle}\boldsymbol{E}_\phi(\boldsymbol{k}', \tau)\right.$$

$$\left. + \frac{2\boldsymbol{v}^0}{\langle\boldsymbol{v}(\boldsymbol{r},t)^2\rangle}\cdot\boldsymbol{E}_v(\boldsymbol{k}', \tau)\right]\cdot(\boldsymbol{k}_\perp - \boldsymbol{k}'_\perp)\hat{G}_0(\boldsymbol{k}_\perp - \boldsymbol{k}'_\perp, \tau) = 0, \tag{3.14}$$

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where

$$A_{\boldsymbol{k}\boldsymbol{k}'} = \exp\left[ik'_{\parallel}v_{\parallel}\tau - k'_{\parallel}(k'_{\parallel} - 2k_{\parallel})\frac{\chi_{\parallel}}{v}\psi(v\tau) - (k'^2_{\perp} - 2\boldsymbol{k}_{\perp}\cdot\boldsymbol{k}'_{\perp})\frac{\chi_{\perp}}{v}\psi(v\tau) - (\boldsymbol{k}'_{\perp}\boldsymbol{k}'_{\perp} - \boldsymbol{k}'_{\perp}\boldsymbol{k}_{\perp} - \boldsymbol{k}_{\perp}\boldsymbol{k}'_{\perp}) : \boldsymbol{\mu}_0(\tau)\right]. \tag{3.15}$$

Employing the spatial Markovian approximation $A_{\boldsymbol{k}\boldsymbol{k}'} \approx A_{\boldsymbol{0}\boldsymbol{k}'}$ in (3.14), we obtain the equation for the inverse Fourier transform of the function $\hat{G}_0(\boldsymbol{k}_{\perp},\tau)$ in the form

$$\partial_\tau \hat{G}_0(\boldsymbol{r}_{\perp} - \boldsymbol{r}'_{\perp},\tau) + \frac{\phi^0}{\langle\phi(\boldsymbol{r},t)^2\rangle}\bar{\boldsymbol{\mathscr{E}}}_{\phi}(\boldsymbol{r}_{\perp} - \boldsymbol{r}'_{\perp},\tau)\cdot\nabla_{\perp}\hat{G}_0(\boldsymbol{r}_{\perp} - \boldsymbol{r}'_{\perp},\tau) + \frac{2\boldsymbol{v}^0}{\langle\boldsymbol{v}(\boldsymbol{r},t)^2\rangle}\cdot\bar{\boldsymbol{\mathscr{E}}}_{v}(\boldsymbol{r}_{\perp} - \boldsymbol{r}'_{\perp},\tau)\cdot\nabla_{\perp}\hat{G}_0(\boldsymbol{r}_{\perp} - \boldsymbol{r}'_{\perp},\tau) = 0, \tag{3.16}$$

where

$$\bar{\boldsymbol{\mathscr{E}}}_{\phi}(\boldsymbol{r}_{\perp},t) = -\frac{1}{B}\boldsymbol{e}_z\times\nabla_{\perp}\bar{\Phi}(\boldsymbol{r}_{\perp},t), \tag{3.17}$$

and

$$\bar{\boldsymbol{\mathscr{E}}}_{v}(\boldsymbol{r}_{\perp},t) = -\frac{1}{B^2}(\boldsymbol{e}_z\times\nabla_{\perp})(\boldsymbol{e}_z\times\nabla_{\perp})\bar{\Phi}(\boldsymbol{r}_{\perp},t) \tag{3.18}$$

with

$$\bar{\Phi}(\boldsymbol{r}_{\perp},t) = \frac{1}{(2\pi)^3}\int d\boldsymbol{k}E(\boldsymbol{k},t) \times\exp\left[i\boldsymbol{k}_{\perp}\cdot\boldsymbol{r}_{\perp} + ik_{\parallel}v_{\parallel}t - \frac{\chi_{\parallel}}{v}\psi(vt)k_{\parallel}^2 - \frac{\chi_{\perp}}{v}\psi(vt)k_{\perp}^2 - \boldsymbol{k}_{\perp}\boldsymbol{k}_{\perp} : \boldsymbol{\mu}_0(t)\right]. \tag{3.19}$$

Then the solution to (3.16) is written as

$$\hat{G}_0(\boldsymbol{r}_{\perp} - \boldsymbol{r}'_{\perp},\tau) = \delta(\boldsymbol{r}_{\perp} - \boldsymbol{r}'_{\perp} - \boldsymbol{X}(\tau)), \tag{3.20}$$

where the function $\boldsymbol{X}(\tau)$ is determined by the equation

$$\frac{d\boldsymbol{X}(\tau)}{d\tau} = \frac{\phi^0}{\langle\phi(\boldsymbol{r},t)^2\rangle}\bar{\boldsymbol{\mathscr{E}}}_{\phi}(\boldsymbol{X}(\tau),\tau) + \frac{2\boldsymbol{v}^0}{\langle\boldsymbol{v}(\boldsymbol{r},t)^2\rangle}\cdot\bar{\boldsymbol{\mathscr{E}}}_{v}(\boldsymbol{X}(\tau),\tau) \tag{3.21}$$

with the condition $\boldsymbol{X}(0) = \boldsymbol{0}$.

According to (3.13) and (3.20), the Fourier transform of the response function $G_0(\boldsymbol{r} - \boldsymbol{r}',\tau)$ can be written as

$$G_0(\boldsymbol{k},\tau) = H(\tau)\exp\left[-i\boldsymbol{k}_{\perp}\cdot\boldsymbol{X}(\tau) - ik_{\parallel}v_{\parallel}\tau - \frac{\chi_{\parallel}}{v}\psi(v\tau)k_{\parallel}^2 - \frac{\chi_{\perp}}{v}\psi(v\tau)k_{\perp}^2 - \boldsymbol{k}_{\perp}\boldsymbol{k}_{\perp} : \boldsymbol{\mu}_0(\tau)\right]. \tag{3.22}$$

Substitution of this expression into (3.12) yields the differential equation for $\boldsymbol{\mu}_0$:

$$\frac{d^2\boldsymbol{\mu}_0}{d\tau^2} = \bar{\boldsymbol{\mathscr{E}}}_{v}(\boldsymbol{X}(\tau),\tau). \tag{3.23}$$

Thus, solving the coupled differentials (3.21) and (3.23) for $\boldsymbol{X}(\tau)$ and $\boldsymbol{\mu}_0(\tau)$, we can calculate the subensemble-averaged response function from (3.22).

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3.2. Running diffusion tensor

The running diffusion tensor is exactly described in terms of the Lagrangian correlation of the stochastic velocity [2, 4]:

$$\boldsymbol{D}_{\perp}^{\mathrm{an}}(t) = \int_0^t dt' \langle \boldsymbol{v}(\boldsymbol{r}, t)\boldsymbol{v}(\boldsymbol{r}'(t'), t')\rangle, \tag{3.24}$$

where $\boldsymbol{r}'(t')$ is the position at time t', which reaches at the position $\boldsymbol{r}$ at time t, by moving along its trajectory determined by the equation of motion $d\boldsymbol{r}/dt = (v_{\parallel} + \eta_{\parallel}(t))\boldsymbol{e}_z + \boldsymbol{v}(\boldsymbol{r}, t) + \boldsymbol{\eta}_{\perp}(t)$. Introducing a fluctuating causal response function $\tilde{G}(\boldsymbol{r} - \boldsymbol{r}', t - t')$ satisfying the equation

$$\left\{\partial_t + [v_{\parallel} + \eta_{\parallel}(t)]\partial_{\parallel} + [\boldsymbol{v}(\boldsymbol{r}, t) + \boldsymbol{\eta}_{\perp}(t)] \cdot \nabla_{\perp}\right\} \tilde{G}(\boldsymbol{r}-\boldsymbol{r}', t-t') = \delta(\boldsymbol{r}-\boldsymbol{r}')\delta(t-t'), \tag{3.25}$$

we can rewrite the exact expression (3.24) in the form

$$\boldsymbol{D}_{\perp}^{\mathrm{an}}(t) = \int_0^t dt' \int d\boldsymbol{r}' \langle \boldsymbol{v}(\boldsymbol{r}, t)\boldsymbol{v}(\boldsymbol{r}', t')\tilde{G}(\boldsymbol{r} - \boldsymbol{r}', t - t')\rangle. \tag{3.26}$$

The expression (2.17) for the running diffusion tensor obtained in the previous section is also derived from the exact one by using the Corrsin approximation [2, 4, 7]. This approximation is the factorization of the joint ensemble average into two separate ensembles:

$$\boldsymbol{D}_{\perp}^{\mathrm{an}}(t) = \int_0^t dt'\, d\boldsymbol{r}' \langle \boldsymbol{v}(\boldsymbol{r}, t)\boldsymbol{v}(\boldsymbol{r}', t')\rangle \langle \tilde{G}(\boldsymbol{r} - \boldsymbol{r}', t - t')\rangle. \tag{3.27}$$

Noting $G(\boldsymbol{r} - \boldsymbol{r}', t - t') = \langle \tilde{G}(\boldsymbol{r} - \boldsymbol{r}', t - t')\rangle$, which is shown in Appendix A, we arrive at the expression (2.17) from (3.27).

We now apply the Corrsin approximation in the subensemble [2]. Let us write the exact running diffusion tensor in the subensemble representation and use the Corrsin approximation in the subensemble. Then we have

$$\begin{aligned}\boldsymbol{D}_{\perp}^{\mathrm{an}}(t) = &\int d\phi^0 d\boldsymbol{v}^0 P_1(\phi^0)P_2(\boldsymbol{v}^0) \\ &\times \int_0^t dt' \int d\boldsymbol{r}' \langle \boldsymbol{v}(\boldsymbol{r}, t)\boldsymbol{v}(\boldsymbol{r}', t')\rangle_0 \langle \tilde{G}(\boldsymbol{r} - \boldsymbol{r}', t - t')\rangle_0,\end{aligned} \tag{3.28}$$

where $P_1(\phi^0)$ and $P_2(\boldsymbol{v}^0)$ are the Gaussian probabilities of ϕ^0 and $\boldsymbol{v}^0$. Using the expression (3.22) and the relation $G_0(\boldsymbol{r} - \boldsymbol{r}', t - t') = \langle \tilde{G}(\boldsymbol{r} - \boldsymbol{r}', t - t')\rangle_0$ (Appendix A), we can have the running diffusion tensor in the form

$$\begin{aligned}\boldsymbol{D}_{\perp}^{\mathrm{an}}(t) = &\int d\phi^0 d\boldsymbol{v}^0 P_1(\phi^0)P_2(\boldsymbol{v}^0) \\ &\times \int_0^t d\tau \boldsymbol{v}^0 \left\{ \frac{\phi^0}{\langle \phi(\boldsymbol{r}, t)^2\rangle} \bar{\boldsymbol{\mathscr{E}}}_{\phi}(\boldsymbol{X}(\tau), \tau) + \frac{2\boldsymbol{v}^0}{\langle \boldsymbol{v}(\boldsymbol{r}, t)^2\rangle} \cdot \bar{\boldsymbol{\mathscr{E}}}_{v}(\boldsymbol{X}(\tau), \tau) \right\},\end{aligned} \tag{3.29}$$

where we have used the relation [2, 10, 11]

$$\langle \boldsymbol{v}(\boldsymbol{r}', t')\rangle_0 = \frac{\phi^0}{\langle \phi(\boldsymbol{r}, t)^2\rangle} \langle \phi(\boldsymbol{r}, t)\boldsymbol{v}(\boldsymbol{r}', t')\rangle + \frac{2\boldsymbol{v}^0}{\langle \boldsymbol{v}(\boldsymbol{r}, t)^2\rangle} \cdot \langle \boldsymbol{v}(\boldsymbol{r}, t)\boldsymbol{v}(\boldsymbol{r}', t')\rangle. \tag{3.30}$$

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The expression (3.29) is rewritten by means of (3.21) as

$$\boldsymbol{D}_{\perp}^{\mathrm{an}}(t) = \int d\phi^0 d\boldsymbol{v}^0 P_1(\phi^0) P_2(\boldsymbol{v}^0) \boldsymbol{v}^0 \boldsymbol{X}(t). \tag{3.31}$$

Thus the running diffusion tensor can be calculated by using $\boldsymbol{X}(t)$, which is obtained by solving the coupled differentials (3.21) and (3.23).

Finally, note in this subsection that the expression (3.29) for the running diffusion tensor reduces to (2.17) with (2.19) in the previous section, when the trapping effect in (3.29) is neglected, i.e. $\boldsymbol{X}(\tau) \approx 0$.

3.3. Linear regime in $\boldsymbol{\mu}_0$

We next consider the linear regime in $\boldsymbol{\mu}_0$, and neglect $\boldsymbol{\mu}_0$ in $\bar{\boldsymbol{\mathscr{E}}}_{\phi}$ and $\bar{\boldsymbol{\mathscr{E}}}_{v}$. Then the equation for $\boldsymbol{X}(\tau)$ becomes

$$\frac{d\boldsymbol{X}(\tau)}{d\tau} = \frac{\phi^0}{\langle \phi(\boldsymbol{r},t)^2 \rangle} \boldsymbol{\mathscr{E}}_{\phi}(\boldsymbol{X}(\tau),\tau) + \frac{2\boldsymbol{v}^0}{\langle \boldsymbol{v}(\boldsymbol{r},t)^2 \rangle} \cdot \boldsymbol{\mathscr{E}}_{v}(\boldsymbol{X}(\tau),\tau), \tag{3.32}$$

where $\boldsymbol{\mathscr{E}}_{\phi}(\boldsymbol{X},\tau) \equiv \bar{\boldsymbol{\mathscr{E}}}_{\phi}(\boldsymbol{X},\tau)\big|_{\boldsymbol{\mu}_0=0}$ and $\boldsymbol{\mathscr{E}}_{v}(\boldsymbol{X},\tau) \equiv \bar{\boldsymbol{\mathscr{E}}}_{v}(\boldsymbol{X},\tau)\big|_{\boldsymbol{\mu}_0=0}$. The solution $\boldsymbol{X}(\tau)$ to expression (3.32) agrees with the decorrelation trajectory proposed by Vlad et al. [2, 10, 11]. The expression (3.29) (or (3.31)) for the running diffusion tensor in this linear regime also agrees with that of Vlad et al. [11].

When the Eulerian correlation function $E(\boldsymbol{r},\tau)$ is factorized as $E(\boldsymbol{r},\tau) = E_{\parallel}(z,\tau) \times E_{\perp}(\boldsymbol{r}_{\perp},\tau)$ and $E_{\perp}(\boldsymbol{r}_{\perp},\tau)$ is isotropic in $\boldsymbol{r}_{\perp}$, we have

$$\boldsymbol{\mathscr{E}}_{\phi}(\boldsymbol{X}(\tau),\tau) = -\frac{(\boldsymbol{e}_z \times \nabla_X)\, \Phi(X(\tau),\tau)}{B}, \tag{3.33}$$

$$\boldsymbol{\mathscr{E}}_{v}(\boldsymbol{X}(\tau),\tau) = -\frac{(\boldsymbol{e}_z \times \nabla_X)(\boldsymbol{e}_z \times \nabla_X)\, \Phi(X(\tau),\tau)}{B^2}, \tag{3.34}$$

with $\nabla_X = \partial/\partial \boldsymbol{X}$ and $X(t) = |\boldsymbol{X}(t)|$, where

$$\Phi(X(\tau),\tau) = \frac{v}{2\chi_{\perp}\psi(v\tau)} \Phi_{\parallel}(\tau) \int_0^{\infty} d\rho \rho E(\rho,\tau) I_0\left(\frac{v}{2\chi_{\perp}\psi(v\tau)} X\rho\right) \times \exp\left[-\frac{v}{4\chi_{\perp}\psi(v\tau)}(X^2+\rho^2)\right] \tag{3.35}$$

with

$$\Phi_{\parallel}(\tau) = \frac{1}{2}\sqrt{\frac{v}{\pi\chi_{\parallel}\psi(v\tau)}} \int_{-\infty}^{\infty} dz E_{\parallel}(z,\tau) \exp\left[-\frac{v}{4\chi_{\parallel}\psi(v\tau)}(z - v_{\parallel}\tau)^2\right], \tag{3.36}$$

and I_0 is the modified Bessel function of the first kind of order 0. The running diffusion coefficient for the isotropic correlation (3.35) with $R(\tau) = \lim_{\epsilon \to +0} \delta(\tau - \epsilon)$ and $\mathscr{E}_{\parallel}(\tau) = 1$ is discussed in [11].

3.4. The invariance of Lagrangian potential in frozen turbulence

We now rewrite the equation for the decorrelation trajectory using the unit vectors $\boldsymbol{e}_1 = \boldsymbol{v}^0/|\boldsymbol{v}^0|$ and $\boldsymbol{e}_2 = \boldsymbol{e}_z \times \boldsymbol{e}_1$. Let us write the decorrelation trajectory as $\boldsymbol{X}(\tau) = \hat{X}(\tau)\boldsymbol{e}_1 + \hat{Y}(\tau)\boldsymbol{e}_2$. Then the equations for $\hat{X}(\tau)$ and $\hat{Y}(\tau)$ have a Hamiltonian structure:

$$\frac{\mathrm{d}\hat{X}}{\mathrm{d}\tau} = \frac{\partial H}{\partial \hat{Y}}, \qquad \frac{\mathrm{d}\hat{Y}}{\mathrm{d}\tau} = -\frac{\partial H}{\partial \hat{X}}, \tag{3.37}$$

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with the Hamiltonian H defined by

$$H(\hat{X},\hat{Y},\tau)=\frac{1}{B}\left[\frac{\phi^0}{\langle\phi(\boldsymbol{r},t)^2\rangle}+\frac{2\boldsymbol{v}^0}{\langle\boldsymbol{v}(\boldsymbol{r},t)^2\rangle}\cdot\frac{1}{B}(\boldsymbol{e}_z\times\nabla_X)\right]\bar{\Phi}(\hat{X},\hat{Y},\tau), \tag{3.38}$$

where $\bar{\Phi}(\hat{X},\hat{Y},\tau)\equiv\bar{\Phi}(\boldsymbol{X},\tau)$. The Hamiltonian is non-autonomous and, therefore, $\mathrm{d}H/\mathrm{d}\tau=\partial H/\partial\tau$. The subensemble-averaged Lagrangian potential obtained by using the Corrsin approximation in the subensemble is found to be

$$\langle\phi(\boldsymbol{r}(t-\tau),t-\tau)\rangle_0=B\,H(\hat{X},\hat{Y},\tau). \tag{3.39}$$

Afterward, in this subsection, we consider the static potential ($\tau_c=\infty$; frozen turbulence) and neglect the particle collisions and the averaged parallel velocity $v_\parallel$. The Lagrangian potential in this frozen turbulence is an invariant of the motion, i.e. $d\phi(\boldsymbol{r}(t))/dt=0$. The DCT method conserves this exact relation in the subensemble, i.e. $d\langle\phi(\boldsymbol{X}(t))\rangle_0/dt=0$. Therefore, the DCT are on the contour lines of subensemble-averaged potential $\langle\phi(\boldsymbol{X}(t))\rangle_0$ instead of the exact trajectories being on the $\phi(\boldsymbol{r}(t))=$ const. This property ensures the permanent trapping of decorrelation trajectories and the resulting subdiffusive behavior of the asymptotic running diffusion coefficient.

Vlad and Spineanu [13] presented the method that takes account of trajectory fluctuations by considering the decorrelation trajectories having the fixed values of second derivatives of the potential in addition to condition (3.1). Their proposed nested DCT method inherits the invariance of the Lagrangian potential and thus ensures the subdiffusive transport for the frozen turbulence, although it consumes enormous amount of computational time compared with the conventional DCT method. In our extension of the DCT method the computational time may be comparable to that in the conventional DCT method. However, our method breaks the conservation of the subensemble-averaged Lagrangian potential (3.39) due to the presence of the time-dependent function $\boldsymbol{\mu}_0(t)$. The DCT method has several variants depending on the choice of the subensemble. In [10], Vlad et al. presented the DCT method using the subensemble with the condition $\phi(\boldsymbol{r},t)=\phi^0$ only instead of (3.1), which has the advantage of simple calculation but needs to externally impose an *ad hoc* condition. For this simplified DCT method, we can find the following invariant:

$$H'=H(\hat{X},\hat{Y},\tau)+B\frac{\phi^0}{\langle\phi(\boldsymbol{r},t)^2\rangle}\left[\frac{\mathrm{d}\mu_{0xx}}{\mathrm{d}\tau}\frac{\mathrm{d}\mu_{0yy}}{\mathrm{d}\tau}-\left(\frac{\mathrm{d}\mu_{0xy}}{\mathrm{d}\tau}\right)^2\right], \tag{3.40}$$

where $\boldsymbol{\mu}_0(\tau)$ is decomposed into $\boldsymbol{\mu}_0(\tau)=\mu_{0xx}\boldsymbol{e}_x\boldsymbol{e}_x+\mu_{0yy}\boldsymbol{e}_y\boldsymbol{e}_y+2\mu_{0xy}\boldsymbol{e}_x\boldsymbol{e}_y$. In the DCT method with (3.1), this modified Hamiltonian H' is not conserved and its time derivative becomes

$$\frac{\mathrm{d}H'}{\mathrm{d}t}=-\frac{2i}{\langle\boldsymbol{v}(\boldsymbol{r},t)^2\rangle}\frac{\mathrm{d}\boldsymbol{\mu}_0}{\mathrm{d}t}:\frac{1}{B^2}\int\frac{d\boldsymbol{k}}{(2\pi)^3}E(\boldsymbol{k})\,\boldsymbol{k}\boldsymbol{k}\,\boldsymbol{v}^0\cdot(\boldsymbol{e}_z\times\boldsymbol{k})$$
$$\times\exp\left[i\boldsymbol{k}\cdot\boldsymbol{X}(t)-\boldsymbol{k}\boldsymbol{k}:\boldsymbol{\mu}_0(t)\right] \tag{3.41}$$

with $E(\boldsymbol{k})=E(\boldsymbol{k},0)$.

It is not apparent due to the lack of the Lagrangian potential invariance that our method can lead us to the subdiffusive transport for the frozen turbulence. We need further study for checking the validity of our theory in the extreme turbulent limit.

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3.5. Gaussian model for the Eulerian correlation

Finally, in this section, we consider a specific model for the Eulerian correlation in order to move further analytically. We now assume the Eulerian correlation function in the form

$$E(\boldsymbol{r},t)=\beta^2\exp\left[-\frac{|t|}{\tau_c}-\frac{\boldsymbol{r}_\perp^2}{2\lambda_\perp^2}-\frac{z^2}{2\lambda_\parallel^2}\right] \tag{3.42}$$

with $\beta=\sqrt{\langle\phi(\boldsymbol{r},t)^2\rangle}$.

The equation of the decorrelation trajectory for the correlation function (3.42) is obtained from (3.21) in the form

$$\frac{\mathrm{d}\bar{\boldsymbol{X}}}{\mathrm{d}\bar{\tau}}=-Ku(\boldsymbol{e}_z\times\bar{\nabla}_X)\left[p+\boldsymbol{e}_1\cdot(\boldsymbol{e}_z\times\bar{\nabla}_X)\right]\hat{\Phi} \tag{3.43}$$

with

$$\hat{\Phi}=\frac{1}{\sqrt{d_xd_yd_z}}\exp\left[-\bar{\tau}-\frac{K_\parallel^2}{2d_z}\bar{\tau}^2-\frac{\bar{X}^2}{2d_x}-\frac{1}{2d_y}\left(\bar{Y}-\frac{2\bar{\mu}_{xy}}{d_x}\bar{X}\right)^2\right], \tag{3.44}$$

where $\bar{\boldsymbol{X}}=\boldsymbol{X}/\lambda_\perp$; $\bar{\tau}=\tau/\tau_c$; $\bar{X}=\hat{X}/\lambda_\perp$ and $\bar{Y}=\hat{Y}/\lambda_\perp$; $\bar{\boldsymbol{\mu}}_0(\tau)=\boldsymbol{\mu}_0(\tau)/\lambda_\perp^2$; $\bar{\nabla}_X=\lambda_\perp\nabla_X$; $\hat{\Phi}=\bar{\Phi}/\beta^2$; $d_x=1+\bar{\chi}_\perp\psi(v\tau)+2\bar{\mu}_{0xx}$ and $d_y=1+\bar{\chi}_\perp\psi(v\tau)+2\bar{\mu}_{0yy}-4\bar{\mu}_{0xy}^2/d_x$ with $\bar{\chi}_\perp=2\chi_\perp/v\lambda_\perp^2$; $d_z=1+\bar{\chi}_\parallel\psi(v\tau)$ with $\bar{\chi}_\parallel=2\chi_\parallel/v\lambda_\parallel^2$; two Kubo numbers $K=\beta\tau_c/B\lambda_\perp^2$ and $K_\parallel=v_\parallel\tau_c/\lambda_\parallel$; and according to Vlad et al. [2, 10, 11] we have used the parameters $p=\phi^0/\beta u$ and $u=|\boldsymbol{v}^0|B\lambda_\perp/\beta$ instead of ϕ^0 and $|\boldsymbol{v}^0|$.

The equation for function $\boldsymbol{\mu}_0(\tau)$ is written as

$$\frac{\mathrm{d}^2\bar{\boldsymbol{\mu}}_0}{\mathrm{d}\bar{\tau}^2}=-K^2\left(\boldsymbol{e}_z\times\bar{\nabla}_\perp\right)\left(\boldsymbol{e}_z\times\bar{\nabla}_\perp\right)\hat{\Phi}. \tag{3.45}$$

The solution to (3.43) has the following property: $\bar{X}(\bar{\tau};u,-p)=\bar{X}(\bar{\tau};u,p)$ and $\bar{Y}(\bar{\tau};u,-p)=-\bar{Y}(\bar{\tau};u,p)$. Using this property, we find that the running diffusion tensor is diagonal, i.e. $\boldsymbol{D}_\perp^{\mathrm{an}}(t)=D_\perp^{\mathrm{an}}(t)\boldsymbol{I}$, and the diagonal component is written as

$$D_\perp^{\mathrm{an}}(t)=\frac{\lambda_\perp^2}{\tau_c}K\frac{1}{\sqrt{2\pi}}\int_0^\infty dp\int_0^\infty duu^3e^{-(1+p^2)u^2/2}\bar{X}(t). \tag{3.46}$$

4. Test-particle approach

In this section, using a test-particle approach, we derive the same concluding results which were obtained in the previous section, i.e. the running diffusion tensor (3.31) and the coupled differentials (3.21) and (3.23).

We consider the motion of a test particle in the fluctuating electrostatic fluctuation, which is described in the guiding-center approximation by the equation

$$\frac{d\boldsymbol{r}(t)}{dt}=v_\parallel\mathbf{e}_z+\sqrt{\epsilon}\left[\boldsymbol{v}(\boldsymbol{r}(t),t)+\boldsymbol{\eta}(t)\right], \tag{4.1}$$

with $\boldsymbol{\eta}(t)=\eta_\parallel(t)\boldsymbol{e}_z+\boldsymbol{\eta}_\perp(t)$. The running diffusion tensor due to fluctuations is expressed using the trajectory of the test particle

$$\boldsymbol{D}_\perp^{\mathrm{an}}(t)=\int_0^t d\tau\left\langle\boldsymbol{v}(\boldsymbol{r}(\tau),\tau)\boldsymbol{v}(\boldsymbol{0},0)\right\rangle. \tag{4.2}$$

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We now introduce the subensemble by the conditions

$$\phi(\boldsymbol{r}=\boldsymbol{0},t=0)=\phi^0,\qquad \boldsymbol{v}(\boldsymbol{r}=\boldsymbol{0},t=0)=\boldsymbol{v}^0 \tag{4.3}$$

instead of (3.1). Then, in the similar method leading to (3.3), the subensemble average for the functional $F[\boldsymbol{r}(t)]$ can be written in the functional representation as

$$\langle F[\boldsymbol{r}]\rangle_0=\mathscr{N}\int\mathscr{D}[\boldsymbol{r}']\mathscr{D}[\hat{\boldsymbol{r}}']F[\boldsymbol{r}']e^{-L_0/\epsilon}, \tag{4.4}$$

where

$$e^{-L_0/\epsilon}=\left\langle e^{-\mathscr{L}_0/\epsilon}\delta(\phi^0-\phi(\boldsymbol{0},0))\delta(\boldsymbol{v}^0-\boldsymbol{v}(\boldsymbol{0},0))\right\rangle \tag{4.5}$$

with

$$\mathscr{L}_0=\int_0^\infty d\tau\hat{\boldsymbol{r}}'(\tau)\cdot\left\{\frac{d\boldsymbol{r}'(\tau)}{d\tau}-v_\parallel\mathbf{e}_z-\sqrt{\epsilon}\left[\boldsymbol{v}(\boldsymbol{r}'(\tau),\tau)+\boldsymbol{\eta}(\tau)\right]\right\}. \tag{4.6}$$

Carrying out the ensemble average in (4.5), we have

$$\begin{aligned}L_0=&\int_0^\infty d\tau\hat{\boldsymbol{r}}'(\tau)\cdot\left[\frac{d\boldsymbol{r}'(\tau)}{d\tau}-v_\parallel\mathbf{e}_z\right]\\&-\frac{1}{2}\int_0^\infty d\tau\int_0^\infty d\bar{\tau}\;\hat{\boldsymbol{r}}'(\tau)\hat{\boldsymbol{r}}'(\bar{\tau}):\boldsymbol{E}_v\left(\boldsymbol{r}'(\tau)-\boldsymbol{r}'(\bar{\tau}),\tau-\bar{\tau}\right)\\&+\frac{1}{2\langle\phi^2(\boldsymbol{0},0)\rangle}\left[\sqrt{\epsilon}\phi^0-\int_0^\infty d\tau\;\hat{\boldsymbol{r}}'(\tau)\cdot\boldsymbol{E}_\phi(\boldsymbol{r}'(\tau),\tau)\right]^2\\&+\frac{1}{\langle v^2(\boldsymbol{0},0)\rangle}\left[\sqrt{\epsilon}\boldsymbol{v}^0-\int_0^\infty d\tau\;\hat{\boldsymbol{r}}'(\tau)\cdot\boldsymbol{E}_v(\boldsymbol{r}'(\tau),\tau)\right]^2\\&-\frac{1}{2}\int_0^\infty d\tau\int_0^\infty d\bar{\tau}\;\hat{\boldsymbol{r}}'(\tau)\hat{\boldsymbol{r}}'(\bar{\tau}):\langle\boldsymbol{\eta}(\tau)\boldsymbol{\eta}(\bar{\tau})\rangle.\end{aligned} \tag{4.7}$$

Let us next introduce a generating functional

$$\begin{aligned}Z[\boldsymbol{a},\hat{\boldsymbol{a}}]&=\mathscr{N}\int\mathscr{D}[\boldsymbol{r}']\mathscr{D}[\hat{\boldsymbol{r}}']\;e^{-S_0/\epsilon}\\&=\langle e^{\theta[\boldsymbol{r}',\hat{\boldsymbol{r}}';\boldsymbol{a},\hat{\boldsymbol{a}}]/\epsilon}\rangle_0\equiv e^{W_0/\epsilon}\end{aligned} \tag{4.8}$$

with $S_0=L_0-\theta[\boldsymbol{r}',\hat{\boldsymbol{r}}';\boldsymbol{a},\hat{\boldsymbol{a}}]$, where

$$\theta[\boldsymbol{r}',\hat{\boldsymbol{r}}';\boldsymbol{a},\hat{\boldsymbol{a}}]=\int_0^\infty d\tau\;[\boldsymbol{a}(\tau)\cdot\boldsymbol{r}'(\tau)+\hat{\boldsymbol{a}}(\tau)\cdot\hat{\boldsymbol{r}}'(\tau)]. \tag{4.9}$$

Using this generating functional, we define the following functions:

$$\boldsymbol{X}(t)=\frac{\delta W_0}{\delta\boldsymbol{a}_\perp(t)},\quad Z(t)=\frac{\delta W_0}{\delta a_\parallel(t)}, \tag{4.10}$$

$$C_{0\parallel}(t)=\epsilon\frac{\delta^2W_0}{\delta a_\parallel(t)\delta a_\parallel(t)},\quad G_{0\parallel}(t-t')=\epsilon\frac{\delta^2W_0}{\delta a_\parallel(t)\delta\hat{a}_\parallel(t')}, \tag{4.11}$$

$$\boldsymbol{C}_{0\perp}(t)=\epsilon\frac{\delta^2W_0}{\delta\boldsymbol{a}_\perp(t)\delta\boldsymbol{a}_\perp(t)},\quad \boldsymbol{G}_{0\perp}(t-t')=\epsilon\frac{\delta^2W_0}{\delta\boldsymbol{a}_\perp(t)\delta\hat{\boldsymbol{a}}_\perp(t')}. \tag{4.12}$$

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Using the identity

$$\int \mathscr{D}[\boldsymbol{r}']\mathscr{D}[\hat{\boldsymbol{r}}']\frac{\delta}{\delta\hat{\boldsymbol{r}}'(t)}e^{-S_0/\epsilon} = \boldsymbol{0}, \tag{4.13}$$

we find $Z(t) = v_\parallel t$ and the equation of motion for $\boldsymbol{X}(t)$

$$\frac{d\boldsymbol{X}(t)}{dt} = \frac{1}{\langle\phi^2(\boldsymbol{0},0)\rangle}\langle\boldsymbol{E}_\phi(\boldsymbol{r}'(t),t)\rangle_0 + \frac{2\boldsymbol{v}^0}{\langle v^2(\boldsymbol{0},0)\rangle}\cdot\langle\boldsymbol{E}_v(\boldsymbol{r}'(t),t)\rangle_0, \tag{4.14}$$

where we have used that the causality leads to the property $\langle\hat{\boldsymbol{r}}(t_1)\cdots\rangle_0 = 0$ if the time t_1 is the latest time.

Functionally differentiating the identity (4.13) with respect to $\boldsymbol{a}(t)$ and $\hat{\boldsymbol{a}}(t')$, we obtain the following equations:

$$\frac{dC_{0\parallel}(t)}{dt} = 2v\chi_\parallel\int_0^\infty d\tau\ R[v(t-\tau)]G_{0\parallel}(t-\tau), \tag{4.15}$$

$$\frac{dG_{0\parallel}(t-t')}{dt} = \delta(t-t'), \tag{4.16}$$

$$\frac{d\boldsymbol{C}_{0\perp}(t)}{dt} = 2v\chi_\perp\int_0^\infty d\tau\ R[v(t-\tau)]\boldsymbol{G}_{0\perp}(t-\tau) + 2\int_0^\infty d\tau\ \boldsymbol{G}_{0\perp}(t-\tau)\cdot\langle\boldsymbol{E}_v(\boldsymbol{r}'(t)-\boldsymbol{r}'(t'),t-t')\rangle_0, \tag{4.17}$$

$$\frac{d\boldsymbol{G}_{0\perp}(t-t')}{dt} = \delta(t-t')\boldsymbol{I}, \tag{4.18}$$

where the terms of order ϕ^4 and $v\phi^2$ are neglected. Solving the (4.16) and (4.18) yields $G_{0\parallel}(t-t') = H(t-t')$ and $\boldsymbol{G}_{0\perp}(t-t') = H(t-t')\boldsymbol{I}$. Substituting these solutions into (4.15) and (4.17), we find

$$C_{0\parallel}(t) = 2\frac{\chi_\parallel}{v}\psi(vt), \tag{4.19}$$

$$\boldsymbol{C}_{0\perp}(t) = 2\frac{\chi_\perp}{v}\psi(vt) + 2\int_0^t d\tau\int_0^\tau d\bar{\tau}\ \langle\boldsymbol{E}_v(\boldsymbol{r}'(\tau)-\boldsymbol{r}'(\bar{\tau}),\tau-\bar{\tau})\rangle_0. \tag{4.20}$$

In the following calculation of $\boldsymbol{C}_{0\perp}(t)$, we use the approximation

$$\langle\boldsymbol{E}_v(\boldsymbol{r}'(\tau)-\boldsymbol{r}'(\bar{\tau}),\tau-\bar{\tau})\rangle_0 \approx \langle\boldsymbol{E}_v(\boldsymbol{r}'(\tau-\bar{\tau}),\tau-\bar{\tau})\rangle_0.$$

Let us define the function $\boldsymbol{\mu}_0(t)$ by

$$\boldsymbol{\mu}_0(t) = \int_0^t d\tau\int_0^\tau d\bar{\tau}\ \langle\boldsymbol{E}_v(\boldsymbol{r}'(\bar{\tau}),\bar{\tau})\rangle_0. \tag{4.21}$$

Then the expression (B 6) shown in Appendix B leads us to

$$\begin{aligned}\left\langle e^{i\boldsymbol{k}\cdot\boldsymbol{r}'(t)}\right\rangle_0 &= \left\langle e^{i\boldsymbol{k}\cdot\boldsymbol{r}'(t)}\ e^{\theta/\epsilon}\right\rangle_0\Big|_{\boldsymbol{a}=\boldsymbol{0},\hat{\boldsymbol{a}}=\boldsymbol{0}}\\ &= \exp\left[i\epsilon\boldsymbol{k}\cdot\frac{\delta}{\delta\boldsymbol{a}(t))}\right]e^{W_0/\epsilon}\Big|_{\boldsymbol{a}=\boldsymbol{0},\hat{\boldsymbol{a}}=\boldsymbol{0}}\\ &= \exp\left[ik_\parallel v_\parallel t + i\boldsymbol{k}_\perp\cdot\boldsymbol{X}(t) - k_\parallel^2\frac{\chi_\parallel}{v}\psi(vt) - k_\perp^2\frac{\chi_\perp}{v}\psi(vt) - \boldsymbol{k}_\perp\boldsymbol{k}_\perp : \boldsymbol{\mu}_0(t)\right].\end{aligned} \tag{4.22}$$

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Applying this formula to (4.14), we obtain the equation for the function $\boldsymbol{X}(t)$ in the form

$$\frac{d\boldsymbol{X}(\tau)}{d\tau} = \frac{\phi^0}{\langle\phi(\boldsymbol{0},0)^2\rangle}\bar{\boldsymbol{\mathscr{E}}}_\phi(\boldsymbol{X}(\tau),\tau) + \frac{2\boldsymbol{v}^0}{\langle\boldsymbol{v}(\boldsymbol{0},0)^2\rangle}\cdot\bar{\boldsymbol{\mathscr{E}}}_v(\boldsymbol{X}(\tau),\tau), \tag{4.23}$$

which agrees with (3.21) because $\langle\phi(\boldsymbol{0},0)^2\rangle = \langle\phi(\boldsymbol{r},t)^2\rangle$ and $\langle\boldsymbol{v}(\boldsymbol{0},0)^2\rangle = \langle\boldsymbol{v}(\boldsymbol{r},t)^2\rangle$ in the homogeneous turbulence. Similarly, we find that the function $\boldsymbol{\mu}_0(t)$ defined by (4.21) satisfies the expression (3.23).

The running diffusion tensor (4.2) is written in the subensemble representation as

$$\begin{aligned}\boldsymbol{D}_\perp^{\mathrm{an}}(t) &= \int d\phi^0 d\boldsymbol{v}^0 P_1(\phi^0)P_2(\boldsymbol{v}^0)\int_0^t d\tau\,\langle\boldsymbol{v}(\boldsymbol{r}(\tau),\tau)\rangle_0\,\boldsymbol{v}^0 \\ &= \int d\phi^0 d\boldsymbol{v}^0 P_1(\phi^0)P_2(\boldsymbol{v}^0)\int_0^t d\tau\int d\boldsymbol{r}\,\langle\boldsymbol{v}(\boldsymbol{r},\tau)\delta(\boldsymbol{r}-\boldsymbol{r}(\tau))\rangle_0\,\boldsymbol{v}^0.\end{aligned} \tag{4.24}$$

By using the Corrsin approximation in the subensemble

$$\langle\boldsymbol{v}(\boldsymbol{r},\tau)\delta(\boldsymbol{r}-\boldsymbol{r}(\tau))\rangle_0 = \langle\boldsymbol{v}(\boldsymbol{r},\tau)\rangle_0\,\langle\delta(\boldsymbol{r}-\boldsymbol{r}(\tau))\rangle_0 \tag{4.25}$$

and the identity (3.30), the following expression for $\boldsymbol{D}_\perp^{\mathrm{an}}(t)$ is obtained:

$$\begin{aligned}\boldsymbol{D}_\perp^{\mathrm{an}}(t) = &\int d\phi^0 d\boldsymbol{v}^0 P_1(\phi^0)P_2(\boldsymbol{v}^0) \\ &\times\int_0^t d\tau\int\frac{d\boldsymbol{k}}{(2\pi)^3}\boldsymbol{v}^0\left\{\frac{\phi^0}{\langle\phi(\boldsymbol{0},0)^2\rangle}\boldsymbol{E}_\phi(\boldsymbol{k},\tau) + \frac{2\boldsymbol{v}^0}{\langle\boldsymbol{v}(\boldsymbol{0},0)^2\rangle}\cdot\boldsymbol{E}_v(\boldsymbol{k},\tau)\right\}\left\langle e^{i\boldsymbol{k}\cdot\boldsymbol{r}(\tau)}\right\rangle_0.\end{aligned} \tag{4.26}$$

Applying the formulae (4.22) to (4.26), we finally arrive at the expressions (3.29) and (3.31). Therefore, we conclude that the kinetic equation approach and the test-particle approach lead us to the same results.

5. Conclusion

Starting with the kinetic equation and using the functional integral method, we have formulated the statistical theory of particle diffusion in the electrostatic turbulent plasma. In the framework of the DIA, the running diffusion coefficient is shown to be determined by the deterministic nonlinear differential expression (2.20). Next we have extended this DIA formulation using the idea of subensemble. Incorporating this idea into the DIA, we have found that the running diffusion tensor is obtained from (3.31) by solving the deterministic nonlinear differential (3.23) for $\boldsymbol{\mu}_0(\tau)$ together with the expression (3.21) for $\boldsymbol{X}(\tau)$. We have also shown that the same results can be obtained in the framework of the test-particle approach.

In the linear regime in $\boldsymbol{\mu}_0$, the $\boldsymbol{X}(\tau)$ agrees with the decorrelation trajectory proposed by Vlad et al., and the expression (3.31) for the running diffusion tensor also agrees with that of Vlad et al.

Our presented formulation is the extension of the conventional DIA relied on the Eulerian description by incorporating the Lagrangian description through decorrelation trajectories, and it is also the extension of the DCT method to the nonlinear regime in $\boldsymbol{\mu}_0$.

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Appendix A

Analogous to (2.11), we introduce a generating functional

$$\tilde{Z}[\zeta,\xi]=\mathscr{N}\int\mathscr{D}[f']\mathscr{D}[\hat{f}]\ \ e^{-\tilde{S}}, \tag{A 1}$$

and define

$$\tilde{G}(\boldsymbol{r}-\boldsymbol{r}',t-t')=\frac{\delta^2\tilde{Z}}{\delta\xi(\boldsymbol{r}',t')\delta\zeta(\boldsymbol{r},t)}, \tag{A 2}$$

where

$$\tilde{S}=\mathscr{L}-\int dt\,d\boldsymbol{r}\left[\zeta(\boldsymbol{r},t)f'(\boldsymbol{r},t)+\xi(\boldsymbol{r},t)\hat{f}(\boldsymbol{r},t)\right]. \tag{A 3}$$

Functionally differentiating the identity $\int\mathscr{D}[f']\mathscr{D}[\hat{f}]\delta/\delta f'(\boldsymbol{r},t)e^{-\tilde{S}}=0$ with respect to $\xi(\boldsymbol{r}',t')$, we find that the function $\tilde{G}(\boldsymbol{r}-\boldsymbol{r}',t-t')$ defined by (A 2) satisfies the expression (3.25) and thus it agrees with the fluctuating response function. Noticing $e^{-S}=\langle e^{-\tilde{S}}\rangle$, we obtain in the limit of $\zeta=\xi=0$

$$\begin{aligned}G(\boldsymbol{r}-\boldsymbol{r}',t-t')&=\frac{\delta^2}{\delta\xi(\boldsymbol{r}',t')\delta\zeta(\boldsymbol{r},t)}\mathscr{N}\int\mathscr{D}[f']\mathscr{D}[\hat{f}]\ \ e^{-S}\\&=\left\langle\frac{\delta^2}{\delta\xi(\boldsymbol{r}',t')\delta\zeta(\boldsymbol{r},t)}\mathscr{N}\int\mathscr{D}[f']\mathscr{D}[\hat{f}]\ \ e^{-\tilde{S}}\right\rangle\\&=\langle\tilde{G}(\boldsymbol{r}-\boldsymbol{r}',t-t').\rangle\end{aligned} \tag{A 4}$$

Similarly, the relation $G_0(\boldsymbol{r}-\boldsymbol{r}',t-t')=\langle\tilde{G}(\boldsymbol{r}-\boldsymbol{r}',t-t')\rangle_0$ is derived using $e^{-S_0}=\langle e^{-\tilde{S}}\rangle_0$.

Appendix B

Let us define the following function:

$$F(\alpha)=e^{-\alpha W_0/\epsilon}\exp\left[i\epsilon\boldsymbol{k}\cdot\frac{\delta}{\delta\boldsymbol{a}(t))}\right]e^{\alpha W_0/\epsilon}. \tag{B 1}$$

Differentiating this function yields

$$\frac{dF(\alpha)}{d\alpha}=\frac{1}{\epsilon}e^{-\alpha W_0/\epsilon}\left\{\exp\left[i\epsilon\boldsymbol{k}\cdot\frac{\delta}{\delta\boldsymbol{a}(t))}\right]W_0-W_0\exp\left[i\epsilon\boldsymbol{k}\cdot\frac{\delta}{\delta\boldsymbol{a}(t))}\right]\right\}e^{\alpha W_0/\epsilon}. \tag{B 2}$$

We now assume

$$\frac{\delta^n W_0}{\delta\boldsymbol{a}(t)\delta\boldsymbol{a}(t)\cdots\delta\boldsymbol{a}(t)}=0\qquad(n=3,4,\cdots). \tag{B 3}$$

Then the right-hand side of (B 2) is written as

$$\frac{dF(\alpha)}{d\alpha}=\left[i\boldsymbol{k}\cdot\frac{\delta W_0}{\delta\boldsymbol{a}(t)}-\frac{1}{2}\epsilon\boldsymbol{k}\boldsymbol{k}:\frac{\delta^2W_0}{\delta\boldsymbol{a}(t)\delta\boldsymbol{a}(t)}\right]F. \tag{B 4}$$

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Solving this differential equation, we obtain

$$F(\alpha) = \exp\left[\alpha\left(i\boldsymbol{k}\cdot\frac{\delta W_0}{\delta \boldsymbol{a}(t)} - \frac{1}{2}\epsilon\boldsymbol{k}\boldsymbol{k} : \frac{\delta^2 W_0}{\delta \boldsymbol{a}(t)\delta \boldsymbol{a}(t)}\right)\right]. \tag{B 5}$$

Setting $\alpha = 1$, we have

$$\exp\left[i\epsilon\boldsymbol{k}\cdot\frac{\delta}{\delta \boldsymbol{a}(t)}\right] e^{W_0/\epsilon} = e^{W_0/\epsilon}\exp\left[i\boldsymbol{k}\cdot\frac{\delta W_0}{\delta \boldsymbol{a}(t)} - \frac{1}{2}\epsilon\boldsymbol{k}\boldsymbol{k} : \frac{\delta^2 W_0}{\delta \boldsymbol{a}(t)\delta \boldsymbol{a}(t)}\right]. \tag{B 6}$$

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