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An analytical study on the existence of solitary structure at $M = M_c$

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Abstract. A general theory for the existence of solitary structure at $M = M_c$ has been discussed, where M_c is the lower bound of the Mach number M, i.e., solitary structures start to exist for $M > M_c$. Three important results have been proved to confirm the existence of solitary structure at $M = M_c$. If $V(\phi)(\equiv V(M,\phi))$ denotes the Sagdeev potential with ϕ being the perturbed field or perturbed dependent variable associated with a specific problem, $V(M,\phi)$ is well defined as a real number for all $M \in \mathscr{M}$ and $\phi \in \Phi_0$, and $V(M,0) = V'(M,0) = V''(M_c,0) = 0$, $V'''(M_c,0) < 0$ ($V'''(M_c,0) > 0$), $\partial V/\partial M < 0$ for all $M(\in \mathscr{M}) > 0$ and $\phi(\in \Phi_0) > 0$ ($\phi(\in \Phi_0) < 0$), where '$'$ $\equiv \partial/\partial\phi$ ', the main analytical results for the existence of solitary wave or double layer solution of the energy integral at $M = M_c$ are as follows. Result 1: If there exists at least one value M_0 of M such that the system supports positive (negative) potential solitary waves for all $M_c < M < M_0$, then there exists either a positive (negative) potential solitary wave or a positive (negative) potential double layer at $M = M_c$. Result 2: If the system supports only negative (positive) potential solitary waves for $M > M_c$, then there does not exist positive (negative) potential solitary wave at $M = M_c$. Result 3: It is not possible to have coexistence of both positive and negative potential solitary structures at $M = M_c$. Apart from the conditions of Result 1, the double layer solution at $M = M_c$ is possible only when there exists a double layer solution in any right neighborhood of M_c. Finally, these analytical results have been applied to a specific problem on dust acoustic (DA) waves in non-thermal plasma in search of new results.

1. Introduction

The Sagdeev potential approach (1966) has been considered to investigate the existence of solitary wave and/or double layer at $M = M_c$, where M_c is the lower bound of the Mach number M, i.e., solitary wave and/or double layer solutions of the energy integral start to exist for $M > M_c$. In most of the earlier works (Shukla and Yu 1978; Popel et al. 1996; Yu et al. 1980; Bharuthram and Shukla 1986; Baboolal et al. 1988, 1989, 1990, 1991; Cairns et al. 1995a,b, 1996; Mamun et al. 1996a,b; Xie et al. 1998; Mendoza-Briceño et al. 2000; Gill et al. 2003; Maharaj et al. 2004, 2006; Hellberg et al. 2006; Baluku and Hellberg 2008; Tanjia and Mamun 2008; Verheest and Pillay 2008; Das et al. 2009, 2010; Djebli and Marif 2009; Verheest 2009, 2010) solitary wave and/or double layer solutions have been investigated for $M > M_c$. However, some recent investigations (Baluku et al. 2010a,b; Verheest and Hellberg 2010) have shown that finite amplitude solitary wave can exist at $M = M_c$ in the parameter regime where solitons of both polarities exist for $M > M_c$. The numerical observations (Baluku et al. 2010a,b; Verheest and Hellberg 2010) of the solitary wave solution of energy integral at $M = M_c$ motivate us to set a general analytical theory for the existence of solitary wave and double layer solution of energy integral at $M = M_c$. The present paper is purely an analytical study on the existence of solitary wave and double layer solution of energy integral at $M = M_c$. Without this analytical study, it is difficult to predict the existence of double layer solution at $M = M_c$. The analytical study also gives the point in the parameter regime where the double layer solution at $M = M_c$ exists. So we consider the following well-known energy integral, which has been commonly used in various plasma environments (Shukla and Yu 1978; Yu et al. 1980; Bharuthram and Shukla 1986; Baboolal et al. 1988, 1989, 1991; Cairns et al. 1995a,b, 1996; Mamun et al. 1996a,b; Popel et al. 1996; Xie et al. 1998; Mendoza-Briceño et al. 2000; Gill et al. 2003; Maharaj et al. 2004, 2006; Hellberg et al. 2006; Baluku and Hellberg 2008; Tanjia and Mamun 2008; Verheest and Pillay 2008; Verheest 2009, 2010; Verheest and Hellberg 2010; Das et al. 2009, 2010; Djebli and Marif 2009; Baluku et al.

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2010a,b):

$$\frac{1}{2}\left(\frac{d\phi}{d\xi}\right)^2 + V(\phi) = 0, \tag{1.1}$$

with $V(\phi) \equiv V(M,\phi)$ is the Sagdeev potential. Here, $V(M,\phi)$ is same as $V(\phi)$, i.e., the Mach number M is omitted from the notation $V(M,\phi)$ when no particular emphasis is put upon it, and let $\mathscr{M} \times \Phi_0$ is the domain of definition of the function $V(M,\phi)$, i.e., $V(M,\phi)$ is well defined as a real number for all $M \in \mathscr{M}$ and for all $\phi \in \Phi_0$. Hereafter, by the phrases ' for all $\phi > 0$ ', ' for all $\phi < 0$ ', and ' for all $M > 0$ ', we mean, respectively, ' all those $\phi \in \Phi_0$ that are strictly positive ', ' all those $\phi \in \Phi_0$ that are strictly negative ', and ' all those $M \in \mathscr{M}$ that are strictly positive'.

The energy integral (1.1) can be regarded as the one-dimensional motion of a particle of unit mass whose position is ϕ at time ξ with velocity $d\phi/d\xi$ in a potential well $V(\phi)$. The first term of the energy integral can be regarded as the kinetic energy of a particle of unit mass at position ϕ and time ξ, whereas $V(\phi)$ is the potential energy of the same particle at that instant. Since kinetic energy is always a non-negative quantity, $V(\phi) \leqslant 0$ for the entire motion, i.e., zero is the maximum value for $V(\phi)$. Again from (1.1), we find $d^2\phi/d\xi^2 + V'(\phi) = 0$, where ' $' \equiv \partial/\partial\phi$ '. Thus, the force acting on the particle of unit mass at position ϕ is $-V'(\phi)$. Suppose, $V(0) = V'(0) = 0$, then the particle is in equilibrium at $\phi = 0$ because the velocity as well as the force acting on the particle at $\phi = 0$ are simultaneously equal to zero. Now if $\phi = 0$ can be made an unstable position of equilibrium, the energy integral can be interpreted as the motion of an oscillatory particle if $V(\phi_m) = 0$ for some $\phi_m \neq 0$, i.e., if the particle is slightly displaced from its unstable position of equilibrium, then it moves away from its unstable position of equilibrium and continues its motion until its velocity is equal to zero, i.e., until ϕ takes the value ϕ_m. Now the force acting on the particle of unit mass at position $\phi = \phi_m$ is $-V'(\phi_m)$. For $\phi_m < 0$, the force acting on the particle at point $\phi = \phi_m$ is directed toward the point $\phi = 0$ if $-V'(\phi_m) > 0$, i.e., if $V'(\phi_m) < 0$. On the other hand, for $\phi_m > 0$, the force acting on the particle at point $\phi = \phi_m$ is directed toward point $\phi = 0$ if $-V'(\phi_m) < 0$, i.e., if $V'(\phi_m) > 0$. Therefore, if $V'(\phi_m) > 0$ (for the positive potential side) or $V'(\phi_m) < 0$ (for the negative potential side), then the particle reflects back again to $\phi = 0$. Again, if $V(\phi_m) = V'(\phi_m) = 0$ then both velocity $d\phi/d\xi$ and force $d^2\phi/d\xi^2$ are simultaneously equal to zero at $\phi = \phi_m$. Consequently, if the particle is slightly displaced from its unstable position of equilibrium ($\phi = 0$), it moves away from $\phi = 0$ and continues its motion until the velocity is equal to zero, i.e., until ϕ takes the value $\phi = \phi_m$. However, it cannot be reflected back again at $\phi = 0$, as the velocity and the force acting on the particle at $\phi = \phi_m$ vanish simultaneously. Actually (Verheest 2000), if $V'(\phi_m) > 0$ (for $\phi_m > 0$) or $V'(\phi_m) < 0$ (for $\phi_m < 0$), the particle takes an infinite long time to move away from the unstable position of equilibrium. After that it continues its motion until ϕ takes the value ϕ_m and again it takes an infinite long time to come back to its unstable position of equilibrium.

Therefore, it is clear from the above discussions that for the existence of a Positive Potential Solitary Wave (PPSW) (Negative Potential Solitary Wave (NPSW)) solution of the energy integral (1.1), the following three conditions must be simultaneously satisfied.

Sa : $\phi = 0$ is the position of unstable equilibrium of the particle, i.e., $V(0) = V'(0) = 0$ and $V''(0) < 0$.

Sb : $V(\phi_m) = 0$, $V'(\phi_m) > 0$ ($V'(\phi_m) < 0$) for some $\phi_m > 0$ ($\phi_m < 0$). This condition is responsible for the oscillation of the particle within the interval $\min\{0, \phi_m\} < \phi < \max\{0, \phi_m\}$.

Sc : $V(\phi) < 0$ for all $0 < \phi < \phi_m$ ($\phi_m < \phi < 0$). This condition is necessary to define the energy integral (1.1) within the interval $\min\{0, \phi_m\} < \phi < \max\{0, \phi_m\}$.

On the other hand, for the existence of a Positive Potential Double Layer (PPDL) (Negative Potential Double Layer (NPDL)) solution of the energy integral (1.1), the following three conditions must be simultaneously satisfied:

Da : $\phi = 0$ is the position of unstable equilibrium of the particle, i.e., $V(0) = V'(0) = 0$ and $V''(0) < 0$.

Db : $V(\phi_m) = 0$, $V'(\phi_m) = 0$, $V''(\phi_m) < 0$ for some $\phi_m > 0$ ($\phi_m < 0$). This condition actually states that the particle cannot be reflected back again at $\phi = 0$.

Dc : $V(\phi) < 0$ for all $0 < \phi < \phi_m$ ($\phi_m < \phi < 0$). This condition is necessary to define the energy integral (1.1) within the interval $\min\{0, \phi_m\} < \phi < \max\{0, \phi_m\}$.

The above discussions for the existence of solitary waves and double layers are valid if $\phi = 0$ is an unstable position of equilibrium, i.e., if $V''(0) < 0$ along with $V(0) = V'(0) = 0$. In other words, $\phi = 0$ can be made an unstable position of equilibrium if the potential energy of the said particle attains its maximum value at $\phi = 0$. Now, the condition $V''(0) < 0$ gives a lower bound M_c of M, i.e., $V''(0) < 0 \Leftrightarrow M > M_c$, $V''(0) > 0 \Leftrightarrow M < M_c$, and $V''(0) = 0 \Leftrightarrow M = M_c \Leftrightarrow V''(M_c, 0) = 0$. This M_c is, in general, a function of the parameters involved in the system, or a constant. Therefore, if $M < M_c$, the potential energy of the said particle attains its minimum value at $\phi = 0$. Consequently, $\phi = 0$ is the position of stable equilibrium of the particle, and in this case it is impossible to make any oscillation of the particle even when it is slightly displaced from its position of stable equilibrium, and therefore there is no question of existence of solitary waves or double layers for $M < M_c$. In other words, for the position of unstable equilibrium of the particle at $\phi = 0$, i.e., for $V''(0) < 0 (\Leftrightarrow M > M_c)$, the function $V(\phi)$ must be convex within a neighborhood of $\phi = 0$ and in this case both types of solitary waves (negative or positive potential) may exist if other conditions are fulfilled. Now, suppose $V''(M_c, 0) = 0$ and also $V'''(M_c, 0) = 0$, then if $V''''(M_c, 0) < 0$, the potential energy of the said particle attains its maximum value at $\phi = 0$ and consequently

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$\phi = 0$ is the position of unstable equilibrium. On the other hand, if $V''(M_c,0) = 0$, $V'''(M_c,0) = 0$, and $V''''(M_c,0) > 0$, the potential energy of the said particle attains its minimum value at $\phi = 0$ and consequently $\phi = 0$ is the position of stable equilibrium of the particle, and in this case there is no question of existence of solitary wave solution as well as double layer solution of the energy integral (1.1). But if $V'''(M_c,0) \neq 0$ along with $V(M_c,0) = V'(M_c,0) = V''(M_c,0) = 0$, then without going through the complete analytical investigation, it is difficult to predict the existence of solitary wave and/or double layer solution of the energy integral (1.1) at $M = M_c$. In this situation, i.e., when $V'''(M_c,0) \neq 0$ along with $V(M_c,0) = V'(M_c,0) = V''(M_c,0) = 0$, to continue the physical interpretation for the existence of solitary wave and/or double layer solution of the energy integral (1.1) at $M = M_c$, we have proved the following theorem in Section 3 with the help of some definitions and well-known theorems of real analysis as given in Section 2.

THEOREM 1. *If $V'''(M_c,0) \neq 0$, then there exists a strictly positive real number ϕ_ϵ (> 0) such that $V'''(M_c,0)$ $V'''(M_c,\phi) > 0$ for all $-\phi_\epsilon < \phi < \phi_\epsilon$.*

Using Theorem 1, some definitions, and well-known theorems of real analysis as given in Section 2, we have proved the following theorem in Section 3.

THEOREM 2. *If $V''(M_c,0) = 0$ and $V'''(M_c,0) \neq 0$, then there exists a strictly positive real number ϕ_ϵ (> 0) such that $V'''(M_c,0)V''(M_c,\phi) > 0$ for all $0 < \phi < \phi_\epsilon$ and $V'''(M_c,0)V''(M_c,\phi) < 0$ for all $-\phi_\epsilon < \phi < 0$.*

From Theorem 2, we observe that whenever $V(M,0) = V'(M,0) = V''(M_c,0) = 0$ and $V'''(M_c,0) < 0$, then $V''(M_c,\phi) > 0$ for all $-\phi_\epsilon < \phi < 0$ and $V''(M_c,\phi) < 0$ for all $0 < \phi < \phi_\epsilon$, i.e., $V(M_c,\phi)$ is a concave function within the interval $-\phi_\epsilon < \phi < 0$ and $V(M_c,\phi)$ is a convex function within the interval $0 < \phi < \phi_\epsilon$, where ϕ_ϵ is strictly a positive quantity but it can be made sufficiently small. Therefore, if the particle placed at $\phi = 0$ be slightly displaced toward the positive potential side, it falls within the interval $0 < \phi < \phi_\epsilon$ and because of the convexity of the function $V(M_c,\phi)$ within the interval $0 < \phi < \phi_\epsilon$, it moves away from $\phi = 0$ and continues its motion until its velocity is equal to zero, i.e., until ϕ takes the value $\phi = \phi_c > 0$, where $V(M_c,\phi_c) = 0$, and in this case if $V'(M_c,\phi_c) > 0$ ($V'(M_c,\phi_c) = 0$ and $V''(M_c,\phi_c) < 0$), one can get a PPSW (PPDL) as a solution of the energy integral (1.1). Again, as $V(M_c,\phi)$ is a concave function within the interval $-\phi_\epsilon < \phi < 0$, if the particle placed at $\phi = 0$ be slightly displaced toward the negative potential side, it falls within the interval $-\phi_\epsilon < \phi < 0$ and because of the concavity of the function $V(M_c,\phi)$ within the interval $-\phi_\epsilon < \phi < 0$, it moves toward the point $\phi = 0$ and consequently there does not exist any solitary wave or double layer solution in the negative potential side. In true physical sense, $\lim_{\phi\to 0+0} V(M_c,\phi)$ is an unstable position that lies in a small right neighborhood of the equilibrium position $\phi = 0$, whereas $\lim_{\phi\to 0-0} V(M_c,\phi)$ is a stable position that lies in a small left neighborhood of the equilibrium position $\phi = 0$, where $\phi \to 0+0 \Leftrightarrow \phi > 0$ and ϕ approaches to zero, i.e., ϕ approaches to zero from the right side of zero and $\phi \to 0-0 \Leftrightarrow \phi < 0$, and ϕ approaches to zero, i.e., ϕ approaches to zero from the left side of zero. Hence, as $\lim_{\phi\to 0+0} V(M_c,\phi)$ is an unstable position that lies in a small right neighborhood of the equilibrium position $\phi = 0$, there may exist either a solitary wave or a double layer in the positive potential side. On the other hand, as $\lim_{\phi\to 0-0} V(M_c,\phi)$ is a stable position that lies in a small left neighborhood of the equilibrium position $\phi = 0$, there does not exist any solitary wave or double layer in the negative potential side. Hence, the above theoretical discussions can be summarized as follows.

THEOREM 3. *If $V(M,0) = V'(M,0) = V''(M_c,0) = 0$ and $V'''(M_c,0) < 0$, then at $M = M_c$, there may exist either a solitary wave or a double layer in the positive potential side but there does not exist any solitary wave or double layer in the negative potential side.*

Again, from Theorem 2, we observe that whenever $V(M,0) = V'(M,0) = V''(M_c,0) = 0$ and $V'''(M_c,0) > 0$, then $V''(M_c,\phi) < 0$ for all $-\phi_\epsilon < \phi < 0$ and $V''(M_c,\phi) > 0$ for all $0 < \phi < \phi_\epsilon$, i.e., $V(M_c,\phi)$ is a convex function within the interval $-\phi_\epsilon < \phi < 0$ and $V(M_c,\phi)$ is a concave function within the interval $0 < \phi < \phi_\epsilon$, where ϕ_ϵ is strictly a positive quantity, but it can be made sufficiently small. Therefore, if the particle placed at $\phi = 0$ be slightly displaced toward the negative potential side, it falls within the interval $-\phi_\epsilon < \phi < 0$ and because of the convexity of the function $V(M_c,\phi)$ within the interval $-\phi_\epsilon < \phi < 0$, it moves away from $\phi = 0$ and continues its motion until its velocity is equal to zero, i.e., until ϕ takes the value $\phi = \phi_c < 0$, where $V(M_c,\phi_c) = 0$ and in this case if $V'(M_c,\phi_c) < 0$ ($V'(M_c,\phi_c) = 0$ and $V''(M_c,\phi_c) < 0$), one can get an NPSW (NPDL) as a solution of the energy integral (1.1). Again, as $V(M_c,\phi)$ is a concave function within the interval $0 < \phi < \phi_\epsilon$, and if the particle placed at $\phi = 0$ is slightly displaced toward the positive potential side, it falls within the interval $0 < \phi < \phi_\epsilon$ and because of the concavity of the function $V(M_c,\phi)$ within the interval $0 < \phi < \phi_\epsilon$, it moves toward the point $\phi = 0$ and consequently there does not exist any solitary wave or double layer solution in the positive potential side. In true physical sense, $\lim_{\phi\to 0-0} V(M_c,\phi)$ is an unstable position that lies in a small left neighborhood of the equilibrium position $\phi = 0$, whereas $\lim_{\phi\to 0+0} V(M_c,\phi)$ is a stable position that lies in a small right neighborhood of the equilibrium position $\phi = 0$. Hence, as $\lim_{\phi\to 0-0} V(M_c,\phi)$ is an unstable position that lies in a small left neighborhood of the equilibrium position $\phi = 0$, there may exist either a solitary wave or a double layer in the negative potential side. On the other hand, as $\lim_{\phi\to 0+0} V(M_c,\phi)$ is a stable

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position that lies in a small right neighborhood of the equilibrium position $\phi = 0$, there does not exist any solitary wave or double layer in the positive potential side. Hence, the above theoretical discussions can be summarized as follows.

THEOREM 4. *If* $V(M,0) = V'(M,0) = V''(M_c,0) = 0$ *and* $V'''(M_c,0) > 0$, *then at* $M = M_c$, *there may exist either a solitary wave or a double layer in the negative potential side but there does not exist any solitary wave or double layer in the positive potential side.*

From Theorems 3 and 4, we note that if $V(0) = V'(0) = 0, V''(M_c,0) = 0$ and $V'''(M_c,0) \neq 0$, the point $\phi = 0$ is the point of inflexion that separates the convex part of $V(M_c,\phi)$ from the concave part, and if $\phi = 0$ is a point of inflexion, then there may exist either a solitary wave or a double layer in either positive potential side or negative potential side, but the solitary structures of both polarities can not coexist at $M = M_c$. Therefore, we have the following.

THEOREM 5. *If* $V(M,0) = V'(M,0) = V''(M_c,0) = 0$ *and* $V'''(M_c,0) \neq 0$, *it is not possible to have coexistence of both positive and negative potential solitary structures at* $M = M_c$.

If $V(M,0) = V'(M,0) = V''(M_c,0) = 0$, using Theorems 3–5, and some well-known theorems of real analysis as given in Sections 2 and 3, we have proved the following theorems to confirm the existence of solitary wave and double layer solution of the energy integral (1.1) at $M = M_c$.

THEOREM 6. *If* $V'''(M_c,0) < 0, \partial V/\partial M < 0$ *for all* $M > 0$ *and for all* $\phi > 0$, *and if there exists at least one value* M_0 *of* M *such that the system supports PPSWs for all* $M_c < M < M_0$, *then there exists either a PPSW or a PPDL at* $M = M_c$.

THEOREM 7. *If* $V'''(M_c,0) > 0, \partial V/\partial M < 0$ *for all* $M > 0$ *and for all* $\phi < 0$, *and if there exists at least one value* M_0 *of* M *such that the system supports NPSWs for all* $M_c < M < M_0$, *then there exists either an NPSW or an NPDL at* $M = M_c$.

THEOREM 8. *If* $V'''(M_c,0) < 0, \partial V/\partial M < 0$ *for all* $M > 0$ *and for all* $\phi > 0$, *and if the system supports only NPSWs for* $M > M_c$, *then there does not exist any PPSW at* $M = M_c$.

THEOREM 9. *If* $V'''(M_c,0) > 0, \partial V/\partial M < 0$ *for all* $M > 0$ *and for all* $\phi < 0$, *and if the system supports only PPSWs for* $M > M_c$, *then there does not exist any NPSW at* $M = M_c$.

THEOREM 10. *If* $V'''(M_c,0) \neq 0, \partial V/\partial M < 0$ *for all* $M > 0$ *and for all* $\phi \neq 0$, *and if there exists at least one value* M_0 *of* M *such that the system supports both PPSWs and NPSWs for all* $M_c < M < M_0$, *i.e., if the system supports the coexistence of both PPSWs and NPSWs for all* $M_c < M < M_0$, *then there exists either a PPSW or a PPDL at* $M = M_c$ *if* $V'''(M_c,0) < 0$, *whereas for* $V'''(M_c,0) > 0$, *there exists either an NPSW or an NPDL at* $M = M_c$.

Combining Theorems 3–10, we have the following three important results to confirm the existence of solitary wave and double layer solutions of the energy integral (1.1) at $M = M_c$. If $V(M,0) = V'(M,0) = V''(M_c,0) = 0$, $V'''(M_c,0) < 0$ ($V'''(M_c,0) > 0$), $\partial V/\partial M < 0$ for all $M > 0$ and for all $\phi > 0$ ($\phi < 0$), the main analytical results for the existence of solitary wave and double layer solution of the energy integral at $M = M_c$ are as follows.

Result 1. If there exists at least one value M_0 of M such that the system supports positive (negative) potential solitary waves for all $M_c < M < M_0$, then there exists either a positive (negative) potential solitary wave or a positive (negative) potential double layer at $M = M_c$.

Result 2. If the system supports only negative (positive) potential solitary waves for $M > M_c$, then there does not exist positive (negative) potential solitary wave at $M = M_c$.

Result 3. It is not possible to have coexistence of both positive and negative potential solitary structures at $M = M_c$.

Apart from the conditions of Theorem 6 (Theorem 7), the PPDL (NPDL) solution at $M = M_c$ is possible only when there exists a PPDL (NPDL) solution in any right neighborhood of M_c, i.e., PPDL (NPDL) solution at $M = M_c$ is possible only when the curve $M = M_D$ tends to intersect the curve $M = M_c$ at some point in the solution space (or the compositional parameter space showing the nature of existence of different solitary structures) of the energy integral where each point of the curve $M = M_D$ corresponds to a PPDL (NPDL) solution of the energy integral whenever $M_D > M_c$. So without going through all qualitatively different solution spaces (or the compositional parameter spaces showing the nature of existence of different solitary structures) of the energy integral by considering the entire range of parameters involved in the system, it is not possible to make a systematic investigation of solitary wave solution and/or double layer solution of energy integral at $M = M_c$. The most interesting fact is the existence of a double layer solution at $M = M_c$.

Finally, we have used the analytical results as presented in this paper to a specific problem of DA wave in non-thermal plasma to investigate new results and new physical ideas.

The present paper is organized as follows: In Section 2, some mathematical preliminaries (including definitions, properties, and some basic theorems) have been given. Using these definitions, properties, and basic theorems we have proved important theorems in Section 3 for the existence of solitary wave and/or double layer solution at $M = M_c$. The analytical results have been verified numerically by considering a problem of DA

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wave in non-thermal plasma in Section 4. Finally, a brief conclusion has been given in Section 5.

2. Mathematical preliminaries

2.1. Definitions and properties

D1: Let a be any point of $\Re$, the set of all real numbers and $\delta > 0$ is a strictly positive real number, then the $\delta-$ neighborhood of the point a is denoted by $N(a,\delta)$ and is defined as a set of all real numbers x such that $a-\delta < x < a+\delta$, i.e., $N(a,\delta) = (a-\delta, a+\delta) = \{x : a-\delta < x < a+\delta\}$, i.e.,

$$x \in N(a,\delta) \Leftrightarrow a-\delta < x < a+\delta \Leftrightarrow |x-a| < \delta. \tag{2.1}$$

P1: If $0 < \delta_1 \leqslant \delta_2$, using (2.1) it is simple to check the following property:

$$N(a,\delta_1) \subseteq N(a,\delta_2). \tag{2.2}$$

D2: Let A be any subset of $\Re$ and $a \in A$, then a is said to be an interior point of A if there exists a $\delta > 0$ such that $N(a,\delta) \subseteq A$, i.e., any point of $N(a,\delta)$ is also a point of A.

D3: $\Re \times \Re$ is the set of all two-dimensional points (x,y), i.e., $\Re \times \Re$ is the set of all ordered pairs (x,y) such that $x \in \Re$ and $y \in \Re$, i.e.,

$$\Re \times \Re = \{(x,y) : x \in \Re \text{ and } y \in \Re\}. \tag{2.3}$$

If A and B are any two subsets of $\Re$, then we define $A \times B$ as the set of all ordered pairs (x,y) such that $x \in A$ and $y \in B$, i.e.,

$$A \times B = \{(x,y) : x \in A \subseteq \Re \text{ and } y \in B \subseteq \Re\}. \tag{2.4}$$

P2: If A, B, C, and D are any four subsets of $\Re$ and $A \times B \subseteq C \times D$, then it is simple to prove the following property:

$$A \subseteq C \text{ and } B \subseteq D. \tag{2.5}$$

P3: For any four subsets A, B, C, and D of $\Re$, it is simple to prove the following property: $(A \times B) \cap (C \times D) = (A \cap C) \times (B \cap D)$.

D4: Let (a,b) be any point of $\Re \times \Re$ and $\delta_1 > 0$, $\delta_2 > 0$ are two strictly positive real numbers, then the $\delta_1\delta_2$-rectangular neighborhood of the point (a,b) is denoted by $N(a,b;\delta_1,\delta_2)$ and is defined as a set of all points (x,y) of $\Re \times \Re$ such that $a-\delta_1 < x < a+\delta_1$ and $b-\delta_2 < y < b+\delta_2$, i.e., $N(a,b;\delta_1,\delta_2) = \{(x,y) : x \in N(a,\delta_1) \text{ and } y \in N(b,\delta_2)\} = \{(x,y) : a-\delta_1 < x < a+\delta_1 \text{ and } b-\delta_2 < y < b+\delta_2\} = \{(x,y) : |x-a| < \delta_1 \text{ and } |y-b| < \delta_2\}$, i.e.,

$$(x,y) \in N(a,b;\delta_1,\delta_2) \Leftrightarrow |x-a| < \delta_1 \text{ and } |y-b| < \delta_2. \tag{2.6}$$

P4: $N(a,b;\delta_1,\delta_2) = N(a,\delta_1) \times N(b,\delta_2)$.

D5: Let $A \times B$ be any subset of $\Re \times \Re$ and (a,b) be a point of $A \times B$, i.e., $a \in A$ and $b \in B$, then (a,b) is said to be an interior point of $A \times B$ if there exists a $\delta_1\delta_2$-rectangular neighborhood of the point (a,b) with $\delta_1 > 0$, $\delta_2 > 0$ such that $N(a,b;\delta_1,\delta_2) \subseteq A \times B$, i.e., any point of $N(a,b;\delta_1,\delta_2)$ is also a point of $A \times B$, i.e., $N(a,\delta_1) \times N(b,\delta_2) \subseteq A \times B$, i.e., $N(a,\delta_1) \subseteq A$ and $N(b,\delta_2) \subseteq B$.

2.2. Some theorems of real analysis

BTh1: Let $f(x)$ be a real-valued function defined on a subset D of the set of real numbers $\Re$ and $f(x)$ be continuous at $x = a \in D$. If $f(a) \neq 0$, then there exists a real number $\delta_1 > 0$ such that $f(a)f(x) > 0$ for all $x \in N(a,\delta_1) \cap D$. Furthermore, if a is an interior point of D, then there exists a real number $\delta > 0$ such that $f(a)f(x) > 0$ for all $x \in N(a,\delta) \subseteq D$.

BTh2: Let $f(x,y)$ be a real-valued function defined on a subset $D = A \times B$ of $\Re \times \Re$ and $f(x,y)$ be continuous at $(x,y) = (a,b) \in D$. If $f(a,b) \neq 0$, then there exist two real numbers $\delta_1 > 0$ and $\delta_2 > 0$ such that $f(a,b)f(x,y) > 0$ for all $(x,y) \in N(a,b;\delta_1,\delta_2) \cap D$. Furthermore, if (a,b) is an interior point of D, then there exist two real numbers $\delta_3 > 0$ and $\delta_4 > 0$ such that $f(a,b)f(x,y) > 0$ for all $(x,y) \in N(a,b;\delta_3,\delta_4) \subseteq D$.

Bolzano's Theorem: Let $f(x)$ be a real-valued function defined on a closed interval $[a,b]$ $(a < b)$. If $f(x)$ is continuous on $[a,b]$ and $f(a)f(b) < 0$, then there exists a point $c \in (a,b)$ such that $f(c) = 0$.

Lagrange's Mean-Value Theorem: Let $f(x)$ be a real-valued function defined on a closed interval $[a,b]$ $(a < b)$. If $f(x)$ is continuous on $[a,b]$ and if it is derivable on (a,b), then there exists a point $c \in (a,b)$ such that

$$[f(b) - f(a)]/(b-a) = f'(c),$$

where ' ' ' indicates derivative with respect to x.

The Implicit Function Theorem of Two Variables (IFT): Let $V(M,\phi)$ be a real-valued function of two variables M and ϕ defined on $\mathscr{M} \times \Phi_0$, i.e., for each $M \in \mathscr{M}$ and for each $\phi \in \Phi_0$, there exists a unique real number $V(M,\phi)$. Let $(M_1,\phi_1) \in \mathscr{M} \times \Phi_0$ be an interior point of $\mathscr{M} \times \Phi_0$ and

A1: $V(M_1,\phi_1) = 0$,

A2: $\frac{\partial V}{\partial \phi}$ is continuous in a rectangular neighborhood of (M_1,ϕ_1),

A3: $\left.\frac{\partial V}{\partial \phi}\right|_{(M_1,\phi_1)} \neq 0$,

there exists a rectangular neighborhood $N(M_1,\phi_1;h_1,k_1) = N(M_1,h_1) \times N(\phi_1,k_1)$ of (M_1,ϕ_1) such that corresponding to every $M \in N(M_1,h_1)$, $\phi(\in N(\phi_1,k_1))$ can be expressed uniquely as a function of M, for which the following conditions are simultaneously satisfied:

C1: $\phi(M_1) = \phi_1$,

C2: $V(M,\phi(M)) = 0$ for all $M \in N(M_1,h_1)$,

C3: $\frac{d\phi}{dM} = -(\frac{\partial V}{\partial M} \div \frac{\partial V}{\partial \phi})$ for all $(M,\phi) \in N(M_1,\phi_1;h_1,k_1) = N(M_1,h_1) \times N(\phi_1,k_1)$.

To prove Theorems 6–10, it is helpful to remember that if a property is true for each point of a set then that property is also true for each point of any subset of that set. Without any loss of generality, we can assume (M,ϕ) as an interior point of $\mathscr{M} \times \Phi_0$.

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3. Proofs of Theorems 1 and 2, and Theorems 6–10

THEOREM 1. *If $V'''(M_c,0)\neq 0$, then there exists a strictly positive real number ϕ_ϵ (>0) such that $V'''(M_c,0)V'''(M_c,\phi)>0$ for all $-\phi_\epsilon<\phi<\phi_\epsilon$.*

Proof. Here $f(\phi)=V'''(M_c,\phi)$ is continuous at $\phi=0$ and consequently from Theorem **BTh1**, we can conclude that there exists a strictly positive real number ϕ_ϵ (>0) such that

$$\begin{aligned} & f(\phi)f(0)>0 \quad \forall \quad \phi\in(0-\phi_\epsilon,0+\phi_\epsilon),\\ \Rightarrow\ & V'''(M_c,\phi)V'''(M_c,0)>0 \quad \forall \quad \phi\in(-\phi_\epsilon,\phi_\epsilon),\\ \Rightarrow\ & V'''(M_c,\phi)V'''(M_c,0)>0 \quad \forall \quad -\phi_\epsilon<\phi<\phi_\epsilon. \end{aligned} \tag{3.1}$$

□

THEOREM 2. *If $V''(M_c,0)=0$ and $V'''(M_c,0)\neq 0$, then there exists a strictly positive real number ϕ_ϵ (>0) such that $V'''(M_c,0)V''(M_c,\phi)>0$ for all $0<\phi<\phi_\epsilon$ and $V'''(M_c,0)V''(M_c,\phi)<0$ for all $-\phi_\epsilon<\phi<0$.*

Proof. From Theorem 1, we see that if $V'''(M_c,0)\neq 0$, then there exists a strictly positive real number ϕ_ϵ (>0) such that

$$V'''(M_c,\phi)V'''(M_c,0)>0 \quad \forall \quad -\phi_\epsilon<\phi<\phi_\epsilon. \tag{3.2}$$

As $V(M_c,\phi)$ and its derivatives of all orders with respect to ϕ exist finitely, $f(\phi)=V''(M_c,\phi)$ is a continuously derivable function of $\phi\in\Phi$. Now, for any $\phi(\neq 0)\in(-\phi_\epsilon,\phi_\epsilon)$, we have

(a) the real-valued function $f(\phi)$ is well defined in the closed interval $[\min\{0,\phi\},\max\{0,\phi\}]$,

(b) the real-valued function $f(\phi)$ is continuous in the closed interval $[\min\{0,\phi\},\max\{0,\phi\}]$, and

(c) the real-valued function $f(\phi)$ is derivable in the open interval $(\min\{0,\phi\},\max\{0,\phi\})$.

Therefore, from the Lagrange's mean-value theorem of differential calculus, we can conclude that there exists a $\phi_1\in(\min\{0,\phi\},\max\{0,\phi\})$ such that

$$\begin{aligned} & \frac{f(\max\{0,\phi\})-f(\min\{0,\phi\})}{\max\{0,\phi\}-\min\{0,\phi\}}=f'(\phi_1),\\ \Rightarrow\ & \frac{f(\phi)-f(0)}{\phi-0}=\frac{f(0)-f(\phi)}{0-\phi}=f'(\phi_1). \end{aligned} \tag{3.3}$$

As $f(0)=V''(M_c,0)=0$ and $\phi\neq 0$, (3.3) assumes the following form:

$$\begin{aligned} & \frac{V''(M_c,\phi)}{\phi}=V'''(M_c,\phi_1),\\ \Rightarrow\ & V''(M_c,\phi)=\phi V'''(M_c,\phi_1),\\ \Rightarrow\ & V'''(M_c,0)V''(M_c,\phi)=\phi V'''(M_c,0)V'''(M_c,\phi_1). \end{aligned} \tag{3.4}$$

Suppose that $\phi<0$, then $\min\{0,\phi\}=\phi$ and $\max\{0,\phi\}=0$, and consequently $\min\{0,\phi\}<\phi_1<\max\{0,\phi\}\Rightarrow\phi<\phi_1<0$, but since $\phi(\neq 0)\in(-\phi_\epsilon,\phi_\epsilon)$, we have $-\phi_\epsilon<\phi<\phi_\epsilon\Rightarrow-\phi_\epsilon<\phi<\phi_1<0<\phi_\epsilon\Rightarrow-\phi_\epsilon<\phi_1<\phi_\epsilon\Rightarrow\phi_1\in(-\phi_\epsilon,\phi_\epsilon)$.

Next, suppose that $\phi>0$, then $\min\{0,\phi\}=0$ and $\max\{0,\phi\}=\phi$, and consequently $\min\{0,\phi\}<\phi_1<\max\{0,\phi\}\Rightarrow 0<\phi_1<\phi$, but since $\phi(\neq 0)\in(-\phi_\epsilon,\phi_\epsilon)$, we have $-\phi_\epsilon<\phi<\phi_\epsilon\Rightarrow-\phi_\epsilon<0<\phi_1<\phi<\phi_\epsilon\Rightarrow-\phi_\epsilon<\phi_1<\phi_\epsilon\Rightarrow\phi_1\in(-\phi_\epsilon,\phi_\epsilon)$.

Therefore, in any case ($\phi<0$ or $\phi>0$), we always have $\phi_1\in(-\phi_\epsilon,\phi_\epsilon)$, and consequently from (3.2) we get

$$V'''(M_c,\phi_1)V'''(M_c,0)>0. \tag{3.5}$$

From (3.5), we get

$$\phi V'''(M_c,\phi_1)V'''(M_c,0)<0 \quad \forall \quad -\phi_\epsilon<\phi<0, \tag{3.6}$$

$$\phi V'''(M_c,\phi_1)V'''(M_c,0)>0 \quad \forall \quad 0<\phi<\phi_\epsilon. \tag{3.7}$$

From (3.6) and (3.7), using (3.4), we get

$$V'''(M_c,0)V''(M_c,\phi)<0 \quad \forall \quad -\phi_\epsilon<\phi<0, \tag{3.8}$$

$$V'''(M_c,0)V''(M_c,\phi)>0 \quad \forall \quad 0<\phi<\phi_\epsilon, \tag{3.9}$$

and consequently Theorem 2 is proved. □

Next, to confirm the existence of the solitary wave and/or double layer solution of the energy integral (1.1) at $M=M_c$, we have proved the following theorems.

THEOREM 6. *If $V(M,0)=V'(M,0)=V''(M_c,0)=0$, $V'''(M_c,0)<0$, $\partial V/\partial M<0$ for all $M>0$ and for all $\phi>0$, and if there exists at least one value M_0 of M such that the system supportsPPSWs for all $M_c<M<M_0$, then there exist either a PPSW or a PPDL at $M=M_c$.*

Proof. From Theorem 3, we have seen that if $V(M,0)=V'(M,0)=V''(M_c,0)=0$ and $V'''(M_c,0)<0$, there may exist either a solitary wave or a double layer in the positive potential side at $M=M_c$ but there does not exist any solitary wave or double layer in the negative potential side at $M=M_c$. Now with the help of IFT we shall confirm the existence of either a solitary wave or a double layer in the positive potential side at $M=M_c$. Let $M_c<M_1<M_0$, then according to the condition of the present theorem, there exists a solitary wave in the positive potential side at $M=M_1$. Let the amplitude of this solitary wave is $\phi_1(>0)$, then from the conditions for existence of PPSW, we have

SM1-1: $V(M_1,\phi_1)=0$,

SM1-2: $\left.\frac{\partial V}{\partial\phi}\right|_{(M_1,\phi_1)}>0$,

SM1-3: $V(M_1,\phi)<0$ for all $0<\phi<\phi_1$.

Again, as $\partial V/\partial\phi$ is a continuous function of (M,ϕ), we get

SM1-4: $\frac{\partial V}{\partial\phi}$ is continuous on a rectangular neighborhood of (M_1,ϕ_1).

Therefore, **SM1-1**, **SM1-4**, and **SM1-2** are, respectively, same as the conditions **A1**, **A2**, and **A3** of IFT, and consequently we can apply IFT in a rectangular neighborhood of the point (M_1,ϕ_1). Therefore, from IFT, we can conclude that there exists a rectangular neighborhood

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$N(M_1,\phi_1;h_1,k_1) = N(M_1,h_1)\times N(\phi_1,k_1)$ of (M_1,ϕ_1) such that corresponding to every $M \in N(M_1,h_1)$, $\phi(\in N(\phi_1,k_1))$ can be expressed uniquely as a function of M, for which the following conditions are simultaneously satisfied:

IFT-C1: $\phi(M_1) = \phi_1$,

IFT-C2: $V(M,\phi(M)) = 0$ for all $M \in N(M_1,h_1)$,

IFT-C3: $\frac{d\phi}{dM} = -(\frac{\partial V}{\partial M} \div \frac{\partial V}{\partial \phi})$ for all $(M,\phi) \in N(M_1,\phi_1;h_1,k_1) = N(M_1,h_1)\times N(\phi_1,k_1)$.

Again from **SM1-2**, using Theorem **BTh2**, we have strictly positive real numbers $h_2(>0)$ and $k_2(>0)$ such that

SM1-5: $\frac{\partial V}{\partial \phi} > 0$ for all $(M,\phi) \in N(M_1,\phi_1;h_2,k_2) = N(M_1,h_2)\times N(\phi_1,k_2)$.

Again from **IFT-C3**, we see that ϕ is a derivable function of M for $M \in (M_1-h_1, M_1+h_1)$, and consequently it is continuous in (M_1-h_1, M_1+h_1). As ϕ is continuous in (M_1-h_1, M_1+h_1) and $\phi(M_1) = \phi_1 > 0$, using Theorem **BTh1**, we have a strictly positive real number $h_3(>0)$ such that

SM1-6: $\phi(M) > 0$ for all $M \in N(M_1,h_3)$.

So if we take $h = \min\{h_1,h_2,h_3\}$ and $k = \min\{k_1,k_2\}$, then $N(M_1,h) \subseteq N(M_1,h_i)$ for all $i = 1,2,3$, and $N(\phi_1,k) \subseteq N(\phi_1,k_j)$ for all $j = 1,2$; consequently, $N(M_1,\phi_1;h,k) = N(M_1,h)\times N(\phi_1,k) \subseteq N(M_1,h_i)\times N(\phi_1,k_j) = N(M_1,\phi_1;h_i,k_j)$ for all $i = 1,2,3$ and $j = 1,2$. Therefore, we can also apply IFT in the rectangular neighborhood $N(M_1,\phi_1;h,k) = N(M_1,h)\times N(\phi_1,k)$, and consequently with respect to the rectangular neighborhood $N(M_1,\phi_1;h,k)$, **IFT-C1**, **IFT-C2**, **IFT-C3**, **SM1-5**, and **SM1-6** can be put in the following form:

mSM1-1: $\phi(M_1) = \phi_1$,

mSM1-2: $V(M,\phi(M)) = 0$ for all $M \in N(M_1,h)$,

mSM1-3: $\frac{d\phi}{dM} = -(\frac{\partial V}{\partial M} \div \frac{\partial V}{\partial \phi})$ for all $(M,\phi) \in N(M_1,\phi_1;h,k) = N(M_1,h)\times N(\phi_1,k)$,

mSM1-4: $\frac{\partial V}{\partial \phi} > 0$ for all $(M,\phi) \in N(M_1,\phi_1;h,k) = N(M_1,h)\times N(\phi_1,k)$,

mSM1-5: $\phi(M) > 0$ for all $M \in N(M_1,h)$.

Now choose two strictly positive real numbers ϵ_1 and ϵ such that $M_1 = M_c+\epsilon_1$ and $N(M_c,\epsilon) \subseteq N(M_1,h)$. These two conditions are simultaneously satisfied if we choose $0 < \epsilon_1 < h$ and $0 < \epsilon < h-\epsilon_1$. As $\phi(M) \in N(\phi_1,k)$ for every $M \in N(M_1,h)$ and as $M_c \in N(M_1,h)$, then we must have $\phi(M_c) \in N(\phi_1,k) \Rightarrow \phi_c \in N(\phi_1,k)$, where we set $\phi_c = \phi(M_c)$. Now choose two strictly positive real numbers δ_1 and δ such that $\phi_1 = \phi_c+\delta_1$ and $N(\phi_c,\delta) \subseteq N(\phi_1,k)$. These two conditions are simultaneously satisfied if we choose $0 < \delta_1 < k$ and $0 < \delta < k-\delta_1$.

Therefore, $N(M_c,\epsilon) \subseteq N(M_1,h)$ and $N(\phi_c,\delta) \subseteq N(\phi_1,k) \Rightarrow N(M_c,\epsilon)\times N(\phi_c,\delta) \subseteq N(M_1,h)\times N(\phi_1,k) \Rightarrow N(M_c,\phi_c;\epsilon,\delta) \subseteq N(M_1,\phi_1;h,k)$, and consequently we can also apply IFT in the rectangular neighborhood $N(M_c,\phi_c;\epsilon,\delta) \subseteq N(M_1,\phi_1;h,k)$. So with respect to the rectangular neighborhood, $N(M_c,\phi_c;\epsilon,\delta)$, **IFT-C1**, **IFT-C2**, **IFT-C3**, **mSM1-4**, and **mSM1-5** can be put in the following form:

fmSM1-1: $\phi(M_c) = \phi_c$,

fmSM1-2: $V(M,\phi(M)) = 0$ for all $M \in N(M_c,\epsilon)$,

fmSM1-3: $\frac{d\phi}{dM} = -(\frac{\partial V}{\partial M} \div \frac{\partial V}{\partial \phi})$ for all $(M,\phi) \in N(M_c,\phi_c;\epsilon,\delta) = N(M_c,\epsilon)\times N(\phi_c,\delta)$,

fmSM1-4: $\frac{\partial V}{\partial \phi} > 0$ for all $(M,\phi) \in N(M_c,\phi_c;\epsilon,\delta) = N(M_c,\epsilon)\times N(\phi_c,\delta)$,

fmSM1-5: $\phi(M) > 0$ for all $M \in N(M_c,\epsilon)$.

As $M_c \in N(M_c,\epsilon)$ and $\phi_c \in N(\phi_c,\delta)$, from **fmSM1-1**, **fmSM1-2**, **fmSM1-4**, and **fmSM1-5**, we get the following conditions:

SMc-1: $V(M_c,\phi_c) = 0$, where $\phi_c = \phi(M_c)$,

SMc-2: $\left.\frac{\partial V}{\partial \phi}\right|_{(M_c,\phi_c)} > 0$,

SMc-3: $\phi_c = \phi(M_c) > 0$.

It is important to note that if we take $M \in (M_c-\epsilon, M_c)$, then also the above three conditions are simultaneously satisfied but $M < M_c \Leftrightarrow V''(M,0) > 0$, and consequently $\phi = 0$ is a position of stable equilibrium. So, there is no question of existence of any solitary wave or double layer solution of the energy integral (1.1) for all $M \in (M_c-\epsilon, M_c)$.

Again, from the condition of the present theorem, we have $\partial V/\partial M < 0$ for all $M > 0$ and for all $\phi > 0$. As our discussion is restricted to positive potential side ($\phi > 0$) only, from **fmSM1-3** and **fmSM1-4**, we get

Mc: $\frac{d\phi}{dM} > 0$ for all $M \in N(M_c,\epsilon)$,

which shows that ϕ is a strictly increasing function of M for all $M_c-\epsilon < M < M_c+\epsilon$.

Now we shall prove that

SMc-4: $V(M_c,\phi) \leqslant 0$ for all $0 < \phi < \phi_c$,

and the following analysis has been developed to prove **SMc-4**.

Consider a value M_2 of M such that $M_c < M_2 < M_c+\epsilon$, then there exists a PPSW of amplitude ϕ_2, and consequently we have

SM2-1: $V(M_2,\phi_2) = 0$,

SM2-2: $\left.\frac{\partial V}{\partial \phi}\right|_{(M_2,\phi_2)} > 0$,

SM2-3: $V(M_2,\phi) < 0$ for all $0 < \phi < \phi_2$.

As $M_c < M_2 < M_c+\epsilon \Rightarrow M_c-\epsilon < M_2 < M_c+\epsilon$, we have $\phi_2 = \phi(M_2)$. Again, as ϕ is an increasing function of M for all $M_c-\epsilon < M < M_c+\epsilon$, and $M_c, M_2 \in N(M_c,\epsilon)$ with $M_c < M_2$, we have $\phi(M_c) < \phi(M_2) \Rightarrow \phi_c < \phi_2$. we can write **SM2-3** as follows:

mSM2-3: $V(M_2,\phi) < 0$ for all $0 < \phi < \phi_c < \phi_2$.

If possible, let there exists a value ϕ_p of ϕ such that $0 < \phi_p < \phi_c$ and

Q1: $V(M_c,\phi_p) > 0$.

Again from **mSM2-3**, as $0 < \phi_p < \phi_c < \phi_2$, we get

Q2: $V(M_2,\phi_p) < 0$.

Considering $V(M,\phi_p)$ as a function of M only, from **Q1**, **Q2**, and the **Bolzano's theorem**, we can conclude that there exists a value M_3 of M such that

$$V(M_3,\phi_p) = 0, \tag{3.10}$$

where $M_c < M_3 < M_2 < M_c+\epsilon$. But as $M_c < M_3 < M_2 < M_c+\epsilon$, there exists a PPSW at $M = M_3$ of amplitude $\phi_3 > 0$, and consequently we have

SM3-1: $V(M_3,\phi_3) = 0$,

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SM3-2: $\left.\frac{\partial V}{\partial \phi}\right|_{(M_3,\phi_3)} > 0$,

SM3-3: $V(M_3,\phi) < 0$ for all $0 < \phi < \phi_3$.

As $M_c < M_3 < M_c + \epsilon \Rightarrow M_c - \epsilon < M_3 < M_c + \epsilon$, we have $\phi_3 = \phi(M_3)$. Again, as ϕ is an increasing function of M for $M_c - \epsilon < M < M_c + \epsilon$, and $M_c, M_3 \in N(M_c, \epsilon)$ with $M_c < M_3$, we have $\phi(M_c) < \phi(M_3) \Rightarrow \phi_c < \phi_3$. So we can write **SM3-3** as follows:

mSM3-3: $V(M_3,\phi) < 0$ for all $0 < \phi < \phi_c < \phi_3$,

Again, from **mSM3-3**, as $0 < \phi_p < \phi_c < \phi_3$, we get

Q3: $V(M_3,\phi_p) < 0$,

which contradicts (3.10), i.e., $V(M_3,\phi_p) = 0$. Therefore, our supposition is wrong, and consequently there does not exist any $\phi_p \in (0,\phi_c)$ such that $V(M_c,\phi_p) > 0$, i.e., $V(M_c,\phi_p) \not> 0$ and consequently we get

mSMc-4: $V(M_c,\phi) \leqslant 0$ for all $0 < \phi < \phi_c$.

Now **mSMc-4** suggests the following two possible cases:

Case 1. There does not exist any $\phi \in (0,\phi_c)$ such that $V(M_c,\phi) = 0$, i.e., $V(M_c,\phi) < 0$ for all $0 < \phi < \phi_c$.

Case 2. There exists at least one $\phi \in (0,\phi_c)$ such that $V(M_c,\phi) = 0$.

Combining **Case 1** with **SMc-1**, **SMc-2**, and **SMc-3**, we get

(1) $V(M_c,\phi_c) = 0$, where $\phi_c = \phi(M_c)$,

(2) $\left.\frac{\partial V}{\partial \phi}\right|_{(M_c,\phi_c)} > 0$,

(3) $\phi_c = \phi(M_c) > 0$,

(4) $V(M_c,\phi) < 0$ for all $0 < \phi < \phi_c$.

These four conditions confirm the existence of the PPSW solution of the energy integral (1.1) at $M = M_c$.

For **Case 2**, there exists at least one $\phi \in (0,\phi_c)$ such that $V(M_c,\phi) = 0$. Let ϕ_{p1} be the smallest strictly positive root of the equation $V(M_c,\phi) = 0$ such that $\phi_{p1} \in (0,\phi_c) (\Leftrightarrow 0 < \phi_{p1} < \phi_c)$ and consequently for this case we have the following subcases:

Subcase 1. $V(M_c,\phi) < 0$ for all $0 < \phi < \phi_{p1}$ and $V(M_c,\phi) = 0$ for all $\phi \in [\phi_{p1},\phi_c]$ (see Fig. 1(a)).

Subcase 2. $V(M_c,\phi) < 0$ for all $0 < \phi < \phi_{p1}$ and $V(M_c,\phi) = 0$ for all $\phi \in [\phi_{p1},\phi_{p2}]$ with $\phi_{p2} \leqslant \phi_c$ (see Fig. 1(b)).

Subcase 3. $V(M_c,\phi) < 0$ for all $0 < \phi < \phi_{p1}$ and $V(M_c,\phi) < 0$ for all $\phi \in (\phi_{p1} - \epsilon_{p1}, \phi_{p1})$ as well as $\phi \in (\phi_{p1}, \phi_{p1} + \epsilon_{p1})$ along with $V(M_c,\phi_{p1}) = 0$ for some $\epsilon_{p1} > 0$ (see Fig. 1(c)).

As $V(M_c,\phi)$ is a continuously differentiable function of ϕ for all $\phi \in \Phi$ and for **Subcase 1**, it is simple to check that $V(M_c,\phi)$ is not differentiable at $\phi = \phi_{p1}$, **Subcase 1** is not possible. With the help of same argument, it is simple to check that **Subcase 2** is not possible when $\phi_{p1} < \phi_{p2}$. When $\phi_{p1} = \phi_{p2}$, then either we have **Subcase 3** or we have two different branches of the curve $V(M_c,\phi)$ originated from $\phi = \phi_{p1}$. The later case is not possible because $V(M_c,\phi)$ is not differentiable at $\phi = \phi_{p1}$. So, **Subcase 3** is the only possible case of **Case 2** and for this subcase we have the following conclusions:

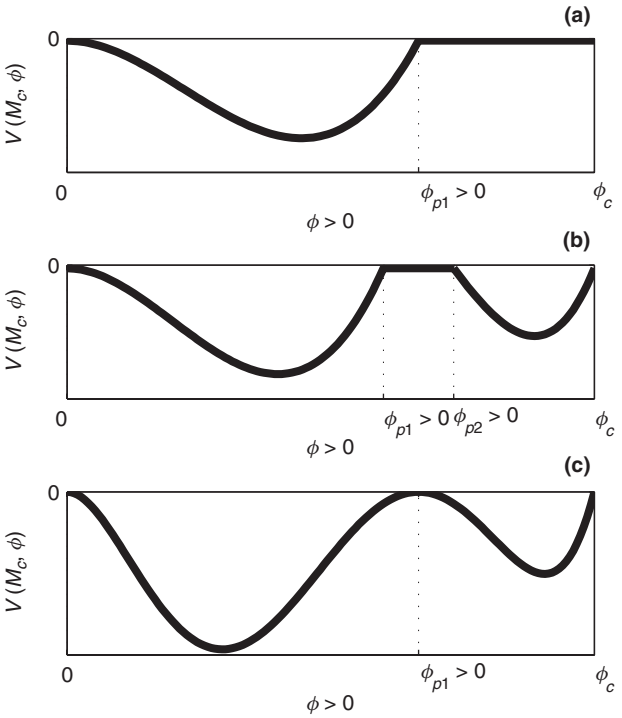


Figure 1. (a) Corresponds to the subcase 1 of case 2 of Theorem 6; (b) corresponds to the subcase 2 of case 2 of Theorem 6; and (c) corresponds to the subcase 3 of case 2 of Theorem 6.

Subcase 3. In this case, as $V(M_c,\phi)$ is a continuously differentiable function of ϕ, $V(M_c,\phi)$ not only attains its maximum value at $\phi = \phi_{p1}$, $V(M_c,\phi)$ is convex for all $\phi \in (\phi_{p1} - \epsilon_{p1}, \phi_{p1} + \epsilon_{p1})$. Therefore, from the elementary theory of convexity of a function, we have

$$\left.\frac{\partial V}{\partial \phi}\right|_{(M_c,\phi_{p1})} = 0 \quad \text{and} \quad \left.\frac{\partial^2 V}{\partial \phi^2}\right|_{(M_c,\phi_{p1})} < 0. \qquad (3.11)$$

Combining these two conditions along with the first condition of **Subcase 3**, i.e., $V(M_c,\phi) < 0$ for all $0 < \phi < \phi_{p1}$, we get

(1) $V(M_c,\phi_{p1}) = 0$,

(2) $\left.\frac{\partial V}{\partial \phi}\right|_{(M_c,\phi_{p1})} = 0$,

(3) $\left.\frac{\partial^2 V}{\partial \phi^2}\right|_{(M_c,\phi_{p1})} < 0$,

(4) $\phi_{p1} > 0$,

(5) $V(M_c,\phi) < 0$ for all $0 < \phi < \phi_{p1}$.

These five conditions confirm the existence of PPDL solution of the energy integral (1.1) at $M = M_c$. Therefore, under the conditions of the present theorem, one can get either a PPSW or a PPDL as a solution of the energy integral (1.1) at $M = M_c$.

THEOREM 7. *If* $V(M,0) = V'(M,0) = V''(M_c,0) = 0$, $V'''(M_c,0) > 0$, $\partial V/\partial M < 0$ *for all* $M > 0$ *and for all* $\phi < 0$, *and if there exists at least one value* M_0 *of* M *such that the system supports NPSWs for all* $M_c < M < M_0$, *then there exist either an NPSW or an NPDL at* $M = M_c$.

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Proof. Same as Theorem 6 with a slight modification for the case of negative potential side. □

THEOREM 8. *If $V(M,0) = V'(M,0) = V''(M_c,0) = 0$, $V'''(M_c,0) < 0$, $\partial V/\partial M < 0$ for all $M > 0$ and for all $\phi > 0$, and if the system supports only NPSWs for $M > M_c$, then there does not exist any PPSW at $M = M_c$.*

Proof. From Theorem 3, we have seen that if $V(M,0) = V'(M,0) = V''(M_c,0) = 0$ and $V'''(M_c,0) < 0$, there may exist solitary wave in the positive potential side at $M = M_c$ but there does not exist any solitary wave in the negative potential side at $M = M_c$. If possible, let there exists a PPSW at $M = M_c$ and for $M > M_c$, the system supports only NPSWs. Therefore, if ϕ_c is the amplitude of PPSW at $M = M_c$, the following conditions are satisfied:

Mc1: $V(M_c,\phi_c) = 0$,

Mc2: $\phi_c > 0$,

Mc3: $\left.\frac{\partial V}{\partial \phi}\right|_{(M_c,\phi_c)} > 0$,

Mc4: $V(M_c,\phi) < 0$ for all $0 < \phi < \phi_c$.

As $\partial V/\partial\phi$ is a continuous function in its domain of definition, we have

Mc5: $\frac{\partial V}{\partial \phi}$ is continuous in some rectangular neighborhood of (M_c,ϕ_c).

Therefore, all the assumptions, viz., **Mc1**, **Mc5**, and **Mc3**, of **IFT** at (M_c,ϕ_c) are satisfied by the function $V(M,\phi)$. Therefore, from IFT, we can conclude that there exists a rectangular neighborhood $N(M_c,\phi_c;\delta_1,\delta_2) = N(M_c,\delta_1) \times N(\phi_c,\delta_2)$ of (M_c,ϕ_c) such that corresponding to every $M \in N(M_c,\delta_1)$, $\phi(\in N(\phi_c,\delta_2))$ can be expressed uniquely as a function of M, for which the following conditions are simultaneously satisfied:

IFT Mc1: $\phi(M_c) = \phi_c$,

IFT Mc2: $V(M,\phi(M)) = 0$ for all $M \in N(M_c,\delta_1)$,

IFT Mc3: $\frac{d\phi}{dM} = -(\frac{\partial V}{\partial M} \div \frac{\partial V}{\partial \phi})$ for all $(M,\phi) \in N(M_c,\phi_c;\delta_1,\delta_2) = N(M_c,\delta_1) \times N(\phi_c,\delta_2)$.

Again from **Mc3**, using Theorem **BTh2**, we have two strictly positive real numbers $\delta_3(>0)$ and $\delta_4(>0)$ such that

Mc6: $\frac{\partial V}{\partial \phi} > 0$ for all $(M,\phi) \in N(M_c,\phi_c;\delta_3,\delta_4) = N(M_c,\delta_3) \times N(\phi_c,\delta_4)$.

So, if we take $\delta = \min\{\delta_1,\delta_3\}$ and $\rho = \min\{\delta_2,\delta_4\}$, then $N(M_c,\delta) \subseteq N(M_c,\delta_i)$ for all $i = 1,3$, and $N(\phi_c,\rho) \subseteq N(\phi_c,\delta_j)$ for all $j = 2,4$; consequently, $N(M_c,\phi_c;\delta,\rho) = N(M_c,\delta) \times N(\phi_c,\rho) \subseteq N(M_c,\delta_i) \times N(\phi_c,\delta_j) = N(M_c,\phi_c;\delta_i,\delta_j)$ for all $i = 1,3$ and $j = 2,4$. Therefore, we can also apply IFT in the rectangular neighborhood, $N(M_c,\phi_c;\delta,\rho) = N(M_c,\delta) \times N(\phi_c,\rho)$, and consequently, **IFT Mc1**, **IFT Mc2**, **IFT Mc3**, and **Mc6** can be rewritten as follows:

mMc1: $\phi(M_c) = \phi_c$,

mMc2: $V(M,\phi(M)) = 0$ for all $M \in N(M_c,\delta)$,

mMc3: $\frac{d\phi}{dM} = -(\frac{\partial V}{\partial M} \div \frac{\partial V}{\partial \phi})$ for all $(M,\phi) \in N(M_c,\phi_c;\delta,\rho) = N(M_c,\delta) \times N(\phi_c,\rho)$,

mMc6: $\frac{\partial V}{\partial \phi} > 0$ for all $(M,\phi) \in N(M_c,\phi_c;\delta,\rho) = N(M_c,\delta) \times N(\phi_c,\rho)$.

Again, from the condition of the present theorem, we have $\partial V/\partial M < 0$ for all $M > 0$ and $\phi > 0$. As our discussion is restricted to positive potential side ($\phi > 0$) only, from **mMc3** and **mMc6**, we get $\phi(M)$, which is an increasing function of M for all $M_c - \delta < M < M_c + \delta$. Consider a value M_1 of M such that $M_c < M_1 < M_c + \delta \Rightarrow M_c - \delta < M_c < M_1 < M_c + \delta \Rightarrow M_c - \delta < M_1 < M_c + \delta \Rightarrow M_1 \in N(M_c,\delta)$, and consequently from **mMc2**, the condition that the function $\phi(M)$ is an increasing function of M for all $M_c - \delta < M < M_c + \delta$ and **mMc6**, we get the following three conditions:

M1-1: $V(M_1,\phi_1) = 0$, with $\phi_1 = \phi(M_1)$,

M1-2: $M_c - \delta < M_c < M_1 < M_c + \delta \Rightarrow M_c, M_1 \in N(M_c,\delta)$ with $M_c < M_1 \Rightarrow \phi(M_c) < \phi(M_1) \Rightarrow \phi_c < \phi_1 \Rightarrow \phi_1 > \phi_c > 0$, and

M1-3: $\left.\frac{\partial V}{\partial \phi}\right|_{(M_1,\phi_1)} > 0$.

(It is important to note that if we take $M \in (M_c-\delta, M_c)$, then also the above three conditions are simultaneously satisfied but $M < M_c \Leftrightarrow V''(M,0) > 0$, and consequently $\phi = 0$ is a position of stable equilibrium. So there is no question of existence of any solitary wave solution of the energy integral (1.1) for all $M \in (M_c - \delta, M_c)$.)

Now if $V(M_1,\phi) < 0$ for all $0 < \phi < \phi_1$, then from **M1-1**, **M1-2**, and **M1-3**, we can conclude that there exists a PPSW at $M = M_1 > M_c$ but this is impossible as the system supports only NPSWs for $M > M_c$. Therefore, there exists at least one value ϕ_2 of ϕ such that

$$V(M_1,\phi_2) \geqslant 0 \quad \text{for} \quad 0 < \phi_2 < \phi_1. \tag{3.12}$$

Now we shall prove that actually ϕ_2 is restricted by the following inequality: $\phi_c < \phi_2 < \phi_1$.

Now from the condition $\partial V/\partial M < 0$ for all $M > 0$ and for all $\phi > 0$, we have $V(M,\phi)$ is a strictly decreasing function of M for all $M > 0$ and for any given value of $\phi > 0$. As $M_1 > M_c$, we get

M1-4: $V(M_1,\phi) < V(M_c,\phi)$ for all $\phi > 0$.

Again, from **Mc4**, we get $V(M_c,\phi) < 0$ for all $0 < \phi < \phi_c \Rightarrow V(M_1,\phi) < V(M_c,\phi) < 0$ for all $0 < \phi < \phi_c$, where we have used the inequality $(V(M_1,\phi) < V(M_c,\phi)$ for all $\phi > 0)$ as given in **M1-4**. Therefore, we have

M1-5: $V(M_1,\phi) < 0$ for all $0 < \phi < \phi_c$.

Again, as $\phi_c > 0$ and $V(M_c,\phi_c) = 0$, putting $\phi = \phi_c$ in **M1-4**, we get

M1-6: $V(M_1,\phi_c) < V(M_c,\phi_c) = 0 \Rightarrow V(M_1,\phi_c) < 0$.

Combining **M1-5** and **M1-6**, we get

M1-7: $V(M_1,\phi) < 0$ for all $0 < \phi \leqslant \phi_c$.

Therefore, from **M1-7**, we can conclude that $\phi_c < \phi_2 < \phi_1$. Now we shall prove that there exists a value M_2 of M such that $\phi_2 = \phi(M_2)$ for $M_c < M_2 < M_1$.

Let us define the function $\psi(M) = \phi(M) - \phi_2$ for all $M \in [M_c,M_1]$. Now as $\phi(M)$ is a continuously differentiable function of M for all $M \in N(M_c,\delta)$, it is automatically continuous for all $M \in N(M_c,\delta)$. Again, as $[M_c,M_1] \subset N(M_c,\delta)$, $\phi(M)$ is also continuous for all $M \in [M_c,M_1]$, and consequently $\psi(M)$ is continuous for all $M \in [M_c,M_1]$. Now as $\phi_c = \phi(M_c)$ and $\phi_1 = \phi(M_1)$, the condition $\phi_c < \phi_2 < \phi_1$ gives $\phi(M_c) < \phi_2 < \phi(M_1)$

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$\Leftrightarrow \phi(M_c) - \phi_2 < 0$ and $\phi(M_1) - \phi_2 > 0 \Leftrightarrow \psi(M_c) < 0$ and $\psi(M_1) > 0$.

Therefore, we have a continuous function $\psi(M) : [M_c, M_1] \longmapsto \Re$ such that $\psi(M_c) < 0$ and $\psi(M_1) > 0$, and consequently by the Bolzano's theorem, we can conclude that there exists a value M_2 of M such that $\psi(M_2) = 0 \Leftrightarrow \phi(M_2) = \phi_2$ for $M_c < M_2 < M_1$. Now as $V(M, \phi)$ is a strictly decreasing function of $M > 0$ for any fixed value of $\phi > 0$, and consequently as $M_2 < M_1$, we have $V(M_2, \phi) > V(M_1, \phi)$ for any given value of $\phi > 0$. If we take $\phi = \phi_2$, we get $V(M_2, \phi_2) > V(M_1, \phi_2) \Leftrightarrow V(M_1, \phi_2) < V(M_2, \phi_2) = V(M_2, \phi(M_2)) = 0$ because $M_c < M_2 < M_1 < M_c + \delta \Rightarrow M_2 \in N(M_c, \delta) \Rightarrow V(M_2, \phi(M_2)) = 0$, where we have used the condition **mMc2** to get $V(M_2, \phi(M_2)) = 0$. So we have $V(M_1, \phi_2) < 0$, which again contradicts (3.12), i.e., $V(M_1, \phi_2) \geqslant 0$. Thus, our original assumption is wrong, and consequently theorem is proved, i.e., if the system supports only NPSWs for $M > M_c$, then there does not exist any PPSW at $M = M_c$, provided the conditions of the present theorem are fulfilled. □

THEOREM 9. *If $V(M,0) = V'(M,0) = V''(M_c,0) = 0$, $V'''(M_c,0) > 0$, $\partial V/\partial M < 0$ for all $M > 0$ and for all $\phi < 0$, and if the system supports only PPSWs for $M > M_c$, then there does not exist any NPSW at $M = M_c$.*

Proof. Same as Theorem 8 with a slight modification for the negative potential side. □

THEOREM 10. *If $V(M,0) = V'(M,0) = V''(M_c,0) = 0$, $V'''(M_c,0) \neq 0$, $\partial V/\partial M < 0$ for all $M > 0$ and for all $\phi \neq 0$, and if there exists at least one value M_0 of M such that the system supports both positive and NPSWs for all $M_c < M < M_0$, i.e., if the system supports the coexistence of both PPSWs and NPSWs for all $M_c < M < M_0$, then if $V'''(M_c,0) < 0$, there exist either a PPSW or a PPDL at $M = M_c$ whereas if $V'''(M_c,0) > 0$ there exist either an NPSW or an NPDL at $M = M_c$.*

Proof. Follows from Theorems 6 and 7. □

From Theorem 6 –10, it is simple to see that the critical Mach number alone cannot determine the existence of particular nonlinear structure and its polarity. This is not even the motivation of our present manuscript. However, our intension is to determine the conditions under which nonlinear structure exists at the critical Mach number. The existence of nonlinear structure at the critical Mach number and its polarity is determined by the nature of the solitary structures for at least one right neighbourhood of the point $M = M_c$ together with the sign of the function $V'''(M_c, 0)$.

Except Theorems 1 and 2, the proof of rest of the theorems (Theorems 6– 10) carry physical sense inherently, since the proofs are based on the physical interpretation of the energy integral. For example, Theorem 6 is proved with the help of Theorem 3, while Theorem 3 is proved on the basis of physical interpretation of the energy integral. One can get new result behind the nonlinear structures at $M = M_c$ by considering any specific problem, only when these theorems have been used on the qualitatively different solution spaces or compositional parameter spaces showing the existence region of different types of solitary structures for $M > M_c$. For this purpose, we have considered the same problem of DA wave in non-thermal plasma of Das et al. (2009) in Section 4.

So, without going through all qualitatively different solution spaces of the energy integral by considering the entire range of parameters involved in the system, it is not possible to make a systematic investigation of the solitary structures at $M = M_c$. To construct the solution space or compositional parameter space showing the existence region of different types of solitary structures for $M > M_c$, it is not necessary to know the existence of solitary structure at $M = M_c$. For $M > M_c$, using the physical interpretation, one can easily set up a numerical scheme to construct the compositional parameter space with respect to any parameter of the system (keeping arbitrary fixed values of other parameters in their respective physically admissible range).

4. Application

Here we consider the specific problem of Das et al. (2009) to apply the analytical results (Theorems 3–10) as presented in this paper. In Das et al. (2009), a computational scheme has been developed to study the arbitrary amplitude Dust Acoustic Solitary (DAS) waves and double layers in a non-thermal plasma consisting of negatively charged dust grains, non-thermal ions, and isothermal electrons, including the effect of dust temperature.

The following are the governing equations describing the nonlinear behavior of DA waves, propagating along the x-axis in a collisionless dusty plasma consisting of negatively charged warm dust grains, non-thermal ions, and isothermal electrons,

$$n_{d,t} + (n_d u_d)_x = 0, \tag{4.1}$$

$$m_d n_d (u_{d,t} + u_d u_{d,x}) = e Z_d n_d \phi_x - p_{d,x}, \tag{4.2}$$

$$p_{d,t} + u_d p_{d,x} + \gamma p_d u_{d,x} = 0, \tag{4.3}$$

$$\phi_{xx} = 4\pi e (Z_d n_d + n_e - n_i). \tag{4.4}$$

Here we have used the notation $\psi_{d,\,q}$ or $(\psi_d)_q$ for $\partial \psi_d/\partial q$ and n_d, n_i, n_e, u_d, p_d, ϕ, x, and t are, respectively, the dust particle number density, ion number density, electron number density, dust fluid velocity, dust fluid pressure, electrostatic potential, spatial variable, and time, $\gamma (= 3)$ is the adiabatic index, m_d is the mass of a dust particle, Z_d is the number of electrons residing on the dust grain surface, and e is the charge of an electron. Here the ions are assumed to be non-thermally distributed and the electrons are assumed to be isothermally distributed. The model of the velocity distribution function of the non-thermal electrons was

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considered for the first time by Cairns et al. (1995a) for the study of ion-acoustic solitary structures in the presence of the population of the fast energetic electrons together with a population of isothermal electrons. This type of velocity distribution is often termed as the Cairns distribution and was considered by many authors in various studies of different collective processes in plasmas and dusty plasmas (Cairns et al. 1995a,b, 1996; Mamun et al. 1996b; Mendoza-Briceño et al. 2000; Gill et al. 2003; Maharaj et al. 2004, 2006; Verheest and Pillay 2008; Das et al. 2009, 2010, 2012; Djebli and Marif 2009; Verheest 2010; Verheest and Hellberg 2010). The number density of non-thermal ions can be written following Cairns et al. (1995a,b) as

$$\frac{n_i}{n_{i0}} = \left[1 + \beta_1\left(\frac{\phi}{\Phi}\right) + \beta_1\left(\frac{\phi}{\Phi}\right)^2\right] exp\left[-\frac{\phi}{\Phi}\right], \quad (4.5)$$

where

$$\beta_1 = \frac{4\alpha_1}{1+3\alpha_1}\ , \text{ with } \alpha_1 \geqslant 0, \quad (4.6)$$

$$\Phi = \frac{K_B T_i}{e}, \quad (4.7)$$

n_{i0} is the unperturbed ion number density, K_B is the Boltzmann constant, and T_i is the average temperature of ions. It can be easily checked that $0 \leqslant \beta_1 < 4/3$. Here α_1 and consequently β_1 is a parameter that determines the proportion of fast energetic ions. However, we cannot take the whole region of β_1 ($0 \leqslant \beta_1 < 4/3$). Plotting the non-thermal velocity distribution of ions against its velocity (v) in phase space, it can be easily shown that the number of ions in phase space in the neighborhood of the point $v = 0$ decreases with increasing β_1 and the number is almost zero when $\beta_1 \to 4/3$. Therefore, for increasing β_1 distribution function develops wings, which become stronger as β_1 increases, and at the same time the center density in phase space drops, the latter as the result of the normalization of the area under the integral. Consequently, we should not take values of $\beta_1 > 4/7$, since that stage might stretch the credibility of the Cairns model too far (Verheest and Pillay 2008). So here we consider the effective range of β_1 as follows: $0 \leqslant \beta_1 \leqslant 0.571429 \approx 0.6$.

For isothermal Boltzmann distributed electrons, their number density can be written as

$$\frac{n_e}{n_{e0}} = \exp\left[\alpha\frac{\phi}{\Phi}\right], \text{ with } \alpha = \frac{T_i}{T_e}, \quad (4.8)$$

where n_{e0} is the unperturbed electron number density, and T_e is the average temperature of electrons.

The charge neutrality condition can be written as

$$\frac{Z_d n_{d0}}{n_{i0}} = 1 - \mu\ , \text{ with } \mu = \frac{n_{e0}}{n_{i0}}, \quad (4.9)$$

where n_{d0} is the unperturbed dust particle number density.

The linear dispersion relation of the DA wave of the present dusty plasma system can be written as

$$\frac{\omega}{k} = c_D\sqrt{\frac{1+3\sigma_d M_s^{-2}\lambda_D^2 k^2}{1+\lambda_D^2 k^2}}, \quad (4.10)$$

where

$$c_D = M_s c_d, \quad (4.11)$$

$$M_s = \sqrt{3\sigma_d + \frac{1}{(f-1)\alpha + f(1-\beta_1)}}, \quad (4.12)$$

$$\sigma_d = \frac{T_d}{Z_d T_i}\ ,\ f = \frac{1}{1-\mu}, \quad (4.13)$$

$$c_d = \sqrt{\frac{Z_d K_B T_i}{m_d}}, \quad (4.14)$$

$$\frac{1}{\lambda_D^2} = \frac{1}{\lambda_{De}^2} + \frac{1-\beta_1}{\lambda_{Di}^2}, \quad (4.15)$$

$$\lambda_{De}^2 = \frac{K_B T_e}{4\pi n_{e0} e^2}, \quad (4.16)$$

$$\lambda_{Di}^2 = \frac{K_B T_i}{4\pi n_{i0} e^2}. \quad (4.17)$$

With the help of (4.15), (4.16), (4.17), and (4.9), we get the following simplified expression for λ_D,

$$\lambda_D = \frac{\lambda_{Dd}}{\sqrt{(f-1)\alpha + f(1-\beta_1)}}, \quad (4.18)$$

$$\lambda_{Dd} = \sqrt{\frac{K_B T_i}{4\pi Z_d n_{d0} e^2}}. \quad (4.19)$$

Now for long-wave length plane wave perturbation, i.e., for $k \to 0$ (Dubinov 2009), from linear dispersion relation (4.10), we have,

$$\lim_{k\to 0}\frac{\omega}{k} = c_D \text{ and } \lim_{k\to 0}\frac{d\omega}{dk} = c_D,$$

and consequently the dispersion relation (4.10) shows that the DA speed of the present plasma system is c_D with λ_D as the Debye length. In the present problem, we have also used the usual dusty plasma approximations: $T_i \approx T_e$ and $n_{e0} \ll n_{i0}$.

The Sagdeev potential approach, which is valid to study the arbitrary amplitude solitary waves and double layers, has been employed. Four basic parameters of the system are μ, α, β_1, and σ_d, which are, respectively, the ratio of unperturbed number density of electrons to that of non-thermal ions, the ratio of average temperature of non-thermal ions to that of isothermal electrons, a parameter of the non-thermal distribution of ions, and the ratio of average temperature of dust particles to that of ions divided by the number of negative charges residing on a dust grain surface.

To study the arbitrary amplitude time independent DAS waves and double layers, we make all the dependent variables depend only on a single variable $\xi = x - Ut$,

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where U is independent of x and t. Thus, in the steady state, in terms of four basic parameters of the system, (4.1)–(4.4) can be written as

$$\frac{d}{d\xi}(-Un_d + n_d u_d) = 0, \tag{4.20}$$

$$\frac{d}{d\xi}\left(-Uu_d + \frac{1}{2}u_d^2 - c_d^2\frac{\phi}{\Phi}\right) + \sigma_d c_d^2 \frac{n_{d0}}{n_d}\frac{d}{d\xi}\left(\frac{p_d}{P}\right) = 0, \tag{4.21}$$

$$\frac{d}{d\xi}\left[(-U + u_d)p_d^{1/\gamma}\right] = 0, \tag{4.22}$$

$$\frac{d^2\phi}{d\xi^2} = \frac{\Phi}{\lambda_{Dd}^2}\left[\frac{n_d}{n_{d0}} + (f-1)\frac{n_e}{n_{e0}} - f\frac{n_i}{n_{i0}}\right], \tag{4.23}$$

where $P = n_{d0}K_B T_d$, n_{d0} is the unperturbed dust particle number density, K_B is the Boltzmann constant, and T_d is the average temperature of dust grains.

Using the boundary conditions,

$$\frac{n_d}{n_{d0}} \to 1, \frac{p_d}{P} \to 1, u_d \to 0, \phi \to 0, \frac{d\phi}{d\xi} \to 0 \text{ as } |\xi| \to \infty, \tag{4.24}$$

and solving (4.20), (4.21), and (4.22), we get a quadratic equation for n_d^2, and the solution of this quadratic equation of n_d^2 can be put in the following form:

$$\frac{n_d^2}{n_{d0}^2} = \frac{1}{6\sigma_d}\left(\sqrt{\frac{\phi}{\Phi} - \frac{\Psi_U}{\Phi}} - \sqrt{\frac{\phi}{\Phi} - \frac{\Phi_U}{\Phi}}\right)^2, \tag{4.25}$$

where

$$\frac{\Psi_U}{\Phi} = -\frac{1}{2}\left(\frac{U}{c_d} - \sqrt{3\sigma_d}\right)^2, \frac{\Phi_U}{\Phi} = -\frac{1}{2}\left(\frac{U}{c_d} + \sqrt{3\sigma_d}\right)^2. \tag{4.26}$$

Now integrating (4.23) with respect to ϕ and using the boundary conditions (4.24), we get the following equation known as energy integral with $V(\phi)$ as the Sagdeev potential or pseudo-potential:

$$\frac{1}{2}\left(\frac{d\phi}{d\xi}\right)^2 + V(\phi) = 0, \tag{4.27}$$

where

$$V(\phi) \equiv V(U,\phi) = \frac{\Phi^2}{\lambda_{Dd}^2}[V_d + (f-1)V_e - fV_i], \tag{4.28}$$

$$V_d = \frac{U^2}{c_d^2} + \sigma_d - \frac{n_d}{n_{d0}}\left(\frac{U^2}{c_d^2} + 3\sigma_d + 2\frac{\phi}{\Phi} - 2\sigma_d\frac{n_d^2}{n_{d0}^2}\right), \tag{4.29}$$

$$V_e = \frac{1}{\alpha}\left(1 - e^{\alpha\frac{\phi}{\Phi}}\right), \tag{4.30}$$

$$V_i = \left(1 + 3\beta_1 + 3\beta_1\frac{\phi}{\Phi} + \beta_1\frac{\phi^2}{\Phi^2}\right)e^{-\frac{\phi}{\Phi}} - (1 + 3\beta_1). \tag{4.31}$$

The necessary conditions for the existence of solitary structures of the energy integral (4.27) are $V(0) = 0$, $V'(0) = 0$, and $V''(0) < 0$. It can be easily checked that $V(0) = 0$, $V'(0) = 0$, and the condition $V''(0) < 0$ gives $U > c_D = M_s c_d$, and consequently the greatest lower bound (glb) of the mach number ($M := U/c_D$, which has been explicitly discussed by Dubinov (2009)) for the existence of solitary structures of the energy integral (4.27) is $M_c = 1$, i.e., the solitary structure of the energy integral (4.27) starts to exist for $U > c_D = M_s c_d$ ($\Leftrightarrow M > M_c = 1$). So we have introduced the following dimensionless quantities: $\overline{x} = x/\lambda_D$, $\overline{\xi} = \xi/\lambda_D$, $\overline{t} = t/(\lambda_D/c_D)$, $\overline{\phi} = \phi/\Phi$, $\overline{n}_d = n_d/n_{d0}$. Now we note the following fact: $\xi = x - Ut = \overline{x}\lambda_D - U\overline{t}(\lambda_D/c_D) = \lambda_D[\overline{x} - U\overline{t}(c_D)^{-1}] = \lambda_D[\overline{x} - (U/c_D)\overline{t}] = \lambda_{Dd}[\overline{x} - M\overline{t}]$ $\Leftrightarrow \xi/\lambda_D = \overline{x} - M\overline{t} \Leftrightarrow \overline{\xi} = \overline{x} - M\overline{t}$. Then with respect to these dimensionless quantities, the energy integral can be simplified as follows:

$$\frac{1}{2}\left(\frac{d\overline{\phi}}{d\overline{\xi}}\right)^2 + \overline{V}(\overline{\phi}) = 0, \tag{4.32}$$

where

$$\overline{V}(\overline{\phi}) \equiv \overline{V}(M,\overline{\phi}) = (M_s^2 - 3\sigma_d)(\overline{V}_d + (f-1)\overline{V}_e - f\overline{V}_i), \tag{4.33}$$

$$\overline{V}_d = M_s^2 M^2 + \sigma_d - \overline{n}_d(M_s^2 M^2 + 3\sigma_d + 2\overline{\phi} - 2\sigma_d\overline{n}_d^2), \tag{4.34}$$

$$\overline{n}_d^2 = \frac{1}{\sqrt{6\sigma_d}}\left(\sqrt{\overline{\phi} - \Psi_M} - \sqrt{\overline{\phi} - \Phi_M}\right)^2, \tag{4.35}$$

$$\Psi_M = -\frac{1}{2}(M_s M - \sqrt{3\sigma_d})^2, \Phi_M = -\frac{1}{2}(M_s M + \sqrt{3\sigma_d})^2, \tag{4.36}$$

$$\overline{V}_e = \frac{1}{\alpha}(1 - e^{\alpha\overline{\phi}}), \tag{4.37}$$

$$\overline{V}_i = \left(1 + 3\beta_1 + 3\beta_1\overline{\phi} + \beta_1\overline{\phi}^2\right)e^{-\overline{\phi}} - (1 + 3\beta_1). \tag{4.38}$$

Obviously, there is one–one correspondence between (4.27) and (4.32). More specifically,

$$\frac{1}{2}\left(\frac{d\overline{\phi}}{d\overline{\xi}}\right)^2 + \overline{V}(\overline{\phi}) = 0 \Leftrightarrow \frac{1}{2}\left(\frac{d\phi}{d\xi}\right)^2 + V(\phi) = 0, \tag{4.39}$$

i.e., one can be deduced from other, i.e., these two equations are dynamically equivalent. Although one can

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take any equation equivalent to the energy integral to discuss the qualitative behavior of solitary structures, but in order to avoid quantitative inaccuracies, the general prescription is to normalize length by λ_D and time by λ_D/c_D, where λ_D is the actual Debye length and c_D is the actual acoustic speed of the plasma system. One can obtain c_D by considering the linear dispersion relation as prescribed by Dubinov (2009). So we take the energy integral (4.32) to study the existence of solitary structures of the present dusty plasma system at the acoustic speed, $U = c_D = M_s c_d$ ($\Leftrightarrow M = M_c = 1$). Now dropping 'overline' on both independent and dependent variables, (4.32)–(4.38) can be written as follows:

$$\frac{1}{2}\left(\frac{d\phi}{d\xi}\right)^2 + V(\phi) = 0, \tag{4.40}$$

where

$$V(\phi) \equiv V(M,\phi) = \left(M_s^2 - 3\sigma_d\right)(V_d + (f-1)V_e - fV_i), \tag{4.41}$$

$$V_d = M_s^2 M^2 + \sigma_d - n_d\left(M_s^2 M^2 + 3\sigma_d + 2\phi - 2\sigma_d n_d^2\right), \tag{4.42}$$

$$n_d^2 = \frac{1}{\sqrt{6\sigma_d}}\left(\sqrt{\phi - \Psi_M} - \sqrt{\phi - \Phi_M}\right)^2, \tag{4.43}$$

$$\Psi_M = -\frac{1}{2}\left(M_s M - \sqrt{3\sigma_d}\right)^2, \Phi_M = -\frac{1}{2}\left(M_s M + \sqrt{3\sigma_d}\right)^2, \tag{4.44}$$

$$V_e = \frac{1}{\alpha}(1 - e^{\alpha\phi}), \tag{4.45}$$

$$V_i = (1 + 3\beta_1 + 3\beta_1\phi + \beta_1\phi^2)e^{-\phi} - (1 + 3\beta_1). \tag{4.46}$$

From the expression of n_d^2 as given by (4.43), we see that n_d^2 is non-negative if and only if both $\sqrt{\phi - \Psi_M}$ and $\sqrt{\phi - \Phi_M}$ are real. Again, both $\sqrt{\phi - \Psi_M}$ and $\sqrt{\phi - \Phi_M}$ are real if and only if both $\phi - \Psi_M$ and $\phi - \Phi_M$ are real and non-negative. But both $\phi - \Psi_M$ and $\phi - \Phi_M$ are real and non-negative if and only if $\phi \geqslant \Psi_M$ and $\phi \geqslant \Phi_M$, and finally the conditions $\phi \geqslant \Psi_M$ and $\phi \geqslant \Phi_M$, hold simultaneously if and only if $\phi \geqslant \mathrm{Max}\{\Psi_M, \Phi_M\}$. Now from (4.44) we see that $\Psi_M \geqslant \Phi_M$ ($\Leftrightarrow \mathrm{Max}\{\Psi_M, \Phi_M\} = \Psi_M$), and consequently (4.43), gives theoretically valid expression of n_d^2 if and only if $\phi \geqslant \Psi_M$. Therefore, for the present problem we are restricted by the inequality: $\phi \geqslant \Psi_M = -(M_s M - \sqrt{3\sigma_d})^2/2$. But numerically it is not possible to get the correct value of n_d from (4.43) for a small value of σ_d because the denominator of the right-hand side of (4.43) is $6\sigma_d$. To remove σ_d from the denominator of the right-hand side of (4.43), we first of all note that $(\sqrt{\phi - \Psi_M} + \sqrt{\phi - \Phi_M})^2 > 0$ for all $\phi \geqslant \Psi_M$ and consequently we can multiply the numerator and denominator of the right-hand side of (4.43) by $(\sqrt{\phi - \Psi_M} + \sqrt{\phi - \Phi_M})^2$ and get the following expression of n_d:

$$n_d = \frac{M_s M\sqrt{2}}{(\sqrt{\phi - \Psi_M} + \sqrt{\phi - \Phi_M})}. \tag{4.47}$$

From (4.47), we see that this equation gives both theoretically and numerically correct expressions of n_d^2 even for small σ_d if $\phi \geqslant \Psi_M$.

The Sagdeev potential, $V(\phi)(\equiv V(M,\phi))$, as given by (4.41) is well defined as a real number for all $M \in \mathscr{M}$ and for all $\phi \in \Phi_0$, where $\mathscr{M}$ is the set of all strictly positive real numbers and $\Phi_0 = \{\phi : \Psi_M \leqslant \phi < +\infty\}$, and Ψ_M is given by the first equation of (4.44). So except the point $\phi = \Psi_M$, any $\phi \in \Phi_0$ is an interior point of Φ_0 and any $M > 0$ is an interior point of $\mathscr{M}$. Following the analysis of Das et al. (2009, 2012) the solution space (i.e., the compositional parameter space showing the nature of existence of different solitary structures) of the energy integral (4.40) with respect to β_1 has been shown in Fig. 2 for fixed values of other parameters. Now we want to define the cut-off values of β_1, which are responsible to delimit the solution space.

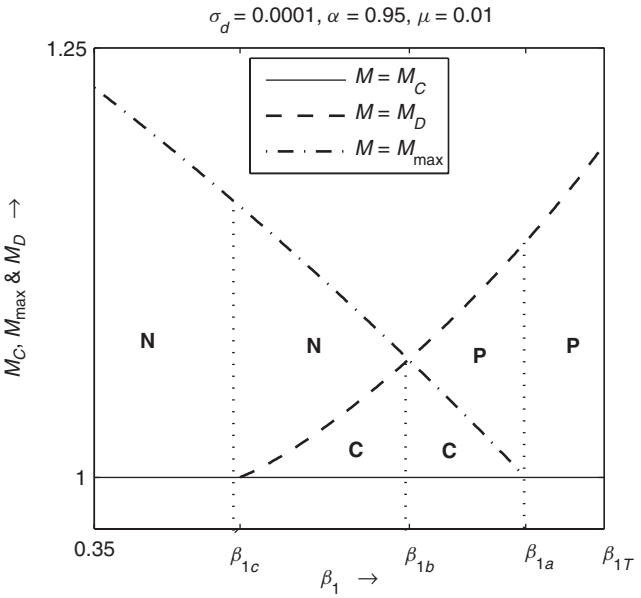


Figure 2. Solution space, i.e., the compositional parameter space showing the nature of existence of different solitary structures for the present system with respect to β_1, where $0.35 \leqslant \beta_1 \leqslant \beta_{1T}$. For $0 \leqslant \beta_1 < 0.35$, the compositional parameter space showing the nature of existence of different solitary structures for the present system with respect to β_1 is qualitatively same as the solution space with respect to β_1 for $0.35 \leqslant \beta_1 < \beta_{1c}$. Here we have used the following terminology. N: Domain (region) of existence of NPSWs, P: domain (region) of existence of PPSWs, and C: domain (region) of coexistence of both NPSWs and PPSWs. At each point on the curve $M = M_D$, there exists a PPDL solution.

β_{1c}: β_{1c} is a cut-off value of β_1 such that PPDL starts to exist whenever $\beta_1 \geqslant \beta_{1c}$ for any value of μ lies within the interval $0 < \mu \leqslant \mu_{1T} < 1$, μ_{1T} is a physically admissible upper bound of μ, i.e., $\beta_1 = \beta_{1c}$ is the lower bound of β_1 for the existence of PPDL solution. Thus,

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β_{1c} is the minimum proportion of fast energetic ions such that maximum potential difference occurs at $\beta_1 = \beta_{1c}$ for fixed values of other parameters of the system. Therefore, for $\beta_1 \geqslant \beta_{1c}$, there exists a value $M_D (> M_c = 1)$ of M such that one can get a PPDL as a solution of the energy integral (4.40) at $M = M_D$. In other words, for $\beta_1 \geqslant \beta_{1c}$, there exists a sequence of PPSW of increasing amplitude restricted by $M_c < M < M_D$ such that this sequence of PPSW converges to PPDL at $M = M_D$, i.e., $M = M_D$ can be regarded as an upper bound of the Mach number for the existence of at least one sequence (class) of PPSW.

β_{1a}: β_{1a} is a cut-off value of β_1 such that $M_{\max}$ exists for all $0 \leqslant \beta_1 \leqslant \beta_{1a}$ whenever $0 < \mu \leqslant \mu_{1T} < 1$. Consequently, $\beta_1 = \beta_{1a}$ is the upper bound of β_1 for the existence of NPSW, i.e., there does not exist any NPSW for $\beta_1 > \beta_{1a}$. Following the same analysis of Das et al. (2009, 2012), it can be easily proved that $M_{\max}$ is the largest positive root of equation $V(M, \Psi_M) = 0$ subject to the condition $V(M, \Psi_M) > 0$ for all $M_c \leqslant M < M_{\max}$.

β_{1b}: β_{1b} is a cut-off value of β_1 such that $M_{\max} = M_D$ at $\beta_1 = \beta_{1b}$ for fixed values of other parameters of the system. Here $M_D (> M_c)$ is the strict upper bound for the existence of PPSW, i.e., PPSW exists for all $M_c < M < M_D$ if $\beta_{1c} \leqslant \beta_1 \leqslant \beta_{1T}$ and $0 < \mu \leqslant \mu_{1T} < 1$ and there does not exist any PPSW if $M > M_D$, and at $M = M_D$ there exists a PPDL solution. Consequently, $\beta_1 = \beta_{1b}$ is the abscissa of the point of intersection between the curves $M = M_{\max}$ and $M = M_D$ in the parametric plane with respect to β_1 for fixed values of other parameters of the system. In other words, there is a coexistence of an NPSW and a PPDL at $\beta_1 = \beta_{1b}$.

Figure 2 shows that for any fixed values of parameters μ, α, and σ_d the entire interval of β_1 can be broken up into four disjoint subintervals – I: $0 \leqslant \beta_1 < \beta_{1c}$, II: $\beta_{1c} \leqslant \beta_1 \leqslant \beta_{1b}$, III: $\beta_{1b} < \beta_1 \leqslant \beta_{1a}$, and IV: $\beta_{1a} < \beta_1 \leqslant \beta_{1T} \approx 0.6 < \beta_M = \min\{1 + \alpha\mu, 4/3\}$. This figure shows that (i) in subinterval I, only NPSWs can exist and the Mach number M for these waves lies within the interval $M_c < M \leqslant M_{\max}$, where M_c is the lower bound of M and $M_{\max}$ is the upper bound of M, which is defined only when the system can support NPSWs, (ii) in subinterval II, both NPSWs and PPSWs can coexist and the Mach number M for these waves lies within the interval $M_c < M < M_D$, whereas only NPSWs are possible if $M_D < M \leqslant M_{\max}$, (iii) in subinterval III, both NPSWs and PPSWs can coexist and the Mach number for these waves lies within $M_c < M \leqslant M_{\max}$, whereas only PPSWs are possible if $M_{\max} < M < M_D$, (iv) in subinterval IV, only PPSWs can exist and the Mach number M for these waves lies within $M_c < M < M_D$. In subintervals II, III, and IV, only PPDL solution is possible along the curve $M = M_D$.

We see from Fig. 2 that the curve $M = M_D$ tends to intersect the curve $M = M_c$ at $\beta_1 = \beta_{1c}$, i.e., there always exists a PPDL solution in any right neighborhood of M_c for some $\beta_1 > \beta_{1c}$. Therefore, at $\beta_1 = \beta_{1c}$, we can expect a PPDL solution at $M = M_c$ if the other conditions of Theorem 6 have been fulfilled. Before going to discuss the existence of solitary wave and/or double layer solutions of the present problem at $M = M_c$, it is necessary to know the sign of $\partial V/\partial M$, $V'''(M_c, 0)$ and the sign of $V''''(M_c, 0)$ when $V'''(M_c, 0) = 0$.

From the expression of $V(\phi) (\equiv V(M, \phi))$ as given by (4.41), we get the following equations:

$$\frac{\partial V}{\partial M} = -(M_s^2 - 3\sigma_d) M_s^2 M \left(\sqrt{n_d} - \frac{1}{\sqrt{n_d}} \right)^2, \quad (4.48)$$

$$V'''(M_c, 0) = -\frac{M_s^2 - 3\sigma_d}{\mu_T^3} \left[12\sigma_d \beta_T^3 + 3\mu_T \beta_T^2 - \mu_T^2 (1 - \alpha^2 \mu) \right], \quad (4.49)$$

where

$$\beta_T = 1 + \alpha\mu - \beta_1, \mu_T = 1 - \mu. \quad (4.50)$$

From (4.48), we see that $\partial V/\partial M < 0$ for all $M > 0$ and for all $\phi \neq 0$, and consequently for all $M > 0$, the condition $\partial V/\partial M < 0$ is satisfied for all $\phi > 0$ as well as $\phi < 0$.

Now from the expression of $V'''(M_c, 0)$ as given by (4.49) of the present paper, it can be easily checked that $(\partial/\partial\beta_1)(V'''(M_c, 0)) > 0$ for all $0 \leqslant \beta_1 < \beta_M$ and for any admissible values of the other parameters consequently, $V'''(M_c, 0)$ is a strictly increasing function of β_1 for all $0 \leqslant \beta_1 < \beta_M$. Again, it can be easily checked that $V'''(M_c, 0) < 0$ for $\beta_1 = 0$ and $V'''(M_c, 0) > 0$ for $\beta_1 = 1 + \alpha\mu + \epsilon \geqslant \beta_M$, where ϵ is any strictly positive real number. Therefore, $V'''(M_c, 0)$ strictly increases with increasing β_1 starting from a negative value and ending with a positive value. So $V'''(M_c, 0)$ intersects the axis of β_1 at $\beta_1 = \beta_c (0 < \beta_c < \beta_M)$ and consequently we have $V'''(M_c, 0) < 0$ for all $0 \leqslant \beta_1 < \beta_c$, $V'''(M_c, 0) > 0$ for all $\beta_c < \beta_1 < \beta_M$ and $V'''(M_c, 0) = 0$ at $\beta_1 = \beta_c$. It is also interesting to note that the equation $V'''(M_c, 0) = 0$ has one and only one real root $\beta_1 = \beta_c$ of β_1 within the admissible range of β_1 and for any fixed values of other parameters. Again, it is simple to check that $V''''(M_c, 0) > 0$ at $\beta_1 = \beta_c$, i.e., $V''''(M_c, 0) > 0$ when $V'''(M_c, 0) = 0$. Therefore, when $V'''(M_c, 0) = 0$, $\phi = 0$ is the position of stable equilibrium and consequently there does not exist any solitary structure of the energy integral (4.40) at $M = M_c$. Now we are in a position to discuss the existence of solitary wave and/or double layer solution of the energy integral (4.40) at $M = M_c$ for any admissible value of β_1 and fixed values of the other parameters.

In subinterval I ($0 \leqslant \beta_1 < \beta_{1c}$), one can expect only NPSW at $M = M_c$, provided $V'''(M_c, 0) > 0$ (from Theorem 4). But it can be easily checked that $V'''(M_c, 0) < 0$ for any value of β_1 lying within subinterval I. So from Theorem 8, we can conclude that there does not exist any solitary wave solution at $M = M_c$ for any value of β_1 lying within subinterval I. Again, in subinterval IV ($\beta_{1a} < \beta_1 \leqslant \beta_{1T}$), one can expect only PPSW at $M = M_c$, provided $V'''(M_c, 0) < 0$ (from Theorem 3). But it can be easily checked that $V'''(M_c, 0) > 0$ for any value of β_1 lying within subinterval IV. So we can conclude from

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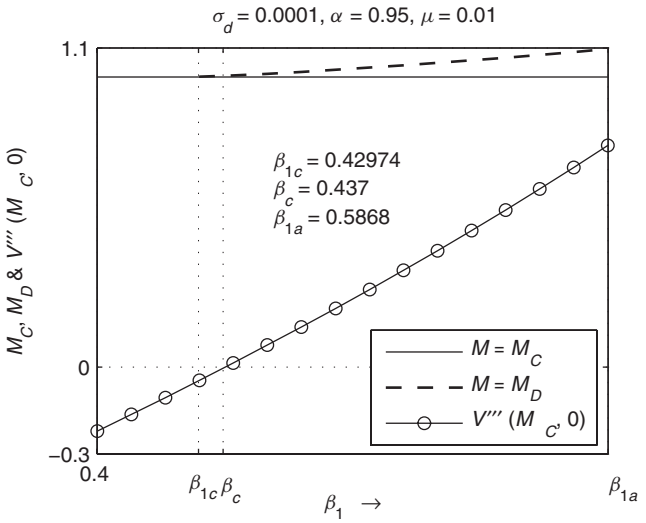


Figure 3. M_c, M_D, and $V'''(M_c, 0)$ are plotted against β_1. At $\beta_1 = \beta_{1c}$, the curve $M = M_D$ tends to intersect the curve $M = M_c$.

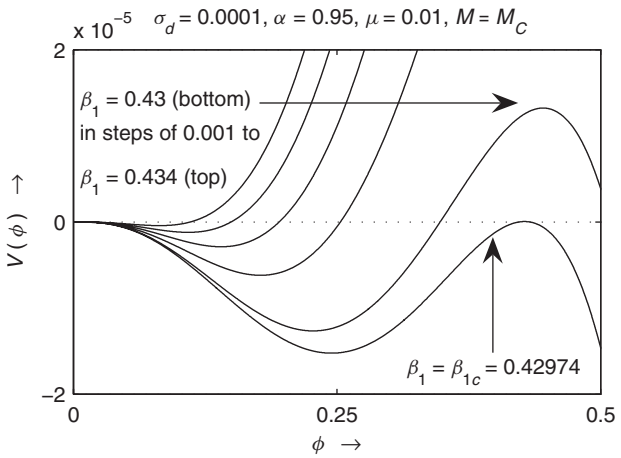


Figure 4. $V(\phi)$ is plotted against ϕ, for different values of β_1 lies within the intervals $\beta_{1c} \leqslant \beta_1 < \beta_c$.

Theorem 9 that there does not exist any solitary wave solution at $M = M_c$ for any value of β_1 lying within subinterval IV. As the system supports coexistence of both NPSWs and PPSWs in subintervals II and III, i.e., for β_1 lying within the interval $\beta_{1c} \leqslant \beta_1 \leqslant \beta_{1a}$, we can conclude from Theorem 10 that there must exist solitary wave or double layer solution at $M = M_c$ when $\beta_{1c} \leqslant \beta_1 \leqslant \beta_{1a}$ and the nature of the solitary wave or double layer solution depends on the sign of $V'''(M_c, 0)$.

In Fig. 3, M_c, M_D and $V'''(M_c, 0)$ have been plotted against β_1 for $\beta_{1c} \leqslant \beta_1 \leqslant \beta_{1a}$. From this figure we see that the double layer solution is not possible for $\beta_{1c} < \beta_1 \leqslant \beta_{1a}$, as the curve $M = M_D$ does not tend to intersect the curve $M = M_c$ for any point of β_1 lying within the interval $\beta_{1c} < \beta_1 \leqslant \beta_{1a}$. So for any β_1 lying within the interval $\beta_{1c} < \beta_1 \leqslant \beta_{1a}$, we get either an NPSW or a PPSW at $M = M_c$. Again, from Fig. 3 we see that the curve $M = M_D$ tends to intersect the curve $M = M_c$ at $\beta_1 = \beta_{1c}$, i.e., there always exists a PPDL solution in any right neighborhood of point M_c for some $\beta_1 > \beta_{1c}$. Therefore, at $\beta_1 = \beta_{1c}$ we have a PPDL solution at $M = M_c$. The existence of solitary wave solution at $M = M_c$ for any β_1 lying within the interval $\beta_{1c} < \beta_1 \leqslant \beta_{1a}$ and the existence of PPDL solution at $M = M_c$ when $\beta_1 = \beta_{1c}$ can be easily verified through Fig. 3. Specifically, we have $V'''(M_c, 0) < 0$ for any β_1 lying within the interval $\beta_{1c} < \beta_1 < \beta_c$ and $V'''(M_c, 0) > 0$ for any β_1 lying within the interval $\beta_c < \beta_1 \leqslant \beta_{1a}$. Consequently, we have PPSW at $M = M_c$ for any β_1 lying within the interval $\beta_{1c} < \beta_1 < \beta_c$, whereas we get NPSW at $M = M_c$ for any β_1 lying within the interval $\beta_c < \beta_1 \leqslant \beta_{1a}$, and at $\beta_1 = \beta_c$ there does not exist any solitary structure at $M = M_c$.

To be more specific, we draw Fig. 4. This figure shows that the amplitude of PPSW increases with decreasing β_1 within $\beta_{1c} < \beta_1 < \beta_c$ and ultimately this sequence of PPSWs end with a PPDL solution at $M = M_c$ when β_1 assumes the value β_{1c}. However, we have seen in the literature that when all the parameters involved in the system assume fixed values in their respective physically admissible range, the amplitude of solitary wave increases with increasing M and these solitary waves end with a double layer of same polarity if it exists. Here it is important to note that M is not a function of the parameters involved in the system but is restricted by the inequality $M_c < M < M_D$, where $M = M_D$ corresponds to a double layer solution. So we cannot compare this case with the case of M_c, since $M_c = 1$. But the solitons and double layer are not two distinct nonlinear structures even when $M = M_c (= 1)$. Actually, if double layer solution exists, it must be the limiting structure of at least one sequence of solitons of same polarity even when $M = M_c$. More specifically, existence of double layer solution implies that there must exist a sequence of solitary waves of same polarity having monotonically increasing amplitude converging to the double layer solution, i.e., the amplitude of the double layer solution acts as an exact upper bound or least upper bound (*lub*) of the amplitudes of the sequence of solitary waves. Therefore, if the double layer solution exists at $M = M_c$ then this double layer solution is also a limiting structure of a sequence of solitary waves of same polarity. So double layer solution can be regarded as a source or sink of solitary waves of same polarity. Particularly for the present problem, the double layer solution at $M = M_c$ when $\beta_1 = \beta_{1c}$ can be regarded as a source for PPSWs for increasing β_1 within $\beta_{1c} < \beta_1 < \beta_c$ along the curve $M = M_c$, whereas the same double layer solution at $M = M_c$ can be regarded as a sink of PPSWs for decreasing β_1 within $\beta_{1c} < \beta_1 < \beta_c$ along the curve $M = M_c$. Similarly, for any other double layer solution, i.e., for the double layer solution at $M = M_D (> M_c)$ can be regarded as sink (source) of PPSWs for increasing (decreasing) M restricted by the inequality $M_c < M < M_D$.

On the other hand, if we take the interval $\beta_c < \beta_1 \leqslant \beta_{1a}$ and move along the curve $M = M_c$, then we have

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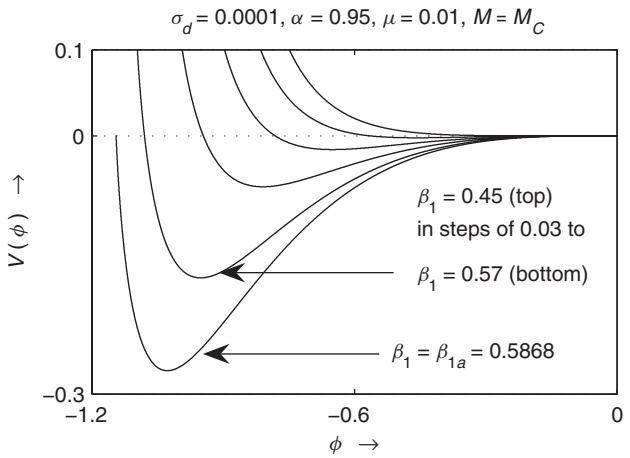


Figure 5. $V(\phi)$ is plotted against ϕ, for different values of β_1 lies within the intervals $\beta_c < \beta_1 \leqslant \beta_{1a}$.

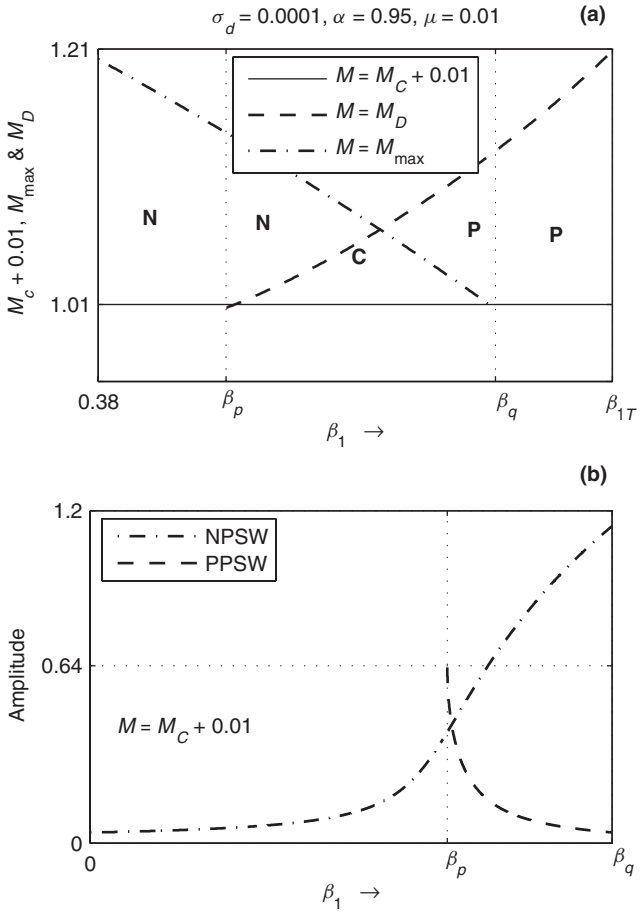


Figure 6. (a) Solution space for the present system with respect to β_1. Here the curve $M = M_c$ is omitted from the solution space as presented in Fig. 2. (b) Variation in amplitude of NPSWs and PPSWs along the curve $M = M_c + 0.01$.

seen from Fig. 5 that the amplitude of NPSW increases with increasing β_1 within $\beta_c < \beta_1 \leqslant \beta_{1a}$ and it attains its maximum value at $\beta_1 = \beta_{1a}$. This is, of course, expected since β_{1a} is the point of intersection of M_c and $M_{\max}$ and for $\beta_1 > \beta_{1a}$ there does not exist any solitary structure at $M = M_c$.

Again, at $\beta_1 = \beta_c$, $V'''(M_c, 0) = 0$ and $V''''(M_c, 0) > 0$, and consequently $\phi = 0$ is the position of stable equilibrium. Therefore, it is not possible to get any solitary structure of the energy integral (4.40) at $M = M_c$ when $\beta_1 = \beta_c$. From Fig. 4, we see that along the curve $M = M_c$ the amplitude of PPSW decreases with increasing β_1 for $\beta_{1c} < \beta_1 < \beta_c$ and ultimately collapses at $\beta_1 = \beta_c$. On the other hand, from Fig. 5, we see that along the curve $M = M_c$ the amplitude of NPSW decreases with decreasing β_1 for $\beta_c < \beta_1 \leqslant \beta_{1a}$ and ultimately it also collapses at $\beta_1 = \beta_c$. So Figs. 4 and 5 are consistent with the physical interpretation for non-existence of solitary structure at $M = M_c$ when $\beta_1 = \beta_c$. Again, PPSWs and NPSWs at $M = M_c$ behave KdV like solitons in the left and right neighbourhood of the point $\beta_1 = \beta_c$, respectively.

Moreover, it can be easily checked that the solitary structures along the curve $M = M_c$ is same as the solitary structures for $M > M_c$ if one can move the solution space through the family of curves parallel to the curve $M = M_c$. To be more specific, solution space for the present system with respect to β_1 has been presented in Fig. 6(a), in which the curve $M = M_c$ is omitted from the solution space as presented in Fig. 2. Now consider the family of curves parallel to $M = M_c$. For instance, consider one such parallel curve for $M = M_c + 0.01$ as shown in Fig. 6(a). In this figure, β_p is the value of β_1 where the curve $M = M_D$ intersects the curve $M = M_c + 0.01$, whereas β_q is the value of β_1 where the curve $M = M_{\max}$ intersects the curve $M = M_c + 0.01$. Figure 6(a) can be interpreted in the same way as Fig. 2 with β_{1c} replaced by β_p and β_{1a} replaced by β_q. However, in Fig. 6(a) the solitary structures exist along the curve $M = M_c + 0.01$, specifically (i) only NPSW exists for $0 \leqslant \beta_1 < \beta_p$, (ii) both NPSW and PPSW coexist for $\beta_p < \beta_1 \leqslant \beta_q$, and (iii) only PPSW exists whenever $\beta_q < \beta_1 \leqslant \beta_{1T}$, and at $\beta_1 = \beta_p$, NPSW coexists with PPDL. The variation of amplitude of these solitary waves (exist along the curve $M = M_c + 0.01$) have been shown in Fig. 6(b). This figure shows that the amplitude of NPSW increases with β_1 having maximum amplitude at $\beta_1 = \beta_q$. Again, the amplitude of PPSW decreases with increasing β_1 for $\beta_1 > \beta_p$, and for any $\beta_1 > \beta_p$ the amplitude of PPSW is bounded by the amplitude of PPDL at $\beta_1 = \beta_p$. This fact is similar for the case when we move along the curve $M = M_c$. So if we move the solution space along the family of curves parallel to the curve $M = M_c$, then one cannot distinguish between the solitary structures that exist along the curve $M = M_c$ and any curve parallel to $M = M_c$. For $\sigma_d = 0.0001$, $\alpha = 0.95$, and $\mu = 0.01$, the values of β_p and β_q lie in the neighborhood of $\beta_1 = 0.44475$ and $\beta_1 = 0.5809$, respectively. To form PPDL, the positive energy must dominate the negative energy at some region of the compositional parameter space of the system and the PPDL will occur at that point of the compositional parameter space where the potential drop is maximum. It is clear from Fig. 6(b) that the positive

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potential dominates the negative potential in the right neighborhood of $\beta_1 = \beta_p$ and again the potential drop is maximum there. The physical reason also suggests the existence of PPDL solution at $\beta_1 = \beta_p$, which is already confirmed in Fig. 6(a). This is the reason for the coexistence of NPSW and PPDL at $\beta_1 = \beta_p$. Actually, the same phenomena happens for the occurrence of PPDL solution on $M = M_c$ at $\beta_1 = \beta_{1c}$, although there does not exist any NPSW along the curve $M = M_c$ for $0 \leqslant \beta_1 < \beta_{1c}$. We will see later that weak nonlinear negative potential solitary structure exists along the curve $M = M_c$ for $0 \leqslant \beta_1 < \beta_{1c}$ and also the amplitude of NPSW tends to zero as $M \to M_c + 0$, whereas PPSWs of finite amplitude exist along the curve $M = M_c$ in the right neighborhood of $\beta_1 = \beta_{1c}$ and consequently the potential drop is maximum at $\beta_1 = \beta_{1c}$. This is the reason for the occurrence of PPDL at $M = M_c$ when $\beta_1 = \beta_{1c}$. This can also be interpreted as follows: Since there is no NPSW solution along the curve $M = M_c$ for $0 \leqslant \beta_1 < \beta_{1c}$, one can simply think of the occurrence of NPSW of amplitude zero along the curve $M = M_c$ for $0 \leqslant \beta_1 < \beta_{1c}$ and in this case also the potential drop is maximum at $\beta_1 = \beta_{1c}$, and consequently we have a PPDL solution on $M = M_c$ at $\beta_1 = \beta_{1c}$. So the solitary structures at $M = M_c$ are not localized. Actually, the solitary structures along the curve $M = M_c$ are same as the solitary structures for $M > M_c$ if we move the solution space through the family of curves parallel to the curve $M = M_c$.

Again, the existence of solitary structures at $M = M_c$ indicates the applicability of the small amplitude theory to study weak nonlinear solitary structures. To be more specific, if we replace $V(\phi)$ by $V_3(\phi)$ ($V_n(\phi)$ represents the power series expansion of the Sagdeev potential or pseudo-potential $V(\phi)$ in ϕ keeping terms up to ϕ^n) to study weak nonlinear solitary structures on the basis of the assumption $|\phi| \ll 1$, the energy integral (4.40) assumes the following form:

$$\frac{1}{2}\left(\frac{d\phi}{d\xi}\right)^2 + V_3(\phi) = 0, \tag{4.51}$$

where $V_3(\phi)$ is given by

$$V_3(\phi) = \frac{1}{2}A\phi^2 + \frac{1}{3}B\phi^3, \tag{4.52}$$

$$A = \left(M_s^2 - 3\sigma_d\right)\left[\frac{1}{M_s^2 M^2 - 3\sigma_d} - \frac{1 - \beta_1 + \alpha\mu}{1 - \mu}\right], \tag{4.53}$$

$$B = \frac{M_s^2 - 3\sigma_d}{2}\left[\frac{1 - \alpha^2\mu}{1 - \mu} - \frac{3\left(M_s^2 M^2 + \sigma_d\right)}{\left(M_s^2 M^2 - 3\sigma_d\right)^3}\right]. \tag{4.54}$$

We do not consider the case $B = 0$, since the expression for $d^2\phi/d\xi^2$, in this case, does not contain any nonlinear term and therefore no solitary structure exists. For $B \neq 0$, we can write $V_3(\phi)$ as

$$V_3(\phi) = -\frac{B}{3}\phi^2(\phi_m - \phi), \tag{4.55}$$

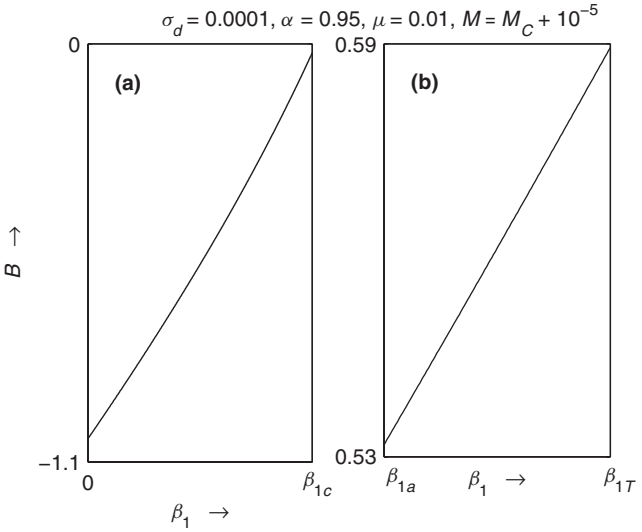


Figure 7. B is plotted against β_1 for (a) $0 \leqslant \beta_1 < \beta_{1c}$, and (b) $\beta_{1a} < \beta_1 \leqslant \beta_{1T}$.

where

$$\phi_m = -\frac{3A}{2B}, \tag{4.56}$$

Now applying conditions for the existence of solitary structures of the energy integral (4.51), we have found that weak nonlinear DAS wave exists if and only if $A < 0$ and $B \neq 0$. The condition $A < 0$ gives

$$M > M_c = 1. \tag{4.57}$$

So M_c is the lower bound of M from which solitary wave solution of (4.51) starts to exist. The condition $B \neq 0$ determines the nature of the solitary wave solution of (4.51). In particular, the solitary wave solution of (4.51) describes either a PPSW or an NPSW according to $B > 0$ or $B < 0$. The amplitude of PPSW or NPSW is given by $|\phi_m| = -(3A)/(2|B|) \to 0$ as $M \to M_c + 0$. Furthermore, solution of (4.51) is unable to describe the existence of both PPSW and NPSW simultaneously, and consequently $V_3(\phi)$ is a representative of $V(\phi)$ to study weak nonlinear solitary structures only when β_1 lies either within the interval $0 \leqslant \beta_1 < \beta_{1c}$ or $\beta_{1a} < \beta_1 \leqslant \beta_{1T}$ because the amplitude of the solitary wave solution of (4.51) tends to zero as $M \to M_c + 0$ and except within a small neighbourhood of β_c there exists a finite amplitude solitary wave or double layer solution of the energy integral (4.40) for any value of β_1 within $\beta_{1c} \leqslant \beta_1 \leqslant \beta_{1a}$. In Fig. 7, B is plotted against β_1 for $M = M_c + 0.00001$. Although from Fig. 7(a) we see that B is equal to zero at $\beta_1 = \beta_{1c}$, one can easily verify that B becomes zero in the neighborhood of $\beta_1 = \beta_c$ when M is close to M_c. Therefore, Figs. 7(a) and (b) actually show that the existence of weak nonlinear NPSW or PPSW according to β_1 lies either within the interval $0 \leqslant \beta_1 < \beta_{1c}$ or $\beta_{1a} < \beta_1 \leqslant \beta_{1T}$. It is also important to note that for $0 \leqslant \beta_1 < \beta_{1c}$ or $\beta_{1a} < \beta_1 \leqslant \beta_{1T}$, there does not exist any solitary structure at $M = M_c$.

To recover the coexistence of small amplitude NPSWs and PPSWs along with a PPDL when β_1 lies within

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the interval $\beta_{1c} \leqslant \beta_1 \leqslant \beta_{1a}$, we consider the following modified energy integral with $n = 4$:

$$\frac{1}{2}\left(\frac{d\phi}{d\xi}\right)^2 + V_4(\phi) = 0. \tag{4.58}$$

Here $V_4(\phi)$ is given by

$$V_4(\phi) = \frac{1}{2}A\phi^2 + \frac{1}{3}B\phi^3 + \frac{1}{4}C\phi^4, \tag{4.59}$$

where

$$C = \frac{M_s^2 - 3\sigma_d}{6}\left[\frac{3\left(5M_s^4M^4 + 30M_s^2M^2\sigma_d + 9\sigma_d^2\right)}{\left(M_s^2M^2 - 3\sigma_d\right)^5} - \frac{1 + 3\beta_1 + \alpha^3\mu}{1 - \mu}\right], \tag{4.60}$$

A and B are given by (4.53) and (4.54), respectively. For $C = 0$, $V_4(\phi)$ is same as $V_3(\phi)$, so we consider the case for $C \neq 0$. For $C \neq 0$, we can write $V_4(\phi)$ as

$$V_4(\phi) = \frac{C}{4}\phi^2(\phi - \phi_{m1})(\phi - \phi_{m2}), \tag{4.61}$$

where

$$\phi_{m1} = -\frac{6A}{2B + \sqrt{4B^2 - 18AC}}, \tag{4.62}$$

$$\phi_{m2} = -\frac{6A}{2B - \sqrt{4B^2 - 18AC}}. \tag{4.63}$$

We note the following facts from (4.62) and (4.63):
(i) Both ϕ_{m1} and ϕ_{m2} are real and distinct if $4B^2 - 18AC > 0$.
(ii) $\phi_{m1}\phi_{m2} = 2A/C$.
(iii) $(\phi - \phi_{m1})(\phi - \phi_{m2}) < 0$ if and only if $\min\{\phi_{m1}, \phi_{m2}\} < \phi < \max\{\phi_{m1}, \phi_{m2}\}$.
(iv) $\phi_{m1} \to 0$ and $\phi_{m2} \to 0$ as $M \to M_c + 0$.

The necessary conditions for the existence of solitary structures of the energy integral (4.58) are $V_4(0) = V_4'(0)$ and $V_4''(0) < 0$. Conditions $V_4(0) = 0$ and $V_4'(0) = 0$ are trivially satisfied, whereas condition $V_4''(0) < 0$ gives $A < 0$. Here also condition $A < 0$ gives $M > M_c$. Therefore, as $A < 0$, from (ii) we see that $\phi_{m1}\phi_{m2} < 0$ if $C > 0$, and in this case $V_4(\phi) < 0$ for all $\min\{\phi_{m1}, \phi_{m2}\} < \phi < 0$ as well as $V_4(\phi) < 0$ for all $0 < \phi < \max\{\phi_{m1}, \phi_{m2}\}$, and consequently both NPSW and PPSW exist simultaneously. On the other hand, if $C < 0$, one cannot get coexistence of weakly nonlinear NPSWs and PPSWs, since for $C < 0$ both ϕ_{m1} and ϕ_{m2} are either positive or negative. Again, as $\phi_{m1} \to 0$ and $\phi_{m2} \to 0$ as $M \to M_c + 0$, $V_4(\phi)$ effectively defines the coexistence of both NPSWs and PPSWs when β_1 lies within the interval $\beta_{1c} \leqslant \beta_1 \leqslant \beta_{1a}$, provided that in this subinterval of β_1, $C > 0$ and $4B^2 - 18AC > 0$. In Fig. 8(a), C and $4B^2 - 18AC$ are plotted against β_1 ($\beta_{1c} \leqslant \beta_1 \leqslant \beta_{1a}$) for $M = M_c + 0.00001$. A quick look of this figure may suggest that $4B^2 - 18AC = 0$ at some point of β_1. To remove this illusion, we draw $4B^2 - 18AC$ against β_1 (in a small neighborhood of β_c) for three different values of M in Fig. 8(b). This figure clearly shows that

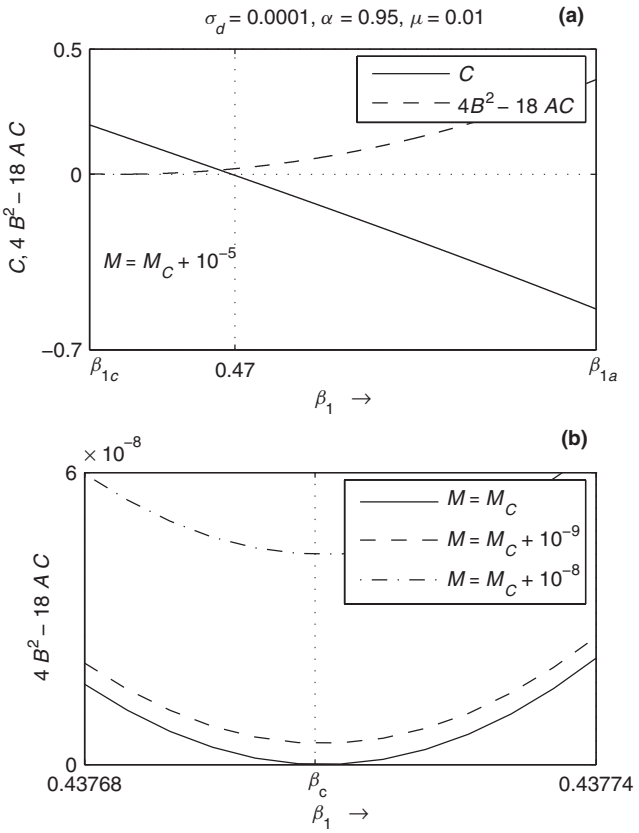


Figure 8. (a) C and $4B^2 - 18AC$ is plotted against β_1 for $\beta_{1c} \leqslant \beta_1 \leqslant \beta_{1a}$. (b) $4B^2 - 18AC$ is plotted against β_1 in a small neighborhood of β_c.

$4B^2 - 18AC = 0$ at $\beta_1 = \beta_c$ only when $M = M_c$. However, we have already found that at $\beta_1 = \beta_c$, $\phi = 0$ is a position of stable equilibrium, and consequently there does not exist any solitary structure of energy integral at $M = M_c$. Therefore, for $M > M_c$, $4B^2 - 18AC > 0$, and consequently these two figures actually show the coexistence of weak nonlinear NPSWs and PPSWs when β_1 lies within the interval $\beta_{1c} \leqslant \beta_1 < 0.47 (< \beta_{1a})$. So $V_4(\phi)$ is unable to produce coexistence of both NPSWs and PPSWs for all β_1 within $\beta_{1c} \leqslant \beta_1 \leqslant \beta_{1a}$ even when we take values of M smaller than $M = M_c + 0.00001$. Again, in this interval of β_1, i.e., $\beta_{1c} \leqslant \beta_1 < 0.47$, the amplitude of both NPSWs and PPSWs tends to zero as $M \to M_c + 0$. But for $\beta_{1c} \leqslant \beta_1 < \beta_c$, we get PPSWs of finite amplitude at $M = M_c$ and for $\beta_c < \beta_1 < 0.47$, we get NPSWs of finite amplitude at $M = M_c$. However, both PPSWs and NPSWs collapse at $\beta_1 = \beta_c$, i.e., except for a very small neighborhood of $\beta_1 = \beta_c$, PPSWs having finite amplitude exist at $M = M_c$, and NPSWs of finite amplitude exist at $M = M_c$. So it is not possible to apply small amplitude method to study the weak nonlinear solitary structures in a parameter regime where finite amplitude solitary structure occurs at $M = M_c$, because in any small amplitude method, amplitude of solitary wave always tends to zero as $M \to M_c + 0$, which contradicts the occurrence of finite amplitude solitary wave at $M = M_c$. There may be another possibility that

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$V_4(\phi)$ is not a perfect match of $V(\phi)$ to study weakly nonlinear solitary structures in $\beta_{1c} \leqslant \beta_1 \leqslant \beta_{1a}$. In any case, one can find inconsistency to study weak nonlinear solitary structures for $\beta_{1c} \leqslant \beta_1 \leqslant \beta_{1a}$. Therefore, our conclusion is that if there exists a region in the parameter space where finite amplitude solitary structures exist at $M = M_c$, it is not possible to study weak nonlinear solitary structures by means of any small amplitude method (viz., replacing $V(\phi)$ by $V_n(\phi)$, where $V_n(\phi)$ represents the power series expansion of the Sagdeev potential or pseudo-potential $V(\phi)$ in ϕ keeping terms up to ϕ^n) in that portion of the parameter space.

5. Conclusion

The paper deals with the existence of solitary wave and double layer at $M = M_c$. The work is relevant to some recent numerical observations (Baluku et al. 2010a,b; Verheest and Hellberg 2010). However, the existence of solitary wave at $M = M_c$ is limited by means of these numerical observations only. In this paper, our motivation is to find the clarifications behind these numerical observations. The analytical theory for the existence of solitary wave and double layer at $M = M_c$ has been provided for the first time in the literature. Moreover, this analytical theory is purely based on the physical interpretation of energy integral. Without having the prior knowledge of physics of energy integral, it is not possible to give the proper mathematical clarification of the above-mentioned numerical observations. Therefore, the theorems presented in this manuscript are based on the physical consequences of energy integral.

On investigating DAS waves in Section 4, we have found the existence of double layer solution at $M = M_c$. We have also discussed in our theoretical section the conditions under which double layer solution occurs at $M = M_c$. This observation is not specific for this problem only. Actually, in any plasma environments if the curve $M = M_D$ tends to intersect the curve $M = M_c$ at some point of compositional parameter space or solution space, then we can always get a double layer solution at $M = M_c$, where $M = M_D (> M_c)$ corresponds to a double layer solution. The existence of double layer solution at $M = M_c$ has been observed for the first time in literature.

The existence of solitary structures at $M = M_c$ indicates where and when one can apply the small amplitude theory to find weak nonlinear solitary structures because of the following reasons:

(a) If in a portion of the solution space there exist solitary waves or double layers of finite amplitude at $M = M_c$, then the amplitude of solitary waves corresponding to all $M > M_c$ is greater than that finite value. Then it is unwise to study weak nonlinear solitary structures by means of any small amplitude methods (viz., replacing $V(\phi)$ by $V_n(\phi)$, where $V_n(\phi)$ represents the power series expansion of the Sagdeev potential or pseudo-potential $V(\phi)$ in ϕ keeping terms up to ϕ^n) in that portion of the solution space, since these methods may lead to erroneous results. Actually, it is not possible to apply small amplitude method to study weak nonlinear solitary structures in a parameter regime where finite amplitude solitary structures exist at $M = M_c$ because in any small amplitude method, amplitude of solitary structure always tends to zero as $M \to M_c + 0 \Leftrightarrow M > M_c$ and $M \to M_c$, which contradicts the occurrence of finite amplitude solitary structure at $M = M_c$. Therefore, to study weak nonlinear solitary structures by means of small amplitude methods for $M > M_c$, the simple prescription is to investigate nonlinear solitary structures at $M = M_c$. So one cannot apply the small amplitude theory to find weak nonlinear solitary structures without investigating the existence of solitary structures at $M = M_c$ in the entire compositional parameter space.

(b) If in a portion of the solution space there exist solitary structures having a finite amplitude at $M = M_c$, then it is simple to find amplitude as well as width of solitary structure for $M = M_c + M_\epsilon$ (for sufficiently small strictly positive quantity M_ϵ) and this can be easily done by plotting $V(\phi)$ against ϕ at $M = M_c + M_\epsilon$. However, if in a portion of the solution space there does not exist solitary structures at $M = M_c$, then it is very difficult to find amplitude as well as width of solitary wave for $M = M_c + M_\epsilon$ by plotting $V(\phi)$ against ϕ. In this situation, the general prescription is to use the small amplitude method for finding weak nonlinear solitary structure at $M = M_c + M_\epsilon$.

Considering any specific problem, it can be easily checked that the solitary structure along the curve $M = M_c$ is same as the solitary structure for $M > M_c$ if one can move the solution space through the family of curves parallel to the curve $M = M_c$.

From the investigation of solitary structures at $M = M_c$ of DA waves in Section 4, we find that solitons and double layer are not two distinct nonlinear structures even when $M = M_c$. Actually, if double layer solution exists, it must be the limiting structure of at least one sequence of solitons of same polarity even when $M = M_c$. Actually, for $M > M_c$, Baboolal et al. (1990) numerically observed that the double layer solution is the ultimate position of solitons, i.e., solitary waves end with a double layer, if exists. This numerical observation of Baboolal et al. (1990) has been referred by several authors in different plasma environments. If we explain any fact with proper justification by means of mathematics, we can apply it in any specific problem. In the present paper, with the help of simple theorems, we have extended the physical interpretation for the existence of solitary structures of the well-known energy integral even when the Mach number M assumes its smallest possible value M_c.

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