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Polygons and gravitons

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1. There has been much interest recently in 'instantons'. These correspond, in differential geometric terms, to connexions in principal bundles over Euclidean 4-space whose curvature satisfies the Yang–Mills equations. The connexion is also required to approach the trivial connexion at infinity. The self-dual solutions have been classified by converting the problem into one of algebraic geometry using the ideas of R. Penrose and R. Ward.

There is a gravitational analogue, called by S. W. Hawking the 'gravitational instanton' which consists of a 4-manifold endowed with a complete, non-singular, positive definite metric which satisfies the Einstein equations. The Riemann curvature tensor of such a metric satisfies the Yang–Mills equations. If the curvature is self-dual, then there is again a complex analytic interpretation – the 'non-linear graviton' of Penrose – and the question of finding such metrics can be converted into one of algebraic, or more generally complex, geometry.

The appropriate conditions at infinity are, however, more subtle. G. Gibbons and C. Pope have shown that if the manifold has the topology of Euclidean space at infinity, that is of $S^3 \times \mathbb{R}$, and approaches the flat Euclidean metric sufficiently fast, then every self-dual solution is in fact isometric to Euclidean space. They propose an alternative asymptotic condition by introducing a fundamental group at infinity: the metric *locally* approaches the Euclidean one but a neighbourhood of infinity has the topology of $S^3/\Gamma \times \mathbb{R}$ where Γ is a finite group of isometries acting freely on S^3.

In this paper we give a class of examples where for Γ we take a cyclic subgroup of $SU(2)$ acting on the unit sphere S^3 in $\mathbb{C}^2$. The image of Γ in $SO(3)$ is the group of rotations of the polygon in the title. We construct the metric by Penrose's non-linear graviton technique using the well-known yet still somewhat mysterious relationship between subgroups of $SU(2)$, simple singularities and Dynkin diagrams. The manifold on which the metric is defined is obtained by resolving the singularity at the origin of $\mathbb{C}^2/\Gamma$. All our metrics are then complete, Ricci–flat Kähler metrics on this manifold relative to some complex structure. We should emphasize, however, that although the approach uses the theory of simple singularities and their resolution, this is only to give an *a priori* justification for the steps involved in constructing the metric. Once we have the metric, we can prove that these singularities can be resolved in a particular way, and even see explicitly the 2-spheres one obtains from the resolution.

The metric so constructed turns out to be the same as one considered recently by Gibbons and Hawking – the 'gravitational multi-instanton'. Our approach, however, remains valid for any finite subgroup of $SU(2)$ and it seems very likely that there are gravitational instantons associated to the binary dihedral, tetrahedral, octahedral and

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icosahedral groups. What is required is an explicit solution in polynomials of a certain algebraic equation. In the case of a cyclic group this is obtained simply by factorization; for the other groups one must deal with a more complicated equation. We hope to consider these cases in a later paper.

In the Yang–Mills theory, one agrees to identify connexions which are related by automorphisms of the principal bundle – this is the notion of gauge equivalence. In Riemannian geometry the analogous notion is that of isometric equivalence: two metrics are isometric if they are equivalent under a diffeomorphism. Using the 'period matrix' of a self-dual metric we determine precisely which of our metrics are equivalent and show that if Γ has order k, then there is a $3k-6$ dimensional moduli space of such metrics.

2. We shall review here briefly Penrose's non-linear graviton construction (7) as it applies to the case of a positive definite metric. We start with a self-dual solution to Einstein's equations. This consists of a 4-dimensional Riemannian manifold M whose Ricci tensor vanishes together with the anti-self-dual part W_- of the Weyl tensor.

The vanishing of W_- enables the *conformal* structure of M to be encoded in complex analytic form (1). The projective spin bundle $Z = \mathbb{P}(V_-)$ has the structure of a complex 3-manifold such that the fibres are holomorphic projective lines with normal bundle $H \oplus H$, H being the standard positive line bundle on $\mathbb{P}_1$. They belong to a family parametrized by a complex 4-manifold M^c. The manifold M^c has a complex conformal structure: two nearby points are null-separated if their corresponding lines intersect. The antipodal map on each fibre $\mathbb{P}_1$ gives Z and M^c a real structure. The real lines are simply the fibres of $\mathbb{P}(V_-)$, parametrized by the base space M, and so M is embedded in M^c as a real submanifold endowed with a real conformal structure. In the case that is important for instantons, $M = S^4$, $Z = \mathbb{P}_3$ and $M^c = \mathbb{Q}_4$, the Klein quadric.

If the Ricci tensor vanishes too, then the bundle V_-, with its induced connexion, is flat. Thus if M is simply connected, we can retract Z onto a particular fibre $\mathbb{P}_1$ by parallel translation and obtain a holomorphic fibration $\pi: Z \to \mathbb{P}_1$. Parallel translation in the vector bundle V_- gives an isomorphism $K \cong \pi^* H^{-4}$ where K is the bundle of holomorphic 3-forms on Z. These two extra pieces of data are sufficient to fix the metric, since for each $u \in \mathbb{P}_1$, the projection from Z onto M identifies M with the fibre $\pi^{-1}(u)$ and the isomorphism $K \cong \pi^* H^{-4}$ defines a non-vanishing holomorphic 2-form on $\pi^{-1}(u)$, hence a volume form, and, together with the conformal structure determined above, a metric.

For each $u \in \mathbb{P}_1$, M has a complex structure, and the metric is a Kähler metric relative to each of these. The non-vanishing holomorphic 2-form is, moreover, covariant constant relative to the Riemannian connexion. It is useful to describe the 3-dimensional space of covariant constant anti-self-dual 2-forms associated to a self-dual metric in this way, namely the space spanned by the Kähler form and the real and imaginary parts of the constant holomorphic 2-form. Note that by our sign convention the orientation given to M by the complex structure is the opposite of the original one.

This reformulation of the self-dual Einstein equations means that to find a solution we must look for the following:

(1) A complex 3-manifold Z, the total space of a holomorphic fibration $\pi: Z \to \mathbb{P}_1$.
(2) A 4-parameter family of sections, each with normal bundle $H \oplus H$.

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(3) A non-vanishing holomorphic section s of $K \otimes \pi^* H^4$.

(4) A real structure on Z such that π and s are real, and Z is fibred by the real sections of the family. Here $\mathbb{P}_1$ is given the real structure of the antipodal map

$$u \mapsto -\bar{u}^{-1}.$$

This set of data is the non-linear graviton.

3. We are going to construct, using the above approach, a solution to the self-dual Einstein equations which at infinity is diffeomorphic to $S^3/\Gamma \times \mathbb{R}$ where $\Gamma \subset SU(2)$ is the cyclic group of matrices

$$\begin{pmatrix} e^{2\pi i n/k} & 0 \\ 0 & e^{-2\pi i n/k} \end{pmatrix} \quad 0 \leqslant n < k.$$

The group Γ acts not just on the unit sphere $S^3 \subset \mathbb{C}^2$, but on the whole of $\mathbb{C}^2$ and we can form the quotient space $\mathbb{C}^2/\Gamma$. This certainly has the topology of $S^3/\Gamma \times \mathbb{R}$ at infinity, but it is not a manifold – it has a singularity at the origin.

We can describe this singularity algebraically as follows. Note that if $(z_1, z_2) \in \mathbb{C}^2$, then the monomials z_1^k, z_2^k, $z_1 z_2$ are invariant under Γ. If we denote them by x, y, z then they satisfy an algebraic relation

$$xy = z^k.$$

This gives an isomorphism between $\mathbb{C}^2/\Gamma$ and the complex surface $xy = z^k$ in $\mathbb{C}^3$. We can now look at deformations of this singularity. There is a universal family of such deformations (the 'universal unfolding' in the sense of Thom) which in this case is given by adding on lower order terms in z:

$$xy = z^k + a_1 z^{k-1} + \ldots + a_k.$$

These still have the topology of $S^3/\Gamma \times \mathbb{R}$ at infinity, but if the discriminant of the right hand side is non-zero, these are non-singular complex surfaces. We shall take M to be the underlying 4-dimensional differentiable manifold of such a surface.

In order to construct Z we need a family of complex stuctures on M parametrized by $u \in \mathbb{P}_1$. As a first approximation to Z, we define $\tilde{Z}$ by the equation:

$$xy = z^k + a_1(u) z^{k-1} + \ldots + a_k(u),$$

where $a_i(u)$ is locally a holomorphic function of u. Since $\mathbb{P}_1$ has no global non-trivial functions we must interpret $a_i(u)$ as a section of a line bundle and then in order for the equation to make sense we must think of $x \in H^l$, $y \in H^m$, $z \in H^n$, $a_i \in \Gamma(H^{ni})$ where $l + m = kn$. The above equation then defines $\tilde{Z}$ as the 3-dimensional hypersurface $f(x, y, z, u) = 0$ in the complex 4-manifold $H^l \oplus H^m \oplus H^n$ where

$$f: H^l \oplus H^m \oplus H^n \to H^{kn}$$

is given by

$$f(x, y, z, u) = xy - \sum_{i=0}^{k} a_i(u) z^{k-i}.$$

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We shall next give $\tilde{Z}$ a real structure. Since $SU(2)$ acts on $\mathbb{C}^2$ as multiplication on the left by unit quaternions, it commutes with right multiplication by j. This is the real structure $(z_1, z_2) \mapsto (-\bar{z}_2, \bar{z}_1)$ (which also induces the antipodal map σ on $\mathbb{P}_1$). The subgroup Γ commutes with the real structure and so we get a real structure on $\mathbb{C}^2/\Gamma$ defined by $x \mapsto (-1)^k \bar{y}$, $y \mapsto \bar{x}$, $z \mapsto -\bar{z}$. We can use this to define a real structure on $H^l \oplus H^m \oplus H^n$:

$$\tau(x, y, z, u) = ((-1)^k \sigma(y),\ \sigma(x),\ -\sigma(z), \sigma(u)), \tag{3·1}$$

where $\sigma: H^i \to \bar{H}^i$ is the real structure on H^i induced by σ. To do this we require of course that $l = m$ since x and y are interchanged. If now the sections a_i of H^{ni} are real (i.e. $a_i(\sigma u) = \sigma(a_i(u))$), then $\tilde{Z}$ has an induced real structure with $\pi: \tilde{Z} \to \mathbb{P}_1$ real.

The hypersurface $\tilde{Z}$ now has a real structure, and a real projection onto $\mathbb{P}_1$, but it is not a manifold: it has singularities over those points $u_j \in \mathbb{P}_1$ at which the discriminant of $\Sigma a_i(u) z^{k-i}$ vanishes. We now use the characteristic property of this type of singularity – a simple singularity, or rational double point – and this is that the singularities can be resolved within the family of non-singular surfaces, if necessary by taking a finite covering of $\mathbb{P}_1$ branched over the points u_j. We refer the interested reader to (2), (3). This procedure would give in general a non-singular manifold Z fibring over a Riemann surface of higher genus. In order to avoid this, we make the assumption that

$$\sum_{i=0}^{k} a_i(u) z^{k-i} = \prod_{j=1}^{k} (z - p_j(u)) \quad (p_j \in \Gamma(H^n)),$$

(The reasoning for this may be found in (3). Replacing the sections a_i by symmetric functions in p_j,is essentially lifting the deformation through the Weyl group of A_{k-1}).

Having done this (and we repeat the fact that the existence of the metric independently shows that this can be done) we obtain a complex 3-manifold Z with a real structure and with a compatible fibration $\pi: Z \to \mathbb{P}_1$.

We have yet to determine the integers l, m, n. They satisfy $l = m$ from the real structure and $l + m = kn$, so $2m = kn$. To determine n we use condition 2 of the graviton: the normal bundle of a section must be $H \oplus H$. Take such a section and project it into $\tilde{Z}$. If it misses the singularities, and a generic one will do so, its normal bundle is just the restriction of the tangent bundle along the fibres, and this is the kernel of the vector bundle homomorphism

$$(f_x, f_y, f_z): H^l \oplus H^m \oplus H^n \to H^{kn}.$$

The first Chern class of the normal bundle $c_1(N)$ is thus $l + m + n - kn$. But if $N = H \oplus H$ then $c_1(N) = 2$, so $n = 2$, $l = m = k$, and $\tilde{Z} \subset H^k \oplus H^k \oplus H^2$.

Condition 3 requires a non-vanishing section s of $K \otimes \pi^* H^4$ on Z. For this we take a constant multiple of the standard form

$$\frac{dx \wedge dy \wedge dz}{f_u} = \frac{dy \wedge dz \wedge du}{f_x} = \ldots,$$

which becomes well defined and non-vanishing on Z, a fact which will again be evident from the existence of the metric.

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The only piece of data missing, and the most important of all, is the 4-parameter family of sections. These define by projection onto $\tilde{Z}$ holomorphic sections of the vector bundle $H^k \oplus H^k \oplus H^2$. If we choose an affine parameter $u \in \mathbb{P}_1$, then a holomorphic section of H^m can be described by a polynomial of degree m in u, hence a line of our family will be given by polynomials $x(u)$, $y(u)$, $z(u)$ of degree k, k, 2 respectively which satisfy the equation

$$xy = \prod_{i=1}^{k} (z - p_i),$$

where $p_i \in \Gamma(H^2)$ is a quadratic polynomial.

Put $z = au^2 + 2bu + c$ and $p_i = a_i u^2 + 2b_i u + c_i$ and let α_i, β_i be the roots of $z - p_i$. Then we can solve the above equation by a simple factorization: we put

$$x = A \prod_{i=1}^{k} (u - \alpha_i); \quad y = B \prod_{i=1}^{k} (u - \alpha_i), \tag{3·2}$$

where $AB = \Pi(a - a_i)$. Locally, these curves are parametrized by the four variables (a, b, c, A) with a finite indeterminacy coming from the choice of roots.

The real lines must be preserved by the real structure on $\tilde{Z}$, so we must have, from (3·1)

$$x(\sigma u) = (-1)^k \sigma(y(u)); \quad y(\sigma u) = \sigma(x(u)); \quad z(\sigma u) = -\sigma(z(u)).$$

This last reality condition on the quadratic polynomial z reduces to $c = -\bar{a}$, $b = \bar{b}$ and so the roots of $z - p_i$ are

$$\frac{-(b - b_i) \pm \sqrt{\{(b - b_i)^2 + |a - a_i|^2\}}}{(a - a_i)}.$$

If we let Δ_i denote the *positive* square root of $(b - b_i)^2 + |a - a_i|^2$ then we have an unambiguous way of assigning roots to x and y: we put

$$x = A \prod_{i=1}^{k} \left(u + \frac{(b - b_i) - \Delta_i}{(a - a_i)} \right); \quad y = B \prod_{i=1}^{k} \left(u + \frac{(b - b_i) + \Delta_i}{(a - a_i)} \right). \tag{3·3}$$

The reality condition on x and y together with the relation $AB = \Pi(a - a_i)$ implies that

$$A\bar{A} = \Pi((b - b_i) + \Delta_i). \tag{3·4}$$

The real lines are thus locally parametrized by a point $(b, \operatorname{Im} a, \operatorname{Re} a)$ in $\mathbb{R}^3$ and an angular variable $\arg A$. Each metric is determined by k real quadratic polynomials p_i, the coefficients of which can be thought of as k points $\mathbf{x}_i$ in $\mathbb{R}^3$.

We now have all the data for the non-linear graviton construction. There are certain properties to be checked – that the normal bundle of our sections is $H \oplus H$, that they extend from $\tilde{Z}$ to Z and that Z is fibred by the real sections. We could proceed in complex analytic language to check these properties, but we prefer to use the data to compute the metric on an open set of M and then show that it is well-defined everywhere.

4. To each point (a, b, c, A) in an open set of $\mathbb{C}^4$ we have associated a section $(x(u), y(u), z(u))$ of $H^k \oplus H^k \oplus H^2$ which lies in $\tilde{Z}$. Taking the derivative of this map, a

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tangent vector (a', b', c', A') goes to a section $(x', y', z') \in \ker (f_x, f_y, f_z)$, a section of the normal bundle of the line $(x(u), y(u), z(u))$. The tangent vector (a', b', c', A') is a null vector in the conformal structure on M^c by definition if and only if the corresponding section of the normal bundle vanishes at some point. This is the infinitesimal version of the condition that two points are null separated if their corresponding lines intersect. Hence the null cone is given by the condition that $x'(u)$, $y'(u)$, $z'(u)$ have a common zero, i.e. by eliminating u from the equations

$$z'(u) = a'u^2 + 2b'u + c' = 0, \tag{4·1}$$

$$\frac{x'(u)}{x(u)} = \frac{A'}{A} - \Sigma \frac{\alpha_i}{u - \alpha_i} = 0, \tag{4·2}$$

obtained by differentiating (3·2). If we write

$$z - p_i = (u - \alpha_i)((a - a_i)u - (c - c_i)\alpha_i^{-1})$$

and differentiate we find

$$0 = z'(u) = \alpha_i'(-(a - a_i)u + (c - c_i)\alpha_i^{-1} + (u - \alpha_i)(c - c_i)\alpha_i^{-2}) + (u - \alpha_i)(a'u - c'\alpha_i^{-1})$$

and hence

$$\frac{\alpha_i'}{u - \alpha_i} = \frac{\alpha_i a'u - c'}{2u\Delta_i},$$

where $2\Delta_i = \alpha_i - \beta_i = 2(b^2 - ac)^{\frac{1}{2}}$.

Substituting in (4·2) we get a linear equation for u:

$$u\left(\frac{2A'}{A} - \Sigma \frac{\alpha_i a'}{\Delta_i}\right) + \Sigma \frac{c'}{\Delta_i} = 0$$

and if we put

$$\gamma = \Sigma \frac{1}{\Delta_i}, \quad \delta = \Sigma \frac{\alpha_i}{\Delta_i}$$

the null cone becomes

$$\gamma^2(a'c' - b'^2) + \left(\frac{2A'}{A} - \delta a' - \gamma b'\right)^2 = 0.$$

If we now consider a *real* tangent vector on the family of real curves, then from (3·4)

$$\mathrm{Re}\left(\frac{2A'}{A}\right) = \Sigma \frac{(b - b_i)' + \Delta_i'}{(b - b_i) + \Delta_i} = \gamma b' + \mathrm{Re}\,(\delta a') \tag{4·3}$$

so the real conformal structure may be defined by the metric

$$\gamma^2(b'^2 + a'\bar{a}') + \left(\mathrm{Im}\left(\frac{2A'}{A} - \delta a'\right)\right)^2 \tag{4·4}$$

This is clearly a positive definite metric. In view of (4·3), we can also write it in the hermitian form

$$\gamma^2 a'\bar{a}' + \left(\frac{2A'}{A} - \delta a'\right)\left(\frac{2\bar{A}'}{\bar{A}} - \bar{\delta} a'\right).$$

We have here the conformal structure written in terms of natural local coordinates in the space of complex sections. We should like to express it in terms of coordinates on a

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non-singular fibre of $\pi\colon Z \to \mathbb{P}_1$. Without loss of generality we may assume that $\pi^{-1}(0)$ is non-singular, i.e. the surface

$$xy = \Pi(z + \bar{a}_i) \quad \text{where} \quad a_i \neq a_j, \quad i \neq j.$$

This is a surface in $\mathbb{C}^3$ and (x, y) are coordinates except where $f_z = 0$, (y, z) except where $f_x = y = 0$. We shall first express the conformal structure in terms of the coordinates (y, z), so we must relate (y, z) to the coordinates (a, b, A). This is easy: at $u = 0$, $z = -\bar{a}$ and

$$y = B\Pi \frac{(b - b_i) + \Delta_i}{(a - a_i)} = \frac{1}{A} \Pi((b - b_i) + \Delta_i) = \bar{A},$$

hence the conformal structure is defined by the hermitian form

$$\gamma^2 dz d\bar{z} + \left(\frac{2dy}{y} + \bar{\delta} dz\right)\left(\frac{2d\bar{y}}{\bar{y}} + \delta d\bar{z}\right). \tag{4·5}$$

To fix the metric within the conformal class we use the holomorphic section

$$s = \frac{dy \wedge dz \wedge du}{f_x} = \frac{dy \wedge dz \wedge du}{y}$$

and this defines the holomorphic 2-form $(dy \wedge dz)/y$ on each fibre and the volume form $\omega = dy \wedge dz \wedge d\bar{y} \wedge d\bar{z}/|y|^2$.

The volume form of the metric (4·5) is $4\gamma^2\omega$, so the metric we are searching for is a constant multiple of

$$\gamma dz d\bar{z} + \gamma^{-1}\left(\frac{2dy}{y} + \bar{\delta} dz\right)\left(\frac{2d\bar{y}}{\bar{y}} + \delta d\bar{z}\right) \tag{4·6}$$

where

$$\gamma = \Sigma((b - b_i)^2 + |\bar{z} + a_i|^2)^{-\frac{1}{2}} = \Sigma \Delta_i^{-1},$$

$$\delta = \Sigma \frac{(b - b_i) - \Delta_i}{\Delta\,(\bar{z} + a_i)},$$

and b is defined implicitly by

$$\Pi((b - b_i) + \Delta_i) = y\bar{y}.$$

If we return to the form of the metric given in (4·4), and the description of the space of real quadratic polynomials as Euclidean 3-space with the metric given by the discriminant, then we obtain the metric which describes the 'gravitational multi-instanton' of Gibbons and Hawking(5):

$$\gamma d\mathbf{x} . d\mathbf{x} + \gamma^{-1}(d\tau + \boldsymbol{\omega} . d\mathbf{x})^2, \tag{4·7}$$

where $\gamma = \Sigma 1/|\mathbf{x} - \mathbf{x}_i|$ and $\operatorname{curl} \boldsymbol{\omega} = \operatorname{grad} \gamma$.

These metrics are themselves generalizations of the metric of Calabi, Eguchi and Hanson(4), which corresponds to the case $k = 2$.

The metric (4·6) appears to become singular if $y = 0$, but this is precisely where (y, z) are not coordinates. If $y = 0$, then $z = -\bar{a}_i$ for some i and so since $a_i \neq a_j$, $f_z \neq 0$ in some neighbourhood of $z = -\bar{a}_i$ and then (x, y) are coordinates. We must now show that the metric is well-defined in such a neighbourhood.

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Now $xy = \Pi(z+\bar{a}_i)$ so since $a_i \neq a_j$, near $z = -\bar{a}_i$ we have

$$z+\bar{a}_i = hxy, \tag{4·8}$$

where h is C^∞ and non-vanishing. Furthermore, $y\bar{y} = \Pi((b-b_j)+\Delta_j)$,

so
$$(b-b_i)+\Delta_i = gy\bar{y}, \tag{4·9}$$

where g is C^∞ and positive. Thus from (4·8)

$$\Delta_i^2 = (b-b_i)^2 + |z+\bar{a}_i|^2 = (b-b_i)^2 + |h|^2|x|^2|y|^2$$

and from (4·9)

$$\Delta_i^2 = g^2|y|^4 + (b-b_i)^2 - 2(b-b_i)g|y|^2.$$

It follows that

$$b-b_i = (g^2|y|^2 - |h|^2|x|^2)/2g$$

and $\Delta_i = (g^2|y|^2 + |h|^2|x|^2)/2g$. Put $r^2 = g^2|y|^2 + |h|^2|x|^2$, then

$$\gamma = 2g/r^2 + \gamma_0,$$

where γ_0 is C^∞ and

$$\delta = \sum_j \frac{-(b-b_j)+\Delta_j}{\Delta_j(-\bar{z}-a_j)} = \frac{-2hx}{r^2\bar{y}} + \delta_0,$$

where δ_0 is C^∞. Furthermore, we have

$$dz = h(x\,dy + y\,dx) + xy\,dh.$$

Using these expressions we can write

$$\frac{2dy}{y} + \bar{\delta}dz = \frac{2}{r^2}(g^2\bar{y}dy - |h|^2\bar{x}\,dx - \bar{h}|x|^2dh) + \bar{\delta}_0\,dz$$

and it is then clear that the terms γ_0 and δ_0 do not affect the regularity of the metric (4·6), and that the metric is regular if and only if

$$\frac{2g}{r^2}dz\,d\bar{z} + \frac{r^2}{2g}\left[\frac{4}{r^4}(g^2\bar{y}\,dy - |h|^2\bar{x}\,dx - \bar{h}|x|^2dh)(\quad)^-\right]$$

is regular. But this one can rewrite as

$$2g^{-1}(h^2dx\,d\bar{x} + g^2dy\,d\bar{y} + \bar{h}x\,dh\,d\bar{x} + h\bar{x}\,dx\,d\bar{h} + |x|^2dh\,d\bar{h})$$

which, since g is non-vanishing, is well defined and we have a non-singular positive definite metric on the whole of the non-singular surface $xy = \Pi(z+\bar{a}_i)$.

The asymptotically locally Euclidean behaviour of the metric can be most easily seen by taking the form of the metric (4·7) given by Gibbons and Hawking, and letting $\mathbf{x}$ go to infinity. Completeness of the metric also follows easily from its asymptotic properties.

5. Now that we have the metric we can return to the question of the resolution of the singularities: the passage from $\tilde{Z}$ to Z. For each point $u \in \mathbb{P}_1$, we have a complex structure on M, relative to which the metric is Kählerian, and a holomorphic map to the fibre $\pi^{-1}(u)$ in $\tilde{Z}$. In particular we have a non-singular complex surface mapping onto the singular fibres of $\tilde{Z}$: a resolution of the singularity.

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These singular surfaces are given by the values of u at which $xy = \Pi(z - p_i(u))$ has a singularity, i.e. those points at which $p_i = p_j$ for some $i \neq j$. At such a point $u = u_0$, the singularity is at $x = 0, y = 0, z = p_i(u_0)$. We shall now use the metric to see what this singular point becomes in the resolution and to do this we consider the real curves in $\tilde{Z}$ which pass through the singularity. These correspond to Z in the real curves passing through the resolution of the singularity, and they trace out in a non-singular fibre an isometric copy of the resolved singularity.

Hence, suppose $(x(u), y(u), z(u))$ passes through the singularity over $u = u_0$. Then $z(u_0) = p_i(u_0)$ where $p_i(u_0) = p_j(u_0)$, but since z, p_i, p_j are real quadratic polynomials they also agree at the antipodal point $-\bar{u}_0^{-1}$, so

$$z - p_i = \lambda(p_j - p_i)$$

where λ is real. The roots of $p_j - p_i$ are

$$\frac{-(b_j - b_i) \pm \Delta_{ij}}{(a_j - a_i)}$$

and according to our choice of real structure (3·3) the corresponding roots of x are

$$\frac{-\lambda(b_i - b_i) + |\lambda|\Delta_{ij}}{\lambda(a_j - a_i)}, \quad \frac{-(\lambda - 1)(b_j - b_i) + |\lambda - 1|\Delta_{ij}}{(\lambda - 1)(a_j - a_i)}.$$

If we assume that this is a generic singularity, i.e. $p_k(u_0) \neq p_l(u_0)$ for any other pair (k, l), then these two roots must be distinct, for x must vanish at both roots of $p_j - p_i$. This means that $\operatorname{Sgn}|\lambda - 1| = -\operatorname{Sgn}|\lambda|$ and hence $0 < \lambda < 1$. On the non-singular surface $u = 0$, the corresponding real lines trace out a 2-sphere of the form:

$$\begin{aligned} x &= \sqrt{\{\lambda(1-\lambda)\}} f(\lambda)\, e^{-i\theta}, \\ y &= \sqrt{\{\lambda(1-\lambda)\}} g(\lambda)\, e^{i\theta}, \\ z &= -\bar{a}_i + \lambda(\bar{a}_i - \bar{a}_j), \end{aligned}$$

as $$0 \leqslant \lambda \leqslant 1 \quad \text{and} \quad 0 \leqslant \theta \leqslant 2\pi,$$

where f and g can be determined by the transformation from coordinates (a, b, A) to (x, y, z).

In the representation of $z(u)$ as a point in $\mathbb{R}^3$, we see that these 2-spheres correspond to the straight line segments joining $\mathbf{x}_i$ to $\mathbf{x}_j$. In general, there are $\frac{1}{2}k(k-1)$ of these, but in certain configurations there are less. The most interesting case is when the points $\mathbf{x}_i$ are collinear. This corresponds, for example, to an equation for $\tilde{Z}$ of the form

$$xy = \Pi(z - a_i(u^2 - 1)) \quad (a_i \in \mathbb{R}).$$

In this case the singularities occur at $u = \pm 1$ and give the singular surface $xy = z^k$ that we started with. The family of curves passing through the singular points now has $(k-1)$ components corresponding to the values $0 < n < k$:

$$x = A(u-1)^{k-n}(u+1)^n; \quad y = B(u-1)^n(u+1)^{k-n}; \quad z = \lambda(u^2 - 1).$$

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The same sort of reasoning as above shows however, that if we order the points on the line $a_1 < a_2 < \ldots < a_k$, then a real curve with exponent n corresponds to the value of λ in the interval (a_n, a_{n+1}) and it follows that the resolved singularity consists of a chain of $(k-1)$ 2-spheres, each intersecting the next.

These 2-spheres define homology classes $\gamma_i \in H_2(M, \mathbb{Z})$, $1 \leqslant i \leqslant k-1$ such that the intersection number $\gamma_i \cdot \gamma_{i+1} = 1$ but otherwise $\gamma_i \cdot \gamma_j = 0$. They are furthermore holomorphic 2-spheres (projective lines) in a surface which has a non-vanishing holomorphic 2-form. This means that the canonical bundle of the surface is trivial, and restricting to each projective line, $c_1(T \oplus N) = 0$, so $c_1(N) = -c_1(T) = -2$ where T and N are the tangent and normal bundles of the line. It follows that the self-intersection number $\gamma_i^2 = -2$.

Now $xy - z^k = 0$ with its singularity resolved is the differentiable manifold M, and it is clear from this point of view that M retracts onto a neighbourhood of the resolved singularity, so $H_2(M)$ is actually generated by $\{\gamma_i\}$. The intersection matrix $\gamma_i \cdot \gamma_j$ can now be identified with the Cartan matrix A_{k-1}, and we have a distinguished class of bases for $H_2(M)$, namely those which satisfy the same relations as the γ_i's, or correspond in Lie algebraic terms to a choice of simple roots. Any two such bases are related by an element of the Weyl group of A_{k-1}, the symmetric group S_k. If we go back to the picture of k points $\mathbf{x}_i$ in $\mathbb{R}^3$, we see that, for a generic metric, a choice of such a basis corresponds to an ordering $\mathbf{x}_1, \ldots, \mathbf{x}_k$ of the points and the homology classes can be represented by the 2-spheres corresponding to the segments $(\mathbf{x}_i, \mathbf{x}_{i+1})$.

Not only does $H_2(M)$ have a distinguished class of bases, but each self-dual metric determines a distinguished 3-dimensional space of closed 2-forms – the covariant constant anti-self-dual 2-forms. An oriented orthonormal basis $\omega_1, \omega_2, \omega_3$ for this space is well-defined modulo the special orthogonal group $SO(3)$. By integrating these 2-forms over the homology classes, we can define the period matrix

$$A_{ij} = \int_{\gamma_j} \omega_i.$$

A_{ij} is a $3 \times (k-1)$ real matrix. Its importance lies in the fact that it is an invariant, modulo the action of $SO(3) \times S_k$, of the isometric equivalence class of the metric, for if $f\colon M \to M$ is a diffeomorphism, the 2-forms $\{f^*\omega_i\}$ form an orthonormal basis of covariant constant 2-forms for the metric f^*g, and $\{f_* \gamma_j\}$ a distinguished basis for $H_2(M)$. Since

$$\int_{\gamma_j} f^*\omega_i = \int_{f_*\gamma_j} \omega_i$$

the period matrices of g and f^*g are equivalent.

We shall next compute A_{ij} for the metrics constructed here and deduce a number of consequences. As mentioned above, we can take the basis $\{\gamma_i\}$ to be represented by the 2-spheres corresponding to the straight line segments $(\mathbf{x}_i, \mathbf{x}_{i+1})$ for some ordering.

If ω is the Kähler form and h the holomorphic 2-form such that $2h \wedge \bar{h} = -\omega \wedge \omega$ then we take for $\{\omega_i\}$:

$$\omega_1 = i\omega,$$
$$\omega_2 = h + \bar{h},$$
$$\omega_3 = i^{-1}(h - \bar{h}).$$

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Hence in our case we take

$$\omega_1 = i\gamma\, dz \wedge d\bar{z} + i\gamma^{-1}\left(\frac{2dy}{y} + \bar{\delta}\, dz\right)\left(\frac{2d\bar{y}}{\bar{y}} + \delta\, d\bar{z}\right),$$

$$\omega_2 + i\omega_3 = 2h = \frac{4dy \wedge dz}{y}.$$

The 2-sphere γ_i is given by

$$z = -\bar{a}_i + \lambda(\bar{a}_i - \bar{a}_{i+1}),$$

$$y = e^{i\theta}h(\lambda), \quad \text{for some function } h,$$

as $0 \leqslant \lambda \leqslant 1$, $0 \leqslant \theta \leqslant 2\pi$.
Hence

$$\int_{\gamma_i} \omega_2 + i\omega_3 = 4\int_{\gamma_i} i\, d\theta \wedge d\lambda(\bar{a}_i - \bar{a}_{i+1}) = 8\pi i(\bar{a}_i - \bar{a}_{i+1})$$

and, restricting ω_1 to γ_i,

$$\omega_1 = i\gamma^{-1}(-2i)\,\mathrm{Re}\left(\frac{2dy}{y} + \bar{\delta}dz\right) \wedge \mathrm{Im}\left(\frac{2dy}{y} + \bar{\delta}dz\right).$$

But from (4·3)

$$\mathrm{Re}\left(\frac{2dy}{y} + \bar{\delta}dz\right) = \gamma\, db = \gamma(b_{i+1} - b_i)\, d\lambda,$$

$$\mathrm{Im}\left(\frac{2dy}{y} + \bar{\delta}dz\right) = 2d\theta + g\, d\lambda,$$

so

$$\int_{\gamma_i} \omega_1 = 2(b_{i+1} - b_i)\int d\lambda \wedge 2d\theta = 8\pi(b_i - b_{i+1}).$$

Thus, up to a constant 8π, the columns of the period matrix are just the vectors $\mathbf{x}_i - \mathbf{x}_{i+1}$. Since the Weyl group acts on these just by permuting the k vectors $\mathbf{x}_i$, we see that, up to a rotation or a translation in Euclidean 3-space, the k vectors, which provide the basic data for one of our metrics, are determined by the period matrix and hence the isometric equivalence class of the metric. It is easy to see that Euclidean motions of the points give equivalent metrics, hence the 'moduli space' of these metrics is the set of k unordered points in $\mathbb{R}^3$, modulo the Euclidean group, a stratified set of dimension $3k-6$.

Suppose now we fix a complex structure on M, say the surface $xy = \Pi(z + \bar{a}_i)$. We can ask how many of our metrics are Kählerian relative to this complex structure. These will correspond to polynomials $p_i(u)$ such that $p_i(u) = -\bar{a}_i$ for some u and all $1 \leqslant i \leqslant k$. By a rotation we can assume $u = 0$, and then p_i is fixed by the real parameter b_i. Hence, identifying equivalent metrics, there is a $(k-1)$ dimensional space of metrics, Kählerian relative to this particular complex structure. The period matrix tells us that the metric is determined uniquely by the cohomology class of the Kähler form.

Finally, we may note that *any* self-dual metric on M has a period matrix and hence determines a set of k point in $\mathbb{R}^3$, and thus, if the points are distinct, a metric of the above family with the same period matrix. In view of this, and the result of Gibbons

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and Pope(6) applied to $\mathbb{R}^4$, it seems reasonable to conjecture that any self-dual metric on M, becoming locally Euclidean sufficiently rapidly at infinity, is in fact equivalent to one of those constructed here. This would be a 'global Torelli theorem' in the terminology of algebraic geometry.

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