Note on Kühne's Theorem. By Mr H. Lob, King's College.

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Given four lines in a plane, we know that the circumcircles of the four triangles formed by them meet in a point. Kühne's theorem states the analogous property for Euclidean hyperspace, viz., that if from $n+2$ primes in space of n dimensions we obtain $n+2$ simplexes by omitting each prime in turn, the hyperspheres circumscribing the simplexes meet in a point when n is even, but not when n is odd. Mr Grace, in a recent reference to the property, gives a short and interesting proof by help of the rational curves which osculate the given primes and the prime at infinity*. It is the purpose of the following note to obtain the result by considering the points of intersection of certain cubic loci.

Throughout the paper, P will denote a prime and S a hypersphere, and in space of n dimensions an $(n-1)$-fold of order 3 passing through all points of the absolute will, for brevity, be termed a "spherical cubic."

1. *Lemma.* In space of n dimensions, n composite spherical cubics, S_1P_1, $S_2P_2, \ldots S_nP_n$, have, in general, $2^{n+1}-1$ common points.

For two hyperspheres meet on a prime and $n-1$ primes meet in a straight line. Thus any selection of the hyperspheres from r and the primes from the remaining $n-r$ of the composite cubics (such, for example, as $S_1S_2S_3P_4P_5 \ldots P_n$) will lead to the two points in which a line cuts one of the hyperspheres, and these two points will be common to the whole set of spherical cubics. But the selection of the n primes gives only a single point. The total number of common points is therefore

$$2(1 + {}_nC_1 + {}_nC_2 + \ldots + {}_nC_{n-1}) + 1 = 2^{n+1} - 1.$$

2. For the plane case the line of reasoning which will be generalised in the sequel is as follows: If C_1 and C_2 are circular cubics through the six intersections of four given lines, the complete family of such cubics is $\lambda_1C_1 + \lambda_2C_2 = 0$. These all pass through the common points of C_1 and C_2, which are seven in number, excluding the circular points at infinity. Four cubics break up into the composite curve formed by one of the lines and

* "Extension of a set of theorems in circle geometry." *Proc. Camb. Phil. Soc.* Vol. xxiv, p. 15 (1928).

I ought to mention that the property was communicated to me about four years ago by Mr R. C. Chevalier, of St John's College, Cambridge, who derived it from the alternate vanishing and non-vanishing of skew-symmetric determinants.

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the circumcircle of the other three lines. Of the seven common points, six are the intersections of the four given lines; the seventh must clearly be common to the four circles.

The Lemma has been proved only for composite spherical cubics, and it is unnecessary in the generalisation to include other kinds before we begin to count up the number of common points.

3. In space of three dimensions the general equation of a spherical cubic is $(x^2+y^2+z^2)(ax+by+cz)$ + a quadratic expression, and it thus contains 12 independent constants. It follows that if C_1, C_2, C_3 are spherical cubics passing through the 10 intersections of five given planes taken by threes, the complete family of spherical cubics passing through the 10 points is

$$\lambda_1 C_1 + \lambda_2 C_2 + \lambda_3 C_3 = 0^*.$$

Let P_1, P_2, ... be any five planes and S_1, S_2, ... the spheres described about the tetrahedra $(P_2P_3P_4P_5)$, ... Then all spherical cubics through the 10 intersections of P_1, P_2, ... taken in threes are included in $\lambda_1 S_1P_1 + \lambda_2 S_2P_2 + \lambda_3 S_3P_3 = 0$, and thus by the Lemma have 15 common points, viz., those common to S_1P_1, S_2P_2, S_3P_3. Also, S_4P_4 and S_5P_5 are two other members of the family.

Let us try to account for these points. The plane P_1 meets the other planes in four lines and the sections of S_2, S_3, S_4, S_5 by P_1 are the circumcircles of the triangles formed by these lines and thus meet in a point (by §2). Hence the surfaces S_1P_1, S_2P_2, ... S_5P_5 have five common points, one on each plane. These with the 10 points of intersection of the planes (the vertices of the tetrahedra), make up the required 15; thus there are no other common points, so that the five spheres themselves have no common point.

4. In four dimensions the general equation of a spherical cubic contains 18 constants, so that the spherical cubics through the 15 points in which six primes intersect by fours are given by

$$\lambda_1 S_1P_1 + \lambda_2 S_2P_2 + \lambda_3 S_3P_3 + \lambda_4 S_4P_4 = 0.$$

By the Lemma the four composite cubics mentioned in this equation have $31 (= 2^5 - 1)$ common points, and these are common to the whole family. Also, S_5P_5 and S_6P_6 are members of the family. Considering all six composites, S_1P_1, S_2P_2, ... S_6P_6, we see that each pair of primes meet in a plane, which cuts the four remaining hyperspheres in circles having a common point. Since the six primes meet by twos in 15 planes we get 15 such points, each common to the six composites, making, with the 15 intersections of the primes (the vertices of the simplexes), a total of

* It is understood, of course, that the five planes are such that no four have a common point.

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30 common points. There is thus one point unaccounted for, and since the sections of five of the hyperspheres by the remaining prime give us the three-dimensional case of five spheres with no common point, this extra point cannot lie on any one of the primes. It can therefore lie only on the six hyperspheres themselves. Hence the six hyperspheres meet in a point.

5. Similarly, in five dimensions we have to account for 63 common points. The seven primes, taken in threes, give us 35 planes, each with a point common to S_1P_1, S_2P_2, ... S_7P_7; each prime cuts the remaining six hyperspheres in a system corresponding to the four-dimensional case and thus contains a common point, giving us another seven points. Counting in the 21 intersections of the primes taken by fives (the vertices of the simplexes), we get the required total of 63 points. There is thus no other point common to the seven spherical cubics, so that the seven hyperspheres do not meet in a point.

6. Proceeding in this way we can build up the general case. In space of n dimensions the equation of a spherical cubic contains $\frac{1}{2}(n^2+5n)$ constants and the $(n+2)$ primes intersect in ${}_{n+2}C_n$ points, giving $\frac{1}{2}(n+2)(n+1)$ vertices of simplexes. The family of spherical cubics through the vertices is thus $\Sigma_1{}^n\lambda_r S_r P_r = 0$. The common points of S_1P_1, S_2P_2, ... S_nP_n amount, by the Lemma, to $2^{n+1}-1$.

(a) When n is odd the location of the common points is as follows:

1 lies at the point in which each n primes intersect, giving ${}_{n+2}C_2$ pts.

1 ... in ... plane.....$(n-2)$..... ${}_{n+2}C_4$ pts.

1 ..... [4] ..... $(n-4)$..... ${}_{n+2}C_6$ pts.

.....

1 lies in the $[n-1]$ formed by each prime, giving ${}_{n+2}C_{n+1}$ pts.

The total number of common points is thus $\frac{1}{2}(2^{n+2}-2)$, i.e. $2^{n+1}-1$.

(b) When n is even the location of the common points is similar to the above, except that the final type of space containing a common point is an $[n-2]$.

The total number of common points is now

$$ {}_{n+2}C_2 + {}_{n+2}C_4 + {}_{n+2}C_6 + \ldots + {}_{n+2}C_n = 2^{n+1}-2. $$

7. It may be seen that the conditions imposed on the spherical cubics, of passing through the vertices of the simplexes, are independent. For the spherical cubic formed by the prime at infinity and any two of the given primes passes through all but

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one of the vertices, that is to say, satisfies all but one of the conditions. Hence this condition cannot be a consequence of the others.

8. The method employed above can clearly be applied to the figure obtained by inverting the system of primes and hyperspheres with respect to any point, O. Each prime becomes a hypersphere through O, and every n such hyperspheres intersect in a further point, A. Through every $(n+1)$ A's a hypersphere can be described, and we thus get a configuration, whether n be even or odd, of points lying in sets of $n+2$ on a hypersphere and of hyperspheres passing by sets of $n+2$ through the same point.

Note. Since writing the present paper I have discovered that Mr Grace used the same principle for proving the Kühne property in space of three and four dimensions*. He does not go beyond four dimensions, however, and there may be some little merit in showing that the method applies to general Euclidean space.

* Grace. "Circles, Spheres and Linear Complexes." *Trans. Camb. Phil. Soc.* Vol. XVI, p. 153 (1898).

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