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doi:10.1017/S026646660809021X

In practical situations it is often not clear whether a linear time trend has to be included in the unit root test equation. If the slope of the trend is small, including a trend reduces the power of the test, whereas the power also drops if a substantial trend is ignored. Similar problems arise if the initial value is large. David Harvey, Stephen Leybourne, and Robert Taylor explore a variety of combination techniques that can be used to improve the power of the test in such situations. The finding that a simple combination of tests such as the union of rejections (*UR*) strategy may substantially improve the power is indeed a very valuable contribution of the paper. In what follows I will confine myself to a discussion of a local trend. Similar comments apply to the problem of choosing the initial conditions.

The Stationary Case

Before considering the unit root case it may be useful to ask whether a similar situation arises in stationary models also or whether this problem is generic to unit root testing. Consider, for example, the linear regression model

$$Y_t = X_{1t}\beta_1 + X_{2t}\beta_2 + \varepsilon_t, \tag{1}$$

where X_{1t} and X_{2t} are mean-zero stationary regressors with $\mathrm{E}(X_{1t}^2) = \sigma_1^2$, $\mathrm{E}(X_{2t}^2) = \sigma_2^2$, and $\mathrm{E}(X_{1t}X_{2t}) = \sigma_{12} \geq 0$. For the errors we assume $\varepsilon_t \overset{i.i.d.}{\sim} N(0, \sigma_\varepsilon^2)$. Consider the situation where we want to test the hypothesis $H_0 : \beta_2 = 0$ against the one-sided alternative $H_1 : \beta_2 < 0$. Let $\beta_1 \equiv \beta_{1T} = T^{-1/2}\kappa$ and $\beta_2 \equiv \beta_{2T} = T^{-1/2}c$. Then, the regression t-statistic computed from the unrestricted regression is asymptotically distributed as $N(c\,\sigma_{2|1}/\sigma_\varepsilon, 1)$, where $\sigma_{2|1}^2 = \sigma_2^2 - \sigma_{12}^2/\sigma_1^2$. The restricted t-statistic based on the simple regression of Y_t on $X2t$ is asymptotically distributed as $N(\lambda_r, 1)$, where

$$\lambda_r = \frac{\sigma_{12}}{\sigma_\varepsilon \sigma_2}\kappa + \frac{\sigma_2}{\sigma_\varepsilon}c.$$

Assume that $\kappa \geq 0$ so that the restricted test using the usual one-sided critical value for $\kappa = 0$ is conservative. A comparison of the asymptotic power shows that the restricted test is more powerful if

$$\kappa < \frac{\sigma_2(\sigma_{2|1} - \sigma_2)}{\sigma_{12}}c, \qquad \text{where } c < 0.$$

In some sense the situation in stationary models is similar to the situation encountered in unit root testing. The power of a (one-sided) test may increase

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https://doi.org/10.1017/S026646660809021X Published online by Cambridge University Press

when dropping the other variable from the regression. Indeed, in empirical practice it is often recommended to remove regressors with insignificant t-statistic before testing the parameter of interest. Therefore, the rationale behind the pretest strategies is not specific to tests for a unit root.

Testing for Unit Roots

Let us first consider the case of a unit root test without any deterministic term, i.e., $H_0 : \rho = 1$ for $y_t = \rho y_{t-1} + \varepsilon_t$. Most unit root statistics can be written as a combination of two sufficient statistics $R_T = T^{-1} \sum_{t=2}^{T} \Delta y_t y_{t-1}$ and $S_T = T^{-2} \sum_{t=2}^{T} y_{t-1}^2$. Elliott, Rothenberg, and Stock (1996) show that the point optimal test statistic against the sequence of alternatives $H_\ell : \rho_T = 1 - c/T$ is a linear combination of the form $R_T + cS_T$. For a significance level of 0.05 the test based on the normalized autoregressive coefficient chooses $\phi_T^\rho = R_T + 8S_T$, and the Dickey–Fuller t-statistic is based on the nonlinear combination $\phi_T^\tau = R_T + 1.95\sqrt{S_T}$, where the test rejects for $\phi_T^j < 0$ ($j \in \{\rho, \tau\}$). The score statistic ignores S_T and the Bhargava–Sargan statistic ignores R_T, respectively. In this simple situation it turns out that test statistics with a large weight for S_T approach the power envelope for all values of c. Thus, the construction of a nearly optimal test is a simple task.

In the model with a local trend the situation becomes much more complicated. This is due to the fact that the point optimal test depends on two local parameters, c and κ, and besides R_T and S_T additional statistics (such as $\sum t y_{t-1}$) enter the point optimal statistic. Therefore, it is difficult to find a simple function of the involved statistics that comes close to the power envelope for all values of c and κ. The approach of Harvey et al. is to reduce the problem by focusing on (linear or nonlinear) combinations of the two statistics $DF\text{-}QD^\mu$ and $DF\text{-}QD^\tau$.

There are various ways to combine the two test statistics. First, we may *combine test decisions*. An example is the *UR* decision rule that rejects the null hypothesis if one of the two test statistics rejects. Second, one may *combine test statistics*. An example is the $WA(\lambda)$ test that results as a weighted average (WA) of the two test statistics, where the weights are functions of another statistic that measures the evidence for the trend specification. It is interesting to note that the *UR* test is equivalent to the test based on $\min_\lambda\{WA(\lambda)\}$. Third, one may compute the test statistic from minimizing a *penalized criterion function*, where the penalty term depends on $\sqrt{T}|\beta|$. A Bayesian approach would be another motivation for this class of test statistics.

There is a considerable degree of arbitrariness in the construction of a combined statistic. Consider the proposed $WA(\lambda)$ test, where λ depends on a test statistic W that represents the evidence for the null hypothesis $\beta = 0$. Harvey et al. suggest using the Vogelsang (1998) test statistic, but any other consistent test may be used. The weight function is specified as $\lambda(W) = \exp(-gT^{-1/2}W)$. Of course, this is not the only function with the property that $\lambda(W) \xrightarrow{p} 1$ for $\beta = 0$ and

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$\lambda(W) \xrightarrow{p} 0$ for $\beta \neq 0$. Instead of the exponential function one may also use, for example, the function $1/(1+gT^{-1/2}W)$. If a step function is applied, the pure pre-test approach results, which is the first procedure considered in Section 3.3 of the Harvey, Leybourne, and Taylor paper. Furthermore, it is not clear how to choose the constant g, and instead of $T^{-1/2}$ one may use some other rate $T^{-\delta}$, where $0 < \delta < 1$. If the pre-test strategy is used with a step function for $\lambda(W)$, it is important to select a suitable significance level for the pre-test. The authors use the traditional size 0.05, but it is not clear whether this is the best choice. To obtain properties similar to the *WA* approach the significance level of the pre-test must converge to zero as $T \to \infty$. This may be achieved by applying an information criterion such as the Schwarz or Hannan–Quinn criteron instead of a fixed significance level.

A somewhat surprising result of the simulation study is that the simple *UR* strategy outperforms more sophisticated pre-test strategies such as the $PT(|t_\beta|)$ and $WA(\lambda)$ tests. It is not clear whether this points to a general superiority of the former approach or whether the performance of the latter tests can be further improved by using different sizes of the pre-tests or by choosing alternative functions $\lambda(W)$. Unfortunately, there does not seem to exist any guidance from statistical theory on how to combine two statistics for the same null hypothesis in an optimal way. Practitioners tend to favor a pre-test approach with an inappropriate pre-test (the t-statistic for the trend in the Dickey–Fuller regression). Clearly, it would be desirable to develop a theory indicating how to combine two (or even more) test statistics for the same null hypothesis in an optimal way. It is however unlikely that the optimal combination of test statistics does not depend on the parameters κ and c, which cannot be estimated consistently.

Further Improvements

Is it possible to further improve the *UR* strategy? An undesirable feature of this approach is that the scaling constant, τ_γ, is applied in all situations even when it is not necessary. If there is strong evidence for a trend, then the probability that $DF\text{-}QD^\mu$ rejects tends to zero. Therefore, the *UR* strategy can be improved by using a pre-test strategy. If there is strong evidence for a trend, the critical value of the $DF\text{-}QD^\tau$ statistic can be used. This suggest the following pre-test *UR* rule: "If the pre-test indicates that there is a significant trend, then the critical value -2.85 is applied to the *UR* test. Otherwise, the critical value is inflated by the scaling factor 1.095." In Table 1 this modified *UR* rule, UR^*, is compared to the original *UR* test. I have used the simple test statistic $T^{-1}(y_T - y_1)^2/\omega^2$ as the pre-test[1] (i.e., test statistic t_0 in Appendix B of Harvey et al.). The null hypothesis $\beta = 0$ is rejected if $\lambda_0 > 3.84$. Of course, more sophisticated test statistics such as the ones discussed in the paper can be applied also and are likely to yield further improvements. The results are based on 10,000 replications of the AR(1) process $y_t = \mu + (\kappa/\sqrt{T})t + u_t$ and $u_t = (1 - c/T)u_{t-1} + \varepsilon_t$. The sample size is $T = 200$.

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TABLE 1. Empirical size and power

κ	$DF\text{-}QD^{\mu}$	$DF\text{-}QD^{\tau}$	UR	UR^{*}
		Size: $c = 0$		
0	0.0628	0.0626	0.0627	0.0626
1	0.0408	0.0642	0.0556	0.0591
2	0.0098	0.0593	0.0336	0.0525
3	0.0008	0.0608	0.0301	0.0579
4	0.0001	0.0626	0.0310	0.0625
		Power: $c = 5$		
0	0.3637	0.1397	0.2797	0.2797
1	0.0036	0.1365	0.0718	0.0719
2	0.0000	0.1387	0.0700	0.1063
3	0.0000	0.1319	0.0681	0.1319
4	0.0000	0.1394	0.0741	0.1394
		Power: $c = 10$		
0	0.7698	0.3526	0.6604	0.6604
1	0.0001	0.3470	0.2055	0.2055
2	0.0000	0.3480	0.2104	0.2809
3	0.0000	0.3493	0.2008	0.3493
4	0.0000	0.3475	0.2068	0.3475
		Power: $c = 15$		
0	0.9567	0.6468	0.9128	0.9128
1	0.0000	0.6464	0.4566	0.4566
2	0.0000	0.6472	0.4504	0.5501
3	0.0000	0.6408	0.4502	0.6408
4	0.0000	0.6543	0.4622	0.6543

Note: Entries report the rejection frequencies of alternative tests computed from 10,000 replications of an AR(1) model with local trend. The sample size is 200.

It turns out that the UR^{*} test has very stable size properties yielding actual sizes close to the nominal size. For small values of κ the power of the UR test is similar to the UR^{*} test. For large values of κ the UR^{*} test is substantially more powerful than the original UR test and approaches the power of the optimal test.

Conclusion

The important contribution of the paper of Harvey et al. is to suggest natural and simple strategies for combining test statistics that have the potential to improve the practice of unit root testing. The authors are to be congratulated for their clear exposition and for providing new insights into this important issue that has been puzzling practitioners for more than two decades.

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NOTE

1. In the simulations, ω is estimated by the sample variance of Δy_t. In cases where the errors are serially correlated, ω is estimated by using the usual estimator for the long-run variance of the residuals from the Dickey–Fuller regression, as otherwise the estimator for the long-run variance tends to zero (cf. Phillips and Oularis, 1990).

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doi:10.1017/S0266466608090221

This is a useful contribution to the solution of two problems that confront every user of unit root tests: how to handle the trend and what to do about the initial condition. The authors provide compelling evidence that in the setting they study, combinations of tests via the "union of rejections" (*UR*) rule work best. Will their work settle the debate?

1. The authors select from the plethora of tests available on the basis of frequency of use and simplicity, which is of course reasonable. However, I'd like to see some comments on the tests not studied in detail. For example, can we be reassured that no omitted procedure is a serious competitor in some plausible but unexamined setting? I'd hope so, but perhaps they could say why they think the list of tests they examine is long enough.
2. The paper ends with two speculations: one concerning more complicated trend functions and a second concerning a fourfold combination, UR^2, perhaps, that could deal with both of their sources of uncertainty simultaneously. In fact, in Ayat and Burridge (2000), which they cite, we considered a quadratic extension of the quasi-differencing (QD) test of Elliott, Rothenberg, and Stock (1996), within a sequential procedure. The *UR* approach might start to become unwieldy even in that simple case; do they have any comment on this? The fourfold combination is quite appealing, but considering it begs an obvious question: what other combinations of circumstances should we factor into our degree of "uncertainty"? Variance shifts, breaks, heavy-tailed distributions? Perhaps the two factors considered are indeed

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