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Dimensional characteristics of invariant measures for circle diffeomorphisms

VICTORIA SADOVSKAYA
Department of Mathematics and Statistics, University of South Alabama, Mobile, AL 36688, USA
(e-mail: sadovska@jaguar1.usouthal.edu)

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Abstract. We consider pointwise, box, and Hausdorff dimensions of invariant measures for circle diffeomorphisms. We discuss the cases of rational, Diophantine, and Liouville rotation numbers. Our main result is that for any Liouville number τ there exists a C^∞ circle diffeomorphism with rotation number τ such that the pointwise and box dimensions of its unique invariant measure do not exist. Moreover, the lower pointwise and lower box dimensions can equal any value $0 \le \beta \le 1$.

1. *Introduction*

The study of dimensional characteristics of invariant sets and measures was originated by physicists and applied mathematicians in the context of strange attractors. Beginning with the work of Eckmann and Ruelle [**3**] it developed into a rigorous mathematical theory. Dimension theory now plays an important role in dynamics [**10**]. Dimensional properties of invariant sets and measures are often related to other characteristics of the dynamical system, such as Lyapunov exponents and entropy.

In this paper we study pointwise, box, and Hausdorff dimensions of invariant measures for circle diffeomorphisms. The notion of pointwise (or local) dimension was introduced by Young [**11**]. It plays an important role in dimension theory of dynamical systems. For a Borel measure μ on a metric space X, its *lower and upper pointwise dimensions* at a point x are defined as

$$\underline{d}_\mu(x) = \liminf_{r\to 0} \frac{\log \mu(B(x,r))}{\log r} \quad \text{and} \quad \overline{d}_\mu(x) = \limsup_{r\to 0} \frac{\log \mu(B(x,r))}{\log r},$$

where $B(x,r)$ is a ball of radius r centered at x. If the two limits coincide, then their common value $d_\mu(x)$ is called the *pointwise dimension* of μ at x. The pointwise dimension describes the local distribution of the measure and of a typical orbit. It serves as an important tool for estimating the Hausdorff and box dimensions of measures and sets,

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see §2.2. For example, if the pointwise dimension of an ergodic measure μ exists almost everywhere, then all dimensional characteristics of μ coincide and give a fundamental characteristic of μ called the fractal dimension.

In [**2**] Barreira, Pesin, and Schmeling showed that pointwise dimension exits for any hyperbolic invariant measure of a $C^{1+\alpha}$ diffeomorphism. Dimensional structures of non-hyperbolic measures can be more complicated. In [**9**] Ledrappier and Misiurewicz constructed an example of a C^r map of an interval preserving an ergodic measure whose pointwise dimension does not exist almost everywhere. A natural class of non-hyperbolic measures is given by invariant measures for circle diffeomorphisms with irrational rotation numbers. In [**8**] we constructed examples of such diffeomorphisms for which pointwise dimension of the measures does not exist almost everywhere.

Dimensional properties of an invariant measure of a circle diffeomorphism f depend significantly on the rotation number of f (see §2.1). First we consider the simpler cases of rational and Diophantine numbers, and then describe our main result for Liouville numbers. The rotation number of f is rational if and only if f has periodic points. Such a diffeomorphism may preserve a variety of measures with different properties, however, any *ergodic* invariant measure for f is a uniform δ-measure on a periodic orbit. This immediately implies the following result.

PROPOSITION 1.1. *Let f be a circle homeomorphism with a rational rotation number and let μ be an ergodic invariant measure for f. Then:*
(1) $d_\mu(x) = 0$ *for μ-almost every x in S^1;*
(2) $\dim_H \mu = \underline{\dim}_B \mu = \overline{\dim}_B \mu = 0$.

In contrast, diffeomorphisms with an irrational rotation number are uniquely ergodic. In this case, the properties of the invariant measure depend on how well the irrational rotation number can be approximated by rational numbers. The numbers that cannot be rapidly approximated by rationals are called Diophantine.

Definition 1.2. A number τ is called *Diophantine* if there exist $\delta > 0$ and $K > 0$ such that

$$|\tau - p/q| > K/|q|^{2+\delta} \quad \text{for any integers } p \text{ and } q. \tag{1.1}$$

Circle diffeomorphisms with Diophantine rotation numbers are smoothly conjugate to rotations. Therefore, the invariant measure for such a diffeomorphism is equivalent to the Lebesgue measure and hence has the same dimensional properties.

PROPOSITION 1.3. *Let f be a C^∞ circle diffeomorphism with a Diophantine rotation number. Then for its unique invariant measure μ:*
(1) $d_\mu(x) = 1$ *for every x in S^1;*
(2) $\dim_H \mu = \underline{\dim}_B \mu = \overline{\dim}_B \mu = 1$.

The most interesting case is that of Liouville rotation numbers. These are irrational numbers that can be rapidly approximated by rationals, more precisely.

Definition 1.4. An irrational number τ is called a *Liouville number* if for any $n \geq 1$ there exist integers p and q, $q > 1$, such that

$$|\tau - p/q| < 1/q^n. \tag{1.2}$$

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Clearly, an irrational number is Liouville if and only if it is not Diophantine. The set of all Liouville numbers is a dense G_δ set in $\mathbb{R}$, and it has zero Lebesgue measure. Our main result, Theorem 1.5, shows that in the case of a Liouville rotation number different types of dimensional properties of the invariant measure are realized. In particular, pointwise and box dimensions may not exist, which is in contrast to the Diophantine case as well as to the case of hyperbolic measures.

THEOREM 1.5. *Let τ be a Liouville number and let $0 \le \beta \le 1$. There exists a C^∞ circle diffeomorphism f with rotation number τ such that for its unique invariant measure μ:*

(1) $\underline{d}_\mu(x) = \beta$ *and* $\overline{d}_\mu(x) = 1$ *for μ-almost every x in S^1;*

(2) $\dim_H \mu = \underline{\dim}_B \mu = \beta$ *and* $\overline{\dim}_B \mu = 1$.

It is an interesting open question whether there exists a circle diffeomorphism with irrational rotation number whose invariant measure has upper pointwise dimension less than one on a set of positive measure.

Our constructions are based on a method developed by Anosov and Katok [**1**] to produce examples of diffeomorphisms with specific ergodic properties. In [**8**] we used this method to construct examples of diffeomorphisms satisfying parts (1) and (2) of Theorem 1.5. However, the qualitative nature of the arguments did not allow us to construct examples for a given rotation number, or even describe explicitly the rotation numbers in our examples. In this paper we use some ideas developed in [**4, 5**] to make an explicit construction with specific quantitative estimates. This allows us to produce the examples for *all* Liouville rotation numbers.

We note that any C^2 circle diffeomorphism f with irrational rotation number is topologically conjugate to the corresponding rotation. The conjugacy gives the distribution function of the invariant measure μ. In the theorem above, μ is singular for $\beta < 1$, and so is the conjugacy. Thus, the theorem implies that for any Liouville rotation number there exists diffeomorphisms with singular conjugacy. Similar methods may be used to construct diffeomorphisms with a specific degree of regularity of the conjugacy for any Liouville rotation number.

2. *Preliminaries*

2.1. *Rotation number of a circle homeomorphism.* See [**7**] for more details of the material covered in this section.

Let f be an orientation-preserving homeomorphism of S^1, let $\pi : \mathbb{R} \to S^1 = \mathbb{R}/\mathbb{Z}$ be the natural projection, and let F be a homeomorphism of $\mathbb{R}$ such that $f \circ \pi = \pi \circ F$. Then the following limit exists and has the same value for all x:

$$\tau(F) = \lim_{|n|\to\infty} \frac{1}{n}(F^n(x) - x).$$

The number $\tau(f) = \pi(\tau(F))$ is called the *rotation number* of f. If $h : S^1 \to S^1$ is a homeomorphism, then $\tau(h^{-1} \circ f \circ h) = \tau(f)$. In particular, if f is topologically conjugate to a rotation by τ then f has rotation number τ.

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2.2. *Hausdorff and box dimensions of sets and measures.* See [**10**] for more details of the material covered in this section.

The *upper and lower box dimensions* of a set $Z \subset \mathbb{R}^k$ are defined as

$$\overline{\dim}_B Z = \limsup_{\varepsilon \to 0} \frac{\log N(Z, \varepsilon)}{\log(1/\varepsilon)} \quad \text{and} \quad \underline{\dim}_B Z = \liminf_{\varepsilon \to 0} \frac{\log N(Z, \varepsilon)}{\log(1/\varepsilon)},$$

where $N(Z, \varepsilon)$ is the lowest number of balls of diameter ε needed to cover Z.

For a number $\alpha \geq 0$, the α-Hausdorff measure of Z is

$$m_H(Z, \alpha) = \lim_{\varepsilon \to 0} \inf_{\mathcal{G}} \sum_{U \in \mathcal{G}} (\operatorname{diam} U)^\alpha,$$

where the infimum is taken over all finite or countable coverings $\mathcal{G}$ of Z by open sets with diameter at most ε. The *Hausdorff dimension* of Z is

$$\dim_H Z = \inf\{\alpha \mid m_H(Z, \alpha) = 0\} = \sup\{\alpha \mid m_H(Z, \alpha) = \infty\}.$$

The Hausdorff and upper and lower box dimensions of a Borel probability measure μ are defined as follows:

$$\begin{aligned} \dim_H \mu &= \inf\{\dim_H Z \mid \mu(Z) = 1\}, \\ \underline{\dim}_B \mu &= \lim_{\varepsilon \to 0} \inf\{\underline{\dim}_B Z \mid \mu(Z) > 1 - \varepsilon\}, \\ \overline{\dim}_B \mu &= \lim_{\varepsilon \to 0} \inf\{\overline{\dim}_B Z \mid \mu(Z) > 1 - \varepsilon\}. \end{aligned} \tag{2.1}$$

It is known that $\dim_H \mu \leq \underline{\dim}_B \mu \leq \overline{\dim}_B \mu$.

The following result by Young [**11**] shows how the pointwise dimension of a measure can be used to estimate its box and Hausdorff dimensions.

THEOREM 2.1. *Let μ be a Borel finite measure on $\mathbb{R}^m$. Then:*

(1) *if $\underline{d}_\mu(x) \geq d$ for μ-almost every x, then $\dim_H \mu \geq d$;*

(2) *if $\overline{d}_\mu(x) \leq d$ for μ-almost every x, then $\overline{\dim}_B \mu \leq d$;*

(3) *if $\underline{d}_\mu(x) = \overline{d}_\mu(x) = d$ for μ-almost every x, then $\dim_H \mu = \underline{\dim}_B \mu = \overline{\dim}_B \mu = d$.*

3. *Proofs*

Throughout this paper we identify the unit circle $S^1 = \mathbb{R}/\mathbb{Z}$ with the interval $[0, 1]$. Let τ be an irrational number and let R_τ be the rotation by τ. The Lebesgue measure is the only measure preserved by R_τ. Suppose that

$$f = h^{-1} \circ R_\tau \circ h,$$

where h is a homeomorphism. Then the unique invariant measure μ for f is the push-forward of the Lebesgue measure λ by h^{-1}, i.e. $\mu(A) = \lambda(hA)$. This means that h is the distribution function for μ, i.e.

$$\mu([x_1, x_2)) = h(x_2) - h(x_1)$$

for any interval $[x_1, x_2) \subset S^1$ and, hence,

$$\mu(B(x, r)) = \Delta h(x, r) \stackrel{\text{def}}{=} h(x + r) - h(x - r). \tag{3.1}$$

In particular, if h is continuously differentiable, then for any x and $r < 1/2$,

$$2r \cdot \min_{[x-r, x+r]} |h'| \leq \mu(B(x, r)) \leq 2r \cdot \max_{[x-r, x+r]} |h'|.$$

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3.1. *Proof of Proposition 1.3.* The following result was established by Herman [**6**]: any $C^{2+\varepsilon}$ circle diffeomorphism whose rotation number τ satisfies the Diophantine condition (1.1) with some $K > 0$ and $0 < \delta < \varepsilon$ is conjugate to the rotation R_τ via a C^1 diffeomorphism. It follows that any C^∞ diffeomorphism with a Diophantine rotation number is smoothly conjugate to the corresponding rotation. This implies that there exist constants m and M such that $2mr \le \mu(B(x, r)) \le 2Mr$ for all x and all $r < 1/2$. Therefore, $d_\mu(x) = 1$ for every $x \in S^1$ and, hence, $\dim_H \mu = \underline{\dim}_B \mu = \overline{\dim}_B \mu = 1$ by Theorem 2.1. □

3.2. *Proof of Theorem 1.5.* Let τ be a Liouville number. First we note that to obtain the result for the case of $\beta = 1$ it suffices to take $f = R_\tau$. The rotation R_τ preserves the Lebesgue measure, which satisfies (1) and (2). From now on we assume that $0 \le \beta < 1$.

We obtain the diffeomorphism f as a limit of a sequence of diffeomorphisms

$$f_n = h_n^{-1} \circ R_{\tau_n} \circ h_n,$$

where h_n are C^∞ diffeomorphisms of S^1 and τ_n are rational numbers that converge to τ. The sequences $\{h_n\}$ and $\{\tau_n\}$ are defined inductively. Once h_n is selected, we construct h_{n+1} in the form

$$h_{n+1} = A_n \circ h_n \quad \text{where } A_n = \mathrm{Id} + a_n$$

is a diffeomorphism and a_n is a C^∞ periodic function.

The diffeomorphisms f_n will converge in the C^∞ topology, and h_n as well as h_n^{-1} will converge in C^0. The homeomorphism $h = \lim_{n\to\infty} h_n$ will give the distribution function of the invariant measure μ for f.

We use the following norm of diffeomorphisms and the corresponding distance.

Definition 3.1. Let g be a C^n diffeomorphism of $[0, 1]$. We define a norm of g

$$\|g\|_n^* = \max |g^{(i)}(x)|,$$

where the maximum is taken over all x and $0 \le i \le n$, and we use the notation

$$\|g\|_n = \max\{\|g\|_n^*, \|g^{-1}\|_n^*\}.$$

For two C^n diffeomorphisms g_1 and g_2, we set

$$d_n(g_1, g_2) = \max\{\|g_1 - g_2\|_n^*, \|g_1^{-1} - g_2^{-1}\|_n^*\}.$$

In the three lemmas below we use Faà di Bruno's formula, which generalizes the chain rule to higher derivatives:

$$\frac{d^n}{dx^n} f(g(x)) = \sum c_{m_1,\dots,m_n} f^{(m_1+\cdots+m_n)}(g(x)) \prod_{j=1}^{n} (g^{(j)}(x))^{m_j}, \tag{3.2}$$

where the constants $c_{m_1,\dots,m_n}$ depend only on $m_1, \dots, m_n$, and the sum is taken over all n-tuples $(m_1, \dots, m_n)$ of integers satisfying

$$1m_1 + 2m_2 + \cdots + nm_n = n \quad \text{and} \quad m_i \ge 0, \quad i = 1, \dots, n. \tag{3.3}$$

The following lemma gives an estimate for the distance between two maps conjugate to two rotations via the same diffeomorphism.

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LEMMA 3.2. *Let R_{τ_1} and R_{τ_2} be two circle rotations, and let h be a C^{n+1} circle diffeomorphism. Then*

$$d_n(h^{-1} \circ R_{\tau_1} \circ h, h^{-1} \circ R_{\tau_2} \circ h) \le c_n |\tau_1 - \tau_2| \cdot \|h\|_{n+1}^{n+1}, \tag{3.4}$$

where the constant c_n depends only on n.

Proof. We estimate

$$\max \left| \frac{d^k}{dx^k}(h^{-1} \circ R_{\tau_1} \circ h) - \frac{d^k}{dx^k}(h^{-1} \circ R_{\tau_2} \circ h) \right|$$

for $0 \le k \le n$. For $k = 0$ we have

$$\max |h^{-1}(h(x) + \tau_1) - h^{-1}(h(x) + \tau_2)| \le \max |(h^{-1})'| \cdot |\tau_1 - \tau_2| \le \|h\|_1 \cdot |\tau_1 - \tau_2|.$$

For $1 \le k \le n$, formula (3.2) yields

$$\begin{aligned} \frac{d^k}{dx^k}(h^{-1} \circ R_\tau \circ h)(x) &= \frac{d^k}{dx^k}(h^{-1}(h(x) + \tau)) \\ &= \sum c_{m_1,\dots,m_k} (h^{-1})^{(m_1+\cdots+m_k)}(h(x) + \tau) \prod_{j=1}^{k} (h^{(j)}(x))^{m_j}. \end{aligned}$$

We estimate the difference between the corresponding terms in $(d^k/dx^k)(h^{-1} \circ R_{\tau_1} \circ h)(x)$ and $(d^k/dx^k)(h^{-1} \circ R_{\tau_2} \circ h)(x)$. By (3.3), $m_1 + \cdots + m_k \le k$, and we have

$$\begin{aligned} &\left| \prod_{j=1}^{k} (h^{(j)}(x))^{m_j} ((h^{-1})^{(m_1+\cdots+m_k)}(h(x) + \tau_1) - (h^{-1})^{(m_1+\cdots+m_k)}(h(x) + \tau_2)) \right| \\ &\quad \le \max \left| \prod_{j=1}^{k} (h^{(j)}(x))^{m_j} \right| \cdot \max |(h^{-1})^{(m_1+\cdots+m_k+1)}| \cdot |\tau_1 - \tau_2| \\ &\quad \le \|h\|_k^k \cdot \|h\|_{k+1} \cdot |\tau_1 - \tau_2| \le \|h\|_{k+1}^{k+1} \cdot |\tau_1 - \tau_2|. \end{aligned}$$

It follows that for any $0 \le k \le n$,

$$\max \left| \frac{d^k}{dx^k}(h^{-1} \circ R_{\tau_1} \circ h)(x) - \frac{d^k}{dx^k}(h^{-1} \circ R_{\tau_2} \circ h)(x) \right| \le c_n |\tau_1 - \tau_2| \cdot \|h\|_{n+1}^{n+1},$$

where c_n is the sum of the coefficients $c_{m_1,\dots,m_n}$ in (3.2). Since $(h^{-1} \circ R_\tau \circ h)^{-1} = h^{-1} \circ R_{-\tau} \circ h$, we have the same estimate for the inverses of the functions, and (3.4) follows. □

When constructing the diffeomorphism $h_{n+1} = A_n \circ h_n$, we choose the function A_n in a specific form, as in the following lemma.

LEMMA 3.3. *Let s and δ be positive numbers such that $1/s$ is an integer and $\delta < s/2$. Then there exists a C^∞ diffeomorphism $A = A_{s,\delta}$ of $[0, 1]$ such that:*

(1) *$A = \mathrm{Id} + a$, where $a = a_{s,\delta}$ is a non-negative C^∞ function of period s;*
(2) *$A(0) = 0$, $A(\delta) = s - \delta$, and $A(s) = s$;*
(3) *$\delta/(2s) \le A'(x) \le 2s/\delta$ for all x;*
(4) *for each $n \ge 0$ there exists a constant ρ_n that does not depend on δ and s such that $\|A\|_n \le \rho_n/\delta^{n^2}$.*

Conditions (1), (2), and (3) guarantee that A is a diffeomorphism.

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Proof. First we construct the function a on the interval $[0, \delta]$. Let g be a C^∞ function on $[0, 1]$ such that $g(x) = 0$ in a neighborhood $[0, \varepsilon)$ of 0, $g(x) = 1$ in a neighborhood $(1 - \varepsilon, 1]$ of one, and $0 \le g'(x) \le 2$ for all x. We obtain the function a on $[0, \delta]$ by rescaling g:

$$a(x) = (s - 2\delta)g(x/\delta) \quad \text{for } x \in [0, \delta].$$

Then $a(x) = 0$ in a neighborhood of zero, $a(x) = s - 2\delta$ in a neighborhood of δ, and $0 \le a'(x) \le 2(s - 2\delta)/\delta = 2s/\delta - 4$. This implies that $A(x) = x$ in a neighborhood of zero, $A(\delta) = s - \delta$, $A'(x) = 1$ in a neighborhood of δ, and $1 \le A'(x) \le 2s/\delta - 3 \le 2s/\delta$ for all x in $[0, \delta]$.

Now we obtain the graph of the function A on $[\delta, s]$ by reflecting its graph on $[0, \delta]$ with respect to the line $y = s - x$, i.e.

$$A(x) = s - A^{-1}(s - x) \quad \text{for } x \text{ in } [s - \delta, s]. \tag{3.5}$$

It follows from the symmetry of the graph that $A(x) = x$ in a neighborhood of s and $1 \ge A'(x) \ge \delta/2s$ for all x in $[s - \delta, s]$. On this interval, we set $a(x) = A(x) - x$. Clearly, $a(x) \ge 0$ for all x and $a(x) = 0$ in a neighborhood of s. Then we extend a to $[0, 1]$ by periodicity, and thus obtain $A = \mathrm{Id} + a$ on $[0, 1]$. Thus, we have constructed a function A satisfying (1), (2), and (3).

Now we verify (4) for A on $[0, \delta]$. Then (3.5) implies that (4) is also satisfied for A on $[\delta, s]$. Since $\max_{[0,\delta]} |a^{(n)}| \le \max_{[0,1]} |g^{(n)}|/\delta^n$, we have

$$\max_{[0,\delta]} |A^{(n)}| \le \left(\max_{[0,1]} |g^{(n)}| + 1\right)/\delta^n \stackrel{\text{def}}{=} \kappa_n/\delta^n \quad \text{for any } n \ge 0. \tag{3.6}$$

Let G be the inverse function for $A|_{[0,\delta]}$. We show using induction that for any $n \ge 0$ there exists a constant ξ_n independent of δ and s such that

$$\max_{[0,1]} |G^{(n)}| \le \xi_n/\delta^{n^2} \quad \text{and} \quad \xi_n \ge \xi_{n-1} \ge \cdots \ge \xi_0.$$

Clearly, $G(x) \le 1$ and $G'(x) \le 1 \le 1/\delta$ for all x. Thus, the statement holds for $k = 0$ and $k = 1$. Suppose that it holds for all $0 \le k \le n - 1$. For $n \ge 2$,

$$\frac{d^n}{dx^n} G(A(x)) = \frac{d^n}{dx^n} x = 0,$$

and it follows from (3.2) that

$$G^{(n)}(A(x)) \cdot (A'(x))^n = -\sum c_{m_1,\ldots,m_n} G^{(m_1+\cdots+m_n)}(A(x)) \prod_{j=1}^{n} (A^{(j)}(x))^{m_j},$$

where the sum is taken over all n-tuples $(m_1, \ldots, m_n)$ such that $1m_1 + \cdots + nm_n = n$ and $m_1 \ne n$. This implies that $m_1 + \cdots + m_n \le n - 1$ and, hence,

$$|G^{(m_1+\cdots+m_n)}(A(x))| \le \xi_{n-1}/\delta^{(n-1)^2}$$

by the induction assumption. Also,

$$\left|\prod_{j=1}^{n} (A^{(j)}(x))^{m_j}\right| \le \prod_{j=1}^{n} (\kappa_j/\delta^j)^{m_j} = \left(\prod_{j=1}^{n} \kappa_j^{m_j}\right)/\delta^{1m_1+2m_2+\cdots+nm_n} \le \left(\prod_{j=1}^{n} \kappa_j^n\right)/\delta^n.$$

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Using the estimates above, we obtain

$$|G^{(n)}(A(x)) \cdot (A'(x))^n| \le \xi_n/\delta^{(n-1)^2+n} \le \xi_n/\delta^{n^2},$$

where $\xi_n \ge \xi_{n-1}$ is a constant independent of s and δ. Since $A'(x) \ge 1$ on $[0, \delta]$, it follows that

$$|G^{(n)}(A(x))| \le \xi_n/\delta^{n^2} \quad \text{for all } x \in [0, \delta].$$

Let $\rho_n = \max\{\kappa_0, \dots, \kappa_n, \xi_n\}$, where $\kappa_0, \dots \kappa_n$ are as in (3.6). Then

$$\|A\|_n^* \le \rho_n/\delta^n \le \rho_n/\delta^{n^2}, \quad \|A^{-1}\|_n^* = \|G\|_n^* \le \rho_n/\delta^{n^2},$$

and, thus, $\|A\|_n \le \rho_n/\delta^{n^2}$. □

LEMMA 3.4. *Let $A = A_{s,\delta}$ be a function as in Lemma 3.3. Then for any $n \ge 0$ and any C^n diffeomorphism h*

$$\|A \circ h\|_n \le \tilde{c}(h, n)/\delta^{n^2}, \tag{3.7}$$

where the constant $\tilde{c}(h, n)$ depends on h and n, but not on δ and s.

Proof. Let $0 \le k \le n$. Each term of the sum representing $(A \circ h)^{(k)}$ is a product of a derivative of A of order at most k and at most k derivatives of h, see (3.2). Therefore, each term can be estimated by $\|A\|_k \cdot \|h\|_k^k \le \|A\|_n \cdot \|h\|_n^n$ and, hence,

$$\|A \circ h\|_n^* \le c_n \|A\|_n \cdot \|h\|_n^n \le c_n \|h\|_n^n \cdot \rho_n/\delta^{n^2},$$

where c_n is the sum of the constants $c_{m_1,\dots,m_n}$ in (3.2) and ρ_n is as in Lemma 3.3(4). Each term of $(h^{-1} \circ A^{-1})^{(k)}$ can be estimated as follows:

$$|(h^{-1})^{(m_1+\cdots+m_k)}(A^{-1}(x)) \prod_{j=1}^{k} ((A^{-1})^{(j)}(x))^{m_j}| \le \|h\|_k \cdot (\rho_1/\delta^{1^2})^{m_1} \cdots (\rho_k/\delta^{k^2})^{m_k}$$

$$\le \|h\|_k \cdot \rho_1 \cdots \rho_k/\delta^{1^2 m_1 + 2^2 m_2 + \cdots + k^2 m_k} \le \|h\|_k \cdot \rho_1 \cdots \rho_k/\delta^{k^2} \le \|h\|_n \cdot \rho_1 \cdots \rho_n/\delta^{n^2}$$

since $1^1 m_1 + 2^2 m_2 + \cdots + k^2 m_k \le k(1m_1 + 2m_2 + \cdots + km_k) = k^2$. It follows that

$$\|h^{-1} \circ A^{-1}\|_n^* \le c_n \|h\|_n \cdot \rho_1 \cdots \rho_n/\delta^{n^2}.$$

Thus $\|A \circ h\|_n \le \tilde{c}(h, n)/\delta^{n^2}$. □

For the rest of the proof we consider the cases of $\beta = 0$ and of $0 < \beta < 1$ separately.

The case of $\beta = 0$. We construct the sequences $\{\tau_n\}_{n=1}^{\infty}$ and $\{h_n\}_{n=1}^{\infty}$ inductively. Let h_1 be the identity map, τ_1 be a rational number close to τ, and $s_1 = 1/2$. Suppose that a number $\tau_{n-1} = p_{n-1}/q_{n-1}$, a function a_{n-1} of period s_{n-1}, and hence diffeomorphisms $A_{n-1} = \mathrm{Id} + a_{n-1}$ and $h_n = A_{n-1} \circ h_{n-1}$ are selected. We denote

$$M_n = \max_{[0,1]} h_n' \quad \text{and} \quad m_n = \min_{[0,1]} h_n' = 1/\max_{[0,1]}(h_n^{-1})'.$$

Clearly, $M_n \ge 1$ and $0 < m_n \le 1$ for all $n \ge 1$, and $M_n \to \infty$, $m_n \to 0$ as $n \to \infty$.

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We choose a rational number $\tau_n = p_n/q_n$, numbers s_n and δ_n, a function a_n of a period s_n, and a function A_n such that

$$\begin{aligned}
&\text{(i) } |\tau - \tau_n| \le |\tau - \tau_{n-1}|,\\
&\text{(ii) } |\tau - \tau_n| = |\tau - p_n/q_n| \le 1/q_n^{3n^4},\\
&\text{(iii) } q_n \ge \max\{1/s_{n-1},\ 1/m_n,\ (3M_n)^n,\ c_n,\ \tilde{c}(h_n, n+1)\},\\
&\quad \text{where } c_n \text{ is as in (3.4) and } \tilde{c}(h_n, n+1) \text{ is as in (3.7)},\\
&\text{(iv) } s_n = s_{n-1}/q_n, \text{ hence } s_n \le s_{n-1}^2 \text{ and } s_n \le 1/2^n,\\
&\text{(v) } \delta_n = s_n^n, \text{ in particular, } \delta_n < s_n/2,\\
&\text{(vi) } a_n \text{ and } A_n \text{ are as in Lemma 3.3 with } \delta = \delta_n \text{ and } s = s_n.
\end{aligned} \tag{3.8}$$

Conditions (i), (ii), and (iii) can be satisfied since τ is a Liouville number. Condition (iv) ensures that the maps A_n and R_{τ_n} commute and, hence,

$$\begin{aligned}
h_n^{-1} \circ R_{\tau_n} \circ h_n &= h_n^{-1} \circ A_n^{-1} \circ A_n \circ R_{\tau_n} \circ h_n\\
&= h_n^{-1} \circ A_n^{-1} \circ R_{\tau_n} \circ A_n \circ h_n = (A_n \circ h_n)^{-1} \circ R_{\tau_n} \circ (A_n \circ h_n).
\end{aligned}$$

Conditions (iii), (iv), and (v) imply that $1/\delta_n = 1/s_n^n = q_n^n/s_{n-1}^n \le q_n^{2n}$. Using this as well as (3.4) and (3.7) we obtain

$$\begin{aligned}
d_n(f_{n+1},\ f_n) &= d_n(h_{n+1}^{-1} \circ R_{\tau_{n+1}} \circ h_{n+1},\ h_n^{-1} \circ R_{\tau_n} \circ h_n)\\
&= d_n((A_n \circ h_n)^{-1} \circ R_{\tau_{n+1}} \circ (A_n \circ h_n),\ (A_n \circ h_n)^{-1} \circ R_{\tau_n} \circ (A_n \circ h_n))\\
&\le c_n|\tau_{n+1} - \tau_n| \cdot \|A_n \circ h_n\|_{n+1}^{n+1} \le 2c_n|\tau - \tau_n| \cdot (\tilde{c}(h_n, n+1)/\delta_n^{(n+1)^2})^{n+1}\\
&\le 2q_n(1/q_n^{3n^4})(q_n \cdot q_n^{2n(n+1)^2})^{n+1} \le 2(1/q_n^{3n^4})q_n^{2n(n+1)^3+n+2} \le 1/2^n
\end{aligned} \tag{3.9}$$

for all sufficiently large n. Since $d_m(f_{n+1},\ f_n) \le d_n(f_{n+1},\ f_n) \le 1/2^n$ for $n \ge m$, it follows that the sequence $\{f_n\}$ converges in the C^m-topology for any m, i.e. it converges in the C^∞ topology.

Now we establish the convergence of the diffeomorphisms h_n. We recall that $A_n = \mathrm{Id} + a_n$ is a diffeomorphism, where a_n is a C^∞ function of period s_n, and $1/s_n$ is an integer. It follows that $\max_{[0,1]} |A_n - \mathrm{Id}| \le s_n$ and $\max_{[0,1]} |A_n^{-1} - \mathrm{Id}| \le s_n$. Since $h_{n+1} = A_n \circ h_n$, we estimate

$$\max_{[0,1]} |h_{n+1} - h_n| = \max_{[0,1]} |(A_n - \mathrm{Id}) \circ h_n| \le s_n, \quad \text{and}$$

$$\max_{[0,1]} |h_{n+1}^{-1} - h_n^{-1}| = \max_{[0,1]} |h_n^{-1} \circ (A_n^{-1} - \mathrm{Id})| \le \max_{[0,1]} (h_n^{-1})' \cdot s_n = (1/m_n)s_n \le s_{n-1}$$

by conditions (iii) and (iv) of (3.8). This implies that $d_0(h_{n+1}, h_n) \le s_{n-1}$, and since $s_n \le 1/2^n$, it follows that the sequence of diffeomorphisms $\{h_n\}$ converges to a homeomorphism h with respect to the distance d_0. Moreover, since $s_n \le s_{n-1}^2$ we have

$$\max_{[0,1]} |h - h_n| \le \sum_{k=n}^{\infty} s_k \le 2s_n. \tag{3.10}$$

Now we prove the dimensional properties of the invariant measure μ with distribution function h. By construction, most of the growth of A_n on the interval [0, 1] is concentrated

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on the union of intervals $[is_n, is_n + \delta_n]$. Let

$$\tilde{E}_n = \bigcup_{i=0}^{(1/s_n)-1} [is_n, is_n + \delta_n] \quad \text{and} \quad E_n = h_n^{-1}(\tilde{E}_n). \tag{3.11}$$

Then $h_{n+1}(E_n) = (A_n \circ h_n)(h_n^{-1}(\tilde{E}_n)) = A_n(\tilde{E}_n)$.

Since $\tilde{E}_n$ consists of $1/s_n$ intervals $[is_n, is_n + \delta_n]$ and $A(is_n + \delta_n) - A(is_n) = s_n - \delta_n$, the total growth of A_n on $\tilde{E}_n$ is

$$(s_n - \delta_n)(1/s_n) = 1 - \delta_n/s_n = 1 - s_n^n/s_n \geq 1 - s_n^{n-1} \geq 1 - s_n,$$

and the total growth of h_{n+1} on E_n is the same. By (3.10), $\Delta h \geq \Delta h_n - 4s_{n+1}$ on each of the $1/s_n$ intervals in E_n. Since $s_{n+1} \leq s_n^2$ and $s_n \leq 1/2^n$ by condition (iv) of (3.8), we estimate that the total growth of h on the set E_n is at least

$$1 - s_n - 4s_{n+1}(1/s_n) \geq 1 - s_n - 4s_n^2/s_n = 1 - 5s_n \geq 1 - 5/2^n.$$

Thus, for the measure μ with distribution function h,

$$\mu(E_n) \geq 1 - 5/2^n. \tag{3.12}$$

Now we show that $\underline{d}_\mu(x)$, the lower pointwise dimension of μ at x, is zero for μ-almost every x. We recall that $m_n = \min_{[0,1]} h_n'$. The length of each interval I in the set E_n is bounded above by δ_n/m_n, since the length of $h_n(I)$ is δ_n. Let

$$r_n = \delta_n/m_n = s_n^n/m_n.$$

Let x be a point in E_n. Then the interval $[x - r_n, x + r_n]$ contains one of the intervals in E and, hence, $\Delta h_{n+1}(x, r_n) \geq s_n - \delta_n$. It follows from (3.10) that

$$\Delta h(x, r_n) \geq \Delta h_{n+1}(x, r_n) - 4s_{n+1} \geq s_n - \delta_n - 4s_{n+1} \geq s_n/2$$

for all sufficiently large n since $\delta_n = s_n^n$ and $s_{n+1} \leq s_n^2$. Therefore,

$$\frac{\log \Delta h(x, r_n)}{\log r_n} \leq \frac{\log(s_n/2)}{\log r_n} = \frac{\log(s_n/2)}{\log(s_n^n/m_n)} \leq \frac{2}{n}.$$

The last inequality is equivalent to $s_n \leq m_n^{2/n}/2$, and it follows from (3.8) that $s_n = s_{n-1}/q_n \leq s_{n-1}m_n \leq m_n/2$.

Thus, for any sufficiently large n there exists $r_n > 0$ such that

$$\frac{\log \mu(B(x, r_n))}{\log r_n} = \frac{\log \Delta h(x, r_n)}{\log r_n} \leq \frac{2}{n} \quad \text{for any } x \in E_n,$$

and $r_n \to 0$ as $n \to \infty$. Let x be a point in $[0, 1]$. If follows that $\underline{d}_\mu(x) = 0$ provided that for any m there exist $n \geq m$ such that $x \in E_n$. Otherwise, x is in $J = \bigcup_{m=1}^{\infty} \bigcap_{n=m}^{\infty}([0, 1] - E_n)$. It follows from (3.12) that $\mu(\bigcap_{n=m}^{\infty}([0, 1] - E_n)) = 0$ and hence $\mu(J) = 0$. We conclude that

$$\underline{d}_\mu(x) = 0 \quad \text{for } \mu\text{-almost every } x \in S^1. \tag{3.13}$$

Now we show that $\overline{d}_\mu(x)$, the upper pointwise dimension of μ at x, equals one for μ-almost all x. We recall that $M_n = \max_{[0,1]} h_n'$, and hence $\Delta h_n(x, r) \leq 2rM_n$. We take

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$\tilde{r}_n = (3M_n)^{-n}$ and note that by (3.8) $s_n = s_{n-1}/q_n < 1/q_n \le 1/(3M_n)^n \le \tilde{r}_n$. It follows that

$$\Delta h(x, \tilde{r}_n) \le \Delta h_n(x, \tilde{r}_n) + 4s_n \le 2\tilde{r}_n M_n + 4\tilde{r}_n \le 3M_n\tilde{r}_n$$

for all sufficiently large n. Hence, for all x,

$$\frac{\log \mu(B(x, \tilde{r}_n))}{\log \tilde{r}_n} = \frac{\log \Delta h(x, \tilde{r}_n)}{\log \tilde{r}_n} \ge \frac{\log(3M_n\tilde{r}_n)}{\log \tilde{r}_n} = 1 + \frac{\log(3M_n)}{\log((3M_n)^{-n})} = 1 - \frac{1}{n}.$$

Clearly, $\tilde{r}_n \to 0$ as $n \to \infty$, and we conclude that $\overline{d}_\mu(x) \ge 1$ for all x. Since μ is a Borel probability measure on S^1, $\overline{d}_\mu(x) \le 1$ for μ-almost every x (see [**8**, Lemma 2.1]). Thus, $\overline{d}_\mu(x) = 1$ for μ-almost all x. Combining this with (3.13) we obtain

$$\underline{d}_\mu(x) = 0 \quad \text{and} \quad \overline{d}_\mu(x) = 1 \quad \text{for } \mu\text{-almost every } x.$$

This completes the proof of the first statement of Theorem 1.5 for the case of $\beta = 0$.

Now we prove the results for the box and Hausdorff dimensions of μ (see §2.2 for the definitions). First we show that $\overline{\dim}_B \mu = 1$. For $\tilde{r}_n$ as above, we have

$$\mu(B(x, \tilde{r}_n)) \le \tilde{r}_n^{\,1-1/n} \quad \text{for any } x \in S^1.$$

Let Z be a set in S^1 with $\mu(Z) > 0$. Then at least $\mu(Z) \cdot \tilde{r}_n^{-(1-1/n)}$ balls of radius $\tilde{r}_n$ are needed to cover Z. Thus,

$$\frac{\log N(Z, \tilde{r}_n)}{\log(1/\tilde{r}_n)} \ge \frac{\log(\mu(Z)\tilde{r}_n^{-(1-1/n)})}{-\log \tilde{r}_n} = 1 - \frac{1}{n} - \frac{\log \mu(Z)}{\log \tilde{r}_n} \underset{n\to\infty}{\longrightarrow} 1.$$

Since $\tilde{r}_n \to 0$ as $n \to \infty$, this implies that $\overline{\dim}_B(Z) = 1$. Thus, $\overline{\dim}_B(Z) = 1$ for any set Z with $\mu(Z) > 0$, and hence $\overline{\dim}_B \mu = 1$ by the definition (2.1).

Now we prove that $\underline{\dim}_B \mu$, the lower box dimension of μ, equals zero. Let $G_k = \bigcap_{n=k}^{\infty} E_n$. By (3.12), $\mu(E_n) \ge 1 - 5/2^n$ for each n and, hence,

$$\mu(G_k) \ge 1 - 5/2^{k-1} \to 1 \text{as } k \to \infty.$$

We recall that for each n the set E_n consists of $1/s_n$ intervals of length at most r_n, and $\log s_n/\log r_n \to 0$ as $n \to \infty$. This implies that each E_n and, hence, G_k can be covered by at most $1/s_n$ balls of diameter r_n, i.e. $N(G_k, r_n) \le 1/s_n$. Therefore,

$$\underline{\dim}_B G_k = \liminf_{\varepsilon\to 0} \frac{\log N(G_k, \varepsilon)}{\log(1/\varepsilon)} \le \liminf_{n\to\infty} \frac{\log N(G_k, r_n)}{\log(1/r_n)} \le \lim_{n\to\infty} \frac{\log s_n}{\log r_n} = 0$$

for any $k > 0$. Thus, for any $\varepsilon > 0$ there exists a set G such that $\mu(G) > 1 - \varepsilon$ and $\underline{\dim}_B G = 0$, which implies that $\underline{\dim}_B \mu = 0$ by the definition (2.1). Since $0 \le \dim_H \mu \le \underline{\dim}_B \mu$, it follows that the Hausdorff dimension of μ is also zero.

This completes the proof of the theorem for the case of $\beta = 0$.

The case of $0 < \beta < 1$. The proof for this case uses the same approach as the proof for the case of $\beta = 0$. However, some modifications are needed to ensure that the lower pointwise dimension is $\beta > 0$.

Let $\gamma = (1 - \beta)/2$, then $\gamma > 0$ and $\beta + \gamma/n < 1$ for any $n \ge 1$. We construct the sequences $\{\tau_n\}_{n=1}^{\infty}$ and $\{h_n\}_{n=1}^{\infty}$ inductively. Let h_1 be the identity map, let τ_1 be a rational

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number close to τ, and let s_1 be a number such that $1/s_1$ is an integer and $\delta_1 = s_1^{1/(\beta+\gamma)} < s_1/2$. Suppose that $\tau_{n-1} = p_{n-1}/q_{n-1}$, a function a_{n-1} of period s_{n-1} and, hence, A_{n-1} and h_n are selected. As before, we set $M_n = \max_{[0,1]} h_n'$ and $m_n = \min_{[0,1]} h_n'$. We choose numbers $\tau_n = p_n/q_n$, s_n and δ_n, and functions a_n of a period s_n and A_n such that

$$\begin{aligned}
&\text{(i) } |\tau - \tau_n| \le |\tau - \tau_{n-1}|,\\
&\text{(ii) } |\tau - \tau_n| = |\tau - p_n/q_n| \le 1/q_n^{n^4},\\
&\text{(iii) } q_n \ge \max\{1/s_{n-1},\, 1/m_n,\, (3M_n)^n,\, c_n,\, \tilde{c}(h_n, n+1)\},\\
&\quad \text{where } c_n \text{ is as in (3.4) and } \tilde{c}(h_n, n+1) \text{ is as in (3.7)},\\
&\text{(iv) } s_n = s_{n-1}/q_n, \text{ hence } s_n \le s_{n-1}^2 \text{ and } s_n \le 1/2^n,\\
&\text{(v) } s_n \le 2^{-n-1} s_{n-1}, \text{ and } s_n^\gamma \le \min\{1/(M_n+1)^n,\, (m_n/(2M_n))^n\},\\
&\text{(vi) } \delta_n = m_n s_n^{1/(\beta+\gamma/n)}, \text{ hence } \delta_n < s_n/2,\\
&\text{(vii) } a_n \text{ and } A_n \text{ are as in Lemma 3.3 with } \delta = \delta_n \text{ and } s = s_n.
\end{aligned} \tag{3.14}$$

Condition (v) is only used in the proof of Lemma 3.5. Clearly, it can be satisfied by choosing a sufficiently large q_n. We note that $q_n \ge \tilde{c}(h_n, n+1) \ge \|h\|_n \ge \max |h^{-1}| = 1/m_n$ and, hence,

$$\frac{1}{\delta_n} = \frac{1}{m_n s_n^{1/(\beta+\gamma/n)}} = \frac{q_n^{1/(\beta+\gamma/n)}}{m_n s_{n-1}^{1/(\beta+\gamma/n)}} \le \frac{q_n^{2/(\beta+\gamma/n)}}{m_n} \le \frac{q_n^{2/\beta}}{m_n} \le q_n^{(2/\beta)+1} \le q_n^{3/\beta}.$$

Thus, we can obtain an estimate similar to (3.9):

$$\begin{aligned}
d_n(f_{n+1}, f_n) &= \cdots \le 2c_n|\tau - \tau_n| \cdot (\tilde{c}(h_n, n+1)/\delta_n^{(n+1)^2})^{n+1}\\
&\le 2q_n(1/q_n^{n^4})(q_n(q_n^{3/\beta})^{(n+1)^2})^{n+1} \le 2(1/q_n^{n^4}) q_n^{(3/\beta)(n+1)^3+n+2} \le 1/2^n
\end{aligned}$$

for all sufficiently large n, and establish convergence of the sequence $\{f_n\}$ in the C^∞ topology. The convergence of $\{h_n\}$ with respect to the distance d_0 can be shown as before.

We define the sets $\tilde{E}_n$ and E_n as in (3.11). The total growth of A_n on $\tilde{E}_n$, and hence the total growth of h_{n+1} on E_n is

$$(s_n - \delta_n)(1/s_n) = 1 - \delta_n/s_n \ge 1 - s_n^{(1/(\beta+\gamma/n))-1}.$$

Since E_n consists of $1/s_n$ intervals, $\max |h - h_{n+1}| \le 2s_{n+1}$, and $s_{n+1} \le s_n^2$, the total growth of h on E_n is at least

$$1 - s_n^{(1/(\beta+\gamma/n))-1} - 4s_{n+1}/s_n \ge 1 - s_n^{(1/(\beta+\gamma/n))-1} - 4s_n \ge 1 - 5s_n^\sigma \ge 1 - 5/2^{n\sigma},$$

where $\sigma = \min\{(1/(\beta+\gamma) - 1,\, 1\} > 0$. Thus, for the measure μ with distribution function h,

$$\mu(E_n) \ge 1 - 5/2^{n\sigma}. \tag{3.15}$$

Now we show that $\underline{d}_\mu(x) = \beta$ for almost all x. We take

$$r_n = \delta_n/m_n = s_n^{1/(\beta+\gamma/n)}, \tag{3.16}$$

then

$$r_n < s_n, \quad s_n = r_n^{\beta+\gamma/n} \quad \text{and hence} \quad \frac{\log s_n}{\log r_n} = \beta + \frac{\gamma}{n}. \tag{3.17}$$

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Let x be a point in E_n. Then

$$\Delta h(x, r_n) \geq \Delta h_{n+1}(x, r_n) - 4s_{n+1} \geq s_n - \delta_n - 4s_{n+1} \geq s_n/2$$

for all sufficiently large n. Therefore,

$$\frac{\log \Delta h(x, r_n)}{\log r_n} \leq \frac{\log(s_n/2)}{\log r_n} = \frac{\log s_n}{\log r_n} - \frac{\log 2}{\log r_n} = \beta - \frac{\log 2}{\log r_n} \leq \beta + \frac{1}{n}$$

since $r_n < s_n \leq 1/2^n$. It follows as before that $\underline{d}_\mu(x) \leq \beta$ for μ-almost every x.

To show that $\underline{d}_\mu(x) \geq \beta$ we prove that the function h is Hölder continuous with exponent β. Then for any x and $r > 0$, $\Delta h(x, r) \leq C(2r)^\beta$ and hence

$$\frac{\log \mu(B(x, r))}{\log r} = \frac{\log \Delta h(x, r)}{\log r} \geq \frac{\log(C2^\beta) + \beta \log r}{\log r} \underset{r \to 0}{\longrightarrow} \beta.$$

This implies that $\underline{d}_\mu(x) \geq \beta$ for all x, and hence $\underline{d}_\mu(x) = \beta$ for μ-almost every x.

LEMMA 3.5. *The function h is Hölder continuous with exponent β.*

Proof. We show using induction that

$$|h_n(x) - h_n(y)| \leq |x - y|^\beta \quad \text{for all } x, y \text{ with } |x - y| \leq s_{n-1},$$

and

$$|h_n(x) - h_n(y)| \leq (3 - 2^{-n})|x - y|^\beta \quad \text{for all } x, y \text{ with } |x - y| \geq s_{n-1}.$$

Clearly, this is true for $h_1 = \mathrm{Id}$. Suppose that it holds for h_n.

Recall that $h_{n+1} = A_n \circ h_n$, and the diffeomorphism A_n is of the form $\mathrm{Id} + a_n$, where $a_n \geq 0$ is a function of period s_n. It follows that $x \leq A_n(x) \leq x + s_n$ for any point x in $[0, 1]$. Hence, $|A_n(x) - A_n(y)| \leq |x - y| + s_n$ for any x and y.

If $|x - y| \geq s_{n-1}$ we obtain

$$\begin{aligned} |h_{n+1}(x) - h_{n+1}(y)| &= |A_n(h_n(x)) - A_n(h_n(y))| \leq |h_n(x) - h_n(y)| + s_n \\ &\leq (3 - 2^{-n})|x - y|^\beta + 2^{-n-1}|x - y|^\beta = (3 - 2^{-n-1})|x - y|^\beta \end{aligned}$$

since by (3.14) $s_n \leq 2^{-n-1}s_{n-1} \leq 2^{-n-1}|x - y| \leq 2^{-n-1}|x - y|^\beta$.

If $s_n \leq |x - y| \leq s_{n-1}$ we have

$$|h_{n+1}(x) - h_{n+1}(y)| \leq |h_n(x) - h_n(y)| + s_n \leq |x - y|^\beta + s_n \leq 2|x - y|^\beta.$$

It follows from (3.14) and (3.17) that $r_n^{\gamma/n} \leq s_n^{\gamma/n} \leq \min\{1/(M_n + 1), m_n/(2M_n)\}$. We use this together with the fact that $\beta + \gamma/n < 1$ for all n in the two estimates below. Suppose that $r_n \leq |x - y| \leq s_n$. Then

$$\begin{aligned} |h_{n+1}(x) - h_{n+1}(y)| &\leq |h_n(x) - h_n(y)| + s_n \leq M_n|x - y| + s_n \\ &\leq (M_n + 1)s_n = (M_n + 1)r_n^{\beta+\gamma/n} \leq r_n^\beta \leq |x - y|^\beta. \end{aligned}$$

Finally, for $|x - y| \leq r_n$ we have

$$\begin{aligned} |h_{n+1}(x) - h_{n+1}(y)| &\leq \max h'_{n+1} \cdot |x - y| = \max A'_n \cdot \max h'_n \cdot |x - y| \\ &\leq \frac{2s_n}{\delta_n} \cdot M_n \cdot |x - y| = \frac{r_n^{\beta+\gamma/n}}{r_n m_n} \cdot 2M_n \cdot |x - y| \leq \frac{2M_n}{m_n} \cdot |x - y|^{\beta+\gamma/n} \\ &\leq \frac{2M_n}{m_n} \cdot r_n^{\gamma/n} \cdot |x - y|^\beta \leq |x - y|^\beta. \end{aligned}$$

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Thus, each function h_n satisfies $|h_n(x) - h_n(y)| \le 3|x - y|^\beta$ for all x and y. Since the sequence $\{h_n\}$ converges to h, it follows that

$$|h(x) - h(y)| \le 3|x - y|^\beta \quad \text{for all } x \text{ and } y. \qquad \square$$

The proof of the fact that $\overline{d}_\mu(x) = 1$ for μ-almost every x does not require any modifications. Thus,

$$\underline{d}_\mu(x) = \beta \quad \text{and} \quad \overline{d}_\mu(x) = 1 \quad \text{for } \mu\text{-almost every } x.$$

The same argument as for $\beta = 0$ shows that $\overline{\dim}_B \mu = 1$ for all x.

Now we show that $\dim_H \mu = \underline{\dim}_B \mu = \beta$. Since $\underline{d}_\mu(x) = \beta$ for almost all x, it follows from Theorem 2.1 that $\dim_H \mu \ge \beta$. So it remains to show that $\underline{\dim}_B \mu \le \beta$. As before, let $G_k = \bigcap_{n=k}^{\infty} E_n$. Since $G_k \subset E_n$ and E_n consists of $1/s_n$ intervals of length at most r_n, we have $N(G_k, r_n) \le 1/s_n$. As $\log s_n / \log r_n \to \beta$ as $n \to \infty$, we obtain

$$\underline{\dim}_B G_k \le \lim_{n\to\infty} \frac{\log N(G_k, r_n)}{\log(1/r_n)} \le \lim_{n\to\infty} \frac{\log s_n}{\log r_n} = \beta$$

for any $k > 0$. It follows from (3.15) that $\mu(G_k) \to 1$ as $k \to \infty$ and, hence, $\underline{\dim}_B \mu \le \beta$.

This completes the proof of the theorem.

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