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Connective Algebraic K-theory

by

SHOUXIN DAI AND MARC LEVINE

Abstract

We examine the theory of connective algebraic K-theory, $\mathcal{CK}$, defined by taking the -1 connective cover of algebraic K-theory with respect to Voevodsky's slice tower in the motivic stable homotopy category. We extend $\mathcal{CK}$ to a bi-graded oriented duality theory $(\mathcal{CK}'_{*,*}, \mathcal{CK}^{*,*})$ when the base scheme is the spectrum of a field k of characteristic zero. The homology theory $\mathcal{CK}'_{*,*}$ may be viewed as connective algebraic G-theory. We identify $\mathcal{CK}'_{2n,n}(X)$ for X a finite type k-scheme with the image of $K_0(\mathcal{M}_{(n)}(X))$ in $K_0(\mathcal{M}_{(n+1)}(X))$, where $\mathcal{M}_{(n)}(X)$ is the abelian category of coherent sheaves on X with support in dimension at most n; this agrees with the (2n,n) part of the theory of connective algebraic K-theory defined by Cai. We also show that the classifying map from algebraic cobordism identifies $\mathcal{CK}'_{2*,*}$ with the universal oriented Borel-Moore homology theory $\Omega^{CK}_* := \Omega_* \otimes_{\mathbb{L}} \mathbb{Z}[\beta]$ having formal group law $u + v - \beta uv$ with coefficient ring $\mathbb{Z}[\beta]$. As an application, we show that every pure dimension d finite type k-scheme has a well-defined fundamental class $[X]_{CK}$ in $\Omega^{CK}_d(X)$, and this fundamental class is functorial with respect to pull-back for l.c.i. morphisms.

Key Words: Algebraic cobordism, algebraic K-theory, oriented homology.
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Introduction

In topology, the theory of *connective K-theory* is represented by the -1-connected cover ku of the topological K-theory spectrum KU; see for example the work of Adams [1] for one of the first discussions of this spectrum and its cohomology. In the setting of motivic stable homotopy theory over a base-scheme S, Voevodsky [32] has constructed the algebraic analog of KU, namely, the algebraic K-theory spectrum $\mathcal{K}_S$. This spectrum represents Quillen's algebraic K-theory on the category of smooth S-schemes, assuming that S itself is a regular scheme (see [23]), in that there are natural isomorphisms $\mathcal{K}_S^{a,b}(X) \cong K_{2b-a}(X)$ for X a smooth finite type S-scheme.

There are a number of possible notions of connectivity in the motivic stable homotopy category over S, $\mathcal{SH}(S)$, but one that has proved quite useful is the one introduced by Voevodsky [30, §2] using T-connectivity ($T = \mathbb{A}^1/\mathbb{A}^1 \setminus \{0\}$), in the sense of the tower of localizing subcategories

$$\cdots \subset \Sigma_T^{n+1}\mathcal{SH}^{eff}(S) \subset \Sigma_T^n\mathcal{SH}^{eff}(S) \subset \ldots \subset \mathcal{SH}(S),$$

where $\mathcal{SH}^{eff}(S)$ is the localizing subcategory of $\mathcal{SH}(S)$ generated by the T-suspension spectra of smooth S-schemes. The associated truncation functors give rise to Voevodsky's *slice tower*

$$\cdots \to f_{n+1}\mathcal{E} \to f_n\mathcal{E} \to \cdots \to \mathcal{E}$$

for each $\mathcal{E} \in \mathcal{SH}(S)$, with $f_n\mathcal{E} \to \mathcal{E}$ the universal morphism from an object in $\Sigma_T^n\mathcal{SH}^{eff}(S)$ to $\mathcal{E}$; one could call this the "T-$(n-1)$-connected cover" of $\mathcal{E}$. It is thus natural to define connective algebraic K-theory as the bi-graded cohomology theory represented by $f_0\mathcal{K}_S$.

In this paper we study connective algebraic K-theory, $\mathcal{CK}^{*,*}$ and its associated oriented homology theory, $\mathcal{CK}'_{*,*}$, this latter for $S = \mathrm{Spec}k$, k a characteristic zero field (see however remark 4.7). The oriented homology theory $\mathcal{CK}'_{*,*}$ is the connective analog of G-theory, that is, the K-theory of coherent sheaves rather than vector bundles. The canonical map $\mathcal{CK}^{*,*} \to \mathcal{K}^{*,*}$ induces the map $\mathcal{CK}'_{*,*} \to G_{*,*}$ (where $G_{a,b}(X) := G_{a-2b}(X)$) and we elucidate here how the connective versions refine the non-connective ones.

Cai [2] has defined a bi-graded oriented Borel-Moore homology theory, which he calls connective algebraic K-theory, by using the Quillen-Gersten spectral

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sequence. For a finite type k-scheme X, let $\mathcal{M}_{(a)}(X)$ be the full subcategory of the category of coherent sheaves on X with objects the sheaves supported in dimension at most a, giving the theory $CK^{\mathrm{Cai}}_{*,*}$ with

$$CK^{\mathrm{Cai}}_{a,b}(X) := \mathrm{im}(K_{a-2b}(\mathcal{M}_{(a)}(X)) \to K_{a-2b}(\mathcal{M}_{(a+1)}(X)))$$

(Cai uses a different indexing convention, see remark 4.6); Cai verifies the properties of an oriented Borel-Moore homology theory for $CK^{\mathrm{Cai}}_{*,*}$. It turns out that Cai's theory does not in general agree with the one given by motivic homotopy theory, but it does agree in the portion corresponding to K_0 or G_0 (see theorem 4.3 and remark 4.6), that is, we have natural isomorphisms

$$\mathcal{CK}'_{2n,n}(X) \cong CK^{\mathrm{Cai}}_{2n,n}(X) := \mathrm{im}(K_0(\mathcal{M}_{(n)}(X)) \to K_0(\mathcal{M}_{(n+1)}(X))).$$

Besides comparison results of this type and other structural properties of connective K-theory, our main result is a comparison with algebraic cobordism. There is a canonical natural transformation

$$\Omega_*(X) \otimes_{\mathbb{L}} \mathbb{Z}[\beta] \to \mathcal{CK}'_{2*,*}(X)$$

which, in the case of characteristic zero, is an isomorphism for all quasi-projective k-schemes X (see theorem 6.3). This allows us to use the natural fundamental classes in G-theory, namely, the class of the structure sheaf, to define a fundamental class $[X] \in \Omega_d(X) \otimes_{\mathbb{L}} \mathbb{Z}[\beta]$ for X of pure dimension d over k, which is functorial with respect to pull-back by l.c.i. morphisms. Note that such fundamental classes in $\Omega_d(X)$ do not exist in general (see [15]).

Here is an outline of the paper: We introduce the main actors, K-theory and connective K-theory and discuss some of their elementary properties, such as ring structure and orientations, in §1. We recall the construction of the homotopy coniveau tower from [13] and relate the homotopy coniveau tower for K-theory to connective K-theory in §2. In §3, we recall from [14] the notion of an oriented duality theory as well as how an oriented motivic ring spectrum gives rise to an oriented duality theory, and we apply these constructions to K-theory and connective K-theory in §4. We also extend the computation of connective K-theory from §2 to give an explicit description of the "geometric part" of connective K-theory in this section. We discuss the classifying map from algebraic cobordism to an oriented duality theory in §5. The main technical tool needed for our main results, the exact sequence (5.5) and commutative diagram (5.6), are also presented in this section. We put this all together in §6, where prove our main results relating algebraic cobordism and connective K-theory. In the concluding section, §7, we

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apply the results of the previous section to construct functorial fundamental classes in connective K-theory.

Notations and conventions: We let **Spc** and $\mathbf{Spc}_\bullet$ denote the categories of simplicial sets and pointed simplicial sets, respectively, with homotopy categories $\mathcal{H}$ and $\mathcal{H}_\bullet$. **Spt** the category of spectra (for the usual suspension operator $\Sigma := (-) \wedge S^1$) and $\mathcal{SH}$ the stable homotopy category.

We denote by $\mathbb{G}_m$ the pointed S-scheme $(\mathbb{A}^1_S - 0_S, 1_S)$. $\mathbb{P}^1_*$ will denote the pointed S-scheme $\mathbb{P}^1_S$, with base-point 1_S.

For a scheme S, $\mathbf{Sch}_S$ will denote the category of quasi-projective schemes over S, $\mathbf{Sm}/S$ the full subcategory of smooth quasi-projective schemes over S. We let $\mathbf{Spc}(S)$ and $\mathbf{Spc}_\bullet(S)$ denote the categories of presheaves on $\mathbf{Sm}/S$ with values in **Spc** and $\mathbf{Spc}_\bullet$, respectively, and $\mathbf{Spt}_{S^1}(S)$ the category of S^1-spectra over S, this being the category of presheaves of spectra on $\mathbf{Sm}/S$. We let $\mathbf{Spt}_T(S)$ denote the category of T-spectra in $\mathbf{Spc}_\bullet(S)$, with $T := \mathbb{A}^1/\mathbb{A}^1\setminus\{0\}$, and $\mathbf{Spt}^\Sigma_T(S)$ the category of symmetric T-spectra. We have as well the category of $S^1-\mathbb{G}_m$ bi-spectra objects in $\mathbf{Spc}_\bullet(S)$, denoted $\mathbf{Spt}_{s,g}(S)$.

The categories $\mathbf{Spc}(S)$, $\mathbf{Spc}_\bullet(S)$, $\mathbf{Spt}_{S^1}(S)$, $\mathbf{Spt}_T(S)$, $\mathbf{Spt}_{s,g}(S)$ and $\mathbf{Spt}^\Sigma_T(S)$ all have so-called *motivic* model structures (the original source for the unstable theory is [21], see also [4, 5] for a compact description. For the stable theory, we refer the reader to [10]), with homotopy categories denoted $\mathcal{H}(S)$, $\mathcal{H}_\bullet(S)$, $\mathcal{SH}_{S^1}(S)$, $\mathcal{SH}(S)$, $\mathcal{SH}_{s,g}(S)$ and $\mathcal{SH}^\Sigma(S)$, respectively. The categories $\mathcal{SH}(S)$, $\mathcal{SH}_{s,g}(S)$ and $\mathcal{SH}^\Sigma(S)$ are equivalent triangulated tensor categories; the tensor structure is induced by a symmetric monodical structure on $\mathbf{Spt}^\Sigma_T(S)$. There is also a symmetric spectra version of $\mathbf{Spt}_{S^1}(S)$ making $\mathcal{SH}_{S^1}(S)$ a tensor triangulated category.

Ord is the category of finite ordered sets; we let $[n] \in \mathbf{Ord}$ denote the set $\{0,\dots,n\}$ with the standard ordering. We let $\mathbb{L}$ denote the Lazard ring, that is, the coefficient ring of the universal rank one commutative formal group law $F_\mathbb{L} \in \mathbb{L}[[u,v]]$. We let $\mathbb{L}^*$ denote $\mathbb{L}$ with the grading determined by $\deg a_{ij} = 1-i-j$ if $F_\mathbb{L}(u,v) = u + v + \sum_{i,j\geq 1} a_{ij}u^i v^j$ and let $\mathbb{L}_*$ denote $\mathbb{L}$ with the opposite grading $\mathbb{L}_n := \mathbb{L}^{-n}$.

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1. K-theory and connective K-theory

We work in the motivic stable homotopy category $\mathcal{SH}(S)$ over a *regular* base-scheme S; we will rather quickly pass to the case $S = \mathrm{Spec}k$ for a field k.

Following Voevodsky [30, §2], we have the Tate-Postnikov tower

$$\ldots \to f_{n+1} \to f_n \to \ldots \to \mathrm{id} \tag{1.1}$$

of endofunctors of $\mathcal{SH}(S)$ associated to the inclusions of full localizing subcategories

$$\cdots \subset \Sigma_T^{n+1}\mathcal{SH}^{eff}(S) \subset \Sigma_T^n\mathcal{SH}^{eff}(S) \subset \cdots \subset \mathcal{SH}(S),$$

with $\Sigma_T^n\mathcal{SH}^{eff}(S)$ the full localizing subcategory of $\mathcal{SH}(S)$ generated by the objects $\Sigma_T^m X_+$ for $X \in \mathbf{Sm}/S$, $m \geq n$. Letting $i_n : \Sigma_T^n\mathcal{SH}^{eff}(S) \to \mathcal{SH}(S)$ be the inclusion, one shows that i_n admits the right adjoint $r_n : \mathcal{SH}(S) \to \Sigma_T^n\mathcal{SH}^{eff}(S)$, and f_n is by definition the composition $i_n \circ r_n$; the natural transformation $\rho_n : f_n \to \mathrm{id}$ is the co-unit of the adjunction. There are functors $s_n : \mathcal{SH}(S) \to \mathcal{SH}(S)$ and triangles $f_{n+1} \to f_n \to s_n \to f_{n+1}[1]$, $n \in \mathbb{Z}$, that are distinguished upon evaluation on each $\mathcal{E} \in \mathcal{SH}(S)$; one can show that these triangles are unique up to unique natural isomorphism. In addition, we have the canonical isomorphism

$$f_{n+1} \circ \Sigma_T \cong \Sigma_T \circ f_n. \tag{1.2}$$

For $n' < n$, we let $\rho_{n',n} : f_n \to f_{n'}$ be the composition of the maps in the tower (1.1), and $\rho_n : f_n \to \mathrm{id}$ the natural transformation given by the same tower. For details, we refer the reader to [13, 30, 31].

Remark 1.1 By [6, proposition 5.3], if $\mathcal{E}$ is given a lifting to an E_∞ ring object in $\mathbf{Spt}_T^\Sigma(S)$, then $f_0\mathcal{E}$ has a canonical lifting to an E_∞ ring object in $\mathbf{Spt}_T^\Sigma(S)$, with $\rho_0(\mathcal{E}) : f_0\mathcal{E} \to \mathcal{E}$ lifting to a map of E_∞ ring objects in $\mathbf{Spt}_T^\Sigma(S)$. In particular, $f_0\mathcal{E}$ is itself a commutative monoid in $\mathcal{SH}(S)$ and thus defines a bi-graded cohomology theory on $\mathbf{Sm}/S$ in the sense of Panin-Smirnov [22, definition 2.0.1].

Following [27, lemma 2.5, theorem 4.1] we have a commutative monoid object $\mathcal{K}_S$ of $\mathbf{Spt}_T^\Sigma(S)$ representing algebraic K-theory in $\mathcal{SH}(S)$, in the sense that, for each $X \in \mathbf{Sm}/S$ and open subscheme U, there is an isomorphism

$$\mathcal{H}om(\Sigma_T^\infty X/U, \mathcal{K}_S) \cong K(X \text{ on } X \setminus U)$$

in $\mathcal{SH}$, natural in pairs (X, U). Here $\mathcal{H}om(-,-)$ is the **Spt**-valued enriched Hom and $K(X \text{ on } X \setminus U)$ is the algebraic K-theory spectrum of the Waldhausen category of perfect complexes on X with support in $X \setminus U$ [28, definition 3.1]. Here we use

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our standing assumption that S is regular; for general S, $\mathcal{K}_S$ represents homotopy invariant K-theory, KH [33].

To fix the conventions, the image of $\mathcal{K}_S$ in $\mathcal{SH}(S)$ under the forgetful functor is isomorphic in $\mathcal{SH}(S)$ to a T-spectrum of the form $(\bar{\mathcal{K}}_S,\bar{\mathcal{K}}_S,\ldots)$, with $\bar{\mathcal{K}}_S$ a commutative monoid in $\mathcal{H}_\bullet(S)$ representing K-theory on $\mathbf{Sm}/S$. The structure map $\bar{\mathcal{K}}_S \wedge T \to \bar{\mathcal{K}}_S$ is isomorphic in $\mathcal{H}_\bullet(S)$ to the composition

$$\bar{\mathcal{K}}_S \wedge T \xrightarrow{\mathrm{id}\wedge\sigma} \bar{\mathcal{K}}_S \wedge \mathbb{P}^1_* \xrightarrow{\mathrm{id}\wedge\gamma} \bar{\mathcal{K}}_S \wedge \bar{\mathcal{K}}_S \xrightarrow{\mu} \bar{\mathcal{K}}_S$$

where μ is the multiplication, $\sigma : T \to \mathbb{P}^1_*$ is the standard isomorphism and $\gamma : \mathbb{P}^1_* \to \bar{\mathcal{K}}_S$ represents the class $[\mathcal{O}] - [\mathcal{O}(-1)]$ in $K_0(\mathbb{P}^1_*)$. This agrees with the convention in [27], and is the negative of the convention used in [23]. We will often write $\mathcal{K}_S$ for the image of $\mathcal{K}_S$ in $\mathcal{SH}(S)$, with the context determining the meaning.

Remark 1.2 The fact that the shift operator leaves the T-spectrum $(\bar{\mathcal{K}}_S,\bar{\mathcal{K}}_S,\ldots)$ unchanged gives us the *Bott periodicity isomorphism* $\Sigma_T\mathcal{K}_S \cong \mathcal{K}_S$ in $\mathcal{SH}(S)$. We note that the map $\gamma : \mathbb{P}^1_* \to \bar{\mathcal{K}}_S$ thus gives rise to an element $[\gamma] \in \mathcal{K}^{2,1}(\mathbb{P}^1_*)$ which corresponds to the unit $1 \in \mathcal{K}^{0,0}(S)$ under the suspension isomorphism $\mathcal{K}^{2,1}(\mathbb{P}^1_*) \cong \mathcal{K}^{0,0}(S)$.

Remark 1.3 Let $t_{\mathcal{K}} \in \mathcal{K}_S^{1,1}(\mathbb{G}_m)$ be the element corresponding to $1 \in \mathcal{K}^{0,0}(S)$ under the suspension isomorphism $\mathcal{K}^{1,1}(\mathbb{G}_m) \cong \mathcal{K}^{0,0}(S)$. Thus $\Sigma_{S^1}t_{\mathcal{K}} = [\gamma]$. Under the isomorphisms $K_1(\mathbb{G}_m) \cong \mathcal{K}^{1,1}(\mathbb{G}_m)$, $K_0(\mathbb{P}^1_*) \cong \mathcal{K}^{2,1}(\mathbb{P}^1_*)$, the isomorphism $\mathcal{K}^{1,1}(\mathbb{G}_m) \cong \mathcal{K}^{2,1}(\Sigma_{S^1}\mathbb{G}_m) \cong \mathcal{K}^{2,1}(\mathbb{P}^1_*)$ is identified with the boundary map in the Mayer-Vietoris sequence

$$\cdots \to K_1(\mathbb{A}^1) \oplus K_1(\mathbb{A}^1) \to K_1(\mathbb{A}^1 \setminus \{0\}) \xrightarrow{\partial} K_0(\mathbb{P}^1) \to \cdots$$

(at least up to a universal sign, see [26, §7, lemma 5.16]). Identifying $\mathbb{A}^1_S \setminus \{0\}$ with $\mathrm{Spec}\mathcal{O}_S[t,t^{-1}]$ gives us the canonical unit t on $\mathbb{A}^1_S \setminus \{0\}$; let $[t] \in K_1(\mathbb{A}^1 \setminus \{0\})$ be the image of t. Then $1 - [O(-1)] = \pm\partial[t]$, and hence $t_{\mathcal{K}} = [t^{\pm 1}]$.

Definition 1.4 Let $\mathcal{CK}_S$ be the object $f_0\mathcal{K}_S$ of $\mathcal{SH}(S)$, with canonical map $\rho : \mathcal{CK}_S \to \mathcal{K}_S$. *Connective algebraic K-theory* over S is the bi-graded cohomology theory on $\mathbf{Sm}/S$ represented by $\mathcal{CK}_S$. We call $\mathcal{CK}_S$ the connective algebraic K-theory T-spectrum (over S).

Remark 1.5 As we have already seen, $\mathcal{K}_S$ is a strict monoid object of $\mathbf{Spt}_T^\Sigma(S)$; thus it follows from [6, theorem 5.14] that the object $\mathcal{CK}_S$ of $\mathcal{SH}(S)$ lifts to an E_∞ object in $\mathbf{Spt}_T^\Sigma(S)$ and the map $\rho : \mathcal{CK}_S \to \mathcal{K}_S$ lifts to a map of E_∞ objects in $\mathbf{Spt}_T^\Sigma(S)$. In particular, the map ρ itself is a monoid map of monoid objects in $\mathcal{SH}(S)$.

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Lemma 1.6 *Suppose $s_0\mathcal{K}_S$ satisfies $(s_0\mathcal{K}_S)^{a,0}(X) = 0$ for $a \neq 0$ and for all $X \in \mathbf{Sm}/S$. For $X \in \mathbf{Sm}/S$, consider the natural map $\rho^{a,1}(X) : \mathcal{CK}_S^{a,1}(X) \to \mathcal{K}_S^{a,1}(X)$. Then $\rho^{2,1}$ is injective, $\rho^{3,1}$ is surjective and $\rho^{a,1}$ is an isomorphism for $a \neq 2,3$.*

Proof: By (1.2) and Bott periodicity, we have $\Sigma_T f_0\mathcal{K}_S = f_1\Sigma_T\mathcal{K}_S = f_1\mathcal{K}_S$. Thus $\mathcal{CK}^{a,1}(X) = (f_1\mathcal{K})^{a-2,0}(X)$. Since $\Sigma_T^\infty X_+$ is in $\mathcal{SH}^{eff}(S)$, we have $\mathcal{K}^{*,0}(X) = (f_0\mathcal{K}_S)^{*,0}(X)$, and we thus have a long exact sequence

$$\ldots \to (s_0\mathcal{K}_S)^{a-3,0}(X) \to \mathcal{CK}^{a,1}(X) \xrightarrow{\rho} \mathcal{K}^{a,1}(X) \to (s_0\mathcal{K}_S)^{a-2,0}(X) \to \ldots.$$

Our assumption $(s_0\mathcal{K}_S)^{a,0}(X) = 0$ for $a \neq 0$ yields the desired result. □

Remark 1.7 For $S = \mathrm{Spec}k$, k a perfect field, we know by [29, theorem 6.6] [13, theorem 10.5.1] that $s_0\mathcal{K}_S$ is isomorphic in $\mathcal{SH}(k)$ to the T-spectrum representing motivic cohomology, $H\mathbb{Z}$. In addition, $H\mathbb{Z}^{a,0}(X) = H^a(X_{\mathrm{Zar}},\mathbb{Z})$ for $X \in \mathbf{Sm}/k$, hence the assumption in lemma 1.6 is satisfied. In addition, the map $\mathcal{K}_k^{2,1}(X) = K_0(X) \to H\mathbb{Z}^{0,0}(X)$ is just the rank homomorphism, hence surjective. As $\mathcal{K}_S^{a,1}(X) = K_{2-a}(X) = 0$ for $a > 2$, we see that $\mathcal{CK}^{a,1}(X) = 0$ for $a > 2$ as well.

The functors f_n and s_n are compatible with pull-back by smooth morphisms [25, theorem 2.12, remark 2.13], and for $f : T \to S$ smooth, we have $f^*\mathcal{K}_S \cong \mathcal{K}_T$. Thus $(s_0\mathcal{K}_S)^{a,0}(X) \cong H^a(X_{\mathrm{Zar}},\mathbb{Z})$ for $X \in \mathbf{Sm}/S$ if S is smooth over a perfect field. Thus, for S smooth over a perfect field, the map $\mathcal{CK}_S^{a,1}(X) \to \mathcal{K}_S^{a,1}(X) = K_{2-a}(X)$ is an isomorphism for $a < 2$, an injection for $a = 2$ and $\mathcal{CK}_S^{a,1}(X) = 0$ for $a > 2$.

Remark 1.8 For $n \geq 1$, let $\mathbb{P}_*^n$ denote the S-scheme $\mathbb{P}_S^n$, pointed by $(1 : 1 : 0 \ldots : 0)$ and let $\mathbb{P}^\infty$ denote the colimit (in $\mathbf{Spc}_\bullet(S)$) of the $\mathbb{P}_*^n$ under the linear embeddings $(x_0 : \ldots : x_n) \mapsto (x_0 : \ldots : x_n : 0)$. Recall from [24, definition 1.2] that an *orientation* on a commutative monoid $\mathcal{E}$ in $\mathcal{SH}(S)$ is given by an element $c \in \mathcal{E}^{2,1}(\mathbb{P}^\infty)$ such that the restriction $c_{|\mathbb{P}^1} \in \mathcal{E}^{2,1}(\mathbb{P}_*^1) = \mathcal{E}^{2,1}(T)$ is $\Sigma_T(1)$ (we use the opposite sign convention from *loc. cit.*).

Consider the sequence of elements $1-[\mathcal{O}_{\mathbb{P}^n}(-1)] \in \mathcal{K}_S^{2,1}(\mathbb{P}_*^n)$, which are clearly compatible with respect to restriction via the hyperplane embeddings $\mathbb{P}^n \to \mathbb{P}^{n+1}$. As the restriction maps $\mathcal{K}_S^{1,1}(\mathbb{P}_S^{n+1}) \to \mathcal{K}_S^{1,1}(\mathbb{P}_S^n)$ are all isomorphisms, it follows that $R^1\lim_n \mathcal{K}_S^{1,1}(\mathbb{P}_S^n) = 0$ and thus the sequence of elements $1 - [\mathcal{O}_{\mathbb{P}^n}(-1)] \in \mathcal{K}_S^{2,1}(\mathbb{P}_*^n)$ defines a unique element $c_{\mathcal{K}} \in \mathcal{K}_S^{2,1}(\mathbb{P}^\infty)$. The fact that $c_{\mathcal{K}}$ is an orientation follows from remark 1.3; we refer to $c_{\mathcal{K}}$ as the *standard orientation* on $\mathcal{K}_S$.

Remark 1.9 Following [32] the *algebraic cobordism spectrum* MGL is the T-spectrum $(\mathrm{MGL}_0,\mathrm{MGL}_1,\ldots)$ with MGL_n the Thom space $Th(E_n)$, and $E_n \to B\mathrm{GL}_n$

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the universal n-plane bundle. MGL_S is a commutative monoid in $\mathcal{SH}(S)$ with an orientation $c_{\mathrm{MGL}} \in \mathrm{MGL}^{2,1}(\mathbb{P}^\infty_S)$ given by the diagram

$$\begin{array}{ccc} E_1 & \longrightarrow & Th(E_1) = \mathrm{MGL}_1 \\ \downarrow & & \\ \mathbb{P}^\infty & & \end{array}$$

noting that $E_1 \to B\mathrm{GL}_1 = \mathbb{P}^\infty_S$ is an isomorphism in $\mathcal{H}(S)$.

We record the following universality property of the pair $(\mathrm{MGL}, c_{\mathrm{MGL}})$:

Theorem 1.10 ([24, theorem 1.1]) *Suppose the base scheme S is* Spec$k$, *k a field. For $\mathcal{E}$ a commutative monoid in $\mathcal{SH}(k)$, sending a monoid morphism $\varphi : \mathrm{MGL} \to \mathcal{E}$ to $\varphi(c_{\mathrm{MGL}})$ gives a bijection of the set of monoid maps φ with the set of orientations $c_{\mathcal{E}} \in \mathcal{E}^{2,1}(\mathbb{P}^\infty_k)$.*

In particular, the orientation $c_{\mathcal{K}}$ gives the canonical monoid morphism $\vartheta_{\mathcal{K}} : \mathrm{MGL} \to \mathcal{K}$ with $\vartheta(c_{\mathrm{MGL}}) = c_{\mathcal{K}}$.

We now consider the question of lifting the standard orientation on $\mathcal{K}_S$ to an orientation on $\mathcal{CK}_S$.

Lemma 1.11 *Suppose that S satisfies the hypothesis of lemma 1.6. Suppose in addition that the map $\mathcal{K}^{0,0}_S(\mathbb{P}^n_S) \to (s_0\mathcal{K}_S)^{0,0}(\mathbb{P}^n_S)$ is the rank homomorphism $K_0(\mathbb{P}^n_S) \to H^0(S_{\mathrm{Zar}}, \mathbb{Z})$ (up to sign). Then there is a unique element $c_{\mathcal{CK}} \in \mathcal{CK}^{2,1}(\mathbb{P}^\infty_S)$ mapping to $c_{\mathcal{K}}$ under the canonical map $\mathcal{CK}^{2,1}_S(\mathbb{P}^\infty_S) \to \mathcal{K}^{2,1}_S(\mathbb{P}^\infty_S)$. Furthermore, $c_{\mathcal{CK}} \in \mathcal{CK}^{2,1}_S(\mathbb{P}^\infty_S)$ defines an orientation for $\mathcal{CK}_S$.*

Proof: By our assumptions on S, we have the exact sequence

$$0 \to \mathcal{CK}^{2,1}_S(\mathbb{P}^n_S) \to K_0(\mathbb{P}^n_S) \xrightarrow{rnk} H^0(S_{\mathrm{Zar}}, \mathbb{Z}) \to 0.$$

Thus $\mathcal{CK}^{2,1}_S(\mathbb{P}^n_*) \to \mathcal{K}^{2,1}_S(\mathbb{P}^n_*)$ is an isomorphism. Furthermore, the canonical map $\mathcal{CK}^{1,1}_S(\mathbb{P}^n_S) \to \mathcal{K}^{1,1}_S(\mathbb{P}^n_S) = K_1(\mathbb{P}^n_S)$ is an isomorphism. By the projective bundle formula, the projective system of groups $n \mapsto K_1(\mathbb{P}^n_S)$ satisfies the Mittag-Leffler condition. Thus, passing to the limit over n gives us the isomorphism

$$\mathcal{CK}^{2,1}_S(\mathbb{P}^\infty_S) \cong \mathcal{K}^{2,1}_S(\mathbb{P}^\infty_S)$$

and the orientation $c_{\mathcal{K}} = (1 - [\mathcal{O}_{\mathbb{P}^n_S}(-1)])_n \in \mathcal{K}^{2,1}_S(\mathbb{P}^\infty_S)$ gives us the element $c_{\mathcal{CK}} \in \mathcal{CK}^{2,1}(\mathbb{P}^\infty_S)$. The fact that $\Sigma_{\mathbb{P}^1}(1_{\mathcal{K}_S}) = c_{\mathcal{K}|\mathbb{P}^1}$ implies $\Sigma_{\mathbb{P}^1}(1_{\mathcal{CK}_S}) = c_{\mathcal{CK}|\mathbb{P}^1}$, hence $c_{\mathcal{CK}}$ is an orientation. □

Remark 1.12 By remark 1.7, if S is smooth over a perfect field, then the hypotheses of lemma 1.11 are fulfilled and hence $\mathcal{CK}_S$ has a unique orientation mapping to the standard orientation of $\mathcal{K}_S$.

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2. An explicit model

We take $S = \mathrm{Spec} k$, k a perfect field. There is a Tate-Postnikov tower in $\mathcal{SH}_{S^1}(k)$, whose construction is similar to that of the tower (1.1). Namely, for $n \geq 0$, let $\Sigma_T^n \mathcal{SH}_{S^1}(k)$ be the localizing subcategory of $\mathcal{SH}_{S^1}(k)$ generated by the objects $\Sigma_T^m \Sigma_{S^1}^\infty X_+$, with $m \geq n$ and $X \in \mathbf{Sm}/k$. Just as for $\mathcal{SH}(k)$, the inclusion $i_n : \Sigma_T^n \mathcal{SH}_{S^1}(k) \to \mathcal{SH}_{S^1}(k)$ admits a right adjoint r_n and one defines $f_n^s := i_n \circ r_n$. This gives the tower

$$\ldots \to f_{n+1}^s \to f_n^s \to \ldots \to f_0^s = \mathrm{id} \tag{2.1}$$

and natural transformations $\rho_{n',n}^s : f_n^s \to f_{n'}^s$, $\rho_n^s := \rho_{0,n}^s : f_n^s \to \mathrm{id}$. The infinite T-loops functor $\Omega_T^\infty : \mathcal{SH}(k) \to \mathcal{SH}_{S^1}(k)$ intertwines towers (1.1) and (2.1) via canonical isomorphisms

$$\Omega_T^\infty \circ f_n \cong f_n^s \circ \Omega_T^\infty \tag{2.2}$$

(this follows from [13, theorems 7.1.1 and 9.0.3]). In case the context makes the meaning clear, we will often write $f_n E$ for $f_n^s E$, $\rho_{n',n}(E)$ for $\rho_{n',n}^s(E)$, etc.

We now describe an explicit model for $f_n^s E$, for $E \in \mathcal{SH}_{S^1}(k)$. Let Δ^* be the cosimplicial scheme $n \mapsto \Delta_k^n$, with

$$\Delta_k^n := \mathrm{Spec} k[t_0, \ldots, t_n] / \sum_i t_i - 1.$$

The morphism $\Delta(g) : \Delta^n \to \Delta^m$ associated to $g : [n] \to [m]$ is given by

$$g^*(t_i) = \sum_{j \in g^{-1}(i)} t_j$$

where as usual the sum over the empty index set is 0. A *face* of Δ^m is a closed subset F defined by equations of the form $t_{i_1} = \ldots t_{i_r} = 0$.

We briefly recall the construction of the homotopy coniveau tower associated to a quasi-fibrant presheaf of S^1-spectra on $\mathbf{Sm}/k$ (for details we refer the reader to [13, §2.1]). For $Y \in \mathbf{Sm}/k$, $W \subset Y$ a closed subset and $E \in \mathbf{Spt}_{S^1}(k)$, we write $E^W(Y)$ for the homotopy fiber of the restriction map $E(Y) \to E(Y \setminus W)$. For $X \in \mathbf{Sm}/k$ and $n, m \geq 0$ integers, let $\mathcal{S}_X^n(m)$ denote the set of closed subsets W of $X \times \Delta^m$ such that

$$\mathrm{codim}_{X \times F} W \cap X \times F \geq n$$

for all faces F of Δ^m. For $E \in \mathbf{Spt}_{S^1}(k)$, we let

$$E^{(n)}(X, m) := \operatorname*{hocolim}_{W \in \mathcal{S}_X^n(m)} E^W(X \times \Delta^m).$$

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This gives us the simplicial spectrum $m \mapsto E^{(n)}(X,m)$, and the associated total spectrum $E^{(n)}(X)$. This construction is contravariantly functorial in X for equi-dimensional morphisms. Letting $\mathbf{Sm}//k \subset \mathbf{Sm}/k$ denote the subcategory of $\mathbf{Sm}/k$ with the same objects, and with morphisms the smooth morphisms, sending X to $E^{(n)}(X)$ defines a presheaf of spectra $E^{(n)}$ on $\mathbf{Sm}//k$. In addition, for $n' < n$, the inclusions $\mathcal{S}^n_X(m) \subset \mathcal{S}^{n'}_X(m)$ induce a natural transformation $\varphi_{n',n}(E) : E^{(n)} \to E^{(n')}$. The identification of $E(X)$ with $E^{(0)}(X,0)$ gives the natural transformation $\epsilon : E \to E^{(0)}$.

We usually apply this construction to a *quasi-fibrant* E in $\mathbf{Spt}_{S^1}(k)$. This means that a fibrant replacement $E \to E^{fib}$ gives a weak equivalence $E(X) \to E^{fib}(X)$ for all $X \in \mathbf{Sm}/k$, or more concretely, that E satisfies the properties of Nisnevich excision and homotopy invariance (see [13, §2.1, conditions A1, A2] or the beginning of §3). In case E is quasi-fibrant, the map $\epsilon(X) : E(X) \to E^{(0)}(X)$ is a weak equivalence for all $X \in \mathbf{Sm}/k$.

According to [13, theorem 4.1.1], for E quasi-fibrant, there is a presheaf of spectra $\tilde{E}^{(n)}$ on $\mathbf{Sm}/k$, whose restriction to $\mathbf{Sm}//k$ is isomorphic (in the appropriate homotopy category) to $E^{(n)}$, and with natural transformations $\tilde{\varphi}_{n,n+1}(E) : \tilde{E}^{(n+1)} \to \tilde{E}^{(n)}$ for all $n \geq 0$, compatible with the natural transformations $\varphi_{n+1,n}(E)$. For $n' < n$, let $\tilde{\varphi}_{n',n}(E) : \tilde{E}^{(n)} \to \tilde{E}^{(n')}$ be the composition of the maps $\tilde{\varphi}_{j,j+1}(E)$, $j = n', \ldots, n-1$. The main result of [13] is:

Theorem 2.1 ([13, theorem 7.1.1]) *Let $E \in \mathbf{Spt}_{S^1}(k)$ be quasi-fibrant. There is a natural isomorphism in $\mathcal{SH}_{S^1}(k)$*

$$f^s_n E \cong \tilde{E}^{(n)}.$$

Via this isomorphism, the maps $\rho^s_{n',n}(E) : f^s_n E \to f^s_{n'} E$ in the tower (2.1) *are compatible with the maps $\tilde{\varphi}_{n',n}(E)$.*

For a category $\mathcal{C}$, we let $\mathcal{C}*$ be $\mathcal{C}$ with a final object $*$ adjoined. Let I be a finite category and $\mathcal{X} : I \to \mathbf{Sm}//k*$ an I-diagram in $\mathbf{Sm}//k$, in other words, a functor. For a presheaf of spectra E on $\mathbf{Sm}//k$, define $E(\mathcal{X})$ as

$$E(\mathcal{X}) := \operatorname*{holim}_{I^{\mathrm{op}}} E \circ \mathcal{X}^{\mathrm{op}},$$

where $E(*)$ is defined to be the initial object in $\mathbf{Spt}$.

We have the I-diagram in $\mathbf{Spt}_{S^1}(k)$,

$$\Sigma^\infty_s \mathcal{X}_+ : I \to \mathbf{Spt}_{S^1}(k),$$

defined by $\Sigma^\infty_{S^1}\mathcal{X}_+(i) := \Sigma^\infty_{S^1}\mathcal{X}(i)_+$ if $\mathcal{X}(i)$ is in $\mathbf{Sm}/k$, and $\Sigma^\infty_{S^1}*_+$ equal to the final S^1-spectrum $(pt, pt, \ldots)$, and the similarly defined I-diagram $\Sigma^\infty_T \mathcal{X}_+ : I \to$

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$\mathbf{Spt}_T(k)$ given by $\Sigma^\infty_T \mathcal{X}_+(i) := \Sigma^\infty_T \mathcal{X}(i)_+$ if $\mathcal{X}(i)$ is in $\mathbf{Sm}/k$, and setting $\Sigma^\infty_T *_+$ equal to the final T-spectrum $(pt, pt, \ldots)$.

Remark 2.2 Let $\mathcal{X} : I \to \mathbf{Sm}//k*$ be an I-diagram. It follows directly from the definitions that, for $E \in \mathbf{Spt}_{S^1}(k)$, we have a canonical isomorphism of spectra

$$E(\mathcal{X}) \cong \mathcal{H}om(\operatorname{hocolim}_I \Sigma^\infty_s \mathcal{X}_+, E).$$

We thus may define $\mathcal{E}(\mathcal{X})$ for $\mathcal{E} \in \mathbf{Spt}_T(k)$ as

$$\mathcal{E}(\mathcal{X}) := \mathcal{H}om(\operatorname{hocolim}_I \Sigma^\infty_T \mathcal{X}_+, \mathcal{E}).$$

Using the functors $\Omega^\infty_T : \mathbf{Spt}_T(k) \to \mathbf{Spt}_{S^1}(k)$, $\Sigma^\infty_T : \mathbf{Spt}_{S^1}(k) \to \mathbf{Spt}_T(k)$, and the adjunction isomorphism $\mathcal{H}om(\Sigma^\infty_T \mathcal{A}, \mathcal{B}) \cong \mathcal{H}om(\mathcal{A}, \Omega^\infty_T \mathcal{B})$, we arrive at the isomorphism

$$\mathcal{E}(\mathcal{X}) \cong (\Omega^\infty_T \mathcal{E})(\mathcal{X}).$$

Definition 2.3 Take $\mathcal{E} \in \mathcal{SH}(k)$ and let $\mathcal{X} : I \to \mathbf{Sm}//k*$ be an I-diagram. For integers a, b, take a fibrant model $\Sigma^{a-b}_{S^1} \Sigma^b_{\mathbb{G}_m} \tilde{\mathcal{E}} \in \mathbf{Spt}_T(k)$ for $\Sigma^{a-b}_{S^1} \Sigma^b_{\mathbb{G}_m} \mathcal{E}$ and define the abelian group $\mathcal{E}^{a,b}(\mathcal{X})$ as

$$\mathcal{E}^{a,b}(\mathcal{X}) := \pi_0((\Sigma^{a-b}_{S^1} \Sigma^b_{\mathbb{G}_m} \tilde{\mathcal{E}})(\mathcal{X})).$$

We note that, for $\mathcal{X}$ the one-point diagram with value $X \in \mathbf{Sm}/k$, we have

$$\begin{aligned} \mathcal{E}^{a,b}(\mathcal{X}) &= \pi_0((\Sigma^{a-b}_{S^1} \Sigma^b_{\mathbb{G}_m} \tilde{\mathcal{E}})(\mathcal{X})) \\ &= \pi_0(\mathcal{H}om(\Sigma^\infty_T X_+, \Sigma^{a-b}_{S^1} \Sigma^b_{\mathbb{G}_m} \mathcal{E})) \\ &= \operatorname{Hom}_{\mathcal{SH}(k)}(\Sigma^\infty_T X_+, \Sigma^{a-b}_{S^1} \Sigma^b_{\mathbb{G}_m} \mathcal{E}) = \mathcal{E}^{a,b}(X). \end{aligned}$$

Example 2.4 Let X be in $\mathbf{Sm}/k$ and $j : U \to X$ an open immersion with closed complement Z. Let I be the category

$$\begin{array}{ccc} 0 & \xrightarrow{a} & 1 \\ \downarrow & & \\ * & & \end{array}$$

and let $X/U : I \to \mathbf{Sm}//k$ be the diagram $0 \mapsto U$, $1 \mapsto X$, $* \mapsto *$, $a \mapsto j$. Then for E a presheaf of spectra on $\mathbf{Sm}//k$, $E(X/U)$ is just the homotopy fiber of $j^* : E(X) \to E(U)$. Similarly, if I is the one-point category 0 and $\mathcal{X}$ is the functor $\mathcal{X} : 0 \to \mathbf{Sm}//k$ with $\mathcal{X}(0) = X$, we have a canonical isomorphism $E(X) \cong E(\mathcal{X})$ in $\mathcal{SH}$.

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Lemma 2.5 *Let $\mathcal{X} : I \to \mathbf{Sm}//k*$ be a finite diagram of smooth k-schemes (possibly with $\mathcal{X}(i) = *$ for some values $i \in I$) and take $\mathcal{E} \in \mathcal{SH}(k)$. Let $E \in \mathbf{Spt}_{S^1}(k)$ be a fibrant model for $\Omega^\infty_T \mathcal{E} \in \mathcal{SH}_{S^1}(k)$. Then $f_n\mathcal{E}(\mathcal{X}) \in \mathcal{SH}$ is represented by the spectrum $E^{(n)}(\mathcal{X})$.*

Proof: As mentioned in remark 2.2, the adjunction

$$\Sigma^\infty_T : \mathcal{SH}_{S^1}(k) \leftrightarrow \mathcal{SH}(k) : \Omega^\infty_T$$

gives the isomorphism in $\mathcal{SH}$

$$f_n\mathcal{E}(\mathcal{X}) \cong \Omega^\infty_T f_n\mathcal{E}(\mathcal{X}).$$

It follows from [13, theorem 7.1.1, theorem 9.0.3] that we have the isomorphism in $\mathcal{SH}_{S^1}(k)$

$$\Omega^\infty_T f_n\mathcal{E} \cong f^s_n \Omega^\infty_T \mathcal{E} = f^s_n E.$$

We thus have the isomorphisms in $\mathcal{SH}$

$$\begin{aligned}\Omega^\infty_T f_n\mathcal{E}(\mathcal{X}) &\cong f_n E(\mathcal{X}) \\ &= \operatorname{holim}_I f^s_n E \circ \mathcal{X}^{\mathrm{op}}.\end{aligned}$$

We give the category $\mathbf{Spt}^I$ of I-diagrams in $\mathbf{Spt}$ the projective model structure, with weak equivalences and fibrations the pointwise ones; this does indeed define a model category structure on $\mathbf{Spt}^I$ since $\mathbf{Spt}$ is cofibrantly generated, see [7, theorem 11.6.1]. Let $\mathbf{Ho}(\mathbf{Spt}^I)$ denote the homotopy category. By [13, theorem 7.1.1], we have the isomorphism in $\mathbf{Ho}(\mathbf{Spt}^I)$

$$f^s_n E \circ \mathcal{X}^{\mathrm{op}} \cong E^{(n)} \circ \mathcal{X}^{\mathrm{op}},$$

giving the isomorphism in $\mathcal{SH}$

$$\Omega^\infty_T f_n\mathcal{E}(\mathcal{X}) \cong \operatorname{holim}_I E^{(n)} \circ \mathcal{X}^{\mathrm{op}} = E^{(n)}(\mathcal{X}).$$

□

We let $K \in \mathbf{Spt}_{S^1}(k)$ be the presheaf of spectra on $\mathbf{Sm}/k$ given by sending X to the Quillen-Waldhausen spectrum $K(X)$ representing the algebraic K-theory of the category of locally free coherent sheaves on X. We note that K is a quasi-fibrant object of $\mathbf{Spt}_{S^1}(k)$.

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Proposition 2.6 *Let $\mathcal{X} : I \to \mathbf{Sm}//k*$ be a finite diagram of smooth k-schemes as in lemma 2.5. Take X in $\mathbf{Sm}/k$.*
1. There are canonical isomorphisms for all $p,q \in \mathbb{Z}$,

$$\mathcal{CK}^{p,q}(\mathcal{X}) \xrightarrow{a^{p,q}_{\mathcal{X}}} \pi_{2q-p}K^{(q)}(\mathcal{X}),$$
$$\mathcal{K}^{p,q}(\mathcal{X}) \xrightarrow{b^{p,q}_{\mathcal{X}}} \pi_{2q-p}K(\mathcal{X})$$

natural in $\mathcal{X}$.
2. Letting $\rho^{p,q}(\mathcal{X}) : \mathcal{CK}^{p,q}(\mathcal{X}) \to \mathcal{K}^{p,q}(\mathcal{X})$ be the map induced by the natural transformation $\rho_0 : \mathcal{CK} \to \mathcal{K}$, and $\varphi^{p,q}(\mathcal{X}) : \pi_{2q-p}K^{(q)}(\mathcal{X}) \to \pi_{2q-p}K(\mathcal{X})$ the map induced by the natural transformation $\varphi_q : K^{(q)} \to K^{(0)}$ and weak equivalence $\epsilon : K \to K^{(0)}$, the diagram

$$\begin{array}{ccc} \mathcal{CK}^{p,q}(\mathcal{X}) & \xrightarrow{a^{p,q}_{\mathcal{X}}} & \pi_{2q-p}K^{(q)}(\mathcal{X}) \\ {\scriptstyle \rho^{p,q}(\mathcal{X})}\downarrow & & \downarrow{\scriptstyle \varphi^{p,q}(\mathcal{X})} \\ \mathcal{K}^{p,q}(\mathcal{X}) & \xrightarrow[b^{p,q}_{\mathcal{X}}]{} & \pi_{2q-p}K(\mathcal{X}) \end{array}$$

commutes.

Proof: Using lemma 2.5 and a variety of isomorphisms, adjunctions and definitions, we have the sequence of isomorphisms

$$\begin{aligned} \mathcal{CK}^{p,q}(\mathcal{X}) &= \mathrm{Hom}_{\mathcal{SH}(k)}(\operatorname*{hocolim}_I \Sigma^\infty_T \mathcal{X}_+, \Sigma^{p,q} f_0 \mathcal{K}) \\ &= \mathrm{Hom}_{\mathcal{SH}(k)}(\operatorname*{hocolim}_I \Sigma^\infty_T \mathcal{X}_+, \Sigma^{p-2q}_{S^1}\Sigma^q_T f_0 \mathcal{K}) \\ &= \mathrm{Hom}_{\mathcal{SH}(k)}(\operatorname*{hocolim}_I \Sigma^\infty_T \mathcal{X}_+, \Sigma^{p-2q}_{S^1} f_q \Sigma^q_T \mathcal{K}) \qquad \text{by (1.2)} \\ &= \mathrm{Hom}_{\mathcal{SH}(k)}(\Sigma^\infty_T \Sigma^{2q-p}_{S^1} \operatorname*{hocolim}_I \Sigma^\infty_s \mathcal{X}_+, f_q \mathcal{K}) \\ &= \mathrm{Hom}_{\mathcal{SH}_{S^1}(k)}(\Sigma^{2q-p}_{S^1} \operatorname*{hocolim}_I \Sigma^\infty_s \mathcal{X}, \Omega^\infty_T f_q \mathcal{K}) \\ &= \mathrm{Hom}_{\mathcal{SH}_{S^1}(k)}(\Sigma^{2q-p}_{S^1} \operatorname*{hocolim}_I \Sigma^\infty_s \mathcal{X}, f^s_q K) \qquad \text{by (2.2)} \\ &= \mathrm{Hom}_{\mathcal{SH}}(S^{2q-p}, \mathcal{H}om(\operatorname*{hocolim}_I \Sigma^\infty_T \mathcal{X}_+, f^s_q K)) \\ &= \pi_{2q-p}K^{(q)}(\mathcal{X}). \end{aligned}$$

This defines the isomorphism $a^{p,q}_{\mathcal{X}}$. The isomorphism $b^{p,q}_{\mathcal{X}}$ is defined similarly,

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using the sequence of isomorphisms (we skip some of the steps given above)

$$\begin{aligned}
\mathcal{K}^{p,q}(\mathcal{X}) &= \mathrm{Hom}_{\mathcal{SH}(k)}(\operatorname*{hocolim}_I \Sigma_T^\infty \mathcal{X}_+, \Sigma_{S^1}^{p-2q}\Sigma_T^q \mathcal{K}) \\
&= \mathrm{Hom}_{\mathcal{SH}(k)}(\operatorname*{hocolim}_I \Sigma_T^\infty \mathcal{X}_+, \Sigma_{S^1}^{p-2q} \mathcal{K}) \\
&= \mathrm{Hom}_{\mathcal{SH}_{S^1}(k)}(\Sigma_{S^1}^{2q-p} \operatorname*{hocolim}_I \Sigma_s^\infty \mathcal{X}, \Omega_T^\infty \mathcal{K}) \\
&= \mathrm{Hom}_{\mathcal{SH}_{S^1}(k)}(\Sigma_{S^1}^{2q-p} \operatorname*{hocolim}_I \Sigma_s^\infty \mathcal{X}, K)) \\
&= \mathrm{Hom}_{\mathcal{SH}}(S^{2q-p}, \mathcal{H}om(\operatorname*{hocolim}_I \Sigma_T^\infty \mathcal{X}_+, K)) \\
&= \pi_{2q-p} K(\mathcal{X}).
\end{aligned}$$

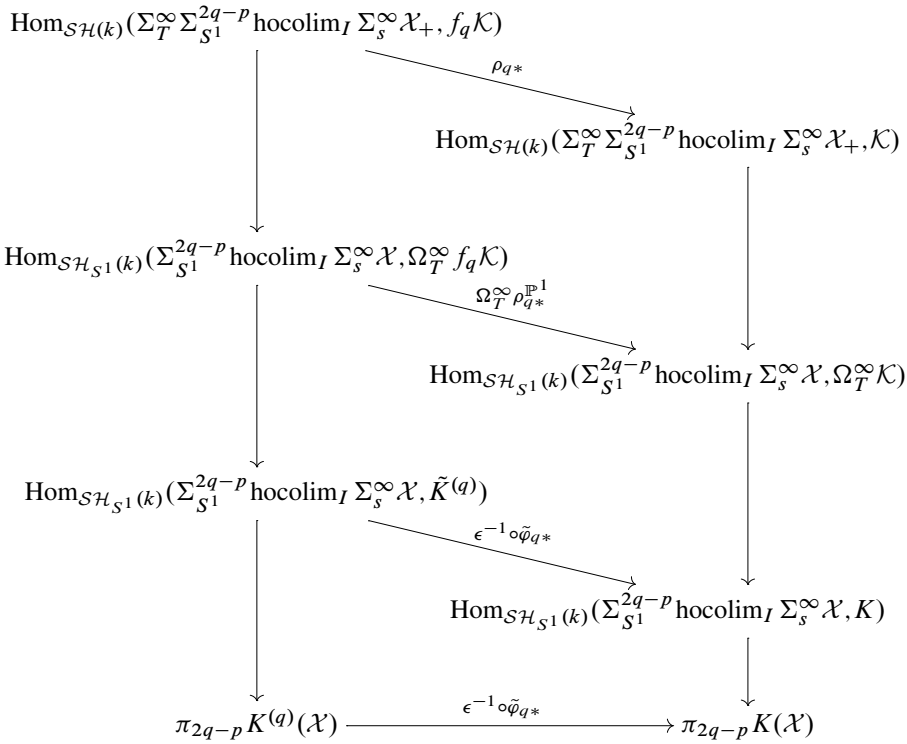
For (2), by [13, theorem 9.0.3], $f_q\mathcal{K}$ may be represented by the $\mathbb{P}^1$-spectrum $(\tilde{K}^{(q)}, \tilde{K}^{(q+1)}, \ldots)$, with bonding map ϵ_n given by a canonical weak equivalence $\tilde{K}^{(q+n)} \to \Omega_{\mathbb{P}^1}\tilde{K}^{(q+n+1)}$ (described in [13, §8.3]). The K-theory $\mathbb{P}^1$-spectrum is similarly represented by $\mathcal{K} := (K, K, \ldots)$, with bonding map ϵ_n given by the weak equivalence $K \to \Omega_{\mathbb{P}^1} K$ ($\mathbb{P}^1$-bundle formula). Furthermore, the map $\rho_q : f_q\mathcal{K} \to \mathcal{K}$ is given by the sequence of maps $\epsilon^{-1} \circ \tilde{\varphi}_{q+n} : \tilde{K}^{(q+n)} \to K$. As the bonding maps are all weak equivalences, the functor $\Omega^\infty_{\mathbb{P}^1}$ applied to these respective models is just the 0th term in the sequence:

$$\Omega^\infty_{\mathbb{P}^1} f_q\mathcal{K} = \tilde{K}^{(q)},\ \Omega^\infty_{\mathbb{P}^1}\mathcal{K} = K$$

and $\Omega^\infty_{\mathbb{P}^1}(\varphi_q^{\mathbb{P}^1}) = \epsilon^{-1} \circ \tilde{\varphi}_q : \tilde{K}^{(q)} \to K$; restricted to $\mathbf{Sm}//k$, this gives us the map

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$\epsilon^{-1} \circ \tilde{\varphi}_q : K^{(q)} \to K$. This gives us the commutative diagram



with all vertical arrows isomorphisms, proving (2). □

Definition 2.7 Let X be a finite type k-scheme. We let $\mathcal{M}_X$ denote the category of coherent sheaves on X. For an integer n, let $\mathcal{M}_{(n)}(X) \subset \mathcal{M}_X$ be the full subcategory with objects those coherent sheaves $\mathcal{F}$ with $\dim_k \mathrm{supp}(\mathcal{F}) \leq n$. If X is locally equi-dimensional over k, let $\mathcal{M}^{(n)}(X) \subset \mathcal{M}_X$ be the full subcategory with objects those coherent sheaves $\mathcal{F}$ with $\mathrm{codim}_X \mathrm{supp}(\mathcal{F}) \geq n$.

Corollary 2.8 *Let X be smooth over k. Then $\mathcal{CK}^{2n,n}(X)$ is equal to $K_0(X)$ for $n \leq 0$. For $n > 0$, $\mathcal{CK}^{2n,n}(X)$ is determined by the exact sequence*

$$K_0^{(n)}(X,1) \xrightarrow{\delta_1^* - \delta_0^*} K_0(\mathcal{M}^{(n)}(X)) \to \mathcal{CK}^{2n,n}(X) \to 0.$$

In particular,

$$\mathcal{CK}^{2n,n}(k) \to \mathcal{K}^{2n,n}(k) = K_0(k) = \mathbb{Z}$$

is an isomorphism for $n \leq 0$; for $n > 0$, $\mathcal{CK}^{2n,n}(k) = 0$.

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Proof: Viewing X as the one-point diagram $* \mapsto X$, it follows from proposition 2.6 that we have a canonical isomorphism $\mathcal{CK}^{2n,n}(X) \cong \pi_0(K^{(n)}(X))$. For $n \leq 0$, $K^{(n)}(X)$ is weakly equivalent to $K(X)$. For $n > 0$, $K^{(n)}(X,m)$ is -1-connected, since $K^W(Y)$ is so for all closed $W \subset Y$, $Y \in \mathbf{Sm}/k$. The spectral sequence

$$E^1_{p,q} = \pi_q K^{(n)}(X,p) \Longrightarrow \pi_{p+q} K^{(n)}(X)$$

for the simplicial spectrum $p \mapsto K^{(n)}(X,p)$ thus gives the desired presentation for $\mathcal{CK}^{2n,n}(X) = \pi_0(K^{(n)}(X))$. □

For $X \in \mathbf{Sm}/k$ with generic point η we define $\mathcal{CK}^{a,b}(\eta)$ as the stalk at η of the presheaf on X_{Zar}, $U \mapsto \mathcal{CK}^{a,b}(U)$.

Corollary 2.9 *Let η be a generic point of some $X \in \mathbf{Sm}/k$. Then $\mathcal{CK}^{p,q}(\eta) = 0$ for $p > q$.*

Proof: By proposition 2.6, $\mathcal{CK}^{p,q}(\eta) \cong \pi_{2q-p} K^{(q)}(\eta)$. But $K^{(q)}(\eta)$ is the total spectrum of the simplicial spectrum $m \mapsto K^{(q)}(\eta,m)$, with

$$K^{(q)}(\eta,m) := \operatorname*{hocolim}_{W \in \mathcal{S}^n_X(m)} K^W(\eta \times \Delta^m).$$

As $\eta \times \Delta^m \cong \Delta^m_{k(\eta)}$, it follows that each closed subset W of $\eta \times \Delta^m$ has $\operatorname{codim} W \leq m$, and hence $K^{(q)}(\eta,m)$ is the 0-spectrum if $m < q$. Furthermore, the K-theory presheaf is a presheaf of -1-connected spectra and for each open immersion $j : U \to V$ in $\mathbf{Sm}/k$, the restriction map $j^* K_0(V) \to K_0(U)$ is surjective. Thus $K^W(\eta \times \Delta^m)$ is a -1-connected spectrum for each m. Using the strongly convergent spectral sequence

$$E^1_{a,b} = \pi_b K^{(q)}(\eta,a) \Longrightarrow \pi_{a+b} K^{(q)}(\eta),$$

we see that $\pi_n K^{(q)}(\eta) = 0$ for $n < q$, hence

$$\mathcal{CK}^{p,q}(\eta) \cong \pi_{2q-p} K^{(q)}(\eta) = 0$$

for $p > q$. □

We conclude this section with a discussion of the functor $\mathcal{CK}^{2q-1,q}$. Let $\mathcal{E} \in \mathcal{SH}(S)$ represent a bi-graded cohomology theory. Let $t_{\mathcal{E}} \in \mathcal{E}^{1,1}(\mathbb{G}_m)$ be the element corresponding to the unit $1 \in \mathcal{E}^{0,0}(S)$ under the suspension isomorphism. By functoriality, $t_{\mathcal{E}}$ gives a map of pointed sets

$$t_{\mathcal{E}}(X) : \mathcal{O}^\times_X(X) \to \mathcal{E}^{1,1}(X);$$

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if $\mathcal{E}$ admits an orientation $c_{\mathcal{E}} \in \mathcal{E}^{2,1}(\mathbb{P}^\infty)$ (which we will from now on assume), then $t_{\mathcal{E}}(X)$ is a group homomorphism.[1] Using the $\mathcal{E}^{*,*}(S)$-module structure on $\mathcal{E}^{*,*}(X)$, $t_{\mathcal{E}}(X)$ extends to a map of $\mathcal{E}^{*,*}(S)$-modules

$$t_{\mathcal{E}}(X) : \mathcal{E}^{2*,*}(S) \otimes_{\mathbb{Z}} \mathcal{O}_X^\times(X) \to \mathcal{E}^{2*+1,*+1}(X).$$

Lemma 2.10 *Suppose $S = \mathrm{Spec} k$. Let η be a generic point of some $X \in \mathbf{Sm}/k$.*

1. Take $\mathcal{E} = \mathcal{K}$. Then $\mathcal{K}^{2*,*}(k) \cong \mathbb{Z}[\beta, \beta^{-1}]$, $\deg\beta = -1$ and $t_{\mathcal{K}}(\eta) : \mathcal{K}^{2*,*}(k) \otimes_{\mathbb{Z}} k(\eta)^\times \to \mathcal{K}^{2*+1,*+1}(\eta)$ is an isomorphism.

2. Take $\mathcal{E} = \mathcal{CK}$. Then $\mathcal{CK}^{2*,*}(k) \cong \mathbb{Z}[\beta]$, $\deg\beta = -1$ and $t_{\mathcal{CK}}(\eta) : \mathcal{CK}^{2*,*}(k) \otimes_{\mathbb{Z}} k(\eta)^\times \to \mathcal{CK}^{2*+1,*+1}(\eta)$ is an isomorphism.

3. Take $\mathcal{E} = \mathrm{MGL}$ and suppose k has characteristic zero. Then $\mathrm{MGL}^{2*,*}(k)$ is canonically isomorphic to the Lazard ring $\mathbb{L}^*$, $t_{\mathrm{MGL}}(\eta) : k(\eta)^\times \to \mathrm{MGL}^{1,1}(\eta)$ is an isomorphism and $t_{\mathrm{MGL}}(\eta) : \mathrm{MGL}^{2*,*}(k) \otimes_{\mathbb{Z}} k(\eta)^\times \to \mathrm{MGL}^{2*+1,*+1}(\eta)$ is surjective.

Proof: We have already seen in remark 1.3 that under the isomorphism $\mathcal{K}^{1,1}(\mathbb{G}_m) \cong K_1(\mathbb{G}_m)$, $t_{\mathcal{K}}$ goes to the class of the canonical unit t (or possibly t^{-1}). By functoriality, the map $t_{\mathcal{K}} : k(\eta)^\times \to K_1(\eta) = \mathcal{K}^{1,1}(\eta)$ is the usual isomorphism $k(\eta)^\times \cong K_1(\eta)$ sending $x \in k(\eta)^\times$ to the class of the automorphism $\times x^\epsilon : k(\eta) \to k(\eta)$, where $\epsilon = \pm 1$ is universal choice of sign (independent of k or η).

The isomorphism $\mathcal{K}^{2*,*}(k) \cong \mathbb{Z}[\beta, \beta^{-1}]$ follows from the Bott periodicity isomorphism $\mathcal{K}^{2n,n}(k) \cong K_0(k) \cong \mathbb{Z}$. Since β is invertible, the fact that $t_{\mathcal{K}} : k(\eta)^\times \to \mathcal{K}^{1,1}(\eta)$ is an isomorphism implies that $t_{\mathcal{K}}(\eta) : \mathcal{K}^{2*,*}(k) \otimes_{\mathbb{Z}} k(\eta)^\times \to \mathcal{K}^{2*+1,*+1}(\eta)$ is an isomorphism.

For (2), it follows from the universal property of $f_0 \to \mathrm{id}$ that $\mathcal{CK}^{a,b}(X) \to \mathcal{K}^{a,b}(X)$ is an isomorphism for all $b \leq 0$, $a \in \mathbb{Z}$, $X \in \mathbf{Sm}/k$. In particular, $\mathcal{CK}^{2b-1,b}(X) = \mathcal{K}^{2b-1,b}(X) = K_1(X)$ for $b \leq 0$. For $b > 1$, $\mathcal{CK}^{2b-1,b}(\eta) = 0$ by corollary 2.9, and $\mathcal{CK}^{1,1}(X) \to \mathcal{K}^{1,1}(X)$ is an isomorphism by lemma 1.6. Similarly, the map $\mathcal{CK}^{2n,n}(X) \to \mathcal{K}^{2n,n}(X) = \mathbb{Z}\beta^{-n}$ is an isomorphism for $n \leq 0$ and by corollary 2.9, $\mathcal{CK}^{2n,n}(\eta) = 0$ for $n > 0$. Thus the map $\mathcal{CK}^{2*,*}(k) \to \mathcal{K}^{2*,*}(k)$ identifies $\mathcal{CK}^{2*,*}(k)$ with the subring $\mathbb{Z}[\beta]$ of $\mathcal{K}^{2*,*}(k) = \mathbb{Z}[\beta, \beta^{-1}]$. Putting this all together, (1) implies (2).

For (3), the orientation $c_{\mathcal{K}}$ gives the canonical monoid morphism $\vartheta_{\mathcal{K}} : \mathrm{MGL} \to$

[1] Letting $\mathbb{S}$ denote the sphere spectrum and writing $[a] := t_{\mathbb{S}}(a)$, this follows from the identity $[ab] = [a] + [b] + H[a][b]$ ($H : \mathbb{S} \wedge \mathbb{G}_m \to \mathbb{S}$ the stable Hopf map) and the fact that H goes to zero in any oriented theory $\mathcal{E}$. Both these facts are proven by Morel in [20, §6].

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$\mathcal{K}$ with $\vartheta(c_{\mathrm{MGL}}) = c_{\mathcal{K}}$ (see remark 1.9), inducing the commutative diagram

$$\begin{array}{ccc} \mathcal{O}_X(X)^\times & \xrightarrow{t_{\mathrm{MGL}}} & \mathrm{MGL}^{1,1}(X) \\ & \searrow^{t_{\mathcal{K}}} & \downarrow{\scriptstyle \vartheta_{\mathcal{K}}} \\ & & \mathcal{K}^{1,1}(X). \end{array}$$

As $t_{\mathcal{K}} : k(\eta)^\times \to \mathcal{K}^{1,1}(\eta) = K_1(\eta)$ is an isomorphism, it follows that $t_{\mathrm{MGL}} : k(\eta)^\times \to \mathrm{MGL}^{1,1}(\eta)$ is injective. The isomorphism $\mathbb{L}^* \to \mathrm{MGL}^{2*,*}(k)$ and the surjectivity of $t_{\mathrm{MGL}} : \mathbb{L}^* \otimes k(\eta)^\times \to \mathrm{MGL}^{2*+1,*+1}(k(\eta))$ follow from the Hopkins-Morel spectral sequence [8, 9]

$$E_2^{p,q} := \mathbb{L}^q \otimes H^{p-q}(Y, \mathbb{Z}(n-q)) \Longrightarrow \mathrm{MGL}^{p+q,n}(Y). \tag{2.3}$$

□

3. Oriented duality theories

Recall from [14, §1] the category $\mathbf{SP}/k$ of *smooth pairs* over k, with objects (M, X), $M \in \mathbf{Sm}/k$ and $X \subset M$ a closed subset (not necessarily smooth); a morphism $f : (M, X) \to (N, Y)$ is a morphism $f : M \to N$ in $\mathbf{Sm}/k$ such that $f^{-1}(Y) \subset X$. For a full subcategory $\mathcal{V}$ of $\mathbf{Sch}_k$, let $\mathcal{V}^{\mathrm{pr}}$ be the subcategory of $\mathcal{V}$ with the same objects as $\mathcal{V}$, but with morphisms the projective morphisms in $\mathbf{Sch}_k$.

Building on work of Mocanasu [19] and Panin-Smirnov [22], the second author has defined in [14, definition 3.1] the notion of a bi-graded *oriented duality theory* (H, A) on $\mathbf{Sch}_k$. Here A is a bi-graded oriented cohomology theory on $\mathbf{SP}/k$, $(M, X) \mapsto A_X^{*,*}(M)$, and H is a functor from $\mathbf{Sch}_k^{\mathrm{pr}}$ to bi-graded abelian groups. The oriented cohomology theory A satisfies the axioms listed in [14, definitions 1.2, 1.5]. For the reader's convenience, we give a partial description of those axioms here:

- (localization) The association $(M, X) \mapsto A_X^{*,*}(M)$ admits a long exact sequence

  $$\ldots \to A_X^{*,*}(M) \to A^{*,*}(M) \to A^{*,*}(M \setminus X) \xrightarrow{\partial} A_X^{*+1,*}(M) \to \ldots$$

  where for instance $A^{*,*}(M) := A_M^{*,*}(M)$ and the boundary map ∂ is part of the data.

- (Nisnevich excision) If $f : M' \to M$ is étale, $X \subset M$ a closed subset and $f : f^{-1}(X) \to X$ an isomorphism of schemes (with reduced structure), then $f^* : A_X^{*,*}(M) \to A_{X'}^{*,*}(M')$ is an isomorphism.

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- (homotopy invariance) For $(M, X) \in \mathbf{SP}/k$, the pull-back map
$$p^* : A_X^{*,*}(M) \to A_{X\times\mathbb{A}^1}^{*,*}(M \times \mathbb{A}^1)$$
is an isomorphism.
- There are natural external products $A_X^{*,*}(M) \otimes A_Y^{*,*}(N) \to A_{X\times Y}^{*,*}(M \times_k N)$, and a ring structure is given by the external products and pull-back by the diagonal.
- There is an *orientation* given by a collection of isomorphisms
$$\mathrm{Th}_X^E : A_X(M) \to A_X(E),$$
for $(M, X) \in \mathbf{SP}/k$ and $E \to M$ a vector bundle, satisfying the axioms of [22, def. 3.1.1].

In [14, theorem 1.12, corollary 1.13], we give an extension of [22, theorem 2.5(1)] to the setting of cohomology with supports. Namely, we show that the data of an orientation is equivalent to giving well-behaved push-forward maps $f_* : A_X(M) \to A_Y(N)$ for $(M, X), (N, Y) \in \mathbf{SP}/k$, with the meaning of "well-behaved" detailed in [14, §1].

The homology theory H comes with the following structures:

- Restriction maps $j^* : H_{*,*}(X) \to H_{*,*}(U)$ for each open immersion $j : U \to X$ in $\mathbf{Sch}_k$
- External products $\times : H_{*,*}(X) \otimes H_{*,*}(Y) \to H_{*,*}(X \times Y)$
- Boundary maps $\partial_{X,Y} : H_{*,*}(X \setminus Y) \to H_{*-1,*}(Y)$ for each closed subset $Y \subset X$,
- Isomorphisms $\alpha_{M,X} : H_{*,*}(X) \to A_X^{2m-*,m-*}(M)$ for each $(M, X) \in \mathbf{SP}/k$, $m = \dim_k M$,
- Cap product maps
$$f^*(-)\cap : A_X^{a,b}(M) \otimes H_{*,*}(Y) \to H_{*-a,*-b}(Y \cap f^{-1}(X))$$
for $(M, X) \in \mathbf{SP}/k$, $f : Y \to X$ a morphism in $\mathbf{Sch}_k$.

These satisfy a number of axioms and compatibilities (see [14, §3] for details), which essentially say that a given structure for $A_X^{*,*}(M)$ is compatible with the corresponding structure for $H_{*,*}(X)$ via the isomorphism $\alpha_{M,X}$. Roughly speaking, this is saying that a given structure for $A_X^{*,*}(M)$ depends only on X and not the choice of embedding $X \hookrightarrow M$.

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Remark 3.1 Let $L \to Y$ be a line bundle on some $Y \in \mathbf{Sm}/k$ with 0-section $0 : Y \to L$. For an oriented cohomology theory A one has the element

$$c_1^A(L) := 0^*(0_*(1_Y^A)),$$

where $1_Y^A \in A^0(Y)$ is the unit element. As pointed out in [22, corollary 3.3.8], or as noted in [14, remark 1.17], for line bundles L, M on some $Y \in \mathbf{Sm}/k$, the elements $c_1(L), c_1(M) \in A^1(Y)$ are nilpotent, and commute with one another, hence for each power series $F(u,v) \in A^*(k)[[u,v]]$ the evaluation $F(c_1(L), c_1(M))$ gives a well-defined element of $A^*(Y)$. In addition, the cohomology theory A has a unique associated formal group law $F_A(u,v) \in A^*(k)[[u,v]]$ with

$$F_A(c_1(L), c_1(M)) = c_1(L \otimes M)$$

for all line bundles L, M on $Y \in \mathbf{Sm}/k$.

For an $X \in \mathbf{Sch}_k$ with line bundle $L \to X$, the quasi-projectivity of X implies that X admits a closed immersion $i : X \to M$ for some $M \in \mathbf{Sm}/k$ such that L extends to a line bundle $\mathcal{L} \to M$. One can then define

$$\tilde{c}_1(L) : H_*(X) \to H_{*-1}(X)$$

via the product

$$(-) \cdot c_1(\mathcal{L}) : A_X^*(M) \to A_X^{*+1}(M)$$

and the isomorphisms $H_*(X) \cong A_X^{d_M - *}(M)$. One shows that this is independent of the choice of $(M, \mathcal{L})$, giving the well-defined operator $\tilde{c}_1(L)$.

The main example of oriented duality theory (H, A) is given by an oriented monoid $\mathcal{E}$ in $\mathcal{SH}(k)$, assuming k is a field admitting resolution of singularities (e.g., characteristic zero), defined by taking

$$\mathcal{E}_X^{a,b}(M) := \mathrm{Hom}_{\mathcal{SH}(k)}(\Sigma_T^\infty(M/M \setminus X), \Sigma^{a,b}\mathcal{E}),$$

that is, the usual bi-graded cohomology with supports. For each $X \in \mathbf{Sch}_k$, choose a closed immersion of X into a smooth M and set $\mathcal{E}'_{a,b}(X) := \mathcal{E}_X^{2m-a,m-b}(M)$, where $m = \dim_k M$. In [14, §4] it was shown that $(M,X) \mapsto \mathrm{MGL}_X^{*,*}(M)$ defines an oriented bi-graded ring cohomology theory. The proof of the fact that $(M,X) \mapsto \mathcal{E}_X^{*,*}(M)$ defines an oriented bi-graded ring cohomology theory for an arbitrary oriented monoid $\mathcal{E}$ in $\mathcal{SH}(k)$ is exactly the same; the main point is theorem of Panin-Smirnov [22, theorem 3.7.4], which says that an orientation for $\mathcal{E}$ (in the sense of remark 1.8) defines an orientation in the sense of ring cohomology theories for the bi-graded $\mathcal{E}$-cohomology with supports.

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The fact that the formula given above for the homology theory $\mathcal{E}'_{*,*}$ is well-defined and extends to make $(\mathcal{E}'_{*,*}(-),\mathcal{E}'^{*,*}_{-}(-))$ a bi-graded oriented duality theory is [14, theorem 3.4]. The essential point is to show that the cohomology with support $\mathcal{E}_X^{2d-*,d-*}(M)$, for $X \hookrightarrow M$ a closed immersion of some X in a smooth M of dimension d, depends (up to canonical isomorphism) only on X, and similarly, given a projective morphism $f : Y \to X$ in $\mathbf{Sch}_k$, there are smooth pairs (M,X), (N,Y), an extension of F to a projective morphism $F : N \to M$ and the map $F_* : \mathcal{E}_Y^{2d_N-*,d_N-*}(N) \to \mathcal{E}_X^{2d_M-*,d_M-*}(M)$ is independent (via the canonical isomorphisms $\mathcal{E}'_{*,*}(Y) \cong \mathcal{E}_Y^{2d_N-*,d_N-*}(N)$, $\mathcal{E}'_{*,*}(X) \cong \mathcal{E}_X^{2d_M-*,d_M-*}(M)$) of the choices. The other structures for $\mathcal{E}'_{*,*}(-)$ are defined similarly via the $\mathcal{E}$-cohomology with supports, and one has the corresponding independence of any choices.

It follows directly from the construction of $\mathcal{E}'$ that the assignment $(\mathcal{E},c_\mathcal{E}) \mapsto (\mathcal{E}',\mathcal{E})$ is functorial in the oriented cohomology theory $(\mathcal{E},c_\mathcal{E})$. In particular, let $\mathrm{ch} : (\mathrm{MGL},\mathrm{c}_{\mathrm{MGL}}) \to (\mathcal{E},\mathrm{c}_\mathcal{E})$ be a morphism of oriented cohomology theories, that is, $\mathrm{ch} : \mathrm{MGL} \to \mathcal{E}$ is a monoid morphism in $\mathcal{SH}(k)$ with $\mathrm{ch}(\mathrm{c}_{\mathrm{MGL}}) = \mathrm{c}_\mathcal{E}$. Then we have an extension of ch to a natural transformation of oriented duality theories

$$(\mathrm{ch}',\mathrm{ch}) : (\mathrm{MGL}',\mathrm{MGL}) \to (\mathcal{E}',\mathcal{E}).$$

Remark 3.2 As shown in lemma 2.10, the coefficient rings for $\mathcal{K}$ and $\mathcal{CK}$ are $\mathcal{K}^{2*,*}(k) = \mathbb{Z}[\beta,\beta^{-1}]$ and $\mathcal{CK}^{2*,*}(k) = \mathbb{Z}[\beta]$, respectively, with β having degree -1. For $\mathcal{K}$, the orientation $c_\mathcal{K}$ restricted to $\mathbb{P}^n$ is given by the class of $1-[\mathcal{O}(-1)] \in K_0(\mathbb{P}^n) \cong \mathcal{K}^{2,1}(\mathbb{P}^n)$. It follows (by functoriality and Jouanoulou's trick) that for a line bundle L on some $X \in \mathbf{Sm}/k$, the 1st Chern class is given by $c_1^\mathcal{K}(L) = \beta^{-1}(1-[L^{-1}])$ (where we consider $1,[L^{-1}] \in \mathcal{K}^{0,0}(X) = K_0(X)$). A direct calculation gives the formal group law for $(\mathcal{K}'_{2*,*},\mathcal{K}^{2*,*})$ as $(F_\mathcal{K}(u,v) = u + v - \beta \cdot uv,\mathbb{Z}[\beta,\beta^{-1}])$. Since the orientation for $\mathcal{K}$ lifts to that of $\mathcal{CK}$, it follows that the formal group law for $(\mathcal{CK}'_{2*,*},\mathcal{CK}^{2*,*})$ is $(u + v - \beta \cdot uv,\mathbb{Z}[\beta])$.

4. G-theory and connective G-theory

We apply the machinery described in §3 to K-theory and connective K-theory, with the orientations given by remark 1.8 (for K-theory) and lemma 1.11 (for connective K-theory). This gives us the oriented duality theories $(\mathcal{K}'_{*,*},\mathcal{K}^{*,*})$ and $(\mathcal{CK}'_{*,*},\mathcal{CK}^{*,*})$; the canonical morphism $\mathcal{CK} \to \mathcal{K}$ is compatible with the orientations and thus gives the map of oriented duality theories $(\mathcal{CK}'_{*,*},\mathcal{CK}^{*,*}) \to (\mathcal{K}'_{*,*},\mathcal{K}^{*,*})$.

We can describe $\mathcal{CK}'_{a,b}(X)$ more explicitly, with the help of the models $K^{(n)}$ for $f_n K$.

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Indeed, for a smooth pair (M,X), Quillen's resolution theorem (see [26, §4], corollary 1 and the comments following corollary 2) and localization theorem [26, §7, prop. 3.2] gives us the natural homotopy fiber sequence

$$G(X) \to K(M) \to K(M \setminus X),$$

where $G(X)$ is the Quillen-Waldhausen K-theory spectrum of the abelian category of coherent sheaves on X. This, together with "algebraic Bott-periodicity"

$$\mathcal{K}^{a,b}(X) \cong K_{2b-a}(X)$$

gives us the canonical isomorphisms

$$\mathcal{K}'_{a,b}(X) \xrightarrow{\beta^X_{a,b}} G_{a-2b}(X);\ \mathcal{K}^{a,b}_X(M) \xrightarrow{\beta^{a,b}_X} K^X_{2b-a}(M) \tag{4.1}$$

where $K^X(M)$ is as usual the homotopy fiber of the restriction map $K(M) \to K(M \setminus X)$.

The construction of the simplicial spectrum $m \mapsto K^{(n)}(X,m)$ can be carried through replacing K-theory with G-theory. For details, we refer the reader to [16, §8.4] and [17]; for the reader's convenience, we give a brief sketch of the construction here.

Let X be a finite type k-scheme. There is a technical problem in the construction of the simplicial spectrum $m \mapsto G^{(n)}(X,m)$, due to the fact that, for $g : \Delta^p \to \Delta^m$ an inclusion of a face of Δ^m, the corresponding restriction map

$$(\mathrm{id}_X \times g)^* : \mathcal{M}_{X\times\Delta^m} \to \mathcal{M}_{X\times\Delta^p}$$

is not exact, hence does not lead to a functor on the corresponding K-theory spectra. To get around this problem, one replaces $\mathcal{M}_{X\times\Delta^m}$ with the full subcategory $\mathcal{M}_{X\times\Delta^m,\partial}$ of coherent sheaves which are Tor-independent with respect to the inclusions of faces; for $U \subset X \times \Delta^m$ an open subset, we let $\mathcal{M}_{U,\partial}$ be the similarly defined full subcategory of $\mathcal{M}_U$. Letting $G(X,m)$ be the corresponding K-theory spectrum of the exact category $\mathcal{M}_{X\times\Delta^m,\partial}$, we have the simplicial spectrum $m \mapsto G(X,m)$.

Since we are not assuming that X is equi-dimensional over k, it is more convenient to index by dimension rather than codimension. Let $\mathcal{S}^X_n(m)$ denote the set of closed subsets W of $X \times \Delta^m$ such that

$$\dim W \cap X \times F \le n + p$$

for all faces $F \cong \Delta^p$ of Δ^m, and let

$$G_{(n)}(X,m) := \operatorname*{hocolim}_{W\in\mathcal{S}^X_n(m)} G_W(X,m)$$

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where $G_W(X,m)$ is by definition the homotopy fiber of the restriction map

$$K(\mathcal{M}_{X\times\Delta^m,\partial}) \to K(\mathcal{M}_{X\times\Delta^m\setminus W,\partial}).$$

This gives us the simplicial spectrum $m \mapsto G_{(n)}(X,m)$ with total spectrum $G_{(n)}(X)$, and the corresponding tower of spectra

$$\ldots \to G_{(n)}(X) \to G_{(n+1)}(X) \to \ldots \\ \to G_{(\dim X)}(X) = G_{(\dim X+1)}(X) = \ldots = G(X\times\Delta^*),$$

where $\dim X$ is by definition the maximum of $\dim_k X'$ as X' runs over the irreducible components of X. The identification of $G(X) := K(\mathcal{M}_X)$ with $G(X,0)$ gives us the map of spectra

$$G(X) \to G_{(\dim X)}(X),$$

which, using the homotopy invariance property of G-theory, is easily seen to be a weak equivalence.

It is easy to see that $X \mapsto G_{(n)}(X)$ is contravariantly functorial with respect to flat maps of finite type k-schemes. As for the usual G-theory construction, there is a canonical push-forward functor

$$i_* : G_{(n)}(W) \to G_{(n)}(X)$$

for $i : W \to X$ a closed immersion; this extends to a push-forward map for arbitrary projective morphisms, but only in the homotopy category $\mathcal{SH}$ as one needs to replace the various categories of coherent sheaves with the corresponding subcategories which have no higher direct images with respect to the given projective morphism (see [17, §4] for details).

In case X is smooth of dimension d over k, the canonical map $K^{(d-n)}(X,m) \to G_{(n)}(X,m)$ is a weak equivalence for all m and n (by Quillen's resolution theorem *loc. cit.*), giving the weak equivalence $\delta_X : K^{(d-n)}(X) \to G_{(n)}(X)$.

The localization theorem for the spectra $G_{(n)}$ from [16, corollary 8.12] is

Theorem 4.1 *Let $j : U \to X$ be an open immersion of finite type k-schemes with closed complement $i : W \to X$. Then the sequence*

$$G_{(n)}(W) \xrightarrow{i_*} G_{(n)}(X) \xrightarrow{j^*} G_{(n)}(U)$$

is a homotopy fiber sequence.

Corollary 4.2 *Let X be in* $\mathbf{Sch}_k$.

1. There are canonical isomorphisms

$$a^X_{a,b} : \mathcal{CK}'_{a,b}(X) \to \pi_{a-2b}(G_{(b)}(X))$$

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compatible with push-forward for projective morphisms and pull-back for open immersions.

2. For $X \in \mathbf{Sm}/k$ of dimension d, the isomorphism $a^X_{a,b}$ is compatible with the isomorphism of proposition 2.6 via the weak equivalence $\delta_X : K^{(d-b)}(X) \to G_{(b)}(X)$ and the isomorphism $\alpha_X : \mathcal{CK}'_{a,b}(X) \to \mathcal{CK}^{2d-a,d-b}(X)$.

3. The isomorphism (4.1) $\beta^X_{a,b} : \mathcal{K}'_{a,b}(X) \to \pi_{a-2b}(G(X))$*, the natural transformation $\rho' : \mathcal{CK}' \to \mathcal{K}'$ and the canonical map $\varphi^X_b : G_{(b)}(X) \to G_{(d)}(X) \sim G(X)$ and the isomorphism $a^X_{a,b}$ give rise to a commutative diagram*

$$\begin{array}{ccc} \mathcal{CK}'_{a,b}(X) & \xrightarrow{a^X_{a,b}} & \pi_{a-2b}(G_{(b)}(X)) \\ {\scriptstyle \rho'_{a,b}(X)}\downarrow & & \downarrow{\scriptstyle \varphi^X_b} \\ \mathcal{K}'_{a,b}(X) & \xrightarrow[\beta^X_{a,b}]{} & \pi_{a-2b}(G(X)). \end{array}$$

Proof: (1) Recalling that $\mathbf{Sch}_k$ is the category of quasi-projective k-scheme, we choose a closed immersion $X \hookrightarrow M$ with $M \in \mathbf{Sm}/k$, giving us the smooth pair $(M, X) \in \mathbf{SP}/k$. For each q, let $K^{(q)}_X(M)$ denote the homotopy fiber of the restriction map

$$j^* : K^{(q)}(M) \to K^{(q)}(M \setminus X).$$

Using example 2.4 and proposition 2.6(1) for the diagram

$$\begin{array}{ccc} M \setminus X & \longrightarrow & M \\ \downarrow & & \\ * & & \end{array}$$

gives us the canonical isomorphism

$$\mathcal{CK}'_{a,b}(X) \xrightarrow[\sim]{\alpha_{M,X}} \mathcal{CK}^{2m-a,m-b}_X(M) \xrightarrow[\sim]{a^{2m-a,m-b}_X} \pi_{a-2b}(K^{(m-b)}_X(M)).$$

By the localization theorem 4.1, we have the canonical weak equivalence

$$\beta_{X,M} : G_{(b)}(X) \to \mathrm{fib}(G_{(b)}(M) \to G_{(b)}(M \setminus X))$$

induced by the functors $(i_X \times \mathrm{id})_* : \mathcal{M}_{X \times \Delta^n} \to \mathcal{M}_{M \times \Delta^n}$ and the canonical isomorphism of $(j_{M \setminus X} \times \mathrm{id})^* \circ (i \times \mathrm{id})_*$ with the 0-functor. As M is regular, Quillen's resolution theorem *loc. cit.* comparing the K-theory and G-theory of

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regular schemes leads to a canonical weak equivalence on the (iterated) homotopy fibers

$$K_X^{(m-b)}(M) = \mathrm{fib}(K^{(m-b)}(M) \to K^{(m-b)}(M \setminus X)) \xrightarrow{\gamma_{X,M}} \mathrm{fib}(G_{(b)}(M) \to G_{(b)}(M \setminus X)).$$

This defines the isomorphism in $\mathcal{SH}$

$$\delta_{X,M} := \beta_{X,M} \circ \gamma_{X,M}^{-1} : K_X^{(m-b)}(M) \to G_{(b)}(X).$$

Both $\gamma_{X,M}$ and $\beta_{X,M}$ are compatible with pull-back by flat maps $f : M' \to M$ (with $X' := f^{-1}(X)$), hence δ is compatible with pull-back by open immersions.

Suppose we have a closed immersion $i : X' \to X$. We use the same ambient smooth scheme M, giving the map of pairs $\mathrm{id}_M : (X,M) \to (X',M)$. The push-forward by i on $K_X^{(m-b)}(M)$ is by definition the map on the homotopy fibers induced by the commutative diagram

$$\begin{array}{ccc} K^{(m-b)}(M) & \xrightarrow{u^*} & K^{(m-b)}(M \setminus X') \\ {\scriptstyle \mathrm{id}}\downarrow & & \downarrow{\scriptstyle w^*} \\ K^{(m-b)}(M) & \xrightarrow{v^*} & K^{(m-b)}(M \setminus X) \end{array}$$

where u,v,w are the evident open immersions. It is not hard to check that the diagram

$$\begin{array}{ccccc} G_{(b)}(X') & \xrightarrow{i_{X'*}} & G_{(b)}(M) & \xrightarrow{u^*} & G_{(b)}(M \setminus X') \\ {\scriptstyle i_*}\downarrow & & {\scriptstyle \mathrm{id}}\downarrow & & \downarrow{\scriptstyle w^*} \\ G_{(b)}(X) & \xrightarrow[i_{X*}]{} & G_{(b)}(M) & \xrightarrow[v^*]{} & G_{(b)}(M \setminus X) \end{array}$$

commutes up to canonical homotopy. From this it follows that $i_*^G \circ \delta_{X,M} = \delta_{X',M} \circ i_*^K$.

For both K-theory and G-theory, the push-forward for a projection $p : Y \times \mathbb{P}^n \to Y$ is given by the projection onto the factor $[\mathcal{O}_{Y \times \mathbb{P}^n}]$ in the respective projective bundle formulas with bases $[\mathcal{O}(-i)]$, $i = 0,\ldots,n$. From this it is not hard to show that $\delta_{X,M}$ is compatible with p_*, and hence compatible with push-forward by an arbitrary projective morphism, completing the proof of (1).

The assertion (2) follows directly from the construction of $\delta_{X,M}$ for $X = M$.

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For (3), we apply proposition 2.6(2) with $\mathcal{X}$ the diagram as above, and note that we have a commutative diagram (in $\mathcal{SH}$)

$$\begin{array}{ccc} K_X^{(m-b)}(M) & \xrightarrow{\delta_{X,M}} & G_{(b)}(X) \\ \downarrow{\scriptstyle \rho_{m-b}} & & \downarrow{\scriptstyle \rho_b} \\ K_X(M) & \xrightarrow[\delta_{X,M}]{} & G(X). \end{array}$$

□

Our main interest is in the "geometric" portion $\mathcal{CK}'_{2*,*}$ of the theory $\mathcal{CK}'_{*,*}$. For a finite type k-scheme X, and integer n, recall from definition 2.7 the full subcategory $\mathcal{M}_{(n)}(X)$ of $\mathcal{M}_X$ of coherent sheaves supported in dimension at most n. The following result improves upon corollary 2.8:

Theorem 4.3 *Let X be a quasi-projective k-scheme.*
1. For each $n \geq 0$, there is a canonical isomorphism

$$a_n^X : \mathcal{CK}'_{2n,n}(X) \to \mathrm{im}(K_0(\mathcal{M}_{(n)}(X)) \to K_0(\mathcal{M}_{(n+1)}(X))),$$

compatible with pull-back by open immersions and push-forward by projective morphisms.

2. For $n \geq 0$, let

$$\rho'_n(X) : \mathrm{im}(K_0(\mathcal{M}_{(n)}(X)) \to K_0(\mathcal{M}_{(n+1)}(X))) \to G_0(X)$$

be the map induced by the inclusions $\mathcal{M}_{(n)}(X) \to \mathcal{M}_{(n+1)}(X) \to \mathcal{M}_X$. Then the diagram

$$\begin{array}{ccc} \mathcal{CK}'_{2n,n}(X) & \xrightarrow{a_n^X} & \mathrm{im}(K_0(\mathcal{M}_{(n)}(X)) \to K_0(\mathcal{M}_{(n+1)}(X))) \\ \downarrow{\scriptstyle \rho'_{2n,n}(X)} & & \downarrow{\scriptstyle \rho'_n(X)} \\ \mathcal{K}'_{2n,n}(X) & \xrightarrow[\beta^X_{2n,n}]{} & G_0(X) \end{array}$$

commutes.

3. $\mathcal{CK}'_{a,b}(X) = 0$ for $a < 2b$.

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Proof: By corollary 4.2, we have the isomorphisms

$$a^X_{a,b} : \mathcal{CK}'_{a,b}(X) \to \pi_{a-2b}(G_{(b)}(X)),$$

compatible with pull-back by open immersions and push-forward by projective morphisms. Thus, for (1) it suffices to define an isomorphism

$$\delta_n(X) : \pi_0(G_{(n)}(X)) \to \mathrm{im}(K_0(\mathcal{M}_{(n)}(X)) \to K_0(\mathcal{M}_{(n+1)}(X))),$$

compatible with ith pull-back by open immersions and push-forward by projective morphisms, and for (3) we need to show that $G_{(b)}(X)$ is -1-connected for all $b \geq 0$.

Recall that $G_{(b)}(X)$ is the simplicial spectrum $m \mapsto G_{(b)}(X,m)$ and

$$G_{(b)}(X,m) = \operatorname*{hocolim}_{W \in \mathcal{S}^X_{(b)}(m)} G_W(\mathcal{M}_{X\times\Delta^m,\partial}).$$

We showed in [16, lemma 8.7] that for all open $U \subset X \times \Delta^m$, the inclusion

$$\mathcal{M}_{U,\partial} \to \mathcal{M}_U$$

induces a weak equivalence on the K-theory spectra. Thus $G_W(\mathcal{M}_{X\times\Delta^m,\partial})$ is weakly equivalent to the homotopy fiber of the restriction map

$$j^* : G(X \times \Delta^m) \to G(X \times \Delta^m \setminus W)$$

which by Quillen's localization theorem is in turn weakly equivalent to $G(W)$. In particular, this shows that $G_{(b)}(X,m)$ is -1-connected and we have a strongly convergent spectral sequence

$$E^1_{p,q} = \pi_q G_{(b)}(X,p) \Longrightarrow \pi_{p+q} G_{(b)}(X).$$

As $E^1_{p,q} = 0$ for $p + q < 0$, this shows that $G_{(b)}(X)$ is -1-connected and hence $\mathcal{CK}'_{a,b}(X) = 0$ for $a < 2b$, proving (3).

For (1), the spectral sequence gives the right exact sequence

$$\pi_0(G_{(n)}(X,1)) \xrightarrow{\partial} \pi_0(G_{(n)}(X,0)) \xrightarrow{\pi_d} \mathcal{CK}'_{2n,n}(X) \to 0.$$

By definition, $G_{(n)}(X,0) = K(\mathcal{M}_{(n)}(X))$, so we need to show that the image M of $\pi_0(G_{(n)}(X,1))$ under ∂ is the same as the kernel N of the map

$$K_0(\mathcal{M}_{(n)}(X)) \to K_0(\mathcal{M}_{(n+1)}(X)).$$

Suppose we have done this. If we take $d \geq \dim X$, then the canonical maps $G_{(d)}(X,m) \to G(X \times \Delta^m)$ are weak equivalences and the canonical map

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$G(X) = G_{(d)}(X,0) \to G_{(d)}(X)$ is a weak equivalence, and thus we may identify $\varphi'_n : G_{(n)}(X) \to G(X)$ with $\varphi'_{d,n} : G_{(n)}(X) \to G_{(d)}(X)$. By corollary 4.2(3), this gives the commutative diagram

$$\begin{array}{ccccc}
\pi_0(G_{(n)}(X,1)) & \xrightarrow{\varphi'_{d,n}(X,1)} & \pi_0(G_{(d)}(X,1)) & & \\
\downarrow{\scriptstyle\partial} & & \downarrow{\scriptstyle\partial} & & \\
\pi_0(G_{(n)}(X,0)) & \xrightarrow{\varphi'_{d,n}(X,0)} & \pi_0(G_{(d)}(X,0)) & = & G_0(X) \\
\downarrow{\scriptstyle\pi_n} & & \downarrow{\scriptstyle\pi_d} & \nearrow{\scriptstyle\beta^X_{2n,n}} & \\
\mathcal{CK}'_{2n,n}(X) & \xrightarrow[\rho'_{2n,n}]{} & \mathcal{K}'_{2n,n}(X) & &
\end{array}$$

from which (2) follows directly.

We now complete the proof of (1). We note that $\mathcal{M}_{(n)}(X)$ is a Serre subcategory of the abelian category $\mathcal{M}_{(n+1)}(X)$; let $\mathcal{M}_{(n+1/n)}(X)$ denote the quotient category. By Quillen's localization theorem, we have the exact sequence

$$\ldots \to K_1(\mathcal{M}_{(n+1/n)}(X)) \xrightarrow{\partial} K_0(\mathcal{M}_{(n)}(X)) \to K_0(\mathcal{M}_{(n+1)}(X)) \to \ldots.$$

Furthermore, by devissage [26, §5, thm. 4] we have

$$K_1(\mathcal{M}_{(n+1/n)}(X)) \cong \oplus_{x \in X_{(n+1)}} K_1(k(x)) = \oplus_{x \in X_{(n+1)}} k(x)^\times$$

where $X_{(n+1)}$ is the set of dimension $n+1$ points of X. Finally, the boundary map $K_1(\mathcal{M}_{(n+1/n)}(X)) \to K_0(\mathcal{M}_{(n)}(X))$ can be described as follows: Take $y \in X_{(n+1)}$ and let $Y \subset X$ be the reduced closure of y. Take $f \in k(y)^\times$ and let $Y' \subset Y \times \mathbb{P}^1$ be the reduced closure of the graph of f. For $t \in \mathbb{P}^1(k)$, let $\mathcal{O}_{Y'}(t) := \mathcal{O}_{Y'} \otimes_{\mathcal{O}_{\mathbb{P}^1}} k(t)$. Then for $f \in k(y)^\times$,

$$\partial(f) = p_{1*}([\mathcal{O}_{Y'}(0)] - [\mathcal{O}_{Y'}(\infty)]).$$

Noting that $\mathcal{O}_{Y'}(t)$ is an $\mathcal{O}_{X\times t}$-module, we can consider $p_{1*}([\mathcal{O}_{Y'}(0)])$ as simply $[\mathcal{O}_{Y'}(0)]$ via the isomorphism $p_1 : X \times 0 \to X$, and similarly for $[\mathcal{O}_{Y'}(\infty)]$. Finally, as f is in $k(y)^\times$ and Y has dimension $n+1$, it follows that $\mathcal{O}_{Y'}(0)$ and $\mathcal{O}_{Y'}(\infty)$ have support of dimension at most n.

Given such a Y and $f \in k(y)^\times$, we may identify $X \times \Delta^1$ with the open subscheme $X \times (\mathbb{P}^1 \setminus \{1\})$ of $X \times \mathbb{P}^1$ via the open immersion $j : \Delta^1 \to \mathbb{P}^1$ defined by $j(t_0,t_1) := t_1/t_0$. Then the pull-back coherent sheaf $(\mathrm{id} \times j)^*(\mathcal{O}_{Y'})$ is in $\mathcal{M}_{(n)}(X \times \Delta^1, \partial)$ and

$$\partial([(\mathrm{id} \times j)^*(\mathcal{O}_{Y'})]) = \pm\partial(f \in k(y)^\times).$$

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Thus, $M \supset N$.

For the reverse inclusion, take $\mathcal{F} \in \mathcal{M}_{(n)}(X \times \Delta^1, \partial)$, let Z be the support of $\mathcal{F}$, let $Y \subset X$ be the closure of $p_1(Z)$ and let $W \subset X \times \Delta^1$ be $p_1^{-1}(Y)$; let $i : Z \to W$ be the inclusion. By devissage, we may assume that $\mathcal{F}$ is an $\mathcal{O}_Z$-module, giving us the class $[\mathcal{F}] \in G_0(Z)$. But $W \cong Y \times \mathbb{A}^1$, so by the homotopy property for G-theory, there is a class $\eta \in G_0(Y)$ with

$$i_*([\mathcal{F}]) = p_1^*(\eta).$$

Letting $i_0, i_1 : Y \to Y \times \Delta^1 = W$ be the inclusions defined by $t_0 = 0$, $t_1 = 0$, respectively, this shows that

$$i_0^*(i_*([\mathcal{F}])) - i_1^*(i_*([\mathcal{F}])) = 0$$

in $G_0(Y)$. As Y has dimension $\leq n+1$, this shows that the image of $\partial([\mathcal{F}])$ in $G_0(\mathcal{M}_{(n+1)}(X))$ is zero, and hence $M \subset N$. □

Corollary 4.4 *Take $X \in \mathbf{Sch}_k$ and let $d = \dim X$. Let $\gamma_n^X : \mathcal{CK}'_{2n,n}(X) \to G_0(X)$ be the composition of $\rho'_{2n,n}(X) : \mathcal{CK}'_{2n,n}(X) \to \mathcal{K}'_{2n,n}(X)$ with the isomorphism $\beta^X_{2n,n} : \mathcal{K}'_{2n,n}(X) \to G_0(X)$. Then γ_n^X is an isomorphism for $n \geq d$ and is injective for $n = d-1$.*

Proof: This follows immediately from theorem 4.3, noting that $\mathcal{M}_{(d)}(X) = \mathcal{M}_X$. □

Remark 4.5 For X smooth of dimension d, using the identity closed immersion $X \to X$ gives the canonical isomorphism

$$\alpha_{X,X} : \mathcal{CK}'_{a,b}(X) \to \mathcal{CK}_X^{2d-a,d-b}(X) = \mathcal{CK}^{2d-a,d-b}(X).$$

In particular, theorem 4.3 shows that

$$\mathcal{CK}^{2n,n}(X) = \mathrm{im}(K_0(\mathcal{M}^{(n)}(X)) \to K_0(\mathcal{M}^{(n-1)}(X))).$$

In fact, the argument for theorem 4.3 works to show this identity for k an arbitrary perfect field without the need for resolution of singularities.

Remark 4.6 Cai [2] has defined the notion of a bi-graded "oriented Borel-Moore functor" on $\mathbf{Sch}_k$ in the sense of [2, §2]; after a suitable reindexing, this notion is similar to, but not the same as the homology part of an oriented duality theory. To conform with our indexing conventions, we change Cai's indexing by the transformation $(a,b) \mapsto (a-b,-b)$. Cai has defined a theory of "connective higher K-theory" as a bi-graded oriented Borel-Moore functor on $\mathbf{Sch}_k$. Denoting Cai's reindexed theory as $CK^{\mathrm{Cai}}_{a,b}$, this is defined as

$$CK^{\mathrm{Cai}}_{a,b}(X) := \mathrm{im}(K_{a-2b}(\mathcal{M}_{(a-b)}(X)) \to K_{a-2b}(\mathcal{M}_{(a-b+1)}(X))).$$

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Thus we have

$$\mathcal{CK}'_{2n,n}(X) = CK^{\mathrm{Cai}}_{2n,n}(X)$$

for $X \in \mathbf{Sch}_k$.

We conclude this section with a description of the 1st Chern class operators on $\mathcal{CK}'_{2n,n}$. Take X in $\mathbf{Sch}_k$. Let L be a line bundle on X, and let $i : X \to M$ be a closed immersion with $M \in \mathbf{Sm}/k$ such that L extends to a line bundle $\mathcal{L}$ on M. We note that the 1st Chern class operator $\tilde{c}_1(L)$ on $\mathcal{CK}'_{2n,n}(X)$ is given by the product with $c_1(\mathcal{L}) \in \mathcal{CK}^{2,1}(M)$ on $\mathcal{CK}^{2*,*}_X(M)$. As $\mathcal{CK}^{2,1}(M) \to \mathcal{K}^{2,1}(M) = K_0(M)$ is injective (corollary 4.4), it follows that the operators $\tilde{c}_1(L)$ satisfy the multiplicative formal group law:

$$\tilde{c}_1(L \otimes L') = \tilde{c}_1(L) + \tilde{c}_1(L') - \beta\tilde{c}_1(L) \circ \tilde{c}_1(L')$$

where $\beta \in \mathcal{CK}'_{2,1}(k)$ is the "Bott element", that is, the element corresponding to $1 \in \mathbb{Z}$ under the sequence of isomorphisms

$$\mathcal{CK}'_{2,1}(k) = \mathcal{CK}^{-2,-1}(k) \to \mathcal{K}^{-2,-1}(k) \cong K_0(k) \xrightarrow{\dim_k} \mathbb{Z}.$$

Thus, we need only describe $\tilde{c}_1(L)$ for L which is very ample on X.

Let α be in $\mathcal{CK}'_{2n,n}(X)$ for some n. If $n \geq \dim_k X + 1$, then both $\mathcal{CK}'_{2n,n}(X)$ and $\mathcal{CK}'_{2n-2,n-1}(X)$ are equal to $G_0(X)$ via the canonical map $\mathcal{CK}'_{2*,*}(X) \to \mathcal{K}'_{2*,*}(X) = G_0(X)$, so $\tilde{c}_1(L)$ is given by the 1st Chern class operator on $\mathcal{K}'_{2*,*}(X) = G_0(X)$. This in turn is given by multiplication by the K-theory 1st Chern class via the $K_0(X)$-module structure on $G_0(X)$, that is,

$$\tilde{c}_1^{G_0}(L)(\alpha) = (1 - L^{-1}) \cdot \alpha.$$

In general, we have our description

$$\mathcal{CK}'_{2n,n}(X) \cong \mathrm{im}(K_0(\mathcal{M}_{(n)}(X)) \to K_0(\mathcal{M}_{(n+1)}(X))),$$

compatible with projective push-forward. Thus, for each $\alpha \in \mathcal{CK}'_{2n,n}(X)$, there is a closed immersion $i : X' \to X$ in $\mathbf{Sch}_k$, with $\dim_k X' \leq n$, and an element $\alpha' \in \mathcal{CK}'_{2n,n}(X')$ with $i_*(\alpha') = \alpha$. Since

$$\tilde{c}_1(L)(i_*\alpha') = i_*(\tilde{c}_1(i^*L)(\alpha')),$$

we reduce to the case $n \geq \dim_k X$; similarly, we may assume X is irreducible. By our above computation, the only remaining case is $n = \dim_k X$.

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In this case, $\mathcal{CK}'_{2n,n}(X) = G_0(X)$ and for $\alpha \in \mathcal{CK}'_{2n,n}(X)$, the element $\tilde{c}_1(\alpha) \in \mathcal{CK}'_{2n-2,n-1}(X)$ is characterized by the identity

$$(1 - L^{-1}) \cdot \alpha = \mathrm{im}(\tilde{c}_1(\alpha) \in G_0(X)),$$

via the inclusion $\mathcal{CK}'_{2n-2,n-1}(X) \to G_0(X)$ (corollary 4.4). We now use the assumption that L is very ample. Write $\alpha = \sum_i n_i[\mathcal{F}_i]$, where the $\mathcal{F}_i$ are coherent sheaves on X, n_i are integers and $[-]$ denotes class in $G_0(X)$. Choose a section s of L that is locally a non-zero divisor on each $\mathcal{F}_i$ and on $\mathcal{O}_X$; this is possible by an application of Kleiman's transversality theorem [11, theorem 2(i), remark 7], using the fact that L is very ample. Let $H \subset X$ be the closed subscheme defined by s, giving us the short exact sequence of sheaves on X

$$0 \to L^{-1} \to \mathcal{O}_X \to \mathcal{O}_H \to 0.$$

For each i, the tensor product $\mathcal{F}_i \otimes \mathcal{O}_H$ is an element of $\mathcal{M}_{(n-1)}(X)$ (its support is contained in H), giving the well-defined class $[\mathcal{F}_i \otimes \mathcal{O}_H]$ in $K_0(\mathcal{M}_{(n-1)}(X))$. In addition, the above exact sequence and the fact that s is a non-zero divisor on $\mathcal{F}_i$ gives the identity

$$[\mathcal{F}_i \otimes \mathcal{O}_H] = (1 - L^{-1}) \cdot [\mathcal{F}_i] \in G_0(X)$$

for each i and hence $\tilde{c}_1(\alpha) \in \mathcal{CK}'_{2n-2,n-1}(X)$ is given by $\sum_i n_i[\mathcal{F}_i \otimes \mathcal{O}_H]$.

Remark 4.7 Set $CG_{a,b}(X) := \pi_{a-2b}G_{(b)}(X)$, for $X \in \mathbf{Sch}_k$. From the proof of corollary 4.2 we have isomorphisms $\delta_{X,M} : \pi_{a-2b}(K_X^{(m-b)}(M)) \to \pi_{a-2b}(G_{(b)}(X))$ for X a closed subscheme of $M \in \mathbf{Sm}/k$, $m = \dim M$. Combined with the natural isomorphisms $\mathcal{CK}_X^{2m-a,m-b}(M) \cong \pi_{a-2b}(K_X^{(m-b)}(M))$ given by theorem 2.1 and lemma 2.5, the properties of $G_{(b)}(-)$ discussed in this section show that we have defined an oriented duality theory $(CG_{*,*}, \mathcal{CK}^{*,*})$ on $\mathbf{Sch}_k$, for k an arbitrary perfect field, except that we have not defined a cap product structure of $\mathcal{CK}^{*,*}$ on $CG_{*,*}$. Possibly this could be supplied by the method used by Cai in [2, §6.3] to define pull back maps in his theory $CK^{Cai}_{*,*}$ for regular embeddings.

The results of this section can be interpreted as saying that, in case k admits resolution of singularities, the oriented duality theory $(\mathcal{CK}'_{*,*}, \mathcal{CK}^{*,*})$ on $\mathbf{Sch}_k$ defined using the results of [14] is isomorphic to $(CG_{*,*}, \mathcal{CK}^{*,*})$ (ignoring the cap product structure).

5. Algebraic cobordism and oriented duality theories

We assume throughout this section and the remainder of the paper that k satisfies resolution of singularities in the sense of [18, appendix A]; at present date, this mean k has characteristic zero.

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We recall the theory of *algebraic cobordism* $X \mapsto \Omega_*(X)$, $X \in \mathbf{Sch}_k$ [18]. For each $X \in \mathbf{Sch}_k$, $\Omega_n(X)$ is an abelian group with generators $(f : Y \to X)$, $Y \in \mathbf{Sm}/k$ irreducible of dimension n over k and $f : Y \to X$ a projective morphism [18, lemma 2.5.11]. Ω_* is the universal *oriented Borel-Moore homology theory* on $\mathbf{Sch}_k$ [18, theorem 7.1.1], where an oriented Borel-Moore homology theory on $\mathbf{Sch}_k$ consists of the data of a functor from $\mathbf{Sch}_k^{\mathrm{pr}}$ to graded abelian groups, external products, first Chern class operators $\tilde{c}_1(L) : \Omega_*(X) \to \Omega_{*-1}(X)$ for $L \to X$ a line bundle, and pull-back maps $g^* : \Omega_*(X) \to \Omega_{*+d}(Y)$ for each l.c.i. morphism $g : Y \to X$ of relative dimension d. These of course satisfy a number of compatibilities and additional axioms, see [18, §5.1] for details.

For an oriented duality theory (H, A) on $\mathbf{Sch}_k$ and Y in $\mathbf{Sm}/k$ of dimension d over k, the *fundamental class* $[Y]_{H,A} \in H_d(Y)$ is the image of the unit $1_Y \in A^0(Y)$ under the inverse of the isomorphism $\alpha_Y : H_d(Y) \to A^0(Y)$. For an oriented Borel-Moore homology theory B on $\mathbf{Sch}_k$, we similarly have the fundamental class $[Y]_B \in B_d(Y)$ defined by $[Y]_B := p^*(1)$, where $1 \in B_0(\mathrm{Spec}k)$ is the unit and $p : Y \to \mathrm{Spec}k$ the structure morphism.

We recall the following result from [14]:

Proposition 5.1 ([14, propositions 4.2, 4.4, 4.5]) *Let k be a field admitting resolution of singularities and let (H, A) be a $\mathbb{Z}$-graded oriented duality theory on $\mathbf{Sch}_k$.*
1. Let **GrAb** *denote the category of graded abelian groups. There is a unique natural transformation $\vartheta_H : \Omega_* \to H_*$ of functors $\mathbf{Sch}_k^{\mathrm{pr}} \to$ **GrAb**, such that $\vartheta_H(Y)([Y]_\Omega) = [Y]_{H,A}$ for all $Y \in \mathbf{Sm}/k$. In addition, ϑ_H is compatible with pull-back maps for open immersions in $\mathbf{Sch}_k$, with 1st Chern class operators, with external products and with cap products.*
2. For $Y \in \mathbf{Sm}/k$, let $\vartheta^A(Y) : \Omega^*(Y) \to A^*(Y)$ be the map induced by ϑ_H, the identity $\Omega^*(Y) = \Omega_{\dim Y - *}(Y)$ and the isomorphism $\alpha_Y : H_{\dim Y - *}(Y) \to A^*(Y)$. Then $\vartheta^A(Y)$ is a ring homomorphism and is compatible with pull-back maps for arbitrary morphisms in $\mathbf{Sm}/k$. Finally, one has

$$\vartheta^A(Y)(c_1^\Omega(L)) = c_1^A(L)$$

for each line bundle $L \to Y$.

Remark 5.2 We have already noted that one has a formal group law $F_A(u, v) \in A^*(k)[[u, v]]$ associated to the oriented cohomology theory A. Similarly, for each oriented Borel-Moore homology theory B on $\mathbf{Sch}_k$, there is an associated formal group law $F_B(u, v) \in B_*(k)[[u, v]]$, characterized by the identity $F_B(c_1(L), c_1(M)) = c_1(L \otimes M)$ for each pair of line bundles L, M on some $Y \in \mathbf{Sm}/k$ (this follows from [18, corollary 4.1.8, proposition 5.2.1, proposition 5.2.6]). Letting $\varphi_A : \mathbb{L}^* \to A^*(k)$, $\varphi_B : \mathbb{L}^* \to B^*(k)$ denote the classifying maps

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associated to F_A, F_B, respectively, suppose that A extends to an oriented duality theory (H, A). Then

$$\vartheta^A(F_\Omega) = F_A. \tag{5.1}$$

Indeed, F_A is characterized by identity $F_A(c_1^A(L), c_1^A(M)) = c_1^A(L \otimes M)$ for each pair of line bundles L, M on some $Y \in \mathbf{Sm}/k$, and since $\vartheta^A(c_1^\Omega(N)) = c_1^A(N)$ for each line bundle $N \to Z$, $Z \in \mathbf{Sm}/k$, the fact that $F_\Omega(c_1^\Omega(L), c_1^\Omega(M)) = c_1^\Omega(L \otimes M)$ combined with proposition 5.1(2) yields the identity (5.1).

Via the universal property of the Lazard ring, the relation (5.1) is equivalent to the identity

$$\vartheta^A \circ \varphi_\Omega = \varphi_A. \tag{5.2}$$

Finally, we recall that the classifying map $\varphi_\Omega : \mathbb{L}_* \to \Omega_*(k)$ is an isomorphism [18, theorem 1.2.7].

Corollary 5.3 *Let $(\mathcal{E}, c_\mathcal{E})$ be a pair consisting of a commutative monoid object $\mathcal{E} \in \mathcal{SH}(k)$ with orientation $c_\mathcal{E}$, and let $(\mathcal{E}'_{*,*}, \mathcal{E}^{*,*})$ be the corresponding bi-graded oriented duality theory. There is a unique natural transformation*

$$\vartheta_{(\mathcal{E}, c_\mathcal{E})} : \Omega_* \to \mathcal{E}'_{2*,*}$$

of functors $\mathbf{Sch}_k^{\mathrm{pr}} \to \mathbf{GrAb}$, such that $\vartheta_{(\mathcal{E}, c_\mathcal{E})}(Y)([Y]_\Omega) = [Y]_{\mathcal{E}', \mathcal{E}}$ for all $Y \in \mathbf{Sm}/k$. In addition, $\vartheta_{(\mathcal{E}, c_\mathcal{E})}$ is compatible with pull-back maps for open immersions in $\mathbf{Sch}_k$, 1st Chern class operators, external products and cap products. For $Y \in \mathbf{Sm}/k$, the map $\vartheta^\mathcal{E}(Y) : \Omega^*(Y) \to \mathcal{E}^{2*,*}(Y)$ induced by $\vartheta_{(\mathcal{E}, c_\mathcal{E})}$ is a ring homomorphism, is compatible with pull-back maps for arbitrary morphisms in $\mathbf{Sm}/k$, and satisfies

$$\vartheta_{(\mathcal{E}, c_\mathcal{E})}(Y)(c_1^\Omega(L)) = c_1^\mathcal{E}(L)$$

for each line bundle $L \to Y$.

Remark 5.4 By [18, lemma 2.5.11], $\Omega_*(X)$ is generated as an abelian group by the cobordism cycles $(f : Y \to X)$, $Y \in \mathbf{Sm}/k$ irreducible, $f : Y \to X$ a projective morphism. Furthermore, the identity $(f : Y \to X) = f_*([Y]_\Omega)$ holds in $\Omega_{\dim Y}(X)$. Thus $\vartheta_{(\mathcal{E}, c_\mathcal{E})}$ is characterized by the formula

$$\vartheta_{(\mathcal{E}, c_\mathcal{E})}(f : Y \to X) := f_*^{\mathcal{E}'}([Y]_{\mathcal{E}', \mathcal{E}}).$$

We may apply corollary 5.3 in the universal case: $\mathcal{E} = \mathrm{MGL}$ with its canonical orientation. This gives us the natural transformation

$$\vartheta_{\mathrm{MGL}} : \Omega_* \to \mathrm{MGL}'_{2*,*}. \tag{5.3}$$

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Theorem 5.5 ([12, theorem 3.1]) *If k is a field of characteristic zero, then the natural transformation* (5.3) *is an isomorphism.*

Remark 5.6 This result relies on the Hopkins-Morel spectral sequence (2.3), see [8, 9].

In the course of the proof of theorem 5.5, we proved another result which we will be using here.

Let X be in $\mathbf{Sch}_k$ and let $d = d_X := \max_{X'} \dim_k X'$, as X' runs over the irreducible components of X. We define $\mathrm{MGL}'^{(1)}_{2*,*}(X)$ by

$$\mathrm{MGL}'^{(1)}_{2*,*}(X) := \varinjlim_{W} \mathrm{MGL}'_{2*,*}(W)$$

as W runs over all (reduced) closed subschemes of X which contain no dimension d generic point of X; $\Omega^{(1)}_*(X)$ is defined similarly. The natural transformation ϑ_{MGL} gives rise to the commutative diagram

$$\begin{array}{ccccccc} \Omega^{(1)}_*(X) & \xrightarrow{i_*} & \Omega_*(X) & \xrightarrow{j^*} & \oplus_{\eta\in X_{(d)}}\Omega_*(k(\eta)) & \longrightarrow & 0 \\ \downarrow{\scriptstyle \vartheta^{(1)}} & & \downarrow{\scriptstyle \vartheta(X)} & & \downarrow{\scriptstyle \vartheta} & & \\ \mathrm{MGL}'^{(1)}_{2*,*}(X) & \xrightarrow[i_*]{} & \mathrm{MGL}'_{2*,*}(X) & \xrightarrow[j^*]{} & \oplus_{\eta\in X_{(d)}}\mathrm{MGL}'_{2*,*}(k(\eta)) & \longrightarrow & 0 \end{array} \tag{5.4}$$

with exact rows and with all vertical arrows isomorphisms. As (MGL′,MGL) is an oriented duality theory, the bottom line extends to the long exact sequence

$$\begin{aligned} \ldots \to \oplus_{\eta\in X_{(d)}}\mathrm{MGL}'_{2*+1,*}(k(\eta)) &\xrightarrow{\partial} \mathrm{MGL}'^{(1)}_{2*,*}(X) \\ &\xrightarrow{i_*} \mathrm{MGL}'_{2*,*}(X) \xrightarrow{j^*} \oplus_{\eta\in X_{(d)}}\mathrm{MGL}'_{2*,*}(k(\eta)) \to 0. \end{aligned}$$

Furthermore, the appropriate edge homomorphism in the Hopkins-Morel spectral sequence (2.3) [8, 9] gives a surjection for each $\eta \in X_{(d)}$

$$t_{\mathrm{MGL}}(\eta) : \mathbb{L}_{*-d+1} \otimes k(\eta)^\times \to \mathrm{MGL}'_{2*+1,*}(k(\eta))$$

(see lemma 2.10).

Let $\mathbb{Z}[k(\eta)^\times]$ denote the free $\mathbb{Z}$-module on the set $k(\eta)^\times$ and let $\gamma_\eta : \mathbb{Z}[k(\eta)^\times] \to k(\eta)^\times$ be the map sending $\sum_i n_i f_i$ to $\prod_i f_i^{n_i}$. We have constructed in [12, §6] a group homomorphism

$$\mathrm{Div} : \mathbb{L}_{*-d+1} \otimes_{\mathbb{Z}} \oplus_{\eta\in X_{(d)}}\mathbb{Z}[k(\eta)^\times] \to \Omega^{(1)}_*(X)$$

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with $\vartheta^{(1)} \circ \mathrm{Div} = \partial \circ \oplus_\eta t_{\mathrm{MGL}}(\eta) \circ \mathrm{id} \otimes \gamma_\eta$. Since the maps $\vartheta^{(1)}$ and $\vartheta(X)$ are isomorphisms, the map Div factors through the surjection

$$\mathbb{L}_{*-d+1} \otimes \oplus_{\eta \in X_{(d)}} \mathbb{Z}[k(\eta)^\times] \xrightarrow{\mathrm{id} \otimes \oplus_\eta \gamma_\eta} \mathbb{L}_{*-d+1} \otimes \oplus_{\eta \in X_{(d)}} k(\eta)^\times.$$

Thus, letting $\mathrm{div} : \oplus_{\eta \in X_{(d)}} \mathbb{L}_{*-d+1} \otimes k(\eta)^\times \to \Omega_*^{(1)}(X)$ be the map induced by Div and defining $\mathrm{div}_{\mathrm{MGL}} := \partial \circ \oplus_\eta t_{\mathrm{MGL}}(\eta)$, we have the exact sequence

$$\oplus_{\eta \in X_{(d)}} \mathbb{L}_{*-d+1} \otimes k(\eta)^\times \xrightarrow{\mathrm{div}} \Omega_*^{(1)}(X) \xrightarrow{i_*} \Omega_*(X) \xrightarrow{j^*} \oplus_{\eta \in X_{(d)}} \Omega_*(k(\eta)) \to 0 \quad (5.5)$$

and the diagram (5.4) extends to the commutative diagram

$$\begin{array}{ccccccccc}
\oplus_\eta \mathbb{L}_{*-d+1} \otimes k(\eta)^\times & \xrightarrow{\mathrm{div}} & \Omega_*^{(1)}(X) & \xrightarrow{i_*} & \Omega_*(X) & \xrightarrow{j^*} & \oplus_\eta \Omega_*(k(\eta)) & \longrightarrow & 0 \\
\| & & \downarrow{\scriptstyle \vartheta^{(1)}} & & \downarrow{\scriptstyle \vartheta(X)} & & \downarrow{\scriptstyle \vartheta} & & \\
\oplus_\eta \mathbb{L}_{*-d+1} \otimes k(\eta)^\times & \xrightarrow[\mathrm{div}_{\mathrm{MGL}}]{} & \mathrm{MGL}'^{(1)}_{2*,*}(X) & \xrightarrow[i_*]{} & \mathrm{MGL}'_{2*,*}(X) & \xrightarrow[j^*]{} & \oplus_\eta \mathrm{MGL}'_{2*,*}(k(\eta)) & \longrightarrow & 0
\end{array} \quad (5.6)$$

with exact rows and with all vertical arrows isomorphisms.

6. The comparison map

We retain the assumption made in §5 that k admits resolution of singularities.

We consider the classifying map

$$\theta_{\mathcal{CK}} : \Omega_* \to \mathcal{CK}'_{2*,*}$$

constructed in proposition 5.1. Let $\varphi_{CK} : \mathbb{L}_* \to \mathbb{Z}[\beta]$ be the ring homomorphism classifying the formal group law $(u+v-\beta uv, \mathbb{Z}[\beta])$ of the theory $\mathcal{CK}$ and $\varphi_\Omega : \mathbb{L}_* \to \Omega_*(k)$ the classifying map for the formal group law for Ω_*. For each $X \in \mathbf{Sch}_k$, the external products make $\Omega_*(X)$ into an $\Omega_*(k)$-module and thus via φ_Ω an $\mathbb{L}_*$-module. It is easy to check that the various structures for Ω_* make the assignment $X \mapsto \Omega_*(X) \otimes_{\mathbb{L}_*} \mathbb{Z}[\beta]$ into an oriented Borel-Moore homology theory on $\mathbf{Sch}_k$, which we denote by $\Omega_* \otimes_{\mathbb{L}_*} \mathbb{Z}[\beta]$; the universality of Ω_* makes $\Omega_* \otimes_{\mathbb{L}_*} \mathbb{Z}[\beta]$ the universal oriented Borel-Moore homology theory on $\mathbf{Sch}_k$ having formal group law $(u + v - \beta uv, \mathbb{Z}[\beta])$.

Lemma 6.1 *$\theta_{\mathcal{CK}}$ factors through the surjection*

$$\Omega_* \to \Omega_* \otimes_{\mathbb{L}_*} \mathbb{Z}[\beta].$$

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Proof: As the natural transformation $\theta_{\mathcal{CK}}$ is compatible with external products, $\theta_{\mathcal{CK}}(k)$ is a ring homomorphism and $\theta_{\mathcal{CK}}(X)$ is a map of $\mathbb{L}_*$-modules, where $\mathbb{L}_*$ acts on $\mathcal{CK}'_{2*,*}$ by using the the action of $\mathcal{CK}'_{2*,*}(k)$ on $\mathcal{CK}'_{2*,*}$ via external products, together with the ring homomorphism

$$\mathbb{L}_* \xrightarrow{\varphi_\Omega} \Omega_*(k) \xrightarrow{\theta_{\mathcal{CK}}(k)} \mathcal{CK}'_{2*,*}(k) = \mathbb{Z}[\beta].$$

Using the universal property of $\Omega_* \otimes_{\mathbb{L}_*} \mathbb{Z}[\beta]$, we need only show that

$$\theta_{\mathcal{CK}}(k) \circ \varphi_\Omega : \mathbb{L}_* \to \mathbb{Z}[\beta]$$

is equal to the classifying homomorphism φ_{CK}. But this is a general property of the classifying map $\theta_{H,A} : \Omega_* \to H_*$ for any oriented duality theory (H, A), as we have already noted in remark 5.2. □

In [18, theorem 1.2.18] it is shown that the classifying map of oriented cohomology theories on $\mathbf{Sm}/k$

$$\vartheta_{K_0} : \Omega^* \otimes_{\mathbb{L}} \mathbb{Z}[\beta, \beta^{-1}] \to K_0[\beta, \beta^{-1}]$$

is an isomorphism, where $\mathbb{L} \to \mathbb{Z}[\beta, \beta^{-1}]$ is the classifying map for the formal group law $(u + v - \beta uv, \mathbb{Z}[\beta, \beta^{-1}])$. Dai [3] extended this result, showing that the classifying map

$$\vartheta_{G_0} : \Omega_* \otimes_{\mathbb{L}} \mathbb{Z}[\beta, \beta^{-1}] \to G_0[\beta, \beta^{-1}]$$

is an isomorphism of oriented Borel-Moore homology theories on $\mathbf{Sch}_k$. We now prove the analogous result for connective K-theory.

Definition 6.2 For $X \in \mathbf{Sch}_k$, write $\Omega_*^{CK}(X)$ for $\Omega_*(X) \otimes_{\mathbb{L}_*} \mathbb{Z}[\beta]$. For $g : Y \to X$ smooth, write g_{CK}^* for $g^* \otimes \mathrm{id}$ and similarly write f_*^{CK} for $f_* \otimes \mathrm{id}$ if $f : Y \to X$ is projective.

Theorem 6.3 *The induced classifying map*

$$\theta_{\mathcal{CK}} : \Omega_*^{CK} \to \mathcal{CK}'_{2*,*}$$

is an isomorphism.

Proof: We proceed by induction on the maximum d of the dimensions of the components of X; we may assume that X is reduced.

For $X \in \mathbf{Sch}_k$, we consider $\theta_{\mathcal{CK}}$ as a map of presheaves on X_{Zar}, so we may evaluate at the generic points of X. For η a dimension d generic point of X, it follows from the description of $\mathcal{CK}'_{2*,*}$ given by theorem 4.3 that

$$\mathcal{CK}'_{2n,n}(\eta) = \begin{cases} K_0(k(\eta)) & \text{for } n \geq d \\ 0 & \text{for } n < d. \end{cases}$$

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By [18, corollary 4.4.3], the pull-back map $p_X^* : \Omega_*(k) \to \Omega_{*+d}(k(\eta))$ is an isomorphism, so by [18, theorem 1.2.7] we have the isomorphism of $\mathbb{L}_*$-modules

$$\Omega_*(k(\eta)) \cong \mathbb{L}_{*-d}.$$

From this it follows easily that

$$\theta_{\mathcal{CK}}(k(\eta)) : \Omega_*^{CK}(k(\eta)) \to \mathcal{CK}'_{2*,*}(k(\eta))$$

is an isomorphism. In particular, $\theta_{\mathcal{CK}}(X)$ is an isomorphism for all reduced X of dimension 0 over k. As the inclusion $X_{red} \to X$ induces an isomorphism on Ω_* and $\mathcal{CK}'_{2*,*}$, the theorem is proven for all X of dimension 0.

Now suppose X has a component of maximal dimension $d > 0$. We use the constructions and notations from theorem 5.5 and the discussion following that theorem. We let $\mathcal{CK}'^{(1)}_{2*,*}(X)$ be the inductive limit

$$\mathcal{CK}'^{(1)}_{2*,*}(X) := \varinjlim_{W} \mathcal{CK}'_{2*,*}(W)$$

as W runs over all (reduced) closed subschemes of X which contain no dimension d generic point of X. Combining the exact sequence (5.5) with the localization sequence for $\mathcal{CK}'_{*,*}$ gives us the commutative diagram with exact rows

$$\begin{array}{ccccccccc}
\oplus_{\eta\in X_{(d)}}\mathbb{L}_{*-d+1}\otimes k(\eta)^\times & \xrightarrow{\text{div}} & \Omega_*^{(1)}(X) & \xrightarrow{i_*} & \Omega_*(X) & \xrightarrow{j^*} & \oplus_{\eta\in X_{(d)}}\Omega_*(k(\eta)) & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \vartheta^{(1)}} & & \downarrow{\scriptstyle \vartheta(X)} & & \downarrow{\scriptstyle \vartheta} & & \\
\oplus_{\eta\in X_{(d)}}\mathcal{CK}'_{2*+1,*}(k(\eta)) & \xrightarrow[\partial]{} & \mathcal{CK}'^{(1)}_{2*,*}(X) & \xrightarrow[i_*]{} & \mathcal{CK}'_{2*,*}(X) & \xrightarrow[j^*]{} & \oplus_{\eta\in X_{(d)}}\mathcal{CK}'_{2*,*}(k(\eta)) & \longrightarrow & 0.
\end{array} \tag{6.1}$$

We apply $\mathbb{Z}[\beta]\otimes_{\mathbb{L}}(-)$ to the top row in (6.1). By lemma 6.1, the vertical maps in (6.1) descend to give the commutative diagram

$$\begin{array}{ccccccccc}
\oplus_\eta\mathbb{Z}[\beta]_{*-d+1}\otimes k(\eta)^\times & \xrightarrow{\text{div}_{CK}} & \Omega_*^{CK(1)}(X) & \xrightarrow{i_*} & \Omega_*^{CK}(X) & \xrightarrow{j^*} & \oplus_{\eta\in X_{(d)}}\Omega_*^{CK}(k(\eta)) & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \vartheta^{(1)}} & & \downarrow{\scriptstyle \vartheta(X)} & & \downarrow{\scriptstyle \vartheta} & & \\
\oplus_\eta\mathcal{CK}'_{2*+1,*}(k(\eta)) & \xrightarrow[\partial]{} & \mathcal{CK}'^{(1)}_{2*,*}(X) & \xrightarrow[i_*]{} & \mathcal{CK}'_{2*,*}(X) & \xrightarrow[j^*]{} & \oplus_{\eta\in X_{(d)}}\mathcal{CK}'_{2*,*}(k(\eta)) & \to & 0.
\end{array} \tag{6.2}$$

By induction on d, the map $\vartheta^{(1)}$ is an isomorphism; we have already seen that ϑ is an isomorphism. We note that the bottom row is a sequence of $\mathbb{Z}[\beta]$-modules via the isomorphism $\mathcal{CK}'_{2*,*}(k) \cong \mathbb{Z}[\beta]$ and the $\mathcal{CK}_{2*,*}(k)$-module structure given by external products.

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Take $\eta \in X_{(d)}$. Then $\mathcal{CK}'_{2*+1,*}(k(\eta)) \cong \mathcal{CK}^{2d-2*-1,d-*}(k(\eta))$. Via lemma 2.10, we have the isomorphism of $\mathbb{Z}[\beta]$-modules

$$t_{\mathcal{CK}} : \mathbb{Z}[\beta]_{*-d+1} \otimes k(\eta)^\times \to \mathcal{CK}'_{2*+1,*}(k(\eta)).$$

Putting this into the diagram (6.2) gives us the commutative diagram

$$\begin{array}{ccccccccc}
\oplus_\eta \mathbb{Z}[\beta]_{*-d+1} \otimes k(\eta)^\times & \xrightarrow{\mathrm{div}_{CK}} & \Omega_*^{CK(1)}(X) & \xrightarrow{i_*} & \Omega_*^{CK}(X) & \xrightarrow{j^*} & \oplus_{\eta \in X_{(d)}} \Omega_*^{CK}(k(\eta)) & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \vartheta^{(1)}} & & \downarrow{\scriptstyle \vartheta(X)} & & \downarrow{\scriptstyle \vartheta} & & \\
\oplus_\eta \mathbb{Z}[\beta]_{*-d+1} \otimes k(\eta)^\times & \xrightarrow[\mathrm{div}_{\mathcal{CK}}]{} & \mathcal{CK}'^{(1)}_{2*,*}(X) & \underset{i_*}{\to} & \mathcal{CK}'_{2*,*}(X) & \underset{j^*}{\to} & \oplus_{\eta \in X_{(d)}} \mathcal{CK}'_{2*,*}(k(\eta)) & \to & 0
\end{array} \tag{6.3}$$

with $\mathrm{div}_{\mathcal{CK}} := \partial \circ t_{\mathcal{CK}}$, the bottom row exact and the top row a complex.

We claim that

$$\mathrm{div}_{\mathcal{CK}} = \vartheta^{(1)} \circ \mathrm{div}_{CK}. \tag{6.4}$$

Assuming this claim, a diagram chase shows that the top row is exact and the map $\vartheta(X)$ is an isomorphism.

To prove the claim, the orientation $c_{\mathcal{CK}}$ for $\mathcal{CK}$ and the universal property of MGL gives the canonical map of oriented cohomology theories

$$\rho_{\mathcal{CK}} : (\mathrm{MGL}, c_{\mathrm{MGL}}) \to (\mathcal{CK}, c_{\mathcal{CK}})$$

which in turn gives us the map of bi-graded oriented duality theories

$$\rho_{\mathcal{CK}} : (\mathrm{MGL}'_{*,*}, \mathrm{MGL}^{*,*}) \to (\mathcal{CK}'_{*,*}, \mathcal{CK}^{*,*}).$$

It follows from the characterization of θ_{MGL}, $\theta_{\mathcal{CK}}$ given in remark 5.4 that

$$\theta_{\mathcal{CK}} = \rho_{\mathcal{CK}} \circ \theta_{\mathrm{MGL}}.$$

As discussed at the end of §2 and in lemma 2.10, the orientations c_{MGL}, $c_{\mathcal{CK}}$ give rise to canonical elements

$$t_{\mathrm{MGL}} \in \mathrm{MGL}^{1,1}(\mathbb{G}_m),\ t_{\mathcal{CK}} \in \mathcal{CK}^{1,1}(\mathbb{G}_m).$$

These in turn give by functoriality canonical homomorphisms for each $X \in \mathbf{Sm}/k$

$$t_{\mathrm{MGL}}(X) : \mathcal{O}_X^\times(X) \to \mathrm{MGL}^{1,1}(X),\ t_{\mathcal{CK}}(X) : \mathcal{O}_X^\times(X) \to \mathcal{CK}^{1,1}(X)$$

with $t_{\mathcal{CK}}(X) = \rho_{\mathcal{CK}}(X) \circ t_{\mathcal{CK}}(X)$. We extend t_{MGL} to

$$t_{\mathrm{MGL}} : \mathcal{O}_X^\times(X) \otimes \mathbb{L}^* \to \mathrm{MGL}^{2*+1,*+1}(X)$$

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using the $\mathbb{L}^*$-module structure, and similarly have the extension of $t_{\mathcal{CK}}(X)$ to

$$t_{\mathcal{CK}} : \mathcal{O}_X^\times(X) \otimes \mathbb{Z}[\beta]^* \to \mathcal{CK}^{2*+1,*+1}(X).$$

The edge homomorphism $k(\eta)^\times \to \mathrm{MGL}^{1,1}(k(\eta))$ arising in the Hopkins-Morel spectral sequence is the map $t_{\mathrm{MGL}}(k(\eta))$. As $\rho_{\mathcal{CK}}$ is a map of $\mathbb{L}_*$-$\mathbb{Z}[\beta]$ modules, we have the commutative diagram

$$\begin{array}{ccccc}
\oplus_{\eta\in X_{(d)}}\mathbb{L}_{*-d+1}\otimes k(\eta)^\times & \xrightarrow{t_{\mathrm{MGL}}} & \oplus_{\eta\in X_{(d)}}\mathrm{MGL}'_{2*+1,*}(k(\eta)) & \xrightarrow{\partial} & \mathrm{MGL}'^{(1)}_{2*,*}(X) \\
\downarrow{\scriptstyle \pi} & & \downarrow{\scriptstyle \oplus_\eta \rho_{\mathcal{CK}}(\eta)} & & \downarrow{\scriptstyle \rho^{(1)}_{\mathcal{CK}}(X)} \\
\oplus_{\eta\in X_{(d)}}\mathbb{Z}[\beta]_{*-d+1}\otimes k(\eta)^\times & \xrightarrow[t_{\mathcal{CK}}]{} & \oplus_{\eta\in X_{(d)}}\mathcal{CK}'_{2*+1,*}(\eta) & \xrightarrow[\partial]{} & \mathcal{CK}'^{(1)}_{2*,*}(X)
\end{array}$$

where π is induced by the classifying map $\mathbb{L}_* \to \mathbb{Z}[\beta]$. The map $\mathrm{div}_{\mathrm{MGL}}$ in diagram (5.6) is the composition $\partial \circ t_{\mathrm{MGL}}$ in the diagram above; as $\mathrm{div}_{\mathcal{CK}} = \partial \circ t_{\mathcal{CK}}$, we have the commutative diagram

$$\begin{array}{ccc}
\oplus_{\eta\in X_{(d)}}\mathbb{L}_{*-d+1}\otimes k(\eta)^\times & \xrightarrow{\mathrm{div}_{\mathrm{MGL}}} & \mathrm{MGL}'^{(1)}_{2*,*}(X) \\
\downarrow{\scriptstyle \pi} & & \downarrow{\scriptstyle \rho^{(1)}_{\mathcal{CK}}(X)} \\
\oplus_{\eta\in X_{(d)}}\mathbb{Z}[\beta]_{*-d+1}\otimes k(\eta)^\times & \xrightarrow[\mathrm{div}_{\mathcal{CK}}]{} & \mathcal{CK}'^{(1)}_{2*,*}(X).
\end{array}$$

Patching in the left-hand square in the commutative diagram (5.6) yields the commutative diagram

$$\begin{array}{ccc}
\oplus_\eta \mathbb{L}_{*-d+1}\otimes k(\eta)^\times & \xrightarrow{\mathrm{div}} & \Omega^{(1)}_*(X) \\
\| & & \downarrow{\scriptstyle \vartheta^{(1)}} \\
\oplus_{\eta\in X_{(d)}}\mathbb{L}_{*-d+1}\otimes k(\eta)^\times & \xrightarrow{\mathrm{div}_{\mathrm{MGL}}} & \mathrm{MGL}'^{(1)}_{2*,*}(X) \\
\downarrow{\scriptstyle \pi} & & \downarrow{\scriptstyle \rho^{(1)}_{\mathcal{CK}(X)}} \\
\oplus_{\eta\in X_{(d)}}\mathbb{Z}[\beta]_{*-d+1}\otimes k(\eta)^\times & \xrightarrow{\mathrm{div}_{\mathcal{CK}}} & \mathcal{CK}'^{(1)}_{2*,*}(X).
\end{array} \tag{6.5}$$

As $\mathrm{div}_{CK} : \mathbb{Z}[\beta] \otimes k(\eta)^\times \to \Omega_*^{CK(1)}(X)$ is just the map formed by applying the functor $\mathbb{Z}[\beta] \otimes_{\mathbb{L}} (-)$ to $\mathrm{div} : \mathbb{L} \otimes k(\eta)^\times \to \Omega_*^{(1)}(X)$, the identity (6.4) follows from the commutativity of (6.5). □

Corollary 6.4 *Take $X \in \mathbf{Sch}_k$, and let $d = \dim X$. Then*

$$\gamma_n^X \circ \theta_{\mathcal{CK}}(X) : \Omega_n^{CK}(X) \to G_0(X)$$

is an isomorphism for $n \geq d$ and an injection for $n = d-1$.

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Proof: This follows from theorem 6.3 and corollary 4.4. □

7. Lci pull-backs and fundamental classes

We maintain our assumption of resolution of singularities. Theorem 6.3 gives us the isomorphism

$$\theta_{\mathcal{CK}} : \Omega_*^{CK}(X) \to \mathcal{CK}'_{2*,*}$$

compatible with projective push-forward, pull-back by open immersions, external products and 1st Chern class operators. However, Ω_*^{CK} is an oriented Borel-Moore homology theory, hence, in addition to these structures, it has functorial pull-back maps for arbitrary l.c.i. morphisms. The isomorphism $\theta_{\mathcal{CK}}$ thus endows the homology theory $\mathcal{CK}'_{2*,*}$ with l.c.i. pull-backs. The theory $X \mapsto \mathcal{K}'_{2n,n}(X) = G_0(X)$ has l.c.i. pull-backs as well; we proceed to compare the two. As the l.c.i. pull-backs in Ω_* are defined using the classes of simple normal crossing divisors (see [18, §6] for details on l.c.i. pull-backs in Ω_*), we first need a result about these classes.

Fix an oriented cohomology theory A^* with formal group law $F = F_A$. As usual, we use the notation $u +_F v$ for $F(u,v)$ and more generally write $[n_1]_F u_1 +_F \ldots +_F [n_r]_F u_r$ for the power series $F_{n_1,\ldots,n_r}(u_1,\ldots,u_r)$ that expresses the evident sum operation. We may write $F_{n_1,\ldots,n_r}(u_1,\ldots,u_r)$ in the following form

$$F_{n_1,\ldots,n_r}(u_1,\ldots,u_r) = \sum_{I\subset\{1,\ldots,r\}, I\neq\emptyset} G_I^{n_1,\ldots,n_r}(u)\cdot u_I$$

with the $G_I^{n_1,\ldots,n_r}(u) \in A_*(k)[[u_1,\ldots,u_r]]$ and $u_I = \prod_{i\in I} u_i$. If we assume (which we will) that for $j \notin I$, u_j does not divide $G_I^{n_1,\ldots,n_r}(u)$, then the expression is unique; for instance, $G_{\{j\}}^{n_1,\ldots,n_r}(u) = n_j$ if $n_j > 0$.

Now let $D = \sum_{i=1}^r n_i D_i$ be an effective simple normal crossing divisor on some $Y \in \mathbf{Sm}/k$, with each D_i smooth and irreducible; let $|D|$ denote the support of D. For $I \subset \{1,\ldots,r\}$, let $D_I := \cap_{i\in I} D_i$ and let $i_I : D_I \to |D|$, $i : |D| \to Y$ be the inclusions. Let $L_j = i^*\mathcal{O}_Y(D_j)$ and let $\tilde{c}_1(L_I)$ stand for the r-tuple $(\tilde{c}_1(i_I^*L_1),\ldots,\tilde{c}_1(i_I^*L_r))$. Suppose Y has dimension n. The divisor class of D, $[D \to |D|]^A \in A_{n-1}(|D|)$ is defined as

$$[D \to |D|]^A := \sum_{I\subset\{1,\ldots,r\}, I\neq\emptyset} i_{I*} G_I^{n_1,\ldots,n_r}(\tilde{c}_1(L_I))(1_{D_I}).$$

Lemma 7.1 *For $A_* = G_0[\beta,\beta^{-1}]$, we have*

$$[D \to |D|]^{G_0} = [\mathcal{O}_D]$$

in $G_0(|D|)$.

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Proof: We write $[D \to |D|]$ for $[D \to |D|]^{G_0}$ and proceed by induction on $\sum_i n_i$, where $D = \sum_{i=1}^r n_i D_i$. For D a smooth irreducible divisor, $[D \to |D|] = 1_D = [\mathcal{O}_D]$, which takes care of the case $\sum_i n_i = 1$.

For the general case, let $D' = D - D_r = \sum_{i=1}^{r-1} n_i D_i + (n_r - 1)D_r$, and let $j : |D'| \to |D|$, $i_r : D_r \to |D|$, $i : |D| \to Y$ be the inclusions.

The formal group law for $G_0[\beta, \beta^{-1}]$ is given by $F(u,v) = u + v - \beta uv \in \mathbb{Z}[\beta, \beta^{-1}][[u,v]]$. From this follows

$$F_{n_1,\dots,n_r}(u) = F_{n_1,\dots,n_r-1}(u) + u_r - \beta \cdot u_r \cdot F_{n_1,\dots,n_r-1}(u),$$

whence follows

$$G_I^{n_1,\dots,n_r}(u) = \begin{cases} G_I^{n_1,\dots,n_r-1}(u) & \text{if } r \notin I \\ G_I^{n_1,\dots,n_r-1}(u) - \beta(G_{I\setminus\{r\}}^{n_1,\dots,n_r-1}(u) + u_r \cdot G_I^{n_1,\dots,n_r-1}(u)) & \text{if } r \in I, |I| > 1 \\ G_I^{n_1,\dots,n_r-1}(u) + 1 = n_r & \text{if } I = \{r\}. \end{cases}$$

The Gysin identity [18, lemma 5.1.11] for the inclusion $D_I \to D_{I\setminus\{r\}}$ implies

$$i_{I*}(G(\tilde{c}_1(L_I))(1_{D_I})) = \tilde{c}_1(L_r)(i_{I\setminus\{r\}*}(G(\tilde{c}_1(L_{I\setminus\{r\}}))(1_{D_{I\setminus\{r\}}}))$$

if $r \in I$, $|I| > 1$, $G \in A_*(k)[[u_1,\dots,u_r]]$; with our identity for $G_I^{n_1,\dots,n_r}(u)$, this yields the identity

$$[D \to |D|] = j_*[D' \to |D'|] + i_{r*}[D_r \to |D_r|] - \beta \cdot \tilde{c}_1(L_r)(j_*[D' \to |D'|])). \quad (7.1)$$

Let $\mathcal{I}_D$, $\mathcal{I}_{D'}$, $\mathcal{I}_{D_r}$ be the respective ideal sheaves in $\mathcal{O}_Y$. We have the exact sequence of $\mathcal{O}_D$-modules

$$0 \to \mathcal{I}_{D_r}/\mathcal{I}_D \to \mathcal{O}_D \to \mathcal{O}_{D_r} \to 0.$$

Furthermore, the multiplication map $\mathcal{O}_D \otimes_{\mathcal{O}_Y} \mathcal{I}_{D_r} \to \mathcal{O}_D$ has image $\mathcal{I}_{D_r}/\mathcal{I}_D$ and descends to an isomorphism of $\mathcal{O}_{D'}$-modules

$$\mathcal{O}_{D'} \otimes_{\mathcal{O}_Y} \mathcal{I}_{D_r} \cong \mathcal{I}_{D_r}/\mathcal{I}_D.$$

This gives us the identities in $G_0(D)[\beta, \beta^{-1}]$:

$$\begin{aligned} [\mathcal{O}_D] &= \mathcal{O}_Y(-D_r) \cdot j_*[\mathcal{O}_{D'}] + i_{r*}[\mathcal{O}_{D_r}] \\ &= j_*[\mathcal{O}_{D'}] + i_{r*}[\mathcal{O}_{D_r}] - (1 - \mathcal{O}_Y(-D_r)) \cdot j_*[\mathcal{O}_{D'}] \\ &= j_*[\mathcal{O}_{D'}] + i_{r*}[\mathcal{O}_{D_r}] - \beta\tilde{c}_1(L_r)(j_*[\mathcal{O}_{D'}]), \end{aligned}$$

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the last identity following from

$$\tilde{c}_1(L)(\alpha) = \beta^{-1}(1 - L^{-1}) \cdot \alpha$$

for L a line bundle on $|D|$ and $\alpha \in G_0(|D|)[\beta, \beta^{-1}]$. Combined with (7.1) and our induction hypothesis, this proves the lemma. □

Proposition 7.2 *Let $f : Y \to X$ be an l.c.i. morphism of relative dimension d in* $\mathbf{Sch}_k$. *Then the diagram*

$$\begin{array}{ccc} \Omega_n^{CK}(X) & \xrightarrow{\gamma_X \circ \theta_X} & G_0(X) \\ {\scriptstyle f^*}\downarrow & & \downarrow{\scriptstyle f^*} \\ \Omega_{n+d}^{CK}(Y) & \xrightarrow[\gamma_Y \circ \theta_Y]{} & G_0(Y) \end{array}$$

commutes.

Proof: It suffices to consider two cases: (a) f a regular embedding, (b) f a smooth morphism.

We first note that $\Omega_n(X)$ is generated as an abelian group by the classes $[g : Z \to X]$ with $Z \in \mathbf{Sm}/k$ irreducible of dimension n over k, and g a projective morphism; here $[g : Z \to X]$ stands for the element $g_*(p_Z^*(1))$, with $1 \in \Omega_*(k)$ the unit in the graded ring $\Omega_*(k)$ and $p_Z : Z \to \mathrm{Spec}k$ the structure morphism. This fact follows from [18, lemma 2.5.11].

The canonical natural transformation $\vartheta_{CK} : \Omega_* \to \Omega_*^{CK}$ is compatible with smooth pull-back, projective push-forward and sends the unit in $\Omega_*(k)$ to the unit in $\Omega_*^{CK}(k)$. In addition, for each $X \in \mathbf{Sch}_k$, the map $\vartheta_{CK}(X) : \Omega_*(X) \to \Omega_*^{CK}(X)$ is just the canonical homomorphism : $\Omega_*(X) \to \Omega_*(X) \otimes_{\mathbb{L}_*} \mathbb{Z}[\beta]$; as $\mathbb{L}_* \to \mathbb{Z}[\beta]$ is surjective, it follows that $\vartheta_{CK}(X)$ is surjective for all $X \in \mathbf{Sch}_k$. Thus, if we define $[g : Z \to X]_{CK}$ to be $g_*^{CK}((p_Z)_{CK}^*(1))$, it follows that $\Omega_n^{CK}(X)$ is generated as an abelian group by the classes $[g : Z \to X]_{CK}$ with $Z \in \mathbf{Sm}/k$ irreducible of dimension n over k, and g a projective morphism.

The image of $[g : Z \to X]_{CK}$ in $G_0(X)$ is given by the analogous formula, that is,

$$(\gamma_X \circ \theta_X)([g : Z \to X]_{CK}) = Rg_*(\mathcal{O}_Z)$$

where $Rg_* : G_0(Z) \to G_0(X)$ is the usual push-forward map on G_0: $Rg_*([\mathcal{F}]) := \sum_i (-1)^i [R^i g_* \mathcal{F}]$. We now compute in cases (a) and (b):

Case (b): $f : Y \to X$ smooth. Then $f^*[g : Z \to X]_{CK} = [p_2 : Z \times_X Y \to Y]_{CK}$

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and

$$\begin{aligned} Rp_{2*}[\mathcal{O}_{Z\times_X Y}] &= Rp_{2*}(p_1^*[\mathcal{O}_Z]) \\ &= f^*(Rg_*[\mathcal{O}_Z]) \end{aligned}$$

the last identity being the base-change isomorphism for the flat morphism f.

Case (a): $f : Y \to X$ a regular embedding, say of codimension c. Since X is quasi-projective, we can factor f through a sequence of regular codimension one embeddings

$$Y = X_c \hookrightarrow X_{c-1} \hookrightarrow \ldots \hookrightarrow X_1 \hookrightarrow X;$$

this reduces us to the case $c = 1$, that is, $f : Y \to X$ is a Cartier divisor on X. In this case, $\Omega_*^{CK}(X)$ is generated by three types of elements (with g, Z as before):

1. $g : Z \to X$ with $g(Z) \subset Y$.
2. $g : Z \to X$ such that $g^{-1}(Y)$ is a simple normal crossing divisor on Z.
3. $g : Z \to X$ with $g^{-1}(Y) = \emptyset$.

Indeed, using the same surjectivity argument as above, this assertion follows from the analogous fact for $\Omega_*(X)$, which in turn is a direct consequence of [18, theorem 6.4.12].

In case (3), $f^*([g : Z \to X]) = 0$ by definition, and clearly $f^*(Rg_*(\mathcal{O}_Z))$ is zero as well. In case (1), let $L = f^*(\mathcal{O}_X(Y))$ and let $\bar{g} : Z \to Y$ be the unique map with $f \circ \bar{g} = g$. Then by definition

$$f^*([g : Z \to X]_{CK}) := \tilde{c}_1(L)([\bar{g} : Z \to Y]_{CK}).$$

On the G-theory side, let $\mathcal{F}$ be a coherent sheaf on Y. Then $Rf_*[\mathcal{F}] = f_*\mathcal{F}$, since f is finite, and $f^*([f_*\mathcal{F}]) := [\mathcal{F}] - [\mathrm{Tor}_1^{\mathcal{O}_X}(f_*\mathcal{F}, \mathcal{O}_Y)]$. Via the exact sequence

$$0 \to \mathcal{O}_X(-Y) \to \mathcal{O}_X \to \mathcal{O}_Y \to 0$$

we see that

$$[\mathcal{F}] - [\mathrm{Tor}_1^{\mathcal{O}_X}(f_*\mathcal{F}, \mathcal{O}_Y)] = (1 - L^{-1}) \cdot [\mathcal{F}]$$

in $G_0(Y)$, so $f^*([f_*\mathcal{F}]) = (1 - L^{-1}) \cdot [\mathcal{F}]$. Applying this to the coherent sheaves $R^i\bar{g}_*\mathcal{O}_Z$ and noting that $\tilde{c}_1^{G_0}(L)$ is multiplication by $\beta^{-1}(1 - L^{-1})$ shows that

$$(\gamma \circ \theta)(f^*[g : Z \to X]_{CK}) = f^*(Rg_*(\mathcal{O}_Z))$$

as desired.

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Finally, in case (2), the diagram

$$\begin{array}{ccc} g^{-1}(Y) & \xrightarrow{\bar{f}} & Z \\ {\scriptstyle \bar{g}}\downarrow & & \downarrow{\scriptstyle g} \\ Y & \xrightarrow[f]{} & X \end{array}$$

is cartesian and Tor-independent, hence we have the base-change identity

$$f^* R g_*[\mathcal{O}_Z] = R\bar{g}_*(\bar{f}^*[\mathcal{O}_Z]) = R\bar{g}_*([\mathcal{O}_{g^{-1}(Y)}]).$$

Thus it suffices to show that

$$[\mathcal{O}_{g^{-1}(Y)}] = (\gamma \circ \theta)(\bar{f}^*(1_Z)) \tag{7.2}$$

in $G_0(g^{-1}(Y))$. But by definition, we have $\bar{f}^*(1_Z) = [g^{-1}(Y) \to |g^{-1}(Y)|]$, so the identity (7.2) follows from lemma 7.1. □

Definition 7.3 Let $X \in \mathbf{Sch}_k$ have pure dimension d. The *fundamental class* $[X]_{CK} \in \Omega_d^{CK}(X)$ is the element corresponding to $[\mathcal{O}_X] \in G_0(X)$ under the isomorphism $\Omega_d^{CK}(X) \to G_0(X)$.

Theorem 7.4 *Let $f : Y \to X$ be an l.c.i. morphism in* $\mathbf{Sch}_k$. *Assume that X has pure dimension d_X and Y has pure dimension d_Y. Then $f^*([X]_{CK}) = [Y]_{CK}$ in $\Omega_{d_Y}^{CK}(Y)$.*

Proof: If f is smooth, then $f^*[\mathcal{O}_X] = [\mathcal{O}_Y]$ in $G_0(Y)$. If f is a regular embedding, then similarly $f^*[\mathcal{O}_X] = [\mathcal{O}_Y]$ in $G_0(Y)$. Thus, for an arbitrary l.c.i. morphism f, $f^*[\mathcal{O}_X] = [\mathcal{O}_Y]$ in $G_0(Y)$. The theorem thus follows from the definition of the fundamental class, corollary 6.4 and proposition 7.2. □

Set $\Omega_*^\times := \Omega_* \otimes_{\mathbb{L}} \mathbb{Z}[\beta, \beta^{-1}]$ and $\Omega_*^+ := \Omega_* \otimes_{\mathbb{L}} \mathbb{Z}$, the first tensor product defined via the classifying map for the formal group law $(u+v-\beta uv, Z[\beta, \beta^{-1}]$, the second via the classifying map for the additive formal group law $(u+v, \mathbb{Z})$. Dai's theorem [3] states that the classifying map $\Omega_*^\times \to G_0[\beta, \beta^{-1}]$ is an isomorphism, and it is shown in [18, theorem 4.5.1] that the classifying map $\Omega_*^+ \to \mathrm{CH}_*$ is an isomorphism.

In particular, inverting β induces the map

$$\vartheta_\times : \Omega_*^{CK} \to \Omega_*^\times,$$

which after the isomorphisms $\Omega_*^\times \cong G_0[\beta, \beta^{-1}] \cong \mathcal{K}'_{2*,*}$ and $\Omega_*^{CK} \cong \mathcal{CK}'_{2*,*}$ is just the canonical map $\rho : \mathcal{CK}'_{2*,*} \to \mathcal{K}'_{2*,*}$. Similarly, taking the quotient by β defines the map

$$\vartheta_+ : \Omega_*^{CK} \to \Omega_*^+$$

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which is the same as the classifying map for the additive theory CH_*.

Both $G_0[\beta,\beta^{-1}]$ and CH_* admit fundamental classes for equi-dimensional finite type k-schemes, functorial with respect to pull-back by l.c.i. morphisms. For $X \in \mathbf{Sch}_k$ of dimension d over k, the fundamental class $[X]_G \in G_0(X)[\beta,\beta^{-1}]_d$ is $\beta^d[\mathcal{O}_X]$, where $[\mathcal{O}_X] \in G_0(X)$ is the class of the structure sheaf. For CH_*, $[X]_{\mathrm{CH}}$ is the cycle class of the scheme X; explicitly, this is $\sum_{i=1}^r n_i[X_i]$ where X has irreducible components $X_1,\dots,X_r$ and n_i is the length of the local ring $\mathcal{O}_{X_i,\eta_i}$ at the generic point $\eta_i \in X_i$.

Proposition 7.5 *Let X be an equi-dimensional finite type k-scheme. Then after canonical identifications $\Omega_*^{\times} \cong G_0[\beta,\beta^{-1}]$, $\Omega_*^{+} \cong \mathrm{CH}_*$, we have*

$$\vartheta_{\times}([X]_{CK}) = [X]_G; \quad \vartheta_{+}([X]_{CK}) = [X]_{CH}.$$

Proof: Suppose X has dimension d over k. For G-theory, it follows by the definition of $[X]_{CK}$ that the corresponding element $[X]_{CK} \in \mathcal{CK}'_{2d,d}(X)$ maps to $[\mathcal{O}_X] \in \mathcal{K}'_{2d,d}(X) = G_0(X)$ under the canonical map $\rho : \mathcal{CK}' \to \mathcal{K}'$. This proves the result for G-theory.

For CH_*, the fundamental class $[X]_{\mathrm{CH}}$ is determined by its restriction to each generic point of X, so we may assume that X is irreducible and X_{red} is smooth over k. Write $[X]_{\mathrm{CH}} = n[X_{\mathrm{red}}]_{\mathrm{CH}}$. Shrinking X if necessary, we may assume that $[\mathcal{O}_X] = n[\mathcal{O}_{X_{\mathrm{red}}}]$ in $G_0(X)$, and thus $[X]_{CK} = n[X_{\mathrm{red}}]_{CK}$ in $\Omega_d^{CK}(X) = \Omega_d^{CK}(X_{\mathrm{red}})$. This reduces us to the case of $X \in \mathbf{Sm}/k$. But then we already have the fundamental class of X in any oriented cohomology theory A, given by

$$[X]_A := p^*(1) \in A^0(X)$$

where $p : X \to \mathrm{Spec}k$ is the structure morphism and $1 \in A^0(k)$ is the unit. Since every natural transformation of oriented cohomology theories preserves the unit and is compatible with pull-back for arbitrary morphisms in $\mathbf{Sm}/k$, we have $\vartheta_+([X]_{CK}) = [X]_{CH}$, as desired. □

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