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100.11 A trinomial identity from both a tiling and an expansion

Introduction

It is well known that the number of ways of tiling a $1 \times n$ rectangle with 1×1 squares and 1×2 dominoes is given by F_{n+1}, where, for $n \geqslant 2$, the nth Fibonacci number F_n is defined via the recurrence relation $F_n = F_{n-1} + F_{n-2}$, with initial values $F_0 = 0$ and $F_1 = 1$ [1, 2]. We use this fact here in order to present a combinatorial proof of the following identity involving trinomial coefficients and Fibonacci numbers:

$$\sum \binom{n}{a,\, b,\, c} F_{m-2}^{a} F_{m-1}^{b} F_{m}^{c} F_{c} = F_{m}^{n} F_{2n}, \tag{1}$$

where the sum is be taken over all triplets of non-negative integers (a, b, c) such that $a + b + c = n$.

Admittedly, (1) might initially appear a somewhat intimidating identity to prove. However, we adapt here an idea given in [3] to provide a remarkably straightforward combinatorial proof of this result. Furthermore, we obtain an algebraic proof of (1) by utilising a suitable trinomial expansion.

A combinatorial proof

Let us consider the set $\mathcal{T}$ of tilings of a $1 \times (2n - 1)$ rectangle using 1×1 squares and 1×2 dominoes in which every element has the property that each of its first n tiles may take any one of F_m colours for some integer $m \geqslant 2$, while the remaining tiles are left uncoloured. Note here that each element of $\mathcal{T}$ will indeed always consist of at least n tiles. In order to prove (1) we will enumerate $\mathcal{T}$ in two different ways.

First, the factor F_{2n} on the right-hand side of (1) counts the number of possible uncoloured tilings of a $1 \times (2n - 1)$ rectangle using 1×1 squares and 1×2 dominoes. The factor F_m^n gives, for each element of $\mathcal{T}$, the number of ways in which the first n tiles may be coloured. It follows from this that the right-hand side of (1) enumerates $\mathcal{T}$.

Now consider the left-hand side of (1). We split the F_m colours into two classes, *light* and *dark* say, such that there are F_{m-2} colours of the former type and hence $F_m - F_{m-2} = F_{m-1}$ of the latter. Subject to the

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condition $n - a - b \geqslant 0$, suppose that for a particular tiling there are, amongst the first n tiles, precisely a light dominoes and b dark dominoes. The remaining $c = n - a - b$ tiles are all squares, and each may take any one of the F_m colours. For a particular triplet (a, b, c) such that $a + b + c = n$, there are

$$\binom{n}{a,\, b,\, c} F_{m-2}^{a} F_{m-1}^{b} F_{m}^{c} \tag{2}$$

possible arrangements of the first n tiles.

Finally, we need to work out the length of the $1 \times (2n - 1)$ rectangle that remains after the first n tiles have been chosen. Bearing in mind that squares and dominoes have length 1 and 2, respectively, the length of the remaining section is given by

$$(2n - 1) - (2a + 2b) - (n - a - b) = n - a - b - 1 = c - 1,$$

which, recalling that anything beyond the first n tiles is uncoloured, may be tiled in F_c ways. On using this in conjunction with (2), and summing over all (a, b, c) such that $a + b + c = n$, we see that the left-hand side of (1) also enumerates $\mathcal{T}$, as required.

An algebraic proof

In order to tackle the proof algebraically, we will make use of the golden ratio ϕ, which is given by

$$\phi = \frac{1 + \sqrt{5}}{2}.$$

The following result may be found in [4, 5], and is straightforward to prove by induction using the Fibonacci recurrence relation and the fact that $\phi^2 = \phi + 1$.

Result 1: $\phi^m = \phi F_m + F_{m-1}$.

We will also use the result that follows. Although it is in fact applicable to any irrational number, we shall be specialising it to ϕ here.

Result 2: Let α be irrational. Then $p\alpha + q = r\alpha + s$ for some $p, q, r, s \in \mathbb{Q}$ if, and only if, $p = r$ and $q = s$.

On expanding and using Result 1, we have

$$\begin{aligned}
(F_{m-2} + F_{m-1} + \phi F_m)^n &= \sum \binom{n}{a,\, b,\, c} F_{m-2}^{a} F_{m-1}^{b} (\phi F_m)^c \\
&= \sum \binom{n}{a,\, b,\, c} F_{m-2}^{a} F_{m-1}^{b} F_m^c \phi^c \\
&= \sum \binom{n}{a,\, b,\, c} F_{m-2}^{a} F_{m-1}^{b} F_m^c (\phi F_c + F_{c-1}) \qquad (3)
\end{aligned}$$

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and

$$\begin{aligned}(F_{m-2} + F_{m-1} + \phi F_m)^n &= (F_m + \phi F_m)^n \\ &= F_m^n (1 + \phi)^n \\ &= F_m^n \left(\phi^2\right)^n \\ &= F_m^n \phi^{2n} \\ &= F_m^n \left(\phi F_{2n} + F_{2n-1}\right). \end{aligned} \tag{4}$$

Then, using (3), (4) and Result 2, we see that (1) has been proved once more.

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100.12 Visual proof of the limit of *f*-mean recurrence sequences

Some of the most common means of positive numbers, like the Arithmetic Mean (AM), the Geometric Mean (GM), the Root-Mean Square (RMS), the Power Mean (PM) and Harmonic Mean (HM), can be seen in a unified way by the use of a continuous function. Here we prove visually the limit of some recurrence sequences obtained by iterative application of any of these means. Let a and b be two positive real numbers. A more general result for similar sequences involving the means of n preceding terms in this sequence may be found in [1].

It is well known that $\mathrm{AM}(a,b) = \frac{1}{2}(a+b)$, $\mathrm{GM}(a,b) = \sqrt{ab}$, $\mathrm{RMS}(a,b) = \sqrt{\frac{1}{2}(a^2+b^2)}$, $\mathrm{PM}(a,b) = \left(\frac{1}{2}\left(a^m+b^m\right)\right)^{1/m}$, while $\mathrm{HM}(a,b) = \dfrac{2}{\frac{1}{a}+\frac{1}{b}}$.

Definition (*f-mean*): Let f be a function which maps an interval I of the real line to the real numbers, and is both continuous and injective. Then the f-mean of two numbers a and b is given by

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