Ergod. Th. & Dynam. Sys. (1999), **19**, 1295–1308
Printed in the United Kingdom © *1999 Cambridge University Press*

Integer sequences with big gaps and the pointwise ergodic theorem

ROGER L. JONES†, MICHAEL LACEY‡ and MÁTÉ WIERDL§

† *Department of Mathematics, DePaul University, 2219 North Kenmore, Chicago, IL 60614, USA*
(e-mail: rjones@condor.depaul.edu)
‡ *Department of Mathematics, Georgia Technical University, Atlanta, GA 30323, USA*
(e-mail: lacey@math.gatech.edu)
§ *Department of Mathematical Sciences, University of Memphis, Memphis, TN 38152, USA*
(e-mail: mw@moni.msci.memphis.edu)

(*Received* 10 *September* 1997 *and accepted in revised form* 12 *May* 1998)

Abstract. First, we show that there exists a sequence (a_n) of integers which is a good averaging sequence in L^2 for the pointwise ergodic theorem and satisfies

$$\frac{a_{n+1}}{a_n} > e^{(\log n)^{-1-\epsilon}}$$

for $n > n(\epsilon)$. This should be contrasted with an earlier result of ours which says that if a sequence (a_n) of integers (or real numbers) satisfies

$$\frac{a_{n+1}}{a_n} > e^{(\log n)^{-\frac{1}{2}+\epsilon}}$$

for some positive ϵ, then it is a bad averaging sequence in L^2 for the pointwise ergodic theorem.

Another result of the paper says that if we select each integer n with probability $1/n$ into a random sequence, then, with probability 1, the random sequence is a bad averaging sequence for the mean ergodic theorem. This result should be contrasted with Bourgain's result which says that if we select each integer n with probability σ_n into a random sequence, where the sequence (σ_n) is decreasing and satisfies

$$\lim_{t\to\infty} \frac{\sum_{n\le t} \sigma_n}{\log t} = \infty,$$

then, with probability 1, the random sequence is a good averaging sequence for the mean ergodic theorem.

https://doi.org/10.1017/S0143385799146819 Published online by Cambridge University Press

1. *Introduction*

Let $a_1 < a_2 < \cdots$ be an infinite sequence of integers. The conditions that imply (a_n) is a good averaging sequence for the pointwise ergodic theorem are arithmetical in nature. At present there is no useful characterization of these conditions. In this paper we investigate the extent to which the size of the successive gaps in the sequence can be used to help determine convergence or divergence of the associated averages. For example, any lacunary sequence is known to be bad for the pointwise ergodic theorem. It is natural to ask how much more slowly the sequence can grow and still necessarily be bad. We consider both pointwise convergence and convergence in mean.

To make the discussion precise we need to introduce some terminology.

Definition 1.1. Let (X, Σ, m) be a non-atomic probability space, and τ a measurable, measure-preserving transformation of X to itself. Let $1 \leq p \leq \infty$.

Almost everywhere convergence. We say that a sequence of positive integers $A = (a_n)$ is a *good averaging sequence for L^p in the system* (X, Σ, m, τ) or just *L^p-good in the system* (X, Σ, m, τ) iff for each $f \in L^p(X)$ the limit

$$\lim_{N\to\infty} M_N(A, f)(x)$$

exists for almost every $x \in X$, where

$$M_N(A, f)(x) = M_N f(x) = \frac{1}{N} \sum_{n\leq N} f(\tau^{a_n} x). \tag{1.1}$$

The sequence $A = (a_n)$ is a *universally good averaging sequence for L^p* or simply *L^p-good,* iff it is L^p-good in every system (X, Σ, m, τ).

The sequence $A = (a_n)$ is *∞-sweeping out* for L^p iff in every *aperiodic* (free) system (X, Σ, m, τ) there is $f \in L^p(X)$ with

$$\limsup_{N\to\infty} M_N(A, f)(x) = \infty$$

for almost every $x \in X$.

The sequence $A = (a_n)$ is *δ-sweeping out* iff in every *aperiodic* (free) system (X, Σ, m, τ) and $\epsilon > 0$ there is an indicator function f with $\int f < \epsilon$ and

$$\limsup_{N\to\infty} M_N(A, f)(x) \geq \delta.$$

Remark 1.2. It is quite easy to see that if a sequence is $(\delta - \epsilon)$-sweeping out for every positive $\epsilon < \delta$ then it is also δ-sweeping out.

A 1-sweeping out sequence is also called *strong sweeping out,* and it has the additional property that in every aperiodic system and $\epsilon > 0$ there is an indicator function f with $\int f < \epsilon$ and

$$\limsup_{N\to\infty} M_N(A, f)(x) = 1, \quad \liminf_{N\to\infty} M_N(A, f)(x) = 0.$$

Mean convergence. We say that the sequence $A = (a_n)$ is *mean-good* iff in every system (X, Σ, m, τ) for every $f \in L^2(X)$ the averages $M_N(A, f)(x)$ converge in L^2-norm.

https://doi.org/10.1017/S0143385799146819 Published online by Cambridge University Press

In this paper, unless we say otherwise, the sequences $A = (a_n)$, $B = (b_n)$, etc. are strictly increasing sequences of positive integers.

Sometimes, when $A = (a_n)$ is strictly increasing, it is more convenient to consider the averages

$$M_t(A, f)(x) = M_t f(x) = \frac{1}{A(t)} \sum_{\substack{a \in A \\ a \le t}} f(\tau^a x), \tag{1.2}$$

where $A(t)$ is the is the counting function of the sequence A:

$$A(t) = \#\{a \mid a \le t, a \in A\}.$$

We distinguish the two kinds of averages by their respective subscripts; if the subscript is a capital letter (N, P, Q, etc.) then we are dealing with the averages in (1.1), and if the subscript is a small letter (s, t, u, etc.) then we think of the averages in (1.2). Since in this paper, the sequence $A = (a_n)$ is always strictly increasing, the convergence–divergence properties of $M_t(A, f)$ and $M_N(A, f)$ are identical.

We seek two kinds of conditions for sequences with large gaps: deterministic conditions, and probabilistic conditions. In discussing pointwise results—to get our point across more quickly—we will consider only $p = 2$, or in some cases, $p = \infty$. When divergence occurs in the case $p = \infty$, we will show not only divergence, but that we in fact have strong sweeping out for the averages we are considering.

1.1. *Deterministic conditions.* In 1983 Bellow [**2**] proved that a lacunary sequence is ∞-sweeping out for L^p for every finite p. Later in [**1**] it was established that every lacunary sequence is strong sweeping out. Consequently we know that, roughly speaking, an L^p-good sequence (a_n) has to satisfy

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = 1.$$

The question arises: how fast is a_{n+1}/a_n is going to one? Two of us proved in [**4**] that if a sequence satisfies

$$\frac{a_{n+1}}{a_n} > e^{(\log n)^{-\frac{1}{2}+\epsilon}}$$

for some positive ϵ then it is ∞-sweeping out for L^2. The first main result is as follows.

THEOREM A. *There is an L^2-good sequence (a_n) of integers which, for every $\epsilon > 0$, satisfies*

$$\frac{a_{n+1}}{a_n} > e^{(\log n)^{-1-\epsilon}}$$

for $n > n(\epsilon)$.

1.2. *Probabilistic conditions.* This set of results are about randomly generated sequences of integers. Here is our scheme of generating random sequences.

Let (σ_n) be a sequence of positive numbers; σ_n is the probability with which we choose n into the random sequence. More precisely, let (Y_n) be a sequence of independent, $0 - 1$

https://doi.org/10.1017/S0143385799146819 Published online by Cambridge University Press

valued random variables on the probability space (Ω, β, P) so that $P(Y_n = 1) = \sigma_n$ and $P(Y_n = 0) = 1 - \sigma_n$. For each $\omega \in \Omega$, let A^ω be the sequence defined by the property that $n \in A^\omega$ if and only if $Y_n(\omega) = 1$.

Bourgain [**3**] proved that if the sequence (σ_n) satisfies

(1) the sequence (σ_n) is decreasing,

(2) $$\lim_{t\to\infty} \frac{\sum_{n\le t} \sigma_n}{\log t} = \infty,$$

then, for almost every $\omega \in \Omega$, the sequence A^ω is mean-good.

For an exposition of the proof, see [**5**, Ch. II], where it is also pointed out that condition (1) above cannot entirely be dropped for such a result. As for the necessity of condition (2), we have our second main result.

THEOREM B. *Let $\sigma_n = 1/n$. Then, for almost every $\omega \in \Omega$, the sequence A^ω is* not *mean-good.*

In fact, the method of proof will also give the following.

THEOREM C. *Let $\sigma_n = 1/n$. Then, for almost every $\omega \in \Omega$, the sequence A^ω is strong sweeping out.*

In the case of almost everywhere convergence, Bourgain [**3**] proved that if $\sigma_n = (\log\log n)^{1+\epsilon}/n$ with some positive ϵ, then for almost every $\omega \in \Omega$, the sequence A^ω is L^2-good. For an exposition of the proof, see [**5**, Ch. IV]. Here, we shall prove the following.

THEOREM D. *If $\sigma_n = (\log\log n)^{\frac{1}{2}-\epsilon}/n$ with some positive ϵ then, for almost every $\omega \in \Omega$, the sequence A^ω is ∞-sweeping out for L^2.*

Notation 1.3. For the functions $f(x), g(x)$ defined on some set of the real numbers, $f \sim g$ means that $f(x)/g(x) \to 1$ as $x \to \infty$ or $x \to 0$ depending on the context.

2. *Proof of Theorem B*

The main idea of the proof is to show that, for a.e. ω, one can extract a lacunary subsequence $B^\omega = (b_n(\omega))$ of the sequence $A^\omega = (a_n(\omega))$, which has the additional property that B^ω has positive density in A^ω. This latter means that

$$\lim_{t\to\infty} \frac{B^\omega(t)}{A^\omega(t)} > 0.$$

This is the content of the proposition below, but notice that we state a sharper result, because we will use this sharpening later in the proof of Theorem C.

PROPOSITION 2.1. *If $\sigma_n = 1/n$ then for a.e. ω and every $\epsilon > 0$ there is a subsequence $B^\omega = (b_n(\omega))$ of $A^\omega = (a_n(\omega))$ with the following two properties:*

- $$\liminf_{n\to\infty} \frac{b_{n+1}(\omega)}{b_n(\omega)} > 1;$$
- $$\liminf_{t\to\infty} \frac{B^\omega(t)}{A^\omega(t)} = \liminf_{t\to\infty} \frac{B^\omega(t)}{\log t} > 1 - \epsilon.$$

https://doi.org/10.1017/S0143385799146819 Published online by Cambridge University Press

Note that by the law of large numbers, for a.e. ω, $A^\omega(t)/\log t \to 1$.

By this proposition, the proof of Theorem B will be finished once we prove Proposition 2.2.

PROPOSITION 2.2. *Suppose there is a lacunary subsequence $B = (b_n)$ of the sequence $A = (a_n)$ which has positive upper density in A, that is,*

$$\limsup_{t\to\infty} \frac{B(t)}{A(t)} > 0.$$

Then A is not mean-good.

Proof of Proposition 2.1. Let us fix $\epsilon > 0$. Choose $\varrho > 1$ close enough to one so that

$$\frac{1 - 1/\varrho}{\log \varrho} > 1 - \epsilon,$$

and then let $\eta > 0$ small enough that we also have

$$\frac{1 - 1/\varrho^{1-\eta}}{\log \varrho} > 1 - \epsilon. \tag{2.1}$$

It is now enough to prove that for a.e. ω there is a subsequence $B^\omega = (b_n(\omega))$ of $A^\omega = (a_n(\omega))$ with the following two properties:

(1) $\liminf_{n\to\infty} \dfrac{b_{n+1}(\omega)}{b_n(\omega)} \geq \varrho^\eta$;

(2) $\liminf_{t\to\infty} \dfrac{B^\omega(t)}{A^\omega(t)} = \liminf_{t\to\infty} \dfrac{B^\omega(t)}{\log t} \geq \dfrac{1 - 1/\varrho^{1-\eta}}{\log \varrho}$.

For each positive integer k let us denote $I_k = \{n \mid \varrho^{k+\eta} \leq u < \varrho^{k+1}\}$. The random sequence $B^\omega = (b_n(\omega))$ will be such that, for each k, it will contain at most one integer from the set I_k.

Indeed, a sequence like that satisfies, with some positive integers $k, k', k < k'$,

$$\varrho^{k+\eta} \leq b_n(\omega) < \varrho^{k+1} \leq \varrho^{k'+\eta} \leq b_{n+1}(\omega),$$

so

$$\frac{b_{n+1}(\omega)}{b_n(\omega)} \geq \frac{\varrho^{k'+\eta}}{\varrho^{k+1}} \geq \frac{\varrho^{k+1+\eta}}{\varrho^{k+1}} = \varrho^\eta,$$

which is the inequality in (1) above.

Let us now see how B^ω is constructed and that the inequalities in (2) are satisfied. For each positive integer k, let

$$X_k = \max\{Y_u \mid \varrho^{k+\eta} \leq u < \varrho^{k+1}\}.$$

The X_k form a sequence of independent, $0-1$ valued random variables. For each $\omega \in \Omega$, let us define $B^\omega = (b_n(\omega))$ by

$$b \in B^\omega \text{ iff for some } k,\ b = \min\{u \mid Y_u = 1, \varrho^{k+\eta} \leq u < \varrho^{k+1}\}.$$

https://doi.org/10.1017/S0143385799146819 Published online by Cambridge University Press

It is clear that

$$\left| B^{\omega}(t) - \sum_{\varrho^k \le t} X_k(\omega) \right| \le 1.$$

Hence, by the strong law of large numbers, for a.e. ω we have,

$$\lim_{t\to\infty} \frac{B^{\omega}(t)}{\sum_{\varrho^k \le t} \mathbb{E}(X_k)} = 1.$$

For a.e. ω, we have

$$\lim_{t\to\infty} \frac{A^{\omega}(t)}{\log t} = 1.$$

Consequently, it will be enough to show

$$\lim_{t\to\infty} \frac{\sum_{\varrho^k \le t} \mathbb{E}(X_k)}{((1-\varrho^{\eta-1})/\log\varrho)\cdot \log t} = 1.$$

Since

$$\lim_{t\to\infty} \frac{\#\{k \mid \varrho^k \le t\}}{\log t/\log\varrho} = 1,$$

we just need to show

$$\lim_{k\to\infty} \mathbb{E}(X_k) = 1 - \varrho^{\eta-1}.$$

We have

$$\begin{aligned}
\mathbb{E}(X_k) &= P(X_k = 1) = 1 - P(X_k = 0) \\
&= 1 - \prod_{u\in I_k} P(Y_u = 0) \\
&= 1 - \prod_{u\in I_k} (1 - 1/u) \\
&= 1 - e^{\sum_{u\in I_k} \log(1-1/u)} \\
&\sim 1 - e^{-\sum_{u\in I_k} 1/u} \\
&\sim 1 - e^{-(\log\varrho^{k+1} - \log\varrho^{k+\eta})} \\
&= 1 - e^{-(1-\eta)\log\varrho} \\
&= 1 - \varrho^{\eta-1}.
\end{aligned}$$

□

Proof of Proposition 2.2. We will show that if the transformation τ is given by an irrational rotation of $\mathbb{T} = [0, 1)$, then for some function $f \in L^2$, the sequence $(M_t f) = (M_t(A, f))$ is not a Cauchy sequence in L^2. More precisely, we will prove the following proposition.

PROPOSITION 2.3. *Under the hypothesis of Proposition 2.2, there is an irrational rotation of* $\mathbb{T} = [0, 1)$, *an indicator function* f *of an interval of the form* $[0, \eta)$, *and* $\epsilon > 0$ *so that for some sequence* (t_k)

$$\|M_{t_{2k}} f - M_{t_{2k+1}} f\| L^2 \ge \epsilon.$$

https://doi.org/10.1017/S0143385799146819 Published online by Cambridge University Press

Proof. Let $\varrho > 1$ be such that

$$\frac{b_{n+1}}{b_n} > \varrho$$

for every n, and let $r > 0$ be such that

$$\limsup \frac{B(t)}{A(t)} > r.$$

Let T contain those t for which $B(t)/A(t) > r$. For the rest of this proof we assume that $t \in T$. We need several lemmas.

LEMMA 2.4. *Let the positive integer q satisfy*

$$\frac{\log \varrho}{\log(2q)} \cdot r > \frac{5}{q}.$$

Then, for a dense set of β, we have

$$\liminf_{t\to\infty} \frac{1}{A(t)} \sum_{\substack{a\in A\\ a\le t}} \chi_{[0,1/q)}(a\beta) > \frac{5}{q}.$$

Proof. Let ℓ be a positive integer so that $\varrho^\ell > 2q$, that is, $\ell > \log(2q)/\log\varrho$. Let the sequence $D = (d_n)$ be defined by $d_n = b_{\ell n}$. Note that the upper density of D in A is greater than $r/\ell > (r\log\varrho)/\log(2q)$. More precisely,

$$\lim_{t\to\infty} \frac{D(t)}{A(t)} > \frac{r\cdot\log\varrho}{\log(2q)}.$$

All we need to prove now is that, for a dense set of β, we have $\chi_{[0,1/q)}(d_n\beta) = 1$ for every large enough n. In other words, $d_n\beta \in [0, 1/q) \bmod 1$ for every large enough n.

The only property of D we use is that, for every n,

$$\frac{d_{n+1}}{d_n} \ge \varrho^\ell > 2q.$$

To make the notation simpler we assume, without loss of generality, that $d_1 > 2q$. Let $I \subset [0,1)$ be an interval. We need to show that there is $\beta \in I$ so that for every n, $d_n\beta \in [0, 1/q)+\mathbb{Z}$. In other words, we need to show that the set $I \cap \bigcap_{n=1}^{\infty}([0,1/q)+\mathbb{Z})/d_n$ is not empty.

Let us denote $J = [0, 1/q]$. Since $d_1 > 2q$, there is an integer k_1 so that $[k_1, k_1+1]/d_1 \subset I$. It follows that

$$(J + k_1)/d_1 \subset I.$$

Since $d_2/d_1 > 2q$, there is an integer k_2 so that

$$(J + k_2)/d_2 \subset ([0, 1/q) + k_1)/d_1.$$

Continuing in this way, for every $n \ge 2$, there is an integer k_n so that

$$(J + k_n)/d_n \subset ([0, 1/q) + k_{n-1})/d_{n-1}.$$

https://doi.org/10.1017/S0143385799146819 Published online by Cambridge University Press

This implies, since the intervals $(J + k_n)/d_n$ are closed, that

$$\bigcap_n (J + k_n)/d_n \neq \varnothing.$$

Combining this with $(J + k_1)/d_1 \subset I$, we obtain $I \cap \bigcap_{n=1}^{\infty}([0, 1/q) + \mathbb{Z})/d_n$ that is not empty. □

LEMMA 2.5. *Let q be a positive integer. Then, for a dense set of β, we have*

$$\lim_{t\to\infty} \frac{1}{A(t)} \sum_{\substack{a\in A\\ a\le t}} \chi_{[0,2/q)}(a\beta) = \frac{2}{q}.$$

Proof. The result follows from an old result of Weyl, which says that the sequence $(a_n\beta)$ is uniformly distributed mod 1 for almost every β, see [**5**, Ch. II, §3]. □

Before the next lemma, we need a definition.

Definition 2.6. A real valued function F defined on $\mathbb{T}$ or $\mathbb{R}$ is called *quasi continuous* iff $\gamma > F(x) > \delta$ implies that there is a closed interval I containing x so that $\gamma > F(I) > \delta$.

LEMMA 2.7. *Let $(F_t)_{t\in T}$ and $(G_t)_{t\in T}$ be two sequences of quasi continuous functions defined on $\mathbb{T}$. Let $\gamma > \delta$, and assume that for a dense set of β,*

$$\liminf_{t\to\infty} F_t(\beta) \ge \gamma,$$

and for a dense set of β,

$$\limsup_{t\to\infty} G_t(\beta) \le \delta.$$

Then for an uncountable, dense set of β, we have

$$\limsup_{t\to\infty} F_t(\beta) \ge \gamma, \quad \liminf_{t\to\infty} G_t(\beta) \le \delta.$$

Proof. The proof is just a slight modification of a well-known result in real analysis where the functions F_t and G_t are assumed to be continuous. □

Here is how we can put together these lemmas to give a proof of Proposition 2.3. Let q be a large enough integer to satisfy

$$\frac{\log \varrho}{\log(2q)} \cdot r > \frac{5}{q}.$$

Denote

$$F_t(\beta) = \frac{1}{A(t)} \sum_{\substack{a\in A\\ a\le t}} \chi_{[0,1/q)}(a\beta), \quad G_t(\beta) = \frac{1}{A(t)} \sum_{\substack{a\in A\\ a\le t}} \chi_{[0,2/q)}(a\beta),$$

and

$$\gamma = \frac{5}{q}, \quad \delta = \frac{2}{q}.$$

https://doi.org/10.1017/S0143385799146819 Published online by Cambridge University Press

By Lemma 2.4, for a dense set of β,

$$\liminf_{t\to\infty} F_t(\beta) > \gamma,$$

and by Lemma 2.5, for a dense set of β,

$$\lim_{t\to\infty} G_t(\beta) \le \delta.$$

It follows, by Lemma 2.7, that for an uncountable dense set D of β,

$$\limsup_{t\to\infty} F_t(\beta) \ge \gamma, \quad \liminf_{t\to\infty} G_t(\beta) \le \delta.$$

In other words, for each $\beta \in D$, there is a infinite sequence $t_1 < t_2 < \cdots$, so that

$$F_{t_{2k}} > \gamma - 1/q, \tag{2.2}$$

and

$$G_{t_{2k+1}} < \delta + 1/q. \tag{2.3}$$

Fix an irrational $\beta \in D$, and let τ be the rotation of $\mathbb{T}$ by β. Let f be the 1-periodic extension of the characteristic function of the interval $[0, 2/q)$. If $x \in [0, 1/q)$ and $a\beta \in [0, 1/q) \bmod 1$, then $x + a\beta \in [0, 2/q) \bmod 1$. Hence for any $x \in [0, 1/q)$ we have, by (2.2),

$$M_{t_{2k}}(A, f)(x) > \gamma - \frac{1}{q} = \frac{4}{q},$$

for every k. If $x \in [0, 1/q)$ and $x + a\beta \in [0, 2/q) \bmod 1$, then $a\beta \in [0, 2/q)$. Hence, for any $x \in [0, 1/q)$ we have, by (2.3),

$$M_{t_{2k+1}}(A, f)(x) < \delta + \frac{1}{q} = \frac{3}{q},$$

for every k. It follows, for every k,

$$\begin{aligned}\|M_{t_{2k}}(A, f)(x) - M_{t_{2k+1}}(A, f)(x)\|_{L^2(\mathbb{T})} &\ge \|M_{t_{2k}}(A, f)(x) - M_{t_{2k+1}}(A, f)(x)\|_{L^1(\mathbb{T})} \\ &\ge 1/q \cdot 1/q,\end{aligned}$$

so we can take $\epsilon = 1/q^2$. □

□

Remark 2.8. Our proof gives a more general result. First we have a definition.

Definition 2.9. The sequence $A = (a_n)$ is called *persistently bad for mean convergence* iff for every sequence $t_1 < t_2 < \cdots$ of indices there is an irrational rotation τ and a function $f \in L^2(\mathbb{T})$ so that the sequence $\left(M_{t_k}(A, f)\right)_{k=1}^{\infty}$ does not converge in L^2.

THEOREM 2.10. *If $\sigma_n = 1/n$ then for a.e. ω the random sequence A^ω is persistently bad for mean convergence.*

https://doi.org/10.1017/S0143385799146819 Published online by Cambridge University Press

3. *Proof of Theorem C*

By Proposition 2.1, we see that for a.e. ω, A^ω is $(1-\epsilon)$-sweeping out. But then, by Remark 1.2, A^ω is also 1- (= strong) sweeping out.

Remark 3.1. This proof applies almost without any change to give a more general result. First a definition.

Definition 3.2. Let $0 < \delta \leq 1$. The sequence $A = (a_n)$ is called *persistently δ-sweeping out* iff for every sequence $t_1 < t_2 < \cdots$ of indices and $\epsilon > 0$ one can find, in any dynamical system, an indicator function f with $\int f < \epsilon$ such that

$$\limsup_{k\to\infty} M_{t_k}(A, f)(x) \geq \delta$$

almost everywhere.

A persistently 1-sweeping out sequence is also called *persistently strong sweeping out*, and it has the additional property that for given $t_1 < t_2 < \cdots$ in every aperiodic system and $\epsilon > 0$ one can find an indicator function f with $\int f < \epsilon$ and

$$\limsup_{k\to\infty} M_{t_k}(A, f)(x) = 1, \quad \liminf_{k\to\infty} M_{t_k}(A, f)(x) = 0.$$

THEOREM 3.3. *If $\sigma_n = 1/n$ then for a.e. ω the random sequence A^ω is persistently 1-sweeping out.*

4. *Proof of Theorem D*

Recall the result mentioned in the introduction: if a sequence (b_n) satisfies

$$\frac{b_{n+1}}{b_n} > e^{(\log n)^{-\frac{1}{2}+\epsilon}}$$

for some $\epsilon > 0$ and for all large enough n then (b_n) is infinity sweeping out for L^2. It follows that we need to prove the following result.

LEMMA 4.1. *Let $\epsilon > 0$. If $\sigma_n = (\log\log n)^{\frac{1}{2}-\epsilon}/n$ then for a.e. ω there is a subsequence $B^\omega = (b_n(\omega))$ of $A^\omega = (a_n(\omega))$ so that*

$$\lim \frac{B^\omega(t)}{A^\omega(t)} = 1,$$

and

$$\frac{b_{n+1}(\omega)}{b_n(\omega)} > e^{(\log n)^{-\frac{1}{2}+\frac{1}{2}\epsilon}}$$

for all large enough n.

Proof. Let $u_n = \min\{t \mid \sum_{u\leq t} \sigma_u \geq n\}$. Since u_n is basically $e^{n\cdot(\log n)^{-\frac{1}{2}+\epsilon}}$, for large enough n we have

$$\frac{u_{n+1}}{u_n} \geq e^{(\log n)^{-\frac{1}{2}+\epsilon}}$$

https://doi.org/10.1017/S0143385799146819 Published online by Cambridge University Press

and, since $\sum_{u\le u_n}\sigma_n \to n$, by the strong law of large numbers, for a.e. ω,

$$\lim_{n\to\infty}\frac{A^\omega(u_n)}{n}=1.$$

Unfortunately, this does not imply $a_{n+1}(\omega)/a_n(\omega)\ge e^{(\log n)^{-\frac12+\epsilon}}$.

Let $v_n=e^{n\cdot(\log n)^{-\frac12+\frac12\epsilon}}$. The properties of (v_n) we need are

(1) $v_{n+1}/v_n\ge e^{(\log n)^{-\frac12+\frac12\epsilon}}$ for large enough n,
(2) with the notation $I_n=[v_n,v_{n+1})\cap\mathbb{Z}$,

$$\lim_{n\to\infty}\sum_{u\in I_n}\sigma_u=0.$$

Now, for a.e. ω, the sequence $B^\omega=(b_n(\omega))$ will have the properties

(i) $|B^\omega\cap I_n|\le 1$ for every large enough n,
(ii) if $|B^\omega\cap I_n|=1$ then $|B^\omega\cap I_{n+1}|=0$.

These two properties would assure that, for a.e. ω, we have $b_{n+1}(\omega)/b_n(\omega)\ge e^{(\log n)^{-\frac12+\frac12\epsilon}}$ for large enough n, and we just need to make sure that

$$\lim_{t\to\infty}\frac{B^\omega(t)}{A^\omega(t)}=1.$$

Let $D^\omega=(d_n(\omega))$ be defined by $d\in D^\omega$ iff, for some n, $d\in I_n\cap A^\omega$ and $I_{n+1}\cap A^\omega=\varnothing$.

The requirements in (i) and (ii) suggest that we need to define $B^\omega=(b_n(\omega))=A^\omega\setminus D^\omega$. It remains to show that for a.e. ω,

$$\lim_{t\to\infty}\frac{D^\omega(t)}{A^\omega(t)}=0.$$

By the strong law of large numbers, we need to show that

$$\lim_{t\to\infty}\frac{\mathbb{E}D^\omega(t)}{\sum_{n\le t}\sigma_n}=0.$$

Note that $D^\omega(t)\le\sum_{v_k\le t}\left(\sum_{n\in I_k}Y_n(\omega)\right)\cdot\sup_{n\in I_{k+1}}Y_n(\omega)$. It follows, since the random variables $X_k=\sum_{n\in I_k}Y_n$, $k=1,2,\dots$, are independent, that

$$\begin{aligned}\mathbb{E}D^\omega(t)&\le\mathbb{E}\sum_{v_k\le t}\left(\sum_{n\in I_k}Y_n\right)\cdot\sup_{n\in I_{k+1}}Y_n\\&\le\mathbb{E}\sum_{v_k\le t}\left(\sum_{n\in I_k}Y_n\right)\cdot\left(\sum_{n\in I_{k+1}}Y_n\right)\\&\le\sum_{v_k\le t}\left(\sum_{n\in I_k}\sigma_n\right)\cdot\left(\sum_{n\in I_{k+1}}\sigma_n\right).\end{aligned}$$

https://doi.org/10.1017/S0143385799146819 Published online by Cambridge University Press

But now, by property (2),

$$\begin{aligned}\limsup_{t\to\infty}\frac{\mathbb{E}D^{\omega}(t)}{\sum_{n\le t}\sigma_n} &\le \limsup_{t\to\infty}\frac{\sum_{v_k\le t}\left(\sum_{n\in I_k}\sigma_n\right)\cdot\left(\sum_{n\in I_{k+1}}\sigma_n\right)}{\sum_{n\le t}\sigma_n}\\ &\le \limsup_{k\to\infty}\left(\sum_{n\in I_{k+1}}\sigma_n\right)\cdot\limsup_{t\to\infty}\frac{\sum_{v_{k+1}\le t}\left(\sum_{n\in I_k}\sigma_n\right)}{\sum_{n\le t}\sigma_n}\\ &= 0,\end{aligned}$$

finishing our proof. □

Remark 4.2. As a particular case of [**5**, Exercise 5, p. 79], we have Theorem 4.3.

THEOREM 4.3. *If $\sigma_n = (\log\log n)^{\frac{1}{2}-\epsilon}/n$ with some positive ϵ then there is a sequence $t_1 < t_2 < \cdots$ so that for a.e. ω the sequence $(M_{t_k}(A^{\omega}, f)(x))_{k=1}^{\infty}$ converges for a.e. x in any dynamical system for $f \in L^2$.*

This is interesting to compare with Theorem 3.3.

5. *Proof of Theorem A*

In the introduction we mentioned Bourgain's result from [**3**]: if $\sigma_n = (\log\log n)^{1+\epsilon}/n$ with some positive ϵ then, for almost every $\omega \in \Omega$, the random sequence A^{ω} is L^2-good.

The idea of our proof is to select a subsequence $B^{\omega} = (b_n(\omega))$ from A^{ω} so that for a.e. ω the density of B^{ω} in A^{ω} is one, and the ratio b_{n+1}/b_n is suitably big.

More precisely, we have the following.

LEMMA 5.1. *Let $\epsilon > 0$. If $\sigma_n = (\log\log n)^{1+\epsilon}/n$ then for a.e. ω there is a subsequence $B^{\omega} = (b_n(\omega))$ of $A^{\omega} = (a_n(\omega))$ so that*

$$\lim\frac{B^{\omega}(t)}{A^{\omega}(t)} = 1,$$

and

$$\frac{b_{n+1}(\omega)}{b_n(\omega)} > e^{(\log n)^{-1-2\epsilon}}$$

for all large enough n.

The proof of this lemma is identical to that of Lemma 4.1, hence we omit it.

The proof of Theorem A is completed with the following simple lemma.

LEMMA 5.2. *Suppose the sequence $A = (a_n)$ is L^2-good, and that a subsequence $B = (b_n)$ of A satisfies*

$$\lim_{t\to\infty}\frac{B(t)}{A(t)} = 1.$$

Then B is also L^2-good.

https://doi.org/10.1017/S0143385799146819 Published online by Cambridge University Press

Proof. Let (X, Σ, τ) be a dynamical system. Because of the estimate, valid for $f \geq 0$,

$$\limsup_{t\to\infty} \frac{1}{B(t)} \sum_{\substack{b\leq t\\ b\in B}} f(\tau^b x) \leq \limsup_{t\to\infty} \frac{A(t)}{B(t)} \frac{1}{A(t)} \sum_{\substack{a\leq t\\ a\in A}} f(\tau^a x)$$
$$= \limsup_{t\to\infty} \frac{1}{A(t)} \sum_{\substack{a\leq t\\ a\in A}} f(\tau^a x)$$

we have that the maximal function of $(M_t(B, f))$ is a.e. finite for all $f \in L^2$. By the Banach principle, it is enough to prove that for a dense (in L^2) set of functions the sequence $(M_t(B, f)(x))$ converges a.e. Hence the following lemma finishes the proof.

LEMMA 5.3. *Let $f : A \to \mathbb{C}$ be a bounded function, and suppose*

$$\lim_{t\to\infty} \frac{1}{A(t)} \sum_{\substack{a\leq t\\ a\in A}} f(a) = \alpha.$$

Then

$$\lim_{t\to\infty} \frac{1}{B(t)} \sum_{\substack{b\leq t\\ b\in B}} f(b) = \alpha.$$

Proof. Since $\lim_{t\to\infty} A(t)/B(t) = 1$, it is enough to show that

$$\lim_{t\to\infty} \frac{1}{A(t)} \sum_{\substack{b\leq t\\ b\in B}} f(b) = \alpha.$$

But we have

$$\limsup_{t\to\infty} \left| \frac{1}{A(t)} \sum_{\substack{b\leq t\\ b\in B}} f(b) - \frac{1}{A(t)} \sum_{\substack{a\leq t\\ a\in A}} f(a) \right| \leq \limsup_{t\to\infty} \frac{1}{A(t)} \sum_{\substack{a\leq t\\ a\in A\setminus B}} |f(a)|$$
$$\leq \sup_{a\in A} |f(a)| \cdot \limsup_{t\to\infty} \frac{1}{A(t)} \sum_{\substack{a\leq t\\ a\in A\setminus B}} 1$$
$$= 0,$$

which finishes our proof. □

□

6. *Unsolved problems*

There are a number of remaining questions. We conclude the paper by listing some of them.

CONJECTURE 1. *If a sequence (a_n) satisfies*

$$\frac{a_{n+1}}{a_n} > e^{(\log n)^{-1}}$$

for every large enough n, then (a_n) is ∞-sweeping out for L^2.

https://doi.org/10.1017/S0143385799146819 Published online by Cambridge University Press

A consequence of the above problem would be the following.

CONJECTURE 2. *Suppose* $\sigma_n = \log\log n/n$. *Then, with probability 1, the random sequence* A^ω *is* ∞*-sweeping out for* L^2.

CONJECTURE 3. *Suppose* $\sigma_n \to 0$ *and* $\sum_n \sigma_n = \infty$. *Then, with probability 1, the random sequence* A^ω *is* ∞*-sweeping out for* L^1.

CONJECTURE 4. *For every* $\epsilon > 0$, *there exists a sequence* $A = (a_n)$ *with the following two properties.*

- *For every* n,
$$\frac{a_{n+1}}{a_n} > e^{(\log n)^{-1-\epsilon}}.$$
- *If* (X, σ, m) *is a probability space,* T *is a positive contraction of* L^2, *and* $f \in L^2$ *then the averages*
$$\frac{1}{N}\sum_{n\le N} T^{a_n} f(x)$$
converge for a.e. $x \in X$.

The above conjecture is true in the special case if T is an L^2-L^∞ contraction, or even if T is an L^2-L^p contraction with some $p > 2$.

REFERENCES

[1] M. Akcoglu, A. Bellow, R. L. Jones, V. Losert, K. Reinhold-Larsson and M. Wierdl. The strong sweeping out property for lacunary sequences, Riemann sums, convolution powers, and related matters. *Ergod. Th. & Dynam. Sys.* **16** (1996), 207–253.

[2] A. Bellow. *On 'Bad Universal' sequences in Ergodic Theory II (Lecture Notes in Mathematics, 1033)*. Springer, 1983.

[3] J. Bourgain. On the maximal ergodic theorem for certain subsets of the positive integers. *Israel J. Math.* **61** (1988), 39–72.

[4] R. L. Jones and M. Wierdl. Convergence and divergence of ergodic averages. *Ergod. Th. & Dynam. Sys.* **14** (1994), 515–535.

[5] J. Rosenblatt and M. Wierdl. Pointwise ergodic theorems via harmonic analysis. *Proc. Conf. on Ergodic Theory, Alexandra, 1993 (London Mathematical Society Lecture Note Series, 205).* Eds K. E. Peterson and I. A. Salama. Cambridge University Press, 1995.

https://doi.org/10.1017/S0143385799146819 Published online by Cambridge University Press