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Piecewise monotone maps without periodic points: rigidity, measures and complexity

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Abstract. We consider piecewise monotone maps of the interval with zero entropy or no periodic points. First, we give a rigid model for these maps: the interval translations mappings, possibly with flips. It follows, for example, that the complexity of a piecewise monotone map of the interval is at most polynomial if and only if this map has a finite number of periodic points up to monotone equivalence. Second, we study the invariant and ergodic measures of a piecewise monotone map with zero entropy and prove that their number is bounded by twice the number of monotony intervals; for a piecewise increasing map their number is at most the number of intervals.

0. *Introduction*

The focus of this paper is to study the dynamics of interval translation mappings (ITMs) and to prove that they are a rigid model for a large class of piecewise monotone maps of the interval with zero entropy.

Definition 1. A map T from an interval I to itself is an ITM if there exists a finite partition of I in subintervals $I_1, \ldots, I_N$ such that for each $j = 1, \ldots, N$ the restriction of T to I_j is a translation.

ITMs were introduced by Boshernitzan and Kornfeld in [**BoKo**] and also studied in [**ST**]. They are a natural generalization of the much-studied interval exchange transformations (IETs). See [**CFS**] for an introduction to this topic and also [**Ve, Ve2, Ve3, Ke**]. Let us remark that the dynamics of ITMs are far more subtle than those of IETs: some ITMs produce examples of exotic pseudo-groups in the sense of Levitt [**le1**]. For instance, Boshernitzan and Kornfeld found an ITM (called of infinite type) for which

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the limit set is a Cantor set of zero Lebesgue measure. The interesting part of the dynamics is concentrated on this set and the dynamical behavior is very different from that of an IET.

We introduce an obvious generalization of ITM.

Definition 2. A map T from an interval I to itself is an interval translation mapping with flips (ITF) if there exists a finite partition of I in subintervals $I_1, \ldots, I_N$ such that for each $j = 1, \ldots, N$ the restriction of T to I_j is an isometry. More precisely, on the interval I_j the map T can be written as $x \to x + c_j$ or $x \to -x + c_j$ for some constant c_j.

ITFs are very interesting because they realize a topological model for a large class of piecewise monotone maps of the interval without periodic points as we will see in the following.

Definition 3. A piecewise monotone map (PMM) is a map $T : I \to I$ such that I is the disjoint union of bounded intervals $I_1, \ldots, I_N$, $N < \infty$, and such that $T : I_j \to T(I_j)$ is a monotone homeomorphism for each $j = 1, \ldots, N$. The minimum N such that this holds is called the number of intervals of T and the corresponding $I_1, \ldots, I_N$ are the intervals of T.

The following theorem is a rigidity result: we extend the classical result of Milnor and Thurston [**MT**], which states that any PMM of positive entropy is topologically semi-conjugate to a map with piecewise constant slope with value $\pm e^{h_{\text{top}}(T)}$, to a PMM with no periodic points (this implies zero entropy).

Obviously, we need to restrict ourselves to maps without wandering intervals as no ITF can have wandering intervals. (Recall that a wandering interval is an interval J of positive length such that $J, T(J), T^2(J), \ldots$ are pairwise disjoint intervals which are not converging towards a periodic orbit.)

The definition of a PMM T is somewhat arbitrary on the endpoints of its intervals. For this reason we say that x is a *periodic limit point* if $x = \lim_{y \to x^+} T^k(y)$ for some $k \geq 1$, possibly with $y \to x^-$ instead of $y \to x^+$.

Irreducibility at infinity, assumed in the following theorem, is defined in §1.3. It roughly means that very small neighborhoods of the endpoints are mapped onto one another under high order iterates.

THEOREM 1. *Let T be a PMM If it has no periodic limit points and no wandering intervals and if it is irreducible at infinity, then T is topologically conjugate to an ITF.*

In particular, any PMM which is minimal on its non-wandering set is topologically conjugate to an ITF.

This generalizes a result of [**HoRa**] (Lemma 15), which was restricted to the case where $T|T^k([0, 1])$ is one-to-one for some $k \geq 1$.

Remarks. (1) It is well-known that such conjugacies are very seldom smooth even if T is very smooth (consider, for instance, the much-studied case of diffeomorphisms of the circle—see, e.g., [**DMVS**, p. 50]).

(2) As we consider discontinuous maps, there is no regularity condition excluding wandering intervals. Namely, Levitt [**Le**] and Camelier and Gutierrez [**CaGu**] gave

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examples of affine interval exchange transformations which have wandering intervals (see also [**Co**] for recent results on this topic).

Perhaps disappointingly, the topological conjugacy with ITF does not hold without assuming 'irreducibility at infinity'.

LEMMA 4. *There exists a PMM with no periodic limit points or wandering intervals which is not topologically conjugate to any ITF.*

To obtain a general result we therefore have to loosen the notion of conjugacy a bit. Say that two PMM are *essentially conjugate* if their restriction to their non-wandering sets define the same symbolic dynamics, perhaps after the removal of countably many points.

THEOREM 2. *If T is a PMM without periodic limit points, then T is essentially conjugate to an ITF.*

The proof of Theorems 1 and 2 rests on the following result, which is interesting in its own right. Recall that a *1-conformal measure* is a probability measure m such that $m(T(A)) = m(A)$ for all measurable sets A such that $T|A$ is one-to-one [**DU**]. We prove the following.

PROPOSITION 5. *Let T be a PMM which is irreducible at infinity (see §1.3). If T has no periodic limit points or wandering intervals, then T has a 1-conformal measure which is atomless and has full support.*

The hypotheses of this theorem are satisfied in particular for any minimal PMM.

This is a rather peculiar existence result for conformal measures as it assumes neither surjectivity (see [**Bo**]) nor a spectral gap (see [**BPS**] and the results quoted there). It would be interesting to have a more classical proof of this result using the transfer operator and avoiding the use of the Markov diagram.

In a second part, we try and generalize to ITMs and ITFs some of the results known for IETs. We consider invariant measures and answer a question of Boshernitzan and Kornfeld [**BoKo**] by a proving a simple bound on the number of invariant and ergodic probability measures carried by an aperiodic ITM.

THEOREM 3. (i) *Let $T : I \to I$ be a PMM with zero entropy. Let N be its number of intervals. Then T has at most $2N$ invariant and ergodic probability measures, excluding those carried by periodic orbits.*

*More precisely, each such measure is of finite rank and the sum of the ranks is at most $2N$ (see Definition 22 below and [**Fe, Ki**] for the notion of finite rank).*

(ii) *Moreover, if T is piecewise increasing then T has at most N invariant and ergodic probability measures, excluding those carried by periodic orbits. More precisely, each such measure is of finite rank and the sum of the ranks is at most N.*

Two immediate consequences are (see [**Fe**] for the first) as follows.

COROLLARY 6. *Let $T : I \to I$ be a PMM with zero entropy on N intervals and μ an T-invariant measure, then the unitary operator U_T adjoint to T in $L^2(\mu)$ has spectral multiplicity less than $2N$.*

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COROLLARY 7. *Let T be an aperiodic ITM on N intervals then T has at most N invariant and ergodic probability measures.*

We note that it was already known that there can be only finitely many such measures: it was proved by Hofbauer and Raith [**HoRa**]. Here we have an explicit bound.

We do not know whether these bounds are sharp. Veech [**Ve**] proved the optimal bound for interval exchange transformations by $N/2$. Numerical experiments of Boshernitzan and Hubert seem to indicate, however, that this $N/2$ bound does not hold for more general PMMs without periodic points.

Remark. This result is interesting only in the non-invertible case. In the invertible case the proof in [**CFS**] for the bound on the number of invariant and ergodic probability measures for IET can be easily applied, proving that invertible PMMs on N intervals have at most N invariant and ergodic probability measures.

The last part of the paper is devoted to the study of the complexity function of PMMs without periodic points.

The rigidity Theorem 1 allows the generalization of some results known for the interval translation mappings. For instance, using the remark of [**BoKo**] which solves the problem for ITFs, we obtain the following.

THEOREM 4. *Let T be a PMM with no periodic limit points. The complexity function of T grows at most polynomially.*

Following Milnor and Thurston [**MT**], let us say that two fixed points $x < y$ of some iterate T^k are monotonically equivalent if T^k maps the entire closed interval $[x, y]$ homeomorphically onto itself.

COROLLARY 8. *Let T be a PMM. T has polynomial complexity if and only if it has finitely many periodic points up to monotone equivalence.*

Remarks. (1) This also extends a corresponding result of Milnor and Thurston (see [**MT**, p. 520]) for *continuous* PMMs.

(2) Baillif [**Ba**] has found another proof of Corollary 8 based on Baladi–Ruelle's formula for zeta functions (see [**BR**]). In fact, he gets a slightly more precise result: the degree of polynomial growth is at most $N + 1$.

1. *Basic definitions*

In this section, we review the basic definitions pertaining to the symbolic dynamics, the Markov diagram of Hofbauer and its dynamics at infinity.

For technical reasons, we generalize the notion of PMMs in the following sense. A PMM will be a map $T : I \to I$ where I is not necessarily a finite union of compact intervals on the real line, but is more generally a totally ordered set which is compact with respect to the order topology (a basis of which is the open intervals) and I is the disjoint union of intervals $I_1, \ldots, I_N$, $N < \infty$, such that:

- the image of any subinterval of an interval in I_j is an interval;
- $T : I_j \to T(I_j)$ is a monotone homeomorphism.

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This notion includes the symbolic dynamics of 'true' PMMs as well as the restrictions of PMMs to invariant finite unions of intervals.

1.1. *Coding and symbolic dynamics.* The symbolic dynamics of a PMM $T : I \to I$ is the following subshift $\Sigma(T)$. Let $I_1, \ldots, I_N$ be as in the definition of PMMs, with N minimal. Let J_i be the interior of I_i. Then

$$\Sigma(T) = \{A \in \{1, \ldots, N\}^{\mathbb{N}} : \forall n \geq 0 \, [A_0 \ldots A_n] \neq \emptyset\}$$

where $[A_0 \ldots A_n] := J_{A_0} \cap T^{-1} J_{A_1} \cap \cdots \cap T^{-n} J_{A_n}$. Such a set is called an $(n+1)$-cylinder. The set of n-cylinders will be denoted by $\mathcal{Z}^n$ and we set $\mathcal{Z} := \mathcal{Z}^1$. We denote the unique element of $\mathcal{Z}^n$ containing x by $\mathcal{Z}^n(x)$ (if it exists).

If the sequence $A_0 A_1 \cdots \in \mathcal{Z}^{\mathbb{N}}$ satisfies $T^k(x) \in J_{A_k}$ for all $k \geq 0$, then A is called the *itinerary* of x. If it exists, it is unique. It only fails to exist for the endpoints of the intervals of T and their preimages to any order—a countable collection of points.

$\sigma : \Sigma(T) \to \Sigma(T)$ is again a PMM if we endow $\Sigma(T)$ with the following variant of the lexicographic ordering:

$$\alpha \leq \beta \iff \alpha = \beta \text{ or } \epsilon(\alpha_0) \ldots \epsilon(\alpha_k)(\beta_k - \alpha_k) > 0$$

where k is maximum with the property that $\alpha_0 \ldots \alpha_k = \beta_0 \ldots \beta_k$ and $\epsilon(i) = +1$ if $T|J_i$ is increasing, $\epsilon(i) = -1$ otherwise.

The invariant and ergodic probability measures of $\sigma : \Sigma(T) \to \Sigma(T)$ and of $T : I \to I$ can be identified, except for the finitely many ones supported by periodic orbits going through endpoints of the intervals I_i. Observe that these periodic measures have no bearing on Theorems 3 and 5.

Thus, for §§1 and 3, we *identify T with its symbolic dynamics*. In particular, T has no *homtervals*, i.e. for every non-trivial interval I, there exists $n \geq 0$ such that $T^n(I)$ is not contained in an interval of $\mathcal{Z}$. Thus, except for the countably many preimages of the endpoints of the I_i, the coding is one-to-one.

1.2. *Hofbauer's definitions and basic results.* We first recall some classical notions in graph theory that we use in the following.

Let S be a set and G be an oriented graph with vertices the elements of S (formally, G is a subset of $S \times S$). A *subgraph* of G is, as a subset of $S \times S$, a subset of G.

Let Σ be a subset of S, the *restriction* of the graph G to the set Σ is the graph G' which is $G \cap (\Sigma \times \Sigma)$ as a subset of $S \times S$. In other words, we only keep the arrows of G starting and ending at elements of Σ.

We say that two graphs $G \subset S \times S$ and $G' \subset S' \times S'$ are isomorphic if there is an invertible map f from S to S' such that $f \times f$ sends G to G'. (The graphs G and G' are the same up to different labelling.)

We recall the results of Hofbauer (see [**Ho1**]) which we use.

Definition 9. Hofbauer's Markov Diagram is the graph $\mathcal{D}$ with vertices

$$\{T^n Z : n \geq 0 \text{ and } Z \in \mathcal{Z}^{n+1}\}$$

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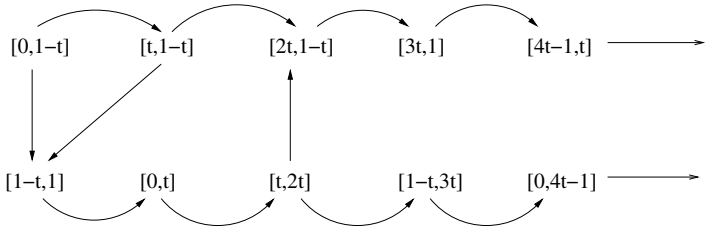


FIGURE 1. The Markov diagram of $x + t \mod 1$. The first (respectively second) line is the branch associated to $0+$ (respectively $1-$).

and arrows

$$D \to E \iff \exists Z \in \mathcal{Z} \quad \text{such that } E = T(D) \cap Z.$$

Hofbauer (see Takahashi [**Tak**] for a special case) discovered that the graph $\mathcal{D}$ has a very special structure. Define $\mathcal{E}$ to be the set of one-sided endpoints $c^{\pm}$ of the intervals in $\mathcal{Z}$. For $c^{\pm} \in \mathcal{E}$, set $D(c^{\pm}, n) := T^n Z$ where Z is the unique interval in $\mathcal{Z}^{n+1}$ having c as an endpoint and containing $c \pm \epsilon$ for small enough $\epsilon > 0$.

The vertices of $\mathcal{D}$ are $\{D(e, n) : e \in \mathcal{E} \text{ and } n \geq 0\}$. Thus, except for coincidences,

$$\mathcal{D} \equiv \mathcal{E} \times \mathbb{N}.$$

The arrows of $\mathcal{D}$ fall exactly into three categories.

(1) Trivial arrows:

$$D(e, n) \to D(e, n + 1)$$

for all $n \geq 0$ and $e \in \mathcal{E}$.

(2) Backward arrows: writing $(e, n) - 1$ for $(e, n - 1)$,

$$D(S(e, i) - 1) \to D(S(Q(e, i)))$$

for all $i \geq 0$ and $e \in \mathcal{E}$ for some auxiliary functions S, Q. These functions $S, Q : \mathcal{E} \times \mathbb{N} \to \mathcal{E} \times \mathbb{N}$ satisfy the following conditions for all $i \geq 0$, $e \in \mathcal{E}$:

- $S(e, i) = (e, n)$ for some integer n depending on e, i;
- $\pi_2 S(e, i + 1) = \pi_2 S(e, i) + \pi_2 S(Q(e, i + 1))$;
- $S(e, 0) > 0$.

(3) Arrows to the basis:

$$D(S(e, i) - 1) \to D(f, 0)$$

for some $(e, i) \in \mathcal{E} \times \mathbb{N}$ and $f \in \mathcal{E}$.

The set $\{D(e, n) : n \geq 0\}$ is the *branch* of $\mathcal{D}$ associated to $e \in \mathcal{E}$. For simplicity we sometimes call the branch associated to $e \in \mathcal{E}$ the branch e.

Figure 1 shows the first vertices of the Markov diagram of the map $x \mapsto x + t \mod 1$ on $[0, 1]$ with $\frac{1}{4} < t < \frac{1}{3}$.

There is a *natural projection* $\pi : \mathcal{D} \to \mathcal{Z}$ defined by $D \subset \pi(D)$. Thus, any finite path of some length n corresponds to a non-empty n-cylinder in an obvious way. In particular, at least one periodic limit point of T corresponds to each loop in $\mathcal{D}$. Conversely, a path of length n starting from a vertex in $\{D(e, 0) : e \in \mathcal{E}\}$, the basis of $\mathcal{D}$, corresponds to each n-cylinder.

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Obviously, for $x \in [0, 1]$ with a well-defined itinerary, there can be many paths on $\mathcal{D}$ projecting to this itinerary. However, there are some uniqueness results. First, there is only one path which starts from a given vertex and projects to a given itinerary. Second, Hofbauer observed the following 'eventual uniqueness'.

LEMMA 10. [**Ho1**] *Let γ and γ' be two paths projecting both to the itinerary of a single point x. If x is in the interior of γ_0 and γ'_0, then $\gamma_n = \gamma'_n$ for all sufficiently large n.*

Proof. The key remark is that $\gamma_n = T^n(\gamma_0 \cap \mathcal{Z}^{n+1}(x))$. □

We also need the following non-trivial fact. A *closed part* G of $\mathcal{D}$ is a subset of $\mathcal{D}$ from which no arrow escapes. We have the following.

LEMMA 11. [**Ho**] *If $\mathcal{C}$ is a closed subset of the Markov diagram $\mathcal{D}$, then there exists $\mathcal{C}_*$ a finite subset of $\mathcal{C}$ such that for every path on $\mathcal{C}$ there is a path starting from $\mathcal{C}_*$ with the same projection.*

The last key property of $\mathcal{D}$ is that for all $(e, i) \in \mathcal{E} \times \mathbb{N}$, writing $(f, j) = Q(e, i + 1)$, the following paths have the same projections (by π):

- $D(S(e, i)) \to \cdots \to D(S(e, i + 1) - 1)$;
- $D(f, 0) \to \cdots \to D(S(f, j) - 1)$.

1.3. *Properties of $\mathcal{D}$ at infinity.* Obviously a closed subset G of $\mathcal{D}$ is of the form

$$G = \bigcup_{e \in \mathcal{E}} \{D(e, n) : n \geq m(e)\}$$

for some $m(e)$, $e \in \mathcal{E}$, possibly infinite.

We denote by $\mathcal{C}(H)$ the smallest closed subset of $\mathcal{D}$ containing H.

An *irreducible part* G of $\mathcal{D}$ is either a non-empty subset $G \subset \mathcal{D}$ such that for all $(D, E) \in G^2$, there is a path from D to E and G is maximal among the subsets of $\mathcal{D}$ with this property or a set of just one vertex through which no loop goes.

If $n \geq 0$, $\mathcal{D}_{\leq n} = \{D(e, k) : e \in \mathcal{E} \text{ and } 0 \leq k \leq n\}$. Also $\mathcal{D}_{>n} = \mathcal{D} \setminus \mathcal{D}_{\leq n}$. For a vertex $D = D(e, n)$, the number n is called the *height* of D. Thus, $D \in \mathcal{D}_{>n}$ is said to be *above level n*.

Definition 12. T is said to be *infinitely renormalizable* around $e \in \mathcal{E}$ if the branch e intersects infinitely many irreducible parts of $\mathcal{D}$.

This is a generalization of the standard notion of renormalizability for unimodal maps. Observe also that if T has no periodic limit points then $\mathcal{D}$ has no loops and, therefore, each vertex is in an irreducible class of its own. Also, all branches are non-eventually periodic. These two facts imply that all branches are infinitely renormalizable.

$e, f \in \mathcal{E}$ are *equivalent at infinity* if, for every $A < \infty$, there exists $B < \infty$ such that $D(f, B) \in \mathcal{C}(D(e, A))$ and $D(e, B) \in \mathcal{C}(D(f, A))$. The equivalence classes defined by this relationship are called *irreducible classes at infinity*.

For simplicity, we use the same symbol $\mathcal{F}$ for both the equivalence class (a subset of $\mathcal{E}$) and the graph obtained by restriction of $\mathcal{D}$ to the branches associated to $e \in \mathcal{F}$.

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Definition 13. The Markov diagram is irreducible at infinity if all $(e, f) \in \mathcal{E}$ are equivalent at infinity.

Definition 14. If T is infinitely renormalizable around $e \in \mathcal{E}$, then the dynamics at infinity around e is the subshift

$$\Sigma_\infty^e(T) = \bigcap_{n\geq 0} \pi(\{\gamma \text{ path on } \mathcal{D}_{>n} : \gamma \text{ stays on branches equivalent at infinity to } e\}).$$

(If T is not infinitely renormalizable around e, we set $\Sigma_\infty^e(T) = \emptyset$.) Set $\Sigma_\infty(T) = \bigcup_{e\in\mathcal{E}} \Sigma_\infty^e(T)$.

Let us investigate some basic properties of this set.

To begin with, we note that, for arbitrary T, $\Sigma_\infty(T)$ contains the transitive subsets without periodic points $N(\Delta)$ defined in Theorem 1 of [**HoRa**]. Indeed, $\Sigma_\infty(T) = \bigcup_\Delta \bigcap_{m\geq 1} W(\mathcal{A}_m, \mathcal{A}_\infty)$ in the notation of Lemma 3 of [**HoRa**].

PROPOSITION 15. *Let T be a PMM. Then $\Sigma_\infty(T) = \bigcup_{\mathcal{F}} \Sigma_\infty^{\mathcal{F}}(T)$ is a finite union of pairwise disjoint closed and minimal subsets not reduced to periodic orbits.*

This proposition is a variant of the results of [**HoRa**]. For the sake of completeness a proof is given in Appendix A.

When the entropy is zero, $\Sigma_\infty(T)$ carries all non-trivial measures.

LEMMA 16. *If T has zero entropy, then all invariant probability measures are carried by $\Sigma_\infty(T)$ with the possible exception of the measures carried by periodic orbits.*

Proof. Let T be a PMM of zero entropy and μ be an invariant and ergodic measure on the symbolic dynamics which is not concentrated on a periodic orbit. First remark that with μ being atomless, we can identify it with a measure on the interval.

Let us prove that $\mu(\Sigma_\infty(T)) = 1$.

Take $x \in I$. Excluding a set of zero measure, we assume that x has a well-defined itinerary and is not eventually periodic. Let γ be the unique path starting from a vertex of the form $D(e, 0)$ and projecting to the itinerary of x (in fact, $\gamma_n = T^n \mathcal{Z}^{n+1}(x)$).

We want to define $c(x)$ to be the irreducible class at infinity containing γ_n for all large n. $c(x)$ is well defined if γ tends to infinite heights. Let us prove this. If that was not the case, γ would go through some vertex infinitely many times, defining infinitely many loops based on this vertex. Some of these loops may coincide, but not all as γ is not eventually periodic. Now taking two distinct loops, we get a subshift with positive entropy, contradicting the assumption that T has zero entropy.

Let us prove that $c(x)$ is almost everywhere constant. As μ is ergodic, it is enough to observe that c is invariant: if γ' is the path defined by $T(x)$, then, except if x or $T(x)$ is an endpoint of some interval of T, $\gamma_{n+1} = \gamma'_n$ for all large n by Lemma 10. Therefore, μ-a.e. $c(T(x)) = c(x)$ which was to be proved. Let c be the almost constant value of $c(x)$.

To prove that the itinerary of μ-a.e. x is in $\Sigma_\infty^c(T)$, we find a path, arbitrarily high in $\mathcal{D}$, contained in the class c and projecting to the itinerary of x.

Fix a large integer N. Let $n(x)$ be the smallest n such that if γ is the path associated above to x, then for all $m \geq n(x)$, γ_m is above level N and is on a branch in the class c.

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$n(x)$ is well defined almost everywhere. Observe that

$$\mu(T^n\{x : n(x) \le n\}) = \mu(T^{-n}T^n\{x : n(x) \le n\}) \ge \mu(\{x : n(x) \le n\}) \to 1$$

when $n \to \infty$. Therefore, for μ-a.e. x, there exist y and $n \ge n(y)$ such that $T^n(y) = x$. However, if γ is the path associated to y, then $\sigma^n\gamma$ is the required path for x. As N was arbitrary, the claim is proved. □

In the following, we will see that the following definition and proposition are very useful tools to reduce the study of a PMM without periodic points to the study of a finite number of PMMs which are irreducible at infinity.

Definition 17. For $e, f \in \mathcal{F}$, write $e \approx f$ iff there exist k_1, k_2 such that $D(e, k_1 + n) = D(f, k_2 + n)$ for all $n \ge 0$. Write $\mathcal{F}/\approx$ for the collection of equivalence classes thus defined.

We denote by $\mathcal{F}_+$ the set of *left* endpoints.

PROPOSITION 18. *Let $\mathcal{F} \subset \mathcal{E}$ be an irreducible class at infinity. Assume that T is infinitely renormalizable around $\mathcal{F}$.*

Then we can find a PMM T' with an irreducible class at infinity $\mathcal{F}'$ such that:

- *$\Sigma_\infty^{\mathcal{F}'}(T')$ is conjugate to $\Sigma_\infty^{\mathcal{F}}(T)$ up to a set which is of zero measure with respect to any invariant probability;*
- *no arrow leaves the branches corresponding to $\mathcal{F}'$;*
- *$\#(\mathcal{F}'/\approx) \le \#\mathcal{F}$—moreover, $\#(\mathcal{F}'_+/\approx) \le \#\mathcal{F}_+$;*
- *the Markov diagram of T' is (isomorphic) to the subgraph of the Markov diagram of T obtained by removing all vertices from which we cannot reach $\mathcal{F}$.*

We defer the proof of this auxiliary result to Appendix A.

2. *Conjugacy with ITF*

We first prove topological conjugacy assuming irreductibility at infinity (Theorem 1). Then we prove the general, essential conjugacy result (Theorem 2). Finally we exhibit a counter-example to topological conjugacy in the general case (Lemma 4).

2.1. *The irreducible case.*

The proof of Theorem 1 is divided in different steps. Let $T : I \to I$ be a PMM without periodic points or wandering intervals which is irreducible at infinity.

Step 1: proof of the theorem assuming the existence of a conformal measure. We assume that there exists a probability measure μ on I which:

(I) is 1-conformal for T;
(II) has full support in I;
(III) has no atom.

We prove in this step that given such a measure, we get the conclusion of Theorem 1. (The existence of such a measure will be proven in the next step.)

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Let h be the repartition function of μ: $h(x) = \int_0^x d\mu = \mu(0, x)$. With the assumptions (II) and (III), h is an homeomorphism from I to the interval $[0, 1]$ (after the obvious identifications of some of the endpoints of the intervals composing I). Let $U = h \circ T \circ h^{-1}$. U is topologically conjugate to T. We claim that U is an interval translation mapping with flips. To prove this, we have to show that U has slope ± 1 except on a finite number of points. Let $\{x_1, \ldots, x_d\}$ be the points of discontinuity of T. The map U is continuous except at the points $\{h(x_1), \ldots, h(x_d)\}$. Let $x \in [0, 1] \setminus \{h(x_1), \ldots, h(x_d)\}$. Let y be a point in the element of the partition for U containing x.

$$\begin{aligned} |h \circ T \circ h^{-1}(x) - h \circ T \circ h^{-1}(y)| &= |\mu[0, T(h^{-1}(x))] - \mu[0, T(h^{-1}(y))]| \\ &= |\mu[T \circ h^{-1}(y), T \circ h^{-1}(x)]| \\ &= |\mu[T[h^{-1}(y), h^{-1}(x)]]|. \end{aligned}$$

As μ is a conformal measure with conformality ratio 1 (assumption (I)) and $h^{-1}(x)$ and $h^{-1}(y)$ are in the same element Z of the partition $\mathcal{Z}$, $\mu[T[h^{-1}(y), h^{-1}(x)]]$ is equal to $\mu[h^{-1}(y), h^{-1}(x)]$. Clearly $\mu[h^{-1}(y), h^{-1}(x)] = |x - y|$, so we have just proven that the derivative of U at the point x is ± 1 depending on whether T is increasing or decreasing on Z, which was our claim.

Step 2: existence of a conformal measure. We prove Proposition 5 assuming the existence of a weight function a on the vertices of the Markov diagram $\mathcal{D}$ satisfying the following properties:

(1) $a(D) \in [0, 1]$, for all $D \in \mathcal{D}$;
(2) $a(D) > 0$ for every $D \in \mathcal{D}$;
(3) $a(D) = \sum_{D \to C} a(C)$;
(4) $a(D)$ tends to zero when the height of D tends to infinity.

(The proof of the existence of the weight function will be done in the next step.)

Let $\mathcal{P}$ be a true partition obtained from $\mathcal{Z}$ by adding to each interval some of its endpoints. To build the measure μ, we first define it on the algebra generated by the dynamical partitions $\mathcal{P}^n$, $n \geq 1$, according to the formula

$$\mu(Z) = a(T^n(Z)), \quad \text{for } Z \in \mathcal{P}^{n+1}.$$

Observe that the partitions $\mathcal{P}^n$ generate the Borel σ-algebra on $[0, 1]$. For μ to define a probability measure, it is enough to satisfy the Lebesgue space criterion [**Ru**, p. 12].

(i) μ is finitely additive over the $\mathcal{Z}^n$.
(ii) For every $\epsilon > 0$, there is a union U of disjoint subsets in $\bigcup_{n \geq 1} \mathcal{P}^n$ with $\mu(U) < \epsilon$ such that for all decreasing sequences $A_1, A_2, \ldots$ with $A_n \in \mathcal{P}^n$ and $\bigcap_{n \geq 1} A_n = \emptyset$, $A_n \subset U$ for all large enough n.

To prove (i), let $Z = Z_1 \cup \cdots \cup Z_k$ with $Z \in \mathcal{Z}^{n+1}$ and $Z_i \in \mathcal{Z}^{n+2}$ for $1 \leq i \leq k$.

By property (3), $a(T^n(Z)) = \sum_{i=1}^k a(T^{n+1}(Z_i))$. The finite additivity easily follows by induction.

To prove (ii), observe that a sequence $A_1, A_2, \ldots, A_n \in \mathcal{P}^n$, with empty intersection must, for large n, be of the form $A_n = (c, x_n]$ or (c, x_n) (or in the reverse order) with $\lim_{n \to \infty} x_n = c$ and c some preimage of some order of some critical point. Therefore, if

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$c_1, c_2, \ldots$ is some sequence enumerating these preimages, then we can take U to be the union, for each $i \geq 1$, of the intervals immediately before and after c_i in $\mathcal{P}^{n_i}$ with n_i such that the μ-size of these two intervals is at most $\epsilon 2^{-i-1}$. We claim that this can be achieved by taking n_i large enough.

Observe that if γ is an infinite path on $\mathcal{D}$, then the height of γ_n must tend to infinity: otherwise γ would go twice through the same vertex and there would be a loop in $\mathcal{D}$.

Apply this to $\gamma = T^n Z_{n+1}$, Z_{n+1} being the element of $\mathcal{Z}^{n+1}$ at the right of c (for instance). The above observation together with property (4) implies that $\mu(Z_{n+1}) = a(T^n Z_{n+1})$ tends to zero when $n \to \infty$, proving the claim.

Thus, we have a Lebesgue probability space and, in particular, a probability measure μ.

Property (1) implies that the measure μ is a finite measure: the finitely many elements of the partition $\mathcal{Z}$ have finite measure. So we can assume that μ is a probability measure.

The measure μ satisfies property (II): the measure μ has full support because, as $\mathcal{Z}$ generates under T, any open set contains an element of some $\mathcal{Z}^n$ and thus has positive measure by property (2).

Let us check (III), i.e. that the measure μ has no atoms. Let $x \in [0, 1]$, $\mu\{x\} = \lim_{n\to+\infty} \mu(Z_{n+1}(x)) = \lim_{n\to+\infty} a(T^n(Z_{n+1}(x)))$, where $Z_{n+1}(x)$ is the element of the partition $\mathcal{Z}^{n+1}$ containing x. As above, property (4) implies that this limit is zero. Thus $\mu\{x\} = 0$ for all $x \in [0, 1]$: μ has no atoms.

Now let us prove (I), i.e. that μ is a conformal measure. We check that $\mu(T(A))/\mu(A) = 1$, for all non-trivial intervals A contained in an element Y of the partition $\mathcal{Z}$ (which is enough to conclude). First, remark that $\mu(A)$ and $\mu(T(A))$ are positive because μ has full support. Given an integer $n \geq 2$, we can split A in the following way:

$$A = A_1^{(n)} \cup Z_1^{(n)} \cup \cdots \cup Z_k^{(n)} \cup A_2^{(n)},$$

where $A_1^{(n)}$ and $A_2^{(n)}$ are intervals contained in some elements of the partition $\mathcal{Z}^{n+1}$ and $Z_1^{(n)} \ldots Z_k^{(n)}$ are elements of $\mathcal{Z}^{n+1}$. Obviously, $\mu(T(Z_i^{(n)})) = \mu(Z_i^{(n)})$. As the elements $T(Z_i)$ $(1 \leq i \leq k)$ are disjoint (T is one-to-one on Y) the previous remark implies that $\mu(Z_1^{(n)} \cup \cdots \cup Z_k^{(n)}) = \mu(T(Z_1^{(n)} \cup \cdots \cup Z_k^{(n)}))$.

To get the equality $\mu(T(A)) = \mu(A)$, it is enough to check that $\mu(A_1^{(n)})$, $\mu(A_2^{(n)})$, $\mu(T(A_1^{(n)}))$, $\mu(T(A_2^{(n)}))$ tend to zero when n tends to infinity. However, this is a direct consequence of property (4).

Step 3: construction of the weight function a. The construction was done by Hofbauer and Raith in [**HoRa**], but we give a proof which makes explicit the generality of this construction.

First of all, we prove properties (1) and (3) in a more general setting. We prove properties (2) and (4) later.

Let Γ be a countable, oriented graph. Recall that Γ has *finite multiplicity* if there is only a finite number of arrows incoming to and outgoing from any vertex.

Definition 19. Let $\mathcal{F}$ be a finite part of the graph Γ, the *boundary* of $\mathcal{F}$ is the set of the vertices of Γ belonging to $\mathcal{F}$ that only have successors in $\Gamma \setminus \mathcal{F}$.

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Definition 20. Let us call a finite part of the graph Γ *backward closed* if there is no incoming arrow to this part.

LEMMA 21. *Let Γ be a countable, oriented graph without loops such that any vertex has finite multiplicity. Let us assume that there exists a sequence $(\mathcal{F}_n)_{n\in\mathbb{N}}$ of finite subsets of the graph Γ such that:*

- *$\mathcal{F}_n \subset \mathcal{F}_{n+1}$ and $\bigcup_{n\in\mathbb{N}} \mathcal{F}_n = \Gamma$;*
- *each $\mathcal{F}_n$ is backward closed;*
- *each element of $\mathcal{F}_m$ can be reached from an element of $\mathcal{F}_n$ for all $n < m$ (reachability property);*
- *the boundary of $\mathcal{F}_n$ is never empty.*

Then there exists a weight function defined on the vertices of the graph Γ satisfying properties (1) and (3).

Proof of Lemma 21. Let C_n be the set of functions f defined on the vertices of Γ with values in $[0, 1]$ satisfying property (3) for all $D \in \mathcal{F}_n$ and such that the maximum of f on $\mathcal{F}_n$ is equal to one. Let us check that the set C_n is non-empty.

Let $(D_i)_{1\le i\le k}$ be the elements of the boundary of $\mathcal{F}_n$ and let $(\alpha_i)_{1\le i\le k}$ be positive numbers. The numbers α_i enable us to define a function g on the set $\mathcal{F}_n$. Given a vertex D belonging to $\mathcal{F}_n$, let n_i the number of different paths from D to D_i $(1 \le i \le k)$. We define $g(D)$ by the formula

$$g(D) = \sum_{i=1}^{k} n_i \alpha_i .$$

As $\mathcal{F}_n$ is backward closed, finite and without loops, each $g(D)$ is finite, so g is well defined. By construction, the function g satisfies (3) on $\mathcal{F}_n$.

We renormalize g. Let f be $g/\max_{\mathcal{F}_n} g$, then f is in C_n, which is therefore non-empty, as we claimed.

For a given m, the set C_m is defined by a finite number of equalities coming from condition (3) and by the renormalization condition which only concerns finitely many vertices ($\mathcal{F}_m$ is finite). All these constraints are closed, so C_m is a closed subset of $[0, 1]^\Gamma$. Hence C_m is compact.

We claim that $C_{m+1} \subset C_m$. Indeed, any $f \in C_{m+1}$ satisfies condition (3) on $\mathcal{F}_{m+1}$, hence on the subset $\mathcal{F}_m$. It remains to see that $\max_{\mathcal{F}_m} f = \max_{\mathcal{F}_{m+1}} f = 1$. Assume by contradiction that there is no $D \in \mathcal{F}_m$ such that $f(D) = \max_{\mathcal{F}_{m+1}} f$. Let $D \in \mathcal{F}_{m+1}$ have this last property. As $\mathcal{F}_{m+1}$ is connected, there exist $D' \in \mathcal{F}_m$ and a path between D' and D. As $\mathcal{F}_m$ is backward closed this path must be from D' to D. However, condition (3) then implies that $f(D') \ge f(D)$, a contradiction.

Let C be the intersection of the C_n for all $n \ge 1$. C is a non-empty compact set. Any element of C is a weight on the graph satisfying properties (1) and (3).

This proves the lemma. □

To apply the previous lemma with $\Gamma = \mathcal{D}$, the Markov diagram, we first observe that $\mathcal{D}$ has no loops and has finite multiplicity (for any PMM the number of outgoing arrows from any vertex is bounded by N; on the other hand, irreducibility at infinity implies that only finitely many arrows point to $\mathcal{D}_{\le n}$ for any n and this implies that there is a finite number of

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incoming arrows at any vertex). Second, we define sets $\mathcal{F}_n$. Let $\mathcal{F}(D)$ be the set of vertices that cannot be reached from a vertex D (the set $\mathcal{F}(D)$ is the complement of the set $\mathcal{C}(\mathcal{D})$ defined in §1.3). For D high enough in the diagram, the set $\mathcal{F}(D)$ is non-empty and its boundary is also non-empty. So, taking vertices D_n higher and higher in the diagram, we can build by induction an increasing sequence $\mathcal{F}_n := \mathcal{F}(D_n)$ of backward closed subsets of the diagram such that $\bigcup_{n\in\mathbb{N}} \mathcal{F}_n = \Gamma$. The reachability property clearly holds because of the trivial arrows. Applying Lemma 21, we get properties (1) and (3).

We now prove that any a obtained in the previous lemma satisfies properties (2) and (4). We prove property (4) by contradiction. As a is decreasing on the branches on the Markov diagram, we can assume that there exists a positive constant α such that, for vertices D arbitrarily high in $\mathcal{D}$, $a(D) \geq \alpha$. As the graph is irreducible at infinity, for all D in the Markov diagram, $a(D)$ is larger than α. Applying property (3) and the fact that the number of vertices that can be reached by a path of length n goes to infinity when $n \to \infty$, we can easily check that a takes infinite values on the base of the diagram, a contradiction.

The irreducibility at infinity of the Markov diagram implies property (2): by contradiction, let D be a vertex in the graph such that $a(D)$ equals zero. According to property (3), the function a equals zero at all the vertices that can be reached from D, i.e. on the complement of $\mathcal{F}(D)$. As the graph is irreducible at infinity, a takes positive values at, at most, a finite number of vertices. Again, by property (3), a equals zero everywhere which contradicts the fact that the maximum of a is one.

2.2. *Essential conjugacy.* We prove Theorem 2, i.e. the essential conjugacy of an arbitrary PMM without periodic points to some ITF The idea is just to consider successively all irreducible classes at infinity and give for each an ITF with the same symbolic dynamics at infinity Σ_∞. Finally, to get the model for T, we just take the disjoint union of all the ITFs previously obtained, as completely invariant subsystems.

Remark first that wandering intervals can intersect the non-wandering set only at their (countably many) endpoints. Thus we can remove these wandering intervals.

The symbolic dynamics of the non-wandering set of a PMM of zero entropy is included in $\Sigma_\infty(T)$, up to the countably many periodic itineraries. Thus it is enough to obtain a conjugacy of $\Sigma_\infty(T)$. Now, $\Sigma_\infty(T) = \bigcup_{\mathcal{F}} \Sigma_\infty^{\mathcal{F}}(T)$ where $\mathcal{F}$ ranges over the irreducible classes at infinity. Therefore, it is enough to find for each irreducible class at infinity $\mathcal{F}$, an ITF S with $\Sigma_\infty^{\mathcal{F}}(T) = \Sigma_\infty(S)$ (and then take the disjoint union of all the ITFs obtained for all the irreducible classes to get an ITF essentially conjugate to T).

By Theorem 1, it is enough to find a PMM U which is irreducible at infinity and such that $\Sigma_\infty(U) = \Sigma_\infty^{\mathcal{F}}(T)$.

To define such a PMM U, begin by considering $J = \bigcup_{n\geq 0} T^n(D(f,\ell))$ for some $f \in \mathcal{F}$ and ℓ such that there is no path from $D(f,\ell)$ to any branch not equivalent to f. J is a finite union of intervals by [**Ho**] (see also the appendix of [**Bu**] for another proof). Thus $T|J$ is a PMM.

Clearly $\Sigma_\infty(T|J) = \bigcup_{\mathcal{G}\geq\mathcal{F}} \Sigma_\infty^{\mathcal{G}}(T)$. Now, destroy all $\mathcal{G}$ classes at infinity distinct from $\mathcal{F}$ that remain in $T|J$ by applying Proposition 18.

The resulting PMM is exactly the claimed U. This proves the theorem.

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2.3. *Counter-example to topological conjugacy.* Let R be the representation of an irrational rotation on $[0, 1]$, i.e. $R(x) = x + \alpha \mod 1$ for some irrational α. Let $T : [0, 2] \to [0, 2]$ be the disjoint union of two copies of R.

Modify T by blowing up the point 0 and each point of its forward and backward orbit to non-trivial intervals. This is possible while keeping T a PMM.

Arrange T by adding a monotony and continuity interval J contained in the interval resulting in the above blowing-up of 0 and such that $T(J) = [1, 2]$.

The Markov diagram of the PMM thus obtained is the disjoint union of two copies of the Markov diagram of the original rotation R *plus* an infinite number of arrows going from vertices arbitrarily high in the diagram to the vertex $[1, 2]$.

The above example cannot be conjugate to an ITF by the following observation: there are only finitely many arrows pointing to any vertex of the Markov diagram of an ITF.

Indeed, in the case of an ITF, the length of $D(e, n)$ is non-increasing as a function of n. Let ℓ be the limit of this length as $n \to \infty$. If $D(e, n) \to D(f, j)$ is a non-trivial arrow, then the length of $D(f, j)$ is at most that of $D(e, n)$ minus that of $D(e, n + 1)$. Hence, the length of $D(f, j)$ tends to zero as $n \to \infty$. Therefore, the height of $D(f, j)$ goes to infinity. This immediately implies the claim and establishes the counter-example, proving Lemma 4.

3. *Invariant measures*

The goal of this section is the proof of the bound on the number of invariant measures 'at infinity' (Theorem 3). More precisely, we prove the following more general statement (see Proposition 15).

THEOREM 5. *Let T be a PMM (possibly with positive entropy). Then $\Sigma_\infty(T)$ carries only finitely many invariant and ergodic measures. Moreover, each of these measures has finite rank (see below) and the sum of their rank is at most $2N$.*

If T is piecewise increasing then the same is true with the bound N instead of $2N$.

The notion of rank used above is defined as follows (see [**Fe, Ki**]).

Definition 22. A subshift Σ is of rank (at most) r if, for arbitrary $\epsilon > 0$, there exist r finite sequences $\omega_1, \ldots, \omega_r$ of lengths $|\omega_1|, \ldots, |\omega_r| \geq \epsilon^{-1}$, such that any sufficiently long sequence in Σ, after discarding at most a proportion ϵ of its symbols, can be written as a concatenation of copies of the finite sequences $\omega_1, \ldots, \omega_r$.

An invariant probability measure μ has rank (at most) r if there is a partition with two elements such that, coding with this partition, μ lives on a subshift of rank (at most) r.

The following says that it is enough to prove the bound on the rank of $\Sigma_\infty(T)$.

PROPOSITION 23. [**Ki**] *A subshift of rank r carries at most r invariant ergodic probability measures.*

Our proof will proceed by reduction. Note that this reduction is not needed if the map is already irreducible at infinity (e.g. if the map is minimal on its non-wandering set).

We recall that $e, f \in \mathcal{F}$ satisfy $e \approx f$ iff there exist k_1, k_2 such that $D(e, k_1 + n) = D(f, k_2 + n)$ for all $n \geq 0$. $\mathcal{F}/\approx$ is the collection of equivalence classes thus defined.

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Proof of Theorem 5—general case. Fix $\mathcal{F}$, an irreducible class at infinity in $\mathcal{E}$. We need to prove that $\Sigma_\infty^{\mathcal{F}}(T)$ has rank $\#\mathcal{F}$.

If T is not infinitely renormalizable around $\mathcal{F}$, then $\Sigma_\infty^{\mathcal{F}}(T) = \emptyset$ and there is nothing to show. Thus we assume that T is infinitely renormalizable around $\mathcal{F}$. Using Proposition 18, we may assume that no arrow leaves the branches $\mathcal{F}$ above height A, setting A large enough. We now have to prove that $\Sigma_\infty^{\mathcal{F}}(T)$ has rank $\#(\mathcal{F}/\approx)$.

Infinite renormalizability gives $B < \infty$ such that there is no path from above B in the branches in $\mathcal{F}$ going to levels lower than A. More precisely, picking $D_* \in \mathcal{D}_{>B}$ on some branch in $\mathcal{F}$, we have

$$\mathcal{C} := \mathcal{C}(D_*) = \bigcup \{D(f,k) : k > n(f)\} \subset \mathcal{D}_{>A}$$

for (some possibly infinite) integers $n(f) \geq A$, $f \in \mathcal{F}$.

A *block* will be the projection of a path between two successive bifurcations, i.e. the projection by π of

$$D(S(e,i)) \to D(S(e,i)+1) \to \cdots \to D(S(e,i+1)-1)$$

for some $(e,i) \in \mathcal{E} \times \mathbb{N}$.

CLAIM. *Any block from $\mathcal{C}$ is a concatenation of copies of the sequences:*

$$\omega_f = \pi(D(f,0)D(f,1)\dots D(f,n(f)-1)) \quad (f \in \mathcal{F}).$$

Recall that by definition the sequences in $\Sigma_\infty^{\mathcal{F}}(T)$ are projections of paths starting arbitrarily high in the graph. The claim clearly implies that the rank of $\Sigma_\infty^{\mathcal{F}}(T)$ is at most $\#\mathcal{F}$. Let us see that it implies the stronger fact that $\Sigma_\infty(T)$ has rank $\#(\mathcal{F}/\approx)$.

For each $\approx$-equivalence class within $\mathcal{F}$, select a representative e. We can select lengths $k(f)$, $f \in \mathcal{F}$, depending only on $\mathcal{F}$ and not on A, such that if f is any other member of the equivalence class, ω_f and ω_e coincide after discarding their $k(e)$ and $k(f)$ first symbols. Let $\bar{\omega}_e$ be ω_e with the first $k(e)$ symbols removed. We can assume that $|\bar{\omega}_e| \geq A$.

Therefore, applying the claim and then discarding the first $k(f)$ symbols of the copies of ω_f—thus removing a negligible fraction—we get a concatenation of copies of $\bar{\omega}_e$ for e ranging over the selected representatives of the classes. However, this is the bound on rank that we had to prove.

It remains to prove the claim. Consider the set of blocks on $\mathcal{C}$ which do not satisfy the claim. If it is not empty, then there exists a block in it with minimal length, say $B = \pi(D(S(e,i))\dots D(S(e,i+1)-1))$ for some $e \in \mathcal{E}$ and $i \geq 0$ with $D(S(e,i)) \in \mathcal{C}$. By §1.2, this block coincides with the projection of $\gamma = D(f,0)D(f,1)\dots D(S(f,j)-1)$ if $(f,j) = Q(e,i)$. As the bifurcation at $D(S(e,i+1)-1)$ is towards $D(S(f,j))$, this last vertex must belong to $\mathcal{C}$. Hence $f \in \mathcal{F}$ and γ is made of $D(f,0)\dots D(f,n(f)-1)$, plus some number of blocks on $\mathcal{C}$. These blocks have length strictly less than the block we started with. Hence these blocks satisfy the claim. However, this implies that the block B also satisfies it. The contradiction proves the claim. □

Proof of Theorem 5—increasing case. We assume additionally that T is piecewise increasing and prove the following strengthening of point (i): the rank of $\Sigma_\infty^{\mathcal{F}}(T)$ is at most $\#(\mathcal{F}'_+/\approx)$. Indeed, $\sum_{\mathcal{F}} \#(\mathcal{F}'_+/\approx) \leq \sum_{\mathcal{F}} \#\mathcal{F}_+ \leq \mathcal{E}_+ \leq N$.

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For this, we have to see that, in the proof of part (i), we can restrict oneself to blocks corresponding to endpoints $e \in \mathcal{F}_+$.

A being given, pick A' very large so that each block $\pi(D(S(f,i))) \dots \pi(D(S(f,i+1)-1))$ above the enormous level A' is made essentially by concatenation of sub-blocks $\pi(D(e,0)) \dots \pi(D(e,S(e,j)-1))$ above the high level A.

The key remark is that these sub-blocks correspond to endpoints e such that there is an arrow from the branch f to the branch e. Geometrically, this means that $T(T^n f, T^m e) = (T^{n+1} f, h) \cup (h, T^{m+1} e)$. As T is piecewise increasing, we see that if f is a right endpoint, then so is $T^n f$ and therefore $T^m g$ and also g itself are left endpoints. Thus, an enormous block on a branch defined by a right endpoint can be essentially replaced by large sub-blocks corresponding to left endpoints.

This remark proves that, when T is piecewise increasing, we can only use blocks corresponding to left endpoints in the above proof. In this way we get the promised bound by $\#(\mathcal{F}'_+/\approx)$. □

4. *Polynomial complexity*

We prove Theorem 4, i.e. that an arbitrary PMM without periodic points has polynomial complexity. To begin with, remark that the complexity of a PMM is the complexity of its Markov diagram, i.e. the maximum number of paths of a given length starting from any vertex.

Write $\mathcal{D} = \bigcup_{\mathcal{F}} \mathcal{D}_F \cup \mathcal{D}_*$, where $\mathcal{D}_{\mathcal{F}} = \{D(e,n) : e \in \mathcal{F} \text{ and } n \geq n_e\}$ with $n_e < \infty$ such that all the arrows in $\mathcal{D}_F$ which point outside of $\mathcal{F}$, point to vertices from which we cannot go back to $\mathcal{F}$. $\mathcal{D}_*$ is the set of the finitely many vertices of $\mathcal{D}$ which remain. Observe that all branches are infinitely renormalizable (there is no loop in $\mathcal{D}$ as a loop would give a limit periodic point).

The following lemma shows that we only have to bound the complexities of the subgraphs $\mathcal{D}_{\mathcal{F}}$.

LEMMA 24. *Let $\mathcal{D}$ be a countable graph without loops with finite outdegree (i.e. the number of arrows leaving any vertex is bounded). Assume that $\mathcal{D}$ is, as a set, the disjoint union of $\mathcal{D}_1 \cup \dots \cup \mathcal{D}_r$ and, as a graph, has exactly the arrows inside each of $\mathcal{D}_i$ plus arbitrary arrows going from D_i to D_j for $i < j$. Assume that for each subgraph $\mathcal{D}_i$, the number of paths starting from any given vertex is polynomially bounded. Then the same is true for $\mathcal{D}$.*

The proof of this is easy and left to the reader.

We now fix $\mathcal{F}$, an infinitely renormalizable equivalence class at infinity, and consider the complexity of $\mathcal{D}_{\mathcal{F}}$.

We apply Proposition 18 to $\mathcal{F}$. We obtain a map T' with a corresponding equivalence class $\mathcal{F}'$ from which no arrows escape. Observe that the Markov diagram of T' restricted to $\mathcal{F}'$ contains $\mathcal{D}'_{\mathcal{F}}$, isomorphic to $\mathcal{D}_{\mathcal{F}}$.

By a result of Hofbauer [**Ho**] any closed subset of the Markov diagram of a PMM projects to the symbolic dynamics of the restriction of the PMM to an invariant finite union of intervals. We apply this to $\mathcal{D}'_{\mathcal{F}}$ and obtain a finite union of intervals which we denote by K. Observe that $T'|K$ is irreducible at infinity: otherwise, K would contain

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two invariant compact subsets, but these would have to be included in $\Sigma_\infty^{\mathcal{F}'}(T')$, which is impossible as this last set is minimal by Proposition 15.

As $T'|K$ is irreducible at infinity, we can conjugate it to an ITF. It therefore has polynomial complexity by the following folklore argument: a group generated by finitely many one-dimensional isometries has polynomial growth. Therefore, $\mathcal{D}'_{\mathcal{F}}$ and also $\mathcal{D}_{\mathcal{F}}$ have polynomial complexity. This concludes the proof of the theorem.

We now prove Corollary 8.

Assume first that T has infinitely many periodic orbits up to monotone equivalence and let us see that this implies super-polynomial complexity, i.e. for any $p < \infty$, for some $C > 0$, the complexity function satisfies $p(n) \geq Cn^p$ for infinitely many n. We can assume that periodic homtervals are collapsed to points, so all periodic points of T are different up to monotone equivalence.

If the topological entropy of T is positive, there is nothing to prove, so let us assume that it is zero.

By assumption, there are infinitely many disjoint loops in the graph. We can arrange them in a sequence $L_1, L_2, \ldots$, such that in each L_i there is a distinguished vertex and for all $i \geq 1$ there is a path ℓ_i from the distinguished vertex in L_i to the distinguished vertex in L_{i+1}.

Fix an arbitrarily large integer p. Consider now the paths of length n going from the distinguished vertex of L_1 to the distinguished vertex of L_{p+1} by making a certain number k_1 of turns in the loop L_1, then following the path ℓ_1, then making k_2 turns in L_2, etc. The number of such paths is given by the number of possible choices of non-negative integers $k_1, \ldots, k_{p+1}$ such that

$$n = k_1|L_1| + \cdots + k_{p+1}|L_{p+1}| + |\ell_1| + \cdots + |\ell_p| \qquad (*)$$

where $|L_1|$ is the length of the loop L_1, etc. Let K be equal to $|l_1| + \cdots + |l_p|$. Let us assume that $n-K$ is a common multiple of $|L_1|, \ldots, |L_{p+1}|$ and let μ be the least common multiple of $|L_1|, \ldots, |L_{p+1}|$ and $n_i = \mu/|L_i|$ for $i = 1, \ldots, p+1$.

From a $(p+1)$-*tuple* of integers $(\lambda_1, \ldots, \lambda_{p+1})$, such that $(n-K)/\mu = \lambda_1 + \cdots \lambda_{p+1}$, we can derive a solution of (*) by $(n_1\lambda_1, \ldots, n_{p+1}\lambda_{p+1})$. So the number of solutions of (*) is larger than the number of $(p+1)$-uples $(\lambda_1, \ldots, \lambda_{p+1})$ such that $(n-K)/\mu = \lambda_1 + \cdots \lambda_{p+1}$. This number is larger than the number of partitions of $(n-K)/\mu$ by $(p+1)$ integers. However, this number is

$$\binom{\frac{n-K}{\mu} + p}{p}.$$

So, there exists a constant C such that for infinitely many integers n,

$$p(n) \geq \binom{\frac{n-K}{\mu} + p}{p} \geq Cn^p.$$

This proves the super-polynomial growth.

The reverse implication was proved as Theorem 4 in the case where there is no periodic orbit. The general case follows along the same lines using that after reduction

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by Proposition 18 (and therefore to an infinitely renormalizable class at infinity which no arrow leaves), we can find a finite and invariant union of intervals such that the restriction of T to that union has no periodic point. Hence, we can apply Theorem 4 to this restriction. This proves that after exclusion of a finite subset with zero entropy (and hence polynomial growth), the remaining invariant subgraph has polynomial complexity. However, this implies polynomial complexity of the initial graph.

A. *Appendix*

A.1. *Reduction to the irreducible case.*

Proof of Proposition 18. We define T' such that its Markov diagram will be that of T minus all vertices from which one cannot reach the branches $\mathcal{F}$.

Step 1: definition of the semi-conjugacy. To begin with, note that $\Sigma_\infty^{\mathcal{F}}(T)$ is an invariant compact set and therefore supports invariant measures. Take one of these measures. By minimality (see Proposition 15), $\mathrm{supp}(\mu) = \Sigma_\infty^{\mathcal{F}}(T)$.

Observe that $\Sigma_\infty(T)$ containing no periodic point μ is atomless.

Let $\nu = \sum_{n\geq 0} e^{-(h_{\mathrm{top}}(T)+1)n} \sum_{A\in\mathcal{Z}^n} (T^n|A)_*^{-1}\mu$. Observe that:

(1) we can normalize ν into a probability measure;
(2) the support of ν contains that of μ;
(3) ν is atomless;
(4) $\nu(TA) \leq C\nu(A)$ for some constant $C < \infty$.

Define $h : I \to [0,1]$ by $h(x) = \nu([0,x])$. h is clearly onto and non-decreasing. Because of point (3) above, h is continuous.

Let us see that h is one-to-one with respect to all invariant probability measures carried by $\Sigma_\infty^{\mathcal{F}}(T)$. Let Z be the union of the closures of the connected components of $I \setminus \mathrm{supp}(\nu)$. The set of the endpoints of the intervals in Z has zero measure with respect to any invariant probability measure on $\Sigma_\infty^{\mathcal{F}}(T)$ because this set is countable and these measures are aperiodic. The rest of Z is clearly negligible. Thus, it is enough to see that for all $x, y \in I \setminus Z, x \neq y \implies h(x) \neq h(y)$.

Let $x < y$ be such that $h(x) = h(y)$. Observe that h, which is monotone, is constant on $[x,y]$. By the definition of h, $[x,y]$ must have zero ν-measure. Therefore, $[x,y]$ is included in Z, proving the claim.

Let us define $T' : [0,1] \to [0,1]$ by $T'(x) = h(T(h^{-1}(x)))$ if $h^{-1}(x)$ is just a point and $T'(x) = \inf h(T(h^{-1}(x)))$ otherwise (the choice of inf is arbitrary). Let us see that $T' \circ h = h \circ T$ over I. This is clear if $h^{-1}(h(x))$ is just $\{x\}$. Otherwise, compute

$$T'(h(x)) = \inf h(T(h^{-1}h(x))) = h(T(x')) = \nu([0, T(x')])$$

where $x' \in h^{-1}h(x)$ and

$$\nu([0,T(x')]) = \nu([0,T(x)]) = h(T(x))$$

as $\nu([T(x),T(x')]) \leq C \cdot \nu([x,x']) = C \cdot |h(x) - h(x')| = 0$. Thus, T' and T are semi-conjugate.

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We prove that T' is a PMM. Let J be an interval of T. First, T' is monotone on $h(J)$. Second, let us see that T' is continuous on the interior of $h(J)$.

Take $x, y \in h(J)$. We have $|T'(x) - T'(y)| = \nu([T(x'), T(y')]) = \nu(T([x', y']))$ for some x', y' in $h^{-1}(x), h^{-1}(y)$. By point (4) above, this implies $|T'(x) - T'(y)| \leq C\nu([x', y']) \leq C \cdot |x - y|$. This proves the continuity of T' inside $h(J)$.

Step 2: the Markov diagram of T'. Let us consider $\mathcal{D}'$ the Markov diagram of T' taking the intervals of T' to be the non-empty images by h of the intervals of T. Observe that, by an immediate induction, $\mathcal{D}' = \{h(D) : D \in \mathcal{D} \text{ and } \nu(D) > 0\}$. Thus $\mathcal{D}'$ is, as claimed, isomorphic to this subgraph of the original Markov diagram of T, obtained by removing the vertices from which one cannot reach the branches $\mathcal{F}$ together with the corresponding arrows.

Let $\mathcal{F}'$ be the equivalence class at infinity of $h(e)$ for some $e \in \mathcal{F}$, where $h(c+) = c'+$ with $c' = \inf h(J)$ if $c = \inf J$ with J the interval of T with left endpoint c ($h(c-)$ is defined in the obvious way). We leave undefined $h(c\pm)$ in the case where the corresponding interval J has no image, i.e. $\nu(J) = 0$.

Observe that by construction we can reach $\mathcal{F}'$ from any vertex in $\mathcal{D}'$. In particular, $\mathcal{F}'$ is the 'last' equivalence class at infinity, i.e. there exists no arrow leaving it, proving claim (2) of the proposition.

Observe also that $\mathcal{F}'$ is infinitely renormalizable, as $\mathcal{F}$ was: no arrow inside the branches $\mathcal{F}$ has been removed.

Clearly $h(\mathcal{F}) \subset \mathcal{F}'$. However, this inclusion will be strict in general and $\#\mathcal{F}'$ may be larger than $\#\mathcal{F}$. Hence the need to bound by $\#(\mathcal{F}'/\approx)$ and not by $\#\mathcal{F}'$.

Let us see an example where $\#\mathcal{F}' > \#\mathcal{F}$. Assume that there is a branch e which 'comes after' $\mathcal{F}$ (there are infinitely many arrows from $\mathcal{F}$ going to the branch e), but such that there are some but finitely many arrows going from the branch e to $\mathcal{F}$. Then $h(e) \in \mathcal{F}'$. Indeed, $\nu(D(e, n)) = 0$ exactly for $n > n_0$ if $D(e, n_0)$ is the last vertex on the branch e from which we can reach $\mathcal{F}$. Therefore, the endpoints of $D(e, n_0 + 1) =: [T^{n_0+1}e, T^q f]$ will be identified by h. However, this means that $D(e, n_0)$ will no longer be a bifurcation of the graph. Instead, we have $D'(e, n) = D'(f, n - n_0 + q)$ (we simplify $D'(h(e), n) =: D'(e, n)$). However, then we see that $h(e) \in \mathcal{F}' \setminus h(\mathcal{F})$.

To control this phenomenon, we are going to show that $e \in \mathcal{F}' \setminus \mathcal{F}$ is always $\approx$-equivalent to some $f \in h(\mathcal{F})$, thus proving $\#(\mathcal{F}'/\approx) \leq \#\mathcal{F}$.

Say that $e > f$ if one the following conditions hold.

(i) There exists $m < n$ such that $\nu(D(e, n)) = 0$ and $D'(e, n + k) = D'(f, m + k)$ for all $k \geq 0$.

(ii) $D'(e, k) = D'(f, k)$ for all $k \geq 0$, whereas $\nu(D(e, n)) = 0$ and $\nu(D(f, n)) > 0$ for some n.

The meaning of this definition is that the branch e, except perhaps for its beginning, is reduced to the branch f. The additional conditions appearing in (i) and (ii) ensure that we cannot have both $e > f$ and $f > e$. Observe that $e > f$ implies $e \approx f$.

We claim that if $e_0 \in \mathcal{F}'$, then it cannot be that $e_0 > e_1 > \cdots > e_r > e_0$ for some $e_1, \ldots, e_r$.

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By the definition of $\approx$, we must have, for all large k,

$$D'(e_0, n_0 + k) = D'(e_1, n_1 + k) = \cdots = D'(e_0, n_{r+1} + k)$$

for some integers $n_0 \geq n_1 \geq \cdots \geq n_{r+1}$. If $n_0 = n_{r+1}$, each relationship $e_i > e_{i+1}$ implies (ii). However, (ii) can only occur if (e_i, e_{i+1}) is some interval of T. Therefore, the cycle reduces to $e_0 > e_1 > e_0$, a contradiction to the antisymmetry of $>$. Thus, we must have $n_0 > n_{r+1}$ and $D'(e_0, n)$ is eventually periodic. In particular, this branch is finite and cannot be infinitely renormalizable. This contradiction proves the claim.

The claim we have just proved implies that to each $e \in \mathcal{F}'$ we can associate some $f \in \mathcal{F}'$ $\approx$-equivalent to e and such that there is no g with $f > g$ (call such a g irreducible).

To conclude, it is enough to see that there is no irreducible $e \in \mathcal{F}' \setminus h(\mathcal{F})$, which implies that any $e \in \mathcal{F}'$ is $\approx$-equivalent to an irreducible f which is in $h(\mathcal{F})$.

We assume by contradiction that we have an irreducible $e' = h(e) \notin h(\mathcal{F})$. If $\nu(D(e, n)) = 0$, then e would be reducible. Therefore, the irreducibility of e, gives that $D'(e', n) = h(D(e, n))$ for all $n \geq 0$. Now all the arrows of $\mathcal{D}'$ were already in $\mathcal{D}$, thus the branch e was already in $\mathcal{F}$ so that $e \in h(\mathcal{F})$, a contradiction. This proves that $\#(\mathcal{F}'/\approx) \leq \#\mathcal{F}$.

We now prove that $\#(\mathcal{F}'_+/\approx) \leq \#\mathcal{F}_+$.

Observe that if $e > f$, then e and f are both left (or both right) endpoints. Therefore, the above proof associates to each $\approx$-equivalence class which contains a left endpoint a left endpoint, of $\mathcal{F}$. However, this is the claim. □

A.2. *Proof of Proposition 15.* We give a proof of these variants of results of [**HoRa**] for the sake of completeness.

Proof of minimality of $\Sigma_\infty^{\mathcal{F}}(T)$. Let $\alpha \in \Sigma_\infty^{\mathcal{F}}(T)$ and $n < \infty$. We claim that for there exist couples $(e, m_e) \in \mathcal{F} \times \mathbb{N}$ with m_e arbitrarily large such that

$$\pi(D(e, m_e) \dots D(e, m_e + n)) = \alpha_0 \dots \alpha_n. \tag{1}$$

First we remark that the claim implies that there exists *some fixed* e such that we can find an arbitrarily large integer m_e (just use that there are only finitely many endpoints).

Second, this for *some* e implies the same for *all* $e \in \mathcal{F}$. Indeed, assume by contradiction that there is some $f \in \mathcal{F}$ such that equation (1) does not hold for an arbitrarily large integer. We may choose f such that if there are arbitrarily high paths from the branch f to some other branch $g \in \mathcal{F}$, then equation (1) is satisfied by g.

Thus there are arbitrarily high paths from the branch f to the branch g and this implies that there is a piece of the branch f which has the same projection as $D(g, 0) \dots D(g, m_g + n)$. In particular, we have $D(f, k) \to D(g, l)$ with k and, therefore, l arbitrary large. However, this implies that $D(f, k - l) \dots D(f, k)$ and $D(g, 0) \dots, D(g, l)$ have the same projection and we may assume $l > m_g + n$. Thus we have found m_f such that equation (1) holds with (f, m_f) instead of (e, m_e).

Let us see that the strengthened claim (for all e there exists an arbitrarily large integer...) implies the minimality of $\Sigma_\infty^{\mathcal{F}}(T)$. Take $\beta \in \Sigma_\infty^{\mathcal{F}}(T)$ and let us prove that there exists q such that $\beta_q \dots \beta_{q+n} = \alpha_0 \dots \alpha_n$.

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Let H be so large that all backwards arrows starting above height H in $\mathcal{F}$ which point to a vertex $D(e, m)$ in $\mathcal{F}$ satisfy $m \geq m_e + n$. This is possible because of the infinite renormalizability of $\mathcal{F}$.

By the definition of $\Sigma_\infty^{\mathcal{F}}(T)$, there exists a path γ staying higher than H on the branches $\mathcal{F}$ which projects to β. This path must go through vertices from which there is a backward arrow staying in $\mathcal{F}$ infinitely often. Let p be the second time this happens. Let h be the number of consecutive vertices without backwards arrows encountered just before time p. h is also the height of the goal of the arrow at time p. Therefore, $h \geq m_e + n$ by the choice of H. Now, $\gamma_{p-h} \dots \gamma_{p-1}$ has the same projection as $D(e, 0) \dots D(e, h)$ and, therefore, this projection contains $\alpha_0 \dots \alpha_n$. We have proved that the orbit of any $\beta \in \Sigma_\infty^{\mathcal{F}}(T)$ comes very close (n is arbitrarily large) to α. Therefore, we have proved the minimality.

We now prove the claim.

Take H_1 so large that any backward arrow included in $\mathcal{F}$ which starts above height H_1 points above height n. Take H_2 so large that any backward arrow included in $\mathcal{F}$ which starts above height H_2 points above height H_1.

Let γ be a path on $\mathcal{F}$ above level H_2 projecting on α. By the choice of H_2, $\gamma_0 \dots \gamma_n$ may go backwards only once as before an arrow pointing to height h there are h vertices without any backwards arrow.

First case. It does not go backwards. Then the claim is obvious and one can take $(e, m_e) := \gamma_0$.

Second case. It does go backwards, say at time t, to $D(f, k)$ for some $f \in \mathcal{F}$ and $k \geq 0$. However, then the projection of $D(f, k-t) \dots D(f, k-1)$ is equal to $\alpha_0 \dots \alpha_{k-1}$. Hence, $D(f, k-t) \dots D(f, k-t+n)$ projects to $\alpha_0 \dots \alpha_t \alpha_{t+1} \dots \alpha_n$, also proving the claim in this case.

This concludes the proof of minimality. □

Proof of the absence of periodic orbits. By minimality, $\Sigma_\infty^{\mathcal{F}}(T)$ can contain a periodic orbit only if it is reduced to it. However, in this case, the branch $\mathcal{F}$ is eventually periodic. In particular it is finite and cannot be infinitely renormalizable, in contrast to the assumption. □

Proof of compactness. We claim that any point in $\Sigma_\infty^{\mathcal{F}}(T)$ which is not isolated is a condensation point, i.e. any neighborhood of it contains *uncountably* many points of $\Sigma_\infty(T)$. Observe that there are infinitely many bifurcations in any branch in $\mathcal{F}$ if $\Sigma_\infty^{\mathcal{F}}(T) \neq \emptyset$. Hence, there are uncountably many paths on $\mathcal{F}$ starting from any given vertex in $\mathcal{F}$. As the paths starting from a given vertex are in a one-to-one correspondence with the projected itineraries, this implies the claim.

Let $\alpha^n \in \Sigma_\infty^{\mathcal{F}}(T)$ be a sequence converging to some α and let x^n be points with itineraries α^n. α must be a condensation point, hence we may assume that the x^n avoid the orbits of the critical points (a countable set). Let us see that $\alpha \in \Sigma_\infty^{\mathcal{F}}(T)$.

Fix $N < \infty$ very large. By definition, for each n, there is a path γ^n on $\mathcal{D}_{>N}$ which stays on the branches of $\mathcal{F}$ and projects to α^n. Let $\mathcal{C}_*$ be the closed subset generated by the intersection of $\mathcal{D}_{>n}$ with the branches in $\mathcal{F}$. By Hofbauer's Lemma 11 above, we can find a finite subset $\mathcal{D}_*$ of $\mathcal{C}_*$ and a path ω^n on $\mathcal{D}_{>n}$ starting from $\mathcal{D}_*$ and projecting to α^n. As $\mathcal{D}_*$ is

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finite and as $\mathcal{D}$ has bounded outdegree, we can assume that the paths ω^n converge to some path ω on $\mathcal{D}_{>n}$ projecting on α. We have to see that the paths ω^n stay in the branches in $\mathcal{F}$.

Observe that vertices in $\mathcal{D}_*$ can be reached from $\mathcal{D}_{>N} \cap \mathcal{F}$. Observe also that $\omega_i^n = \gamma_i^n$ for large enough i by Lemma 10 as x^n, not being on a critical orbit, is in the interior of both γ_0^n and ω_0^n. If some ω^n leaves the branches in $\mathcal{F}$, we have a path which stays high (in $\mathcal{D}_{>N}$) and which goes from inside $\mathcal{F}$ to some branch outside $\mathcal{F}$ and back (as $\omega_i^n = \gamma_i^n$ for i large). As N is arbitrarily large, this implies that the other branch is also in $\mathcal{F}$, a contradiction. □

Proof of disjointness. Let α be in the intersection. Let x be the point in I associated to α. For $i = 1, 2$, let γ^i be a path on $\mathcal{D}_{>N}$ which stays in the branches $\mathcal{F}_i$ and projects to α. Let us see that x is not an endpoint of the intervals γ_0^1, γ_0^2. Otherwise, $x = T^n(c\pm)$ for some $c\pm \in \mathcal{E}$ and some $n \geq N$. However, N is arbitrarily large, so that we can avoid this case except if $x = T^n(c\pm)$ is periodic. However, $\Sigma_\infty^{\mathcal{F}_i}(T)$ contains no periodic point.

By Lemma 10, this implies that $\gamma_n^1 = \gamma_n^2$ for all large n. Hence, $\mathcal{F}_1 = \mathcal{F}_2$. □

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