Math. Proc. Camb. Phil. Soc. (1986), **99**, 151
Printed in Great Britain

Stopping non-commutative processes

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(*Received* 28 *November* 1984; *revised* 26 *April* 1985)

0. *Introduction*

Stopping times are a powerful tool in the theory of stochastic processes, so it is natural to ask whether they have a counterpart in the theory of non-commutative processes. This paper is a *part* answer to that question. We show that the '*formalism*' of stopping times carries over to a non-commutative context and prove an Optional Stopping Theorem.

1·0. *Stopping times*

Let $F = (\Omega, \Sigma, P, (\Sigma_\alpha), \mathbb{R}^+)$ be a stochastic base [**10**]. We consider right continuous real-valued processes adapted to F. A stopping time is a Σ-measurable map $\tau\colon \Omega \to \mathbb{R}^+$ such that $\{\tau \leqslant \alpha\} \in \Sigma_\alpha$. For a process $X = (X_\alpha)$ adapted to F the stopped random variable and the process stopped by τ are defined by

$$X_\tau\colon \omega \mapsto X_{\tau(\omega)}(\omega) \quad \text{and} \quad X_{\tau \wedge \alpha}\colon (\alpha, \omega) \mapsto X_{\tau(\omega) \wedge \alpha}(\omega) \tag{1·1}$$

respectively, where $\alpha \in \mathbb{R}^+$ and $\omega \in \Omega$ and $\tau(\omega) \wedge \alpha = \min\{\tau(\omega), \alpha\}$. We consider a class of very simple stopping times – those that take only finitely many values. We shall call these finite discrete times. They will be our point of departure for the definition of a 'non-commutative time'.

Remark. The terminology 'non-commutative time' is intended to convey the context in which we work rather than some mysterious property of time.

When τ is a finite discrete time we can write $\tau = \sum_{i=0}^n t_i \chi_{E_i}$ with $E_i = \{\tau = t_i\} \in \Sigma_{t_i}$. The sets E_i are disjoint and without loss of generality we can assume

$$0 \leqslant t_0 < t_1 < \ldots < t_n.$$

So we can regard τ as a finite sequence of characteristic functions with sum χ_Ω which have associated with them a finite sequence $\{t_i\} \subseteq \mathbb{R}^+$ such that $\chi_{E_i} \in \mathscr{L}^\infty(\Sigma_{t_i})$. We can reformulate (1·1) now as

$$\chi_\tau = \sum_{i=0}^{n} \chi_{E_i} X_{t_i} \quad \text{and} \quad X_{\tau \wedge \alpha} = \sum_{t_i < \alpha} \chi_{E_i} X_{t_i} + X_\alpha (\chi_\Omega - \sum_{t_i < \alpha} \chi_{E_i}) \tag{1·2}$$

with the usual convention for empty sums. We make one more reformulation, the point of which is to display stopping as a kind of vector integral. Write $P_j = \bigcup_{i \leqslant j} E_i$ with the convention that $P_{-1} = \varnothing$. Then $E_j = P_j \backslash P_{j-1}$ and (1·2) becomes

$$X_\tau = \sum_{i=0}^{n} X_{t_i} (\chi_{P_i} - \chi_{P_{i-1}}),$$

$$X_{\tau \wedge \alpha} = \sum_{t_i < \alpha} X_{t_i} (\chi_{P_i} - \chi_{P_{i-1}}) + X_\alpha (\chi_\Omega - \sum_{t_i < \alpha} \chi_{P_i \backslash P_{i-1}}).$$

https://doi.org/10.1017/S0305004100064045 Published online by Cambridge University Press

So X_τ looks like

$$\int_0^\infty X_\beta d(\chi_{P_\beta}):$$

we shall make proper sense of this later.

At this point it is polite to note that quite a number of people have arrived at some reformulation of stopping with a view to a non-commutative generalization; among them are Hudson[9], Barnett[1], and in personal communications P. E. Kopp, F. J. Yeadon and G. Vincent-Smith.

2·0. *The non-commutative context*

We shall restrict ourselves to the Clifford probability gauge space [3]. With modification, some of our results could be extended to arbitrary filtrations of probability gauge spaces.

Let $\wedge(L^2(\mathbb{R}^+))$ be the antisymmetric Fock space over $L^2(\mathbb{R}^+)$. For $u \in L^2(\mathbb{R}^+)$, $\psi(u) \in \mathscr{B}(\wedge(L^2(\mathbb{R}^+)))$ denotes the fermion field at u. The fields satisfy the canonical anticommutation relations, i.e. for $u, v \in L^2(\mathbb{R}^+)$

$$\psi(u)\psi(v) + \psi(v)\psi(u) = 2\langle \bar{v}, u\rangle \mathbb{1}.$$

For $\alpha \geqslant 0$, A_α denotes the W^*-algebra acting on $\wedge(L^2(\mathbb{R}^2))$ generated by the fields $\psi(u)$ with u supported in $[0, \alpha]$. The W^*-algebra $A_\infty = (\bigcup_\alpha A_\alpha)''$ is called the weakly closed Clifford algebra over $L^2(\mathbb{R}^+)$ (with conjugation derived from $\mathbb{C}$) [3]. It is the uniformly hyperfinite II_1 factor. The Fock vacuum $\Omega \in \wedge(L^2(\mathbb{R}^+))$ defines a faithful normal finite tracial (vector) state ϕ on A_∞. For $1 \leqslant p < \infty$, $L^p(A_\alpha)$ $(0 \leqslant \alpha \leqslant \infty)$, will denote the non-commutative Lebesgue spaces associated with (A_α, ϕ). We note that for a fixed $p \in [1, \infty)$, $\{L^p(A_\alpha): \alpha \in \mathbb{R}^+\}$ forms an increasing family of closed subspaces of $L^p(A_\infty)$. For each fixed $\alpha \in \mathbb{R}^+$ and $p \in [1, \infty)$ we have a map $M_\alpha: L^p(A_\infty) \to L^p(A_\alpha)$ – the conditional expectation. We do not distinguish between expectations for a fixed $\alpha \in \mathbb{R}^+$ and differing $p \in [1, \infty)$ because they are all the appropriate $(\|\cdot\|_p)$ extensions of their (common) restriction to A_∞ (i.e. the conditional expectation of A_∞ onto A_α). For a fixed $p \in [1, \infty)$ a (non-commutative) L^p *process* is a family (X_α), $\alpha \in \mathbb{R}^+$, with $X_\alpha \in L^p(A_\alpha)$. A process is *simple* if it has the form $\sum_{i=1}^n X_{\alpha_{i-1}} \chi_{[\alpha_{i-1}, \alpha_i)}(\alpha)$ for

$$0 = \alpha_0 < \alpha_1 < \ldots < \alpha_n \; n \in \mathbb{N}.$$

We will be concerned mainly with L^2 processes. L^p martingales are L^p processes for which $M_\alpha(X_\beta) = X_\alpha$ for $\beta \geqslant \alpha$. The parity operator $\mathscr{P}: L^2(A_\infty) \to L^2(A_\infty)$ is defined as in 1·2 of [4]. $L^2(A_\infty)$ is a Hilbert space and $\mathscr{P}$ is a self adjoint unitary operator on $L^2(A_\infty)$.

In the next part of this section we describe some of the integrals that appear subsequently and prove a few Lemmas about certain types of processes. In order to simplify the notation we note that whilst we have defined a process as a *family*, indexed by $\mathbb{R}^+$, we will often regard a process as a *function*. So if $X = (X_\beta)$ we will write X for the function $\beta \mapsto X_\beta$. This makes sense of the symbol $X\chi_{[0,\alpha]}$ used below.

2·1. *Integration of processes*

The Itô–Clifford integral [3], [4], [5]

Let $\psi_\beta = \psi(\chi_{[0,\beta]})$; then (ψ_β) is a martingale. Let $\mathscr{H}[0, \alpha]$ denote the Hilbert space

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of (equivalence classes of) λ-measurable L^2 processes $X = (X_\beta)$ for which $X_\beta = 0$ for $\beta > \alpha$ and $\beta \mapsto \|X_\beta\|_2^2$ is λ-integrable on $[0, \alpha]$, with $\|X\|^2 = \int_0^\alpha \|X_\beta\|_2^2 d\lambda$. If X is such that $X\chi_{[0,\alpha]} \in \mathscr{H}[0,\alpha]$ for all $\alpha \in \mathbb{R}^+$ then for each $\alpha \in \mathbb{R}^+$, $\int_0^\alpha X_\beta d\psi_\beta$ exists and $(\int_0^\alpha X_\beta d\psi_\beta)$ is a martingale. For a fixed $\alpha > 0$ the integral $\int_0^\alpha X_\beta d\psi_\beta$ is obtained by approximating in the norm of $\mathscr{H}[0,\alpha]$ by simple processes and using the *isometry property*

$$\left\| \int_0^\alpha X_\beta d\psi_\beta \right\|_2^2 = \int_0^\alpha \|X_\beta\|_2^2 d\lambda.$$

This is an important feature of Itô–Clifford theory. It shows that the stochastic integral is an isometry of $\mathscr{H}[0,\alpha]$ into $L^2(\mathscr{A}_\alpha)$. There is a 'left' integral too and it is related to the 'right' integral by the parity operator, viz.

$$\int_0^\alpha d\psi_\beta X_\beta = \int_0^\alpha \mathscr{P}(X_\beta)\, d\psi_\beta.$$

Another important result is the *Martingale Representation Theorem*: If (X_α) is an L^2 martingale then there is an L^2 process $\tilde{X} = (\tilde{X}_\alpha)$ such that $\tilde{X} \cdot \chi_{[0,\alpha]} \in \mathscr{H}[0,\alpha]$ for each $\alpha \in \mathbb{R}^+$ and, for each $\alpha \in \mathbb{R}^+$,

$$X_\alpha = \int_0^\alpha \tilde{X}_\beta d\psi_\beta.$$

The representation theorem allows us to define stochastic integrals with respect to any L^2-martingale, (X_α), as in [4]. In [4] it was shown that an A_∞-valued process f which is the $\|\tilde{X}_\beta\|_2^2 d\lambda(\beta)$ – a.e. limit (in $\|\cdot\|_\infty$) of uniformly bounded processes (f_n) has a 'dX integral' given, for $\alpha \in \mathbb{R}^+$, by

$$\int_0^\alpha f(\beta)\, dX_\beta = \int_0^\alpha f(\beta)\, \tilde{X}_\beta d\psi_\beta = \lim_{n\to\infty} \int_0^\alpha f_n(\beta)\, dX_\beta,$$

where

$$\int_0^\alpha f_n(\beta) dX_\beta$$

is a finite sum of the form $\sum_i f_n(\beta_{i-1})\,(X_{\beta_i} - X_{\beta_{i-1}})$.

Our first result will be to extend the integral just (briefly) described. We denote the strong operator topology on A_∞ by σ, and the self-adjoint part of A_∞ by A_∞^h.

2·2. Lemma. *Let f: $\mathbb{R}^+ \to A^h$ be a process which is the λ – a.e. limit in σ of a sequence of uniformly bounded A_∞^h valued simple processes f_n $n = 1, 2, \ldots$. Suppose that (X_α) is an L^2 martingale and that for $\alpha \in \mathbb{R}^+$*

$$X_\alpha = \int_0^a \tilde{X}_\beta d\psi_\beta.$$

Then (a) $f\tilde{X}$ is an element of $\mathscr{H}[0,\alpha]$ for each $\alpha \in \mathbb{R}^+$.

(b) $\int_0^\alpha f_n dX \to \int_0^\alpha f dX$ *as* $n \to \infty$.

Proof. Fix $\alpha \geqslant 0$. Multiplication is jointly sequentially continuous on A_∞^1 (the unit ball) equipped with σ and hence (scaling if necessary) $(f_n(\beta) - f(\beta))^2 \to 0$ in σ for λ-almost every $\beta \in [0,\alpha]$. Since σ-convergence implies weak operator convergence and for any $\beta \in [0,\alpha]$ the functional $\phi(|\tilde{X}_\beta^*|^2 \cdot)$ is weak operator continuous on A_∞^1 then

$$\|(f_n(\beta) - f(\beta))\, \tilde{X}_\beta\|_2^2 = \phi(|\tilde{X}_\beta^*|^2 (f_n(\beta) - f(\beta))^2) \to 0 \quad \text{as} \quad n \to \infty.$$

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So $f\tilde{X}$ is the λ a.e. limit of $(f_n\tilde{X})$. Clearly each $f_n\tilde{X}$ is λ-measurable because $\tilde{X}$ is, and f_n is piecewise constant, and multiplication by an element of A_∞ is a continuous linear operation $L^2(A_\infty)\to L^2(A_\infty)$. Further, $\|f_n\tilde{X}\|_2 \leqslant M\|\tilde{X}\|_2$, where M is a uniform bound for the f_n. It follows from the dominated convergence theorem III 3·7 of [7] that $f\tilde{X}\in\mathscr{H}[0,\alpha]$ and that $\int_0^\alpha \|f_n\tilde{X}-f\tilde{X}\|_2^2\,d\lambda\to 0$ as $n\to\infty$. Hence

$$\left\|\int_0^\alpha f_n\tilde{X}\,d\psi-\int_0^\alpha f\tilde{X}\,d\psi\right\|_2=\left(\int_0^\alpha\|(f_n-f)\tilde{X}\|_2^2\,d\lambda\right)^{\frac{1}{2}}\xrightarrow[n\to\infty]{}0.$$

Using 4·2, 4·3 of [4], we have

$$\int_0^\alpha f_n(\beta)\,dX_\beta=\int_0^\alpha f_n(\beta)\,\tilde{X}_\beta\,d\psi_\beta\xrightarrow[n\to\infty]{}\int_0^\alpha f(\beta)\,\tilde{X}_\beta\,d\psi_\beta=\int_0^\alpha f(\beta)\,dX_\beta.$$

2·3. LEMMA. *Let $\tau=(P_\alpha)$ be an increasing projection-valued process (i.e. with $P_{\alpha_1}\leqslant P_{\alpha_2}$ for $\alpha_1\leqslant\alpha_2$). Then $\alpha\mapsto P_\alpha$ is λ-measurable and τ is the $\lambda-$a.e. σ-limit of* simple *projection-valued processes τ_n, each τ_n is* increasing *in that if $\tau_n=(P_\alpha^{(n)})$ then*

$$\alpha_1\leqslant\alpha_2\Rightarrow P_{\alpha_1}^{(n)}\leqslant P_{\alpha_2}^{(n)}.$$

Moreover (as a map) $\tau\in\mathscr{H}[0,\beta]$ for $\beta\geqslant 0$.

Proof. Consider $\alpha\mapsto\phi(P_\alpha)$. Since ϕ is faithful, normal and positive, and $\phi(I)=1$, it follows that $D=\{t\in[0,\beta]\colon \lim_{s\to t+}P_s\neq\lim_{s\to t-}P_s\}$ is at most countable. Let $x_1,\ldots,x_n,\ldots$ be an enumeration of D and put $D_n=\{x_1,\ldots,x_n\}$. To simplify the notation in what follows, let $k/2^n=(k)_n$ and for $\beta\in\mathbb{R}^+$ and $n\in\mathbb{N}$ let $M=[2^n\beta]$. Let (R_α) be *any* process. For each $n\in\mathbb{N}$ and $1\leqslant k\leqslant M$, we define $f_n(R_\alpha)$ to be $R_{(k-1)_n}$ on $[(k-1)_n,\ (k)_n)$ if $[(k-1)_n,\ (k)_n)\cap D_n=\varnothing$, and if this intersection is not empty, containing say $y_1,\ldots,y_j$ (in order), then we put

$$f_n(R_\alpha)=\sum_{i=0}^{j}R_{y_i}\chi_{[y_i,\,y_{i+1})}(\alpha),$$

where we have set $y_0=(k-1)_n$, $y_{j+1}=(k)_n$, and we define $f_n(R_\alpha)$ to be $R_{(M)_n}$ in the interval $[(M)_n,\ \beta]$. Let $\tau_n(\alpha)=f_n(P_\alpha)$. Clearly τ_n is a simple projection-valued process, increasing in the sense described above. Now if $\alpha\in D$ there is $n(\alpha)$ such that if $n\geqslant n(\alpha)$ then $\tau_n(\alpha)=P_\alpha$, by construction. If $\alpha\in[0,\beta]\backslash D$ then it is a point of σ-continuity and so $\tau_n(\alpha)\to P_\alpha$ in σ as $n\to\infty$ (the dyadic approximation comes from below). So $\tau_n(\alpha)\to\tau(\alpha)$ in σ for $\alpha\in[0,\beta]$. Finally $\alpha\mapsto\phi(P_\alpha)$ is clearly λ-measurable, because $\{\alpha\colon\phi(P_\alpha)\leqslant\epsilon\}$ is either a half-open interval or a closed interval, and is bounded, hence integrable, on $[0,\beta]$, for $\beta\geqslant 0$.

2·4. COROLLARY. *Any projection-valued process, increasing in the sense of* 2·3, *is locally integrable with respect to any L^2 martingale.*

2·5. *Remark.* The lemmas show that $\int_0^\alpha P_\beta\,dX_\beta$, $\int_0^\alpha dX_\beta P_\beta$ are limits in $\|\cdot\|_2$ of sums of the form $\sum_i P_{i-1}(X_i-X_{i-1})$ and $\sum_i(X_i-X_{i-1})P_{i-1}$ respectively.

Integration under standing hypotheses [5]

This theory was originally worked out by McShane [11] and extended to the non-commutative context in [5]. Let $p\in[2,\infty]$. An L^2 process $X=(X_\alpha)$ is said to satisfy

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the *standing hypotheses* (p) if there is $0 < K \in \mathbb{R}^+$ such that for each $s \in [0, T]$ and each $t \in [s, T]$ with $t - s < 1$ we have

$$\|M_s(X_t - X_s)\|_p \leqslant K(t-s), \tag{1}$$

$$\|M_s|X_t - X_s|^2\|_{p/2} \leqslant K(t-s). \tag{2}$$

For such a process, stochastic integrals of certain L^q processes, $1/p + 1/q = \frac{1}{2}$, can be defined. As usual for a *simple* process $h\colon [0,T] \to L^q(A_\infty)$ the *left* stochastic integral is defined by $\sum_i (X_{\beta_i} - X_{\beta_{i-1}})\, h(\beta_{i-1})$. The integral is now extended to the set of processes in $L^2([0,T], ds, L^q(A_\infty))$ by completion (see [5], § 2 in particular). We shall see that an increasing projection-valued process is an element of $L^2([0,T], ds, L^q(A_\infty))$ for $q \neq \infty$.

2·6. Lemma. *Let (P_n) be a sequence of projections converging in σ to the projection P. Then $\|P_n - P\|_q \to 0$ as $n \to \infty$ for $q \in [1, \infty)$.*

Proof. We have $P_n - P \to 0$ in σ. The function $z \mapsto |z|^q$ is continuous and therefore a bounded Borel function on bounded discs in $\mathbb{C}$. By theorem 2 X: 7 of [8], $|P_n - P|^q \to 0$ in σ. And since ϕ is σ-continuous, $\phi(|P_n - P|^q) \to 0$.

2·7. Lemma. (cf. 2·2 above). *Let (P_α) be an increasing projection-valued process. Then $\alpha \mapsto P_\alpha$ is an element of $L^2([0,T], ds, L^q(A_\infty))$. If $(Q_\alpha^{(n)})$ is a sequence of simple projection-valued processes converging to (P_α) pointwise in σ, then*

$$\int_0^T dx_\beta Q_\beta^n \to \int_0^T dx_\beta P_\beta \quad \text{in} \quad \|\cdot\|_2 \quad \text{as} \quad n \to \infty.$$

Proof. Choosing (P_α^n) as in 2·3 and using 2·6, we see that $\alpha \mapsto P_\alpha$ is λ-measurable (the limits being taken in $\|\cdot\|_q$) and, since $\|P_\alpha\|_q^2 = \phi(P^q)^{2/q} = \phi(P)^{2/q}$, that it is an element of $L^2([0,T], ds, L^q(A_\infty))$. So the left dX integral will exist. Once again III: 3·7 of [7] can be used to show that $\int_0^T dX_\beta P_\beta \cdot$ the $\|\cdot\|_2$ limit of $\int_0^T dX_\beta Q_\beta^0$.

Integrating with respect to an increasing projection-valued process

Let (P_α) be an increasing projection-valued process. Our reformulation of Stopping in § 1 leads us to want to define stopping in the non-commutative context as an integral with respect to some family such as (P_α). For an $L^1(A_\infty)$ valued simple process $X = (X_\alpha)$, $X_\alpha = \sum_i X_{\beta_{i-1}} \chi_{[\beta_{i-1}, \beta_i)}(\alpha)$, 1·3 indicates that we should define $\int_0^\infty X_\alpha dP_\alpha$ by

$$\sum_i X_{\beta_i}(P_{\beta_i} - P_{\beta_{i-1}}).$$

Note that the integrand is 'to the future' of the increment of (P_α). If (X_α) is not simple then we require that $\int_0^\infty X_\alpha dP_\alpha$ be a limit of sums like $\sum_i X_{\beta_i}(P_{\beta_i} - P_{\beta_{i-1}})$. The following observation shows one way in which this can be achieved. Recall the definition of $\int_0^\infty X_\alpha dP_\alpha$ when $X = (X_\alpha)$ is simple:

$$\begin{aligned}\int_0^\infty X_\beta dP_\beta = \sum_{i=1}^n X_{\beta_i}(P_{\beta_i} - P_{\beta_{i-1}}) &= X_{\beta_n} P_{\beta_n} - X_{\beta_0} P_{\beta_0} - \sum_{i=0}^{n-1} (X_{\beta_{i+1}} - X_{\beta_i}) P_{\beta_i} \\ &= [X_{\beta_n} P_{\beta_N} - X_{\beta_0} P_{\beta_0}] - \int_0^\infty dX_\beta P_\beta.\end{aligned}$$

Here we have 'integrated by parts'. The integral on the right-hand side is the 'left' stochastic integral of (P_α) with respect to (X_α).

We have indicated above that there are classes of processes for which the left

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stochastic integral can be defined. The formula above suggests that for a process (X_α) for which the $\|\cdot\|_2 - \lim_{\alpha\to\infty} X_\alpha = X$ exists and an increasing projection-valued process (P_α) we should *define* $\int_0^\infty X_\beta dP_\beta$ by

$$\int_0^\infty X_\beta dP_\beta \overset{\text{def}}{=} [XP_\infty - X_0P_0] - \int_0^\infty dX_\beta P_\beta \quad \text{where} \quad P_\infty = \sup_\alpha P_\alpha, \tag{2·8}$$

so long as the left stochastic integral on the right exists.

Integrals over $[0, T]$, $T \in \mathbb{R}^+$, are defined similarly.

In what follows, we will write $(X_\alpha) \in SH(p)$ to indicate that a process satisfies the standing hypotheses (p). If (X_α) is a martingale, all the integrals we consider will be Itô–Clifford integrals.

2·9. COROLLARY. *Let* (P_α) *and* (P^n_α), $n \in \mathbb{N}$, *be increasing projection-valued processes. Let* $(X_\alpha) \in SH(p)$. *If* $\forall \alpha P^n_\alpha \overset{\sigma}{\to} P_\alpha$, *then for* $\alpha \geqslant 0$ $\int_0^\alpha X_\beta dP^n_\beta \to \int_0^\alpha X_\beta dP_\beta$ *in* $L^2(A_\infty)$ *as* $n \to \infty$.

Proof. Given 2·8, it is enough to consider $\int_0^\alpha dX_\beta P^n_\beta$. But now this is just 2·7 rewritten.

3·0. *Non-commutative times*

The reformulation 1·3 displays stopping as a kind of vector integral with respect to (χ_{P_α}). The analogues for characteristic functions in the non-commutative context are projections, i.e. self-adjoint idempotents. Bearing this and the latter part of §2 in mind, the following definitions make some sense.

3·1. *Definitions*

(i) A (stopping) time, τ, is an increasing family of projections, (P_α), $\alpha \in \mathbb{R}^+$, such that $P_\alpha \in A_\alpha$. We say that τ is discrete if $P_\alpha = \sum_{i=0}^\infty P_{\beta_i} \chi_{[\beta_i, \beta_{i+1})}(\alpha)$ for some sequence (β_i) with $0 = \beta_0 < \beta_1 < \ldots < \beta_n \uparrow \infty$.

(ii) *Let* $X = (X_\alpha)$ be a process; then for any time τ,

$$X_\tau \overset{\text{def}}{=} \int_0^\alpha X_\beta dP_\beta \quad X_{\tau\wedge\alpha} \overset{\text{def}}{=} \int_0^\alpha X_\beta dP_\beta + X_\alpha(1 - P_\alpha) \quad \alpha \in \mathbb{R}^+.$$

(iii) Let $\sigma = (Q_\alpha)$ and $\tau = (P_\alpha)$ be times. We write $\sigma \leqslant \tau$ if $\forall \alpha$, $P_\alpha \leqslant Q_\alpha$.

(iv) Let $\sigma = (Q_\alpha)$ be a time.

$$A_\sigma \underset{\text{def}}{=} \{P \in A_\infty : P \text{ is a projection and } PQ_\alpha \in A_\alpha \forall \alpha\}'',$$

where for a subset B of $\mathscr{B}(\wedge(R^+))$, B'' denotes the second commutant of B in $\mathscr{B}(\wedge(\mathbb{R}^+))$. If $B = B^*$ it is the W^* algebra generated by B.

3·2. *Remarks.* (*a*) We call X_τ the stopped operator and $(X_{\tau\wedge\alpha})$ the process X stopped by the time τ.

(*b*) The rationale for (iii) is that if σ and τ are stopping times associated to the stochastic base F in §1 then $\sigma(w) \leqslant \tau(w)\, \forall w \in \Omega$ iff $\{\tau \leqslant \alpha\} \subseteq \{\sigma \leqslant \alpha\}$, $\forall \alpha$.

(*c*) A_σ corresponds to the sigma-algebra of 'events known at the time σ', i.e. $\Sigma_\sigma = S\{A \in \Sigma : A \cap \{\sigma \leqslant \alpha\} \in \Sigma_\alpha\}$ ($S(M)$ is the sigma-algebra generated by M) so A_σ is an *algebraic* generalization rather than a lattice-theoretic one.

(*d*) In [9] stopping is defined (essentially) as in 3·1 (ii) above. The notion of a time is

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a little more restrictive, requiring that the projections form a resolution of the identity.

The integrands are unitary operator valued and satisfy some commutation relation with the time. The emphasis in [9] was to regard the integral with respect to a time as a kind of spectral integral and so the connection with stochastic integration was not made.

3·3. PROPOSITION. *Let* $\sigma = (Q_\alpha)$ *and* $\tau = (P_\alpha)$ *be times. Suppose that* $\sup_\alpha P_\alpha = I$.

(i) *If* $\sigma \leqslant \tau$ *then* $A_\sigma \subseteq A_\tau$.

(ii) *Let* $X = (X_\alpha)$ *be an* L^2 *martingale. Then* $y_\alpha = X_{\tau \wedge \alpha}$ $(\alpha \in R^+)$ *defines an* L^2 *martingale. If* X *is* L^2 *bounded, then* $X_{\tau \wedge \alpha} \to X_\tau$ *as* $\alpha \to \infty$ *in* $\|\cdot\|_2$.

Proof. (i) Let $R \in \{R \in A_\infty,\ R = R^* = R^2,\ \forall \alpha RQ_\alpha \in A_\alpha\}$. Then for all α

$$RP_\alpha = RQ_\alpha P_\alpha = (RQ_\alpha) P_\alpha \in A_\alpha$$

since each of RQ_α and P_α are in A_α. So

$$\{R \in A_\infty : R = R^* = R^2, \quad \forall \alpha RQ_\alpha \in A_\alpha\} \subseteq \{R \in A_\infty,\ R = R^* = R^2,\ \forall \alpha RP_\alpha \in A_\alpha\}.$$

Thus $A_\sigma \subseteq A_\tau$ (take commutants twice).

(ii) Since X is an L^2 martingale, we have by definition

$$y_\alpha = \int_0^\alpha X_\beta dP_\beta + X_\alpha(1 - P_\alpha) = X_\alpha P_\alpha - X_0 P_0 - \int_0^\alpha dX_\beta P_\beta + X_\alpha(1 - P_\alpha)$$

$$= X_\alpha - X_0 P_0 - \int_0^\alpha dX_\beta P_\beta.$$

So if $\gamma \leqslant \alpha$

$$M_\gamma(y_\alpha) = M_\gamma(X_\alpha) - X_0 P_0 - M_\gamma\left(\int_0^\alpha dX_\beta P_\beta\right)$$

$$= X_\gamma - X_0 P_0 - \int_0^\gamma dX_\beta P_\beta = y_\gamma,$$

since (X_α) and $(\int_0^\alpha dX_\beta P_\beta)$ are martingales. Thus (y_α) is a martingale. If X is L^2-bounded, then the map $\alpha \mapsto \tilde{X}_\alpha$ is in $\mathscr{H}(\mathbb{R}^+)$ and hence the map $\alpha \to \tilde{X}_\alpha P_\alpha$ is in $\mathscr{H}(\mathbb{R}^+)$. So $\lim_{\alpha \to \infty} \int_0^\alpha dX_\beta P_\beta = \int_0^\infty dX_\beta P_\beta$. Since $P_\alpha \uparrow I$, $\|X_\alpha(1 - P_\alpha)\|_2 \to 0$ as $\alpha \to \infty$, and the result follows.

3·4. *A remark about the definition of* A_σ

Recall definition 3·1 (iv). Since this definition preserves the algebraic character of Σ_σ, why not, as in [9], define

$$A_{\sigma^2} = \{P \in A_\infty,\ P = P^* = P^2,\ PQ_\alpha = Q_\alpha P \in A_\alpha \forall_\alpha\}''$$

as the analogue of Σ_σ? The consequences of adopting this definition suggest it may not be appropriate. Consider a discrete parameter filtration such as (A_n) $n \in N$ and think of a time as a sequence on *disjoint* adapted projections with sum I. A_{σ^2} is defined as above with the obvious trivial modifications. Let $\tau = (I, 0, \ldots)$ and $\sigma = (I - Q, Q, 0, \ldots)$. Clearly $\tau \leqslant \sigma$ (by adding up the projections up to the kth place and comparing $k = 1, 2, \ldots$). Now A_1 *is* non-abelian and $A_1 = A_{\tau^2}$. Thus there is $R \in A_1$ and $Q \in A_1$, Q a projection, with $RQ \neq QR$ (for if not, A_1 is abelian). With this choice of Q we have $\tau \leqslant \sigma$ but $A_{\tau^2} \not\subseteq A_{\sigma^2}$. Another feature appears when the filtration has some degeneracy.

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Suppose $A_n = A_3$ for $n \geqslant 3$. Let $\tau = (P_1, P_2, P_3, 0, \ldots)$ and $\sigma = (P_1, P_3, P_2, 0, 0, \ldots)$. Then $\sigma \neq \tau$ but $A_{\sigma^2} = A_{\tau^2}$.

3·5. THEOREM (Optional Stopping for L^2 martingales). *Let $\sigma = (Q_\alpha)$ and $\tau = (P_\alpha)$ be times with $\sigma \leqslant \tau$ and $\sup_\alpha P_\alpha = P_\infty = Q_\infty = \sup_\alpha Q_\alpha$. Then for each L^2-bounded martingale $X = (X_\alpha)$ we have*

$$M_\sigma(X_\tau) = X_\sigma \quad \textit{where} \quad M_\sigma(A) = \int_0^\infty M_\beta(A)\, dQ_\beta \quad \textit{for} \quad A \in L^2(A_\infty).$$

Proof. By definition M_σ is a linear operation, so as $X_\tau = (XP_\infty - X_0P_0) - \int_0^\infty dX_\gamma P_\gamma$ then

$$M_\sigma(X_\tau) = \int_0^\infty M_\beta(XP_\infty)\, dQ_\beta - \int_0^\infty M_\beta(X_0P_0)\, dQ_\beta - \int_0^\infty M_\beta\left(\int_0^\infty dX_\gamma P_\gamma\right) dQ_\beta.$$

Now

$$\begin{aligned}\int_0^\infty M_\beta(XP_\infty)\, dQ_\beta &= (XP_\infty)\, Q_\infty - M_0(XP_\infty)\, Q_0 - \int_0^\infty dM_\beta(XP_\infty)\, Q_\beta \\ &= XQ_\infty - X_0Q_0 - \int_0^\infty dM_\beta(X)\, Q_\beta,\end{aligned}$$

because $P_\infty = Q_\infty$ and $P_\infty Q_0 = Q_0$ and $P_\infty Q_\beta = Q_\beta$ (the change in the integral is achieved via Remark 2·5). So $\int_0^\infty M_\beta(XP_\infty)\, dQ_\beta = X_\sigma$ and it remains to show that the other terms are zero. As $P_0 \leqslant Q_0 \leqslant Q_\infty$ we have $\int_0^\infty M_\beta(X_0P_0)\, dQ_\beta = 0$. Also

$$\int_0^\infty \left(\int_0^\beta dX_\gamma P_\gamma\right) dQ_\beta = \left(\int_0^\infty dX_\gamma P_\gamma\right) Q_\infty - 0 \cdot Q_0 - \int_0^\infty dX_\beta P_\beta Q_\beta$$

by the Radon–Nikodým theorems 4·2, 4·3 of [4]. Now $P_\beta Q_\beta = P_\beta$, and using Remark 2·5 we see that $(\int_0^\infty dX_\gamma P_\gamma)\, Q_\infty = \int_0^\infty dX_\gamma P_\gamma$. Hence $\int_0^\infty (\int_0^\beta dX_\gamma P_\gamma)\, dQ_\beta$ is zero.

3·6. *Remark.* In the classical ('commutative') case the map $M_\tau(\cdot)$ is precisely the conditional expectation on to A_τ (see below). We note that, for an L^2-bounded martingale $X = (X_\alpha)$ with limit X, stopping by τ is identical with the action of $M_\tau(\cdot)$ on X.

The following result seems to say that if a sequence of times τ_n converge in distribution to τ then X_{τ_n} converges to X_τ in L^2 norm. We assume that all the times τ, τ_n have a common 'supremum' (which may as well be I).

3·7. THEOREM. *Let $\tau, \tau_n, n = 1, 2, \ldots,$ be times with $\tau = (P_\alpha)$, $\tau_n = (P_\alpha^{(n)})$ and $\forall \alpha\ P_\alpha^{(n)} \to P_\alpha$. Then for each L^2-bounded martingale $X = (X_\alpha)$, $X_{\tau_n} \to X_\tau$, in $L^2(A_\infty)$.*

Proof. $\forall \alpha\ P_\alpha^{(n)} \to P_\alpha$, so $\forall \alpha\ \|P_\alpha^{(n)} - P_\alpha\|_2 \to 0$ as $n \to \infty$; thus $\int_0^\infty dX_\beta P_\beta^{(n)} \to \int_0^\infty dX_\beta P_\beta$ by 'dominated convergence' (see 2·7). Hence

$$\begin{aligned}X_{\tau_n} &= X(\sup_\alpha P_\alpha^{(n)}) - X_0P_0^{(n)} - \int_0^\infty dX_\beta P_\beta^{(n)} \\ &\to X(\sup_\alpha P_\alpha) - X_0P_0 - \int_0^\infty dX_\beta P_\beta \quad \text{in} \quad \|\ \|_2 \quad \text{as} \quad n \to \infty;\end{aligned}$$

but this is just X_τ.

The Map $M_\sigma(\cdot)$

We recall that $M_\sigma(B) = \int_0^\infty M_\beta(B)\, dQ_\beta$ if $\sigma = (Q_\alpha)$. We now show that although M_σ fails to be a conditional expectation it does share some of the features of an expectation.

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3·8. PROPOSITION. *Let $\sigma = (Q_\alpha)$ be a time. Then*

(i) *$M_\sigma(\cdot)$ is an L^2 contraction, and $M_\sigma(I) = I(\sup_\alpha Q_\alpha - Q)$.*

(ii) *M_σ is self-adjoint.*

(iii) *M_σ is a projection.*

(iv) *Let $C_\sigma = \{T \in A_\infty : \forall \alpha\, TQ_\alpha = Q_\alpha T\}$. Then $M_\sigma \restriction C_\sigma$ is a conditional expectation provided $\sup_\alpha Q_\alpha = I$.*

(v) *Let $B \in L^2(A_\infty)$. Then $\{M_\tau(B) : \tau$ is a time$\}$ is weakly relatively compact in $L^1(A_\infty)$.*

Proof. (i) By definition $M_\sigma(B) = \int_0^\infty M_\beta(B)\, dQ_\beta$ and the right-hand side is the limit in $\| \|_2$ of a sequence (S^n) of sums of the form $S^n = \sum_i M_{\beta_i}(B)(Q_{\beta_i} - Q_{\beta_{i-1}})$. Now

$$\begin{aligned}\|S^n\|_2^2 = \phi(S^{n*}S^n) &= \phi((\sum_j M_{\beta_j}(B)(Q_{\beta_j} - Q_{\beta_{j-1}}))^*(\sum_i M_{\beta_i}(B)(Q_{\beta_i} - Q_{\beta_{i-1}})))\\ &= \phi(\sum_j \sum_i (Q_{\beta_j} - Q_{\beta_{j-1}}) M_{\beta_j}(B^*) M_{\beta_i}(B)(Q_{\beta_i} - Q_{\beta_{i-1}}))\\ &= \sum_i \phi((Q_{\beta_i} - Q_{\beta_{i-1}}) M_{\beta_i}(B^*) M_{\beta_i}(B)(Q_{\beta_i} - Q_{\beta_{i-1}})),\end{aligned}$$

because when $i \neq j$ $(Q_{\beta_i} - Q_{\beta_{i-1}})(Q_{\beta_j} - Q_{\beta_{j-1}}) = 0$ (recall that $\phi(BT) = \phi(TB)$ for B, $T \in L^2(A_\infty)$). Since $M_{\beta_i}(B^*) M_{\beta_i}(B) \leqslant M_{\beta_i}(B^*B)$ and $\phi(\cdot)$ is positive, we have

$$\|S^n\|_2^2 < \sum_i \phi((Q_{\beta_i} - Q_{\beta_{i-1}}) M_{\beta_i}(|B|^2)(Q_{\beta_i} - Q_{\beta_{i-1}})),$$

which, as $Q_{\beta_i} - Q_{\beta_{i-1}}$ is a projection in A_{β_i}, is equal to

$$\sum_i \phi((Q_{\beta_i} - Q_{\beta_{i-1}})|B|^2 = \phi(|B|^2(\sup_\alpha Q_\alpha - Q_0)) \leqslant \phi(|B|^2) = \|B\|_2^2,$$

so that $\|M_\sigma(B)\|_2 < \|B\|_2$. Clearly $M_\sigma(I) = \sup_\alpha Q_\alpha - Q_0$.

(ii) By considering sums such as S^n in (i), it is not difficult to see with the help of 2·3 that, for $Y \in L^2(A_\infty)$,

$$\begin{aligned}\langle M_\sigma(B), Y\rangle = \phi(Y^* M_\sigma(B)) &= \phi\left(Y^* \int_0^\infty M_\beta(B)\, dQ_\beta\right)\\ &= \lim_n \phi\left(Y^* \int_0^\infty f_n(B)\, dQ_\beta\right),\end{aligned}$$

where $f_n(\cdot)$ is the *function* that defines τ_n in 2·3. The integration by parts dualizes this into $-\int_0^\infty dM_\beta(B) f_n(Q_\beta)$, which we know to converge to $-\int_0^\infty dM_\beta(B)\, Q_\beta$.

Now

$$\begin{aligned}\phi\left(Y^* \int_0^\infty f_n(B)\, dQ_\beta\right) = \phi\left(B \int_0^\infty dQ_\beta f_n(Y^*)\right) &= \phi\left(B\left(\int_0^\infty f_n(Y)\, dQ_\beta\right)^*\right)\\ &= \left\langle B, \int_0^\infty f_n(Y)\, dQ_\beta\right\rangle \to \langle B, M_\sigma(Y)\rangle \quad \text{as} \quad n \to \infty.\end{aligned}$$

So M_σ is self-adjoint.

(iii) $M_\sigma(B) = BQ_\infty - B_0 \cdot Q_0 - \int_0^\infty dB_\beta Q_\beta$ (B_α is just $M_\alpha(B)$). So

$$M_\alpha(M_\sigma(B)) = M_\alpha(BQ_\infty) - B_0 \cdot Q_0 - \int_0^\alpha dB_\beta Q_\beta.$$

Now let $S = \sum_i M_{\alpha_i}(M_\sigma(B))(Q_{\alpha_i} - Q_{\alpha_{i-1}})$; then

$$S = \sum_i \left\{M_{\alpha_i}(BQ_\infty)(Q_{\alpha_i} - Q_{\alpha_{i-i}}) - B_0 \cdot Q_0(Q_{\alpha_i} - Q_{\alpha_{i-1}}) - \left(\int_0^{\alpha_i} dB_\beta Q_\beta\right)(Q_{\alpha_i} - Q_{\alpha_{i-1}})\right\}.$$

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But

$$M_{\alpha_i}(BQ_\infty)(Q_{\alpha_i}-Q_{\alpha_{i-1}}) = B_{\alpha_i}(Q_{\alpha_i}-Q_{\alpha_{i-1}})$$

and

$$B_0\cdot Q_0\cdot(Q_{\alpha_i}-Q_{\alpha_{i-1}}) = 0,$$

so

$$S = \sum_i B_{\alpha_i}(Q_{\alpha_i}-Q_{\alpha_{i-1}}) - \sum_i \left(\int_0^{\alpha_i} dB_\beta Q_\beta\right)(Q_{\alpha_i}-Q_{\alpha_{i-1}}).$$

Now $M_\sigma(M_\sigma(B))$ is a limit of sums just like this. So it is enough to show that

$$\left(\int_0^\infty dB_\beta Q_\beta\right)Q_\infty - 0\cdot Q_0 - \int_0^\infty d\left(\int_0^\infty dB_\gamma Q_\gamma\right)Q_\beta = 0.$$

First observe that $(\int_0^\infty dB_\beta Q_\beta)\,Q_\infty = \int_0^\infty dB_\beta Q_\beta$; then use the Radon–Nikodým theorems 4·2 and 4·3 of [4] to confirm that

$$\int_0^\infty d\left(\int_0^\beta dB_\gamma Q_\gamma\right)Q_\beta = \int_0^\infty dB_\beta Q_\beta\cdot Q_\beta = \int_0^\infty dB_\beta Q_\beta.$$

(iv) If $TQ_\alpha = Q_\alpha T\ \forall\alpha$, then for a fixed α and every $\beta\leqslant\alpha$ we have

$$M_\alpha(T)\,Q_\beta = M_\alpha(TQ_\beta) = M_\alpha(Q_\beta T) = Q_\beta M_\alpha(T),$$

and hence $M_\alpha(T^*)\,Q_\beta = Q_\beta M_\alpha(T^*)$ for $\beta\leqslant\alpha$.

$$M_\sigma(B^*) = \lim_n \sum_{i=1}^{j(n)} M_{\beta_i^n}(B^*)(Q_{\beta_i^n}-Q_{\beta_{i-1}^n})$$

say, and this is equal to

$$\lim_n \left(\sum_{i=1}^{j(n)}(Q_{\beta_i^n}-Q_{\beta_{i-1}^n})\right)M_{\beta_i^n}(B^*)$$

by the first part, which is

$$\lim_n \left(\sum_{i=1}^{j(n)} M_{\beta_i^n}(B)(Q_{\beta_i^n}-Q_{\beta_{i-1}^n})\right)^* = M_\sigma(B)^*,$$

so $M_\sigma(\cdot)$ preserves adjoints. Clearly $M(I) = I$ and is an L^2 contraction. If $X, Y\in C_\sigma$ then $M_\sigma(X)$ and $M_\sigma(Y)$ are both limits of sums of the form

$$\sum_{i=1}^{j(n)} M_{\beta_i^n}(T)(Q_{\beta_i^n}-Q_{\beta_{i-1}^n})$$

where we put $T = X$ or Y and we need not change the β_i^n. So

$$M_{\sigma_n}(X)\,M_{\sigma_n}(Y) = \left(\sum_{i=1}^{j(n)}(M_{\beta_i^n}(X)(Q_{\beta_i^n}-Q_{\beta_{i-1}^n})\right)\left(\sum_{k=1}^{j(n)} M_{\beta_k^n}(Y)(Q_{\beta_k^n}-Q_{\beta_{k-1}^n})\right)$$
$$= \sum_{i=1}^{j(n)} M_{\beta_i^n}(M_{\beta_i^n}(X)\,Y)(Q_{\beta_i^n}-Q_{\beta_{i-1}^n})$$

by the first part and the 'orthogonality of the increments of (Q_α)'. But this is

$$M_{\sigma_n}(M_{\sigma_n}(X)\,Y) = \sum_{i=1}^{j(n)} M_{\beta_i^n}\left(\left(\sum_{k=1}^{j(n)} M_{\beta_k^n}(X)(Q_{\beta_k^n}-Q_{\beta_{k-1}^n})\right)Y\right)(Q_{\beta_i^n}-Q_{\beta_{i-1}^n}).$$

Now the situation we have is precisely this: we can choose discrete times σ_n as in 2·3 so that those sums above correspond to $M_{\sigma_n}(\,.\,)$ and the σ_n converge pointwise to σ in

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the strong operator topology. By 3·7 $M_{\sigma_n}(T) \to M_\sigma(T)$ for $T \in L^2(A_\infty)$. But each M_{σ_n} is $\| \|_2$ continuous. So put $\Delta = \|M_{\sigma_n}(M_{\sigma_n}(X)\, Y) - M_\sigma(M_\sigma(X)\, Y)\|_2$. Then

$$\begin{aligned}\Delta &\leqslant \|M_{\sigma_n}(M_{\sigma_n}(X)\, Y) - M_{\sigma_n}(M_\sigma(X)\, Y)\|_2 + \|M_{\sigma_n}(M_\sigma(X)\, Y) - M_\sigma(M_\sigma(X)\, Y\|_2 \\ &\leqslant \|M_{\sigma_n}(X)\, Y - M(X)\, Y\|_2 + \|M_{\sigma_n}(M_\sigma(X)\, Y) - M_\sigma(M_\sigma(X)\, Y)\|_2 \to 0 \quad \text{as} \quad n \to \infty.\end{aligned}$$

Thus $M_\sigma(M_\sigma(X)\, Y) = M_\sigma(X)\, M_\sigma(Y)$ (the latter is the $\| \|_1$-limit of $M_{\sigma_n}(X)\, M_{\sigma_n}(Y)$, which must agree with the $\| \|_2$ limit just established). So M_σ is an identity and adjoint preserving average, hence [2] it is a conditional expectation.

(v) Since $M_\tau(\cdot)$ is an L^2 contraction $\{M_\tau(B)\colon \tau \text{ is a time}\}$ is L^2-bounded and hence it is weakly relatively compact in L^1.

4·0. *An example of a time*

4·1. Proposition (See [6]). *Let $Q \in A_\infty$ be a projection. For $0 < \alpha < \infty$ let*

$$Q_\alpha = \chi_{\{1\}}(M_\alpha(Q));$$

then (Q_α) is a time. Moreover, Q_α is the greatest projection in A_α which is smaller than Q.

Proof. First note that $M_\alpha(Q_\alpha(I-Q)Q_\alpha) = Q_\alpha(I - M_\alpha(Q))Q_\alpha = 0$ because the function $X \to X\chi_{\{1\}}(X)$ is just the function $\chi_{\{1\}}(X)$. Since $Q_\alpha(I-Q)Q_\alpha \geqslant 0$ and M_α is faithful we have $Q_\alpha(I-Q)Q_\alpha = 0$, hence $(I-Q)Q_\alpha = 0$. We conclude that $Q_\alpha \leqslant Q$. If $F \in A_\alpha$ and $F \leqslant Q$ then $FM_\alpha(Q) = M_\alpha(FQ) = F$, so F commutes with $M_\alpha(Q)$. We can represent F and $M_\alpha(Q)$ with respect to a common spectral measure U say. We have $F = U_{\chi_B}$ and $M_\alpha(Q) = U_f$ and $U_{\chi_B} U_{\chi_f} = U_{\chi_B}$; thus $\chi_B f = \chi_B$ (U − a.e.) and hence $B \subseteq \{f = 1\}$ (up to U-null sets). Hence $U_{\chi_B} \leqslant U_{\{f=1\}}$ as projections, but $U_{\{f=1\}} = \chi_{\{1\}}(M_\alpha(Q))$ and so $F \leqslant Q_\alpha$. So it remains to show that (Q_α) increases. But this is obvious, since the A_α increase and Q_α is maximal in A_α subject to being smaller than Q.

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