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Isometries of self-adjoint complex function spaces

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1. *Introduction*

By a *complex function space* A we will mean a uniformly closed linear space of continuous complex-valued functions on a compact Hausdorff space X, such that A contains constants and separates the points of X. We denote by S the *state-space*

$$\{\phi \in A^* : \phi(1) = 1 = \|\phi\|\}$$

endowed with the w^*-topology. If A is self-adjoint then it is well known (cf. [1]) that A is naturally isometrically isomorphic to $A_{\mathbf{C}}(S)$, and re A is naturally isometrically isomorphic to $A(S)$, where $A_{\mathbf{C}}(S)$ (respectively $A(S)$) denotes the Banach space of all complex-valued (respectively real-valued) continuous affine functions on S with the supremum norm.

We consider the question whether the existence of an isometric isomorphism between two self-adjoint complex function spaces A_1, A_2 implies the existence of an isometric isomorphism between re A_1 and re A_2. In other words, if $A_{\mathbf{C}}(S_1)$ and $A_{\mathbf{C}}(S_2)$ are isometrically isomorphic are $A(S_1)$ and $A(S_2)$ isometrically isomorphic? We show that in general the answer is negative and we give additional conditions on A_j in order to get a positive answer.

For a complex function space A we write $Z = \operatorname{co}(S \cup -iS)$ in A^* for the *complex state space* of A. We also consider whether the existence of an isometric isomorphism between $A(Z_1)$ and $A(Z_2)$ for two self-adjoint complex function spaces A_1, A_2 implies that re A_1 and re A_2 are isometrically isomorphic.

2. *Isometries between* A_1 *and* A_2

If K_1 and K_2 are compact convex subsets of a locally convex Hausdorff space then $A_{\mathbf{C}}(K_1)$ and $A_{\mathbf{C}}(K_2)$ are self-adjoint complex function spaces. If K_1 and K_2 are affinely homeomorphic then it is easy to show that $A_{\mathbf{C}}(K_1)$ and $A_{\mathbf{C}}(K_2)$ (also $A(K_1)$ and $A(K_2)$) are isometrically isomorphic; an example of J. T. Chan (see Lemma 2 below) shows that the converse statement is false even in finite dimension.

By the *centre* of $A_{\mathbf{C}}(K)$ we will mean the set of functions f in $A_{\mathbf{C}}(K)$ such that whenever $g \in A_{\mathbf{C}}(K)$ there exists an $h \in A_{\mathbf{C}}(K)$ with $h(x) = f(x)\,g(x)$ for all $x \in \overline{\partial K}$, where ∂K denotes the set of extreme points of K. The centre of $A(K)$ is defined analogously, and it is clear that the centre of $A_{\mathbf{C}}(K)$ is just the complex-linear span of the centre of $A(K)$. If the centres consist only of the constant functions then $A(K)$ and $A_{\mathbf{C}}(K)$ are called *antisymmetric*. The sets of constancy for the centre of $A(K)$ restricted to $\overline{\partial K}$ form a decomposition of $\overline{\partial K}$ into closed sets, called the *Silov decomposition* for $A(K)$ (see [1], p. 212).

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We note that Rao[5] has studied the above problem in the case of simplexes K_1, K_2 (see Corollary 5 below), and for general K_1, K_2 has shown that if an isometric isomorphism maps the centre of $A_C(K_1)$ into the centre of $A_C(K_2)$ then K_1 and K_2 are affinely homeomorphic. That some restriction on the isometry is necessary is shown by the following result.

THEOREM 1. *If $A(K_1)$ and $A(K_2)$ are isometrically isomorphic then $A_C(K_1)$ and $A_C(K_2)$ are isometrically isomorphic. There exist compact convex sets K_1, K_2 such that $A_C(K_1)$ and $A_C(K_2)$ are isometrically isomorphic while $A(K_1)$ and $A(K_2)$ are not isometrically isomorphic.*

Proof. Given an isometric isomorphism $T: A(K_1) \to A(K_2)$, by standard procedures we may construct an isometric isomorphism $T': A_C(K_1) \to A_C(K_2)$ by defining $T'(u+iv) = Tu + iTv$ for $u, v \in A(K_1)$. T' is clearly a complex-linear bijection. Given u, $v \in A(K_1)$, there exists a point $y_0 \in \partial K_2$ such that $\|T'(u+iv)\| = |Tu(y_0) + iTv(y_0)|$, and since $T^*y_0 = \pm x_0$ for some $x_0 \in \partial K_1$ we obtain

$$\|T'(u+iv)\| = |u(x_0) + iv(x_0)| \leqslant \|u+iv\|.$$

On the other hand $\|u+iv\| = |u(x_1)+iv(x_1)|$ for some $x_1 \in \partial K_1$, and since $x_1 = \pm T^*y_1$ for some $y_1 \in \partial K_2$ we obtain

$$\|u+iv\| = |Tu(y_1) + iTv(y_1)| \leqslant \|T'(u+iv)\|.$$

Consequently T' is an isometry.

Before constructing the example required for the second part of the theorem we will prove an elementary lemma.

LEMMA 2. *Let K be the compact convex set in $\mathbb{R}^3$ with extreme points 0, e_1, e_2, e_3, e_1+e_3, e_2+e_3, where e_j denote the coordinate vectors. Let $l_1(x, y, z) = 1-2z$, $l_2(x, y, z) = 1-2x$ and let $L_j = \{l_j f : f \in A(K) \mid \partial K\}$ for $j = 1, 2$. Then the state spaces of L_1 and L_2 are not affinely homeomorphic.*

Proof. K is a triangular prism in $\mathbb{R}^3$ and the functions in $A(K)$ have the form

$$f(x, y, z) = \alpha + \beta x + \gamma y + \delta z,$$

α, β, γ, δ being real constants. Hence at the vertices of K, in the order given, f takes the values $\alpha, \alpha+\beta, \alpha+\gamma, \alpha+\delta, \alpha+\beta+\delta, \alpha+\gamma+\delta$. Since $l_j \in A(K)$ and $l_j^2 = 1$ on ∂K for $j = 1, 2$, the function spaces L_j contain constants and separate points of ∂K, and each L_j is isometrically isomorphic to $A(K) \mid \partial K$.

For f of the form given above, $l_1 f$ takes the respective values α, $\alpha+\beta$, $\alpha+\gamma$, $-\alpha-\delta$, $-\alpha-\beta-\delta$, $-\alpha-\gamma-\delta$ on ∂K, while $l_2 f$ takes the respective values α, $-\alpha-\beta$, $\alpha+\gamma$, $\alpha+\delta$, $-\alpha-\beta-\delta$, $\alpha+\gamma+\delta$ on ∂K. Writing $\delta' = -2\alpha-\delta$, we see that $l_1 f$ takes the values $\alpha, \alpha+\beta, \alpha+\gamma, \alpha+\delta', \alpha-\beta+\delta', \alpha-\gamma+\delta'$, and writing $\beta' = -2\alpha-\delta$, we see that $l_2 f$ takes the values α, $\alpha+\beta'$, $\alpha+\gamma$, $\alpha+\delta$, $\alpha+\beta'-\delta$, $\alpha+\gamma+\delta$. Consequently it is evident that L_j may be represented as $A(K_j)$, where K_1 has extreme points 0, e_1, e_2, e_3, $-e_1+e_3$, $-e_2+e_3$, and K_2 has extreme points 0, e_1, e_2, e_3, e_1-e_3, e_2+e_3.

It may now be verified that all two-dimensional faces of K_1 are triangles while K_2 contains square faces. Therefore K_1 and K_2 are not affinely homeomorphic. Similarly K and K_1 are not affinely homeomorphic.

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Returning to the proof of Theorem 1, we write $X = \Gamma_1 \cup \Gamma_2 \cup \Gamma_3$, where the Γ_j are disjoint copies of the unit circle in the complex plane. The points in Γ_1, Γ_2, Γ_3 corresponding to $e^{i\alpha}$ we write as x_α, y_α, z_α respectively. For each α in $[0, \pi)$ we denote by K_α a triangular prism affinely homeomorphic to K in Lemma 2, for which the vertices x_α, y_α, z_α, $x_{\alpha+\pi}$, $y_{\alpha+\pi}$, $z_{\alpha+\pi}$ correspond to 0, e_1, e_2, e_3, e_1+e_3, e_2+e_3 respectively. We denote by L_α the restriction of $A_{\mathbb{C}}(K_\alpha)$ to the vertex set ∂K_α.

Define $L = \{f \in C_{\mathbb{C}}(X) : f|K_\alpha \in L_\alpha,\ 0 \leqslant \alpha < \pi\}$. Then L is a self-adjoint complex function space on X. Since $A(K_\alpha)$ is antisymmetric it is clear that the sets of the Šilov decomposition for L are $\{\partial K_\alpha : 0 \leqslant \alpha < \pi\}$.

Let $\lambda \in C_{\mathbb{C}}(X)$ satisfy

$$\lambda(x_\alpha) = \lambda(y_\alpha) = \lambda(z_\alpha) = e^{i\alpha}$$

for $0 \leqslant \alpha < 2\pi$ and define $M = \{\lambda f : f \in L\}$. Since $\lambda \in L$, $|\lambda| = 1$ and $\bar{\lambda}^2$ belongs to the centre of L it is straightforward to verify that M is a self-adjoint complex function space isometrically isomorphic to L. Moreover the sets in the Šilov decomposition for M are also $\{\partial K_\alpha : 0 \leqslant \alpha < \pi\}$. However, in the state space of M the points in ∂K_2 correspond to the vertices of a compact convex set K'_α which is affinely homeomorphic to the set K_1 in Lemma 2.

The theorem will be proved if we show that $\operatorname{re} L$ and $\operatorname{re} M$ are not isometrically isomorphic. If an isometric isomorphism between $\operatorname{re} L$ and $\operatorname{re} M$ exists, then there exist a homeomorphism $\rho : X \to X$ and a function $l \in C_{\mathbb{R}}(X)$ with $|l| = 1$ such that $M = \{l(f \circ \rho) : f \in L\}$. Without loss of generality, using connectedness, we may assume that either $l = 1$ on X or $l = 1$ on two of the circles Γ_j with $l = -1$ on the remaining circle.

If $l = 1$ on X then $M = \{f \circ \rho : f \in L\}$, and so the mapping $Tf = f \circ \rho$ is an isometric isomorphism between $\operatorname{re} L$ and $\operatorname{re} M$ mapping 1 into 1. Consequently T^* restricts to an affine homeomorphism between the state spaces of M and L. But then K_α and K'_α must be affinely homeomorphic, contradicting Lemma 2.

Suppose then that $l = 1$ on $\Gamma_1 \cup \Gamma_3$ with $l = -1$ on Γ_2, for example. Then $M = \{lg : g \in N\}$, where the complex function space $N = \{f \circ \rho : f \in L\}$ has a state space affinely homeomorphic to that of L. Since ρ is a homeomorphism of X, it maps each Γ_j onto some Γ_k and so, for example, l has the same sign on $\rho^{-1}(x_\alpha)$ and $\rho^{-1}(x_{\alpha+\pi})$. A typical face K''_α for the Šilov decomposition for N has vertices $\rho^{-1}(x_\alpha)$, $\rho^{-1}(y_\alpha)$, $\rho^{-1}(z_\alpha)$, $\rho^{-1}(x_{\alpha+\pi})$, $\rho^{-1}(y_{\alpha+\pi})$, $\rho^{-1}(z_{\alpha+\pi})$, and ρ extends to an affine homeomorphism between K''_α and K_α. Hence $\rho^{-1}(x_\alpha)$, $\rho^{-1}(y_\alpha)$, $\rho^{-1}(z_\alpha)$ form a triangular face of K''_α because x_α, y_α, z_α form a triangular face of K_α. These considerations lead to the conclusion that the state space of M is affinely homeomorphic to the state space of $P = \{lf : f \in L\}$. However, the faces for the Šilov decomposition for P are affinely homeomorphic to the set K_2 in Lemma 2, showing that K'_α is affinely homeomorphic to K_2. This contradiction completes the proof of the theorem.

In the example given in the above proof the Šilov decompositions for the compact convex sets in question are infinite. We now show that for finite Šilov decompositions we can obtain a positive result.

THEOREM 3. *If $A(K_1)$ is antisymmetric, or more generally if the Šilov decomposition for $A(K_1)$ is finite, then $A(K_1)$ and $A(K_2)$ are isometrically isomorphic if and only if $A_{\mathbb{C}}(K_1)$ and $A_{\mathbb{C}}(K_2)$ are isometrically isomorphic.*

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Proof. Let $T: A_{\mathbf{C}}(K_1) \to A_{\mathbf{C}}(K_2)$ be an isometric isomorphism, so that there exist a homeomorphism $\tau: \overline{\partial K_2} \to \overline{\partial K_1}$ and a function $\lambda \in A_{\mathbf{C}}(K_2)$ with $|\lambda(x)| = 1$ for $x \in \overline{\partial K_2}$ such that $Tf = \lambda(f \circ \tau)$ on $\overline{\partial K_2}$. Write $X = \overline{\partial K_1}$ and define $L = \{f \circ \tau : f \in A_{\mathbf{C}}(K_1) \mid X\}$ and $M = \{\lambda g : g \in L\}$. Then L and M are isometrically isomorphic, and $\operatorname{re} L$ and $A(K_1)$ are isometrically isomorphic, and also $\operatorname{re} M$ and $A(K_2)$ are isometrically isomorphic. The spaces L and M are complex function spaces on $Y = \overline{\partial K_2}$ and $L = \{\bar{\lambda} g : g \in M\}$. We also note that if $f \in L$ then $\lambda f \in M$ so that $\bar{\lambda} \bar{f} \in M$, which implies that $\bar{\lambda}^2 \bar{f}$ and $\lambda^2 f$ belong to L; consequently λ^2 belongs to the centre of L. It is easily verified that the Šilov decompositions of Y for L and M coincide.

If the Šilov decomposition for $A(K_1)$ is finite then so is the Šilov decomposition for L; denote by $E_1, \ldots, E_n$ the sets in this latter decomposition. Then λ^2 is constant on each E_j, say $\lambda^2 = e^{i\theta_j}$ for $1 \leqslant j \leqslant n$. Write $E_j = C_j \cup D_j$, where $\lambda = e^{i\theta_j/2}$ on C_j and $\lambda = -e^{i\theta_j/2}$ on D_j, and put $w = 1$ on C_j, $w = -1$ on D_j. The function w belongs to $C_{\mathbf{R}}(Y)$, $|w| = 1$ on Y and $\lambda = e^{i\theta_j/2} w$ on E_j. If $u \in \operatorname{re} L$ then on E_j we have

$$uw = e^{-i\theta_j/2} \lambda u \in \operatorname{re} M \mid E_j$$

and consequently uw belongs to $\operatorname{re} M$. On the other hand if $v \in \operatorname{re} M$ then $v = \lambda f$ for some $f \in L$, so that $v \mid E_j = (e^{i\theta_j/2} wf) \mid E_j$. Putting $g = e^{i\theta_j/2} f$ on E_j, for $1 \leqslant j \leqslant n$, we have $g \in \operatorname{re} L$ and $v = gw$. Therefore the mapping $Tu = uw$ is an isometric isomorphism between $\operatorname{re} L$ and $\operatorname{re} M$ as required.

If K_1 is finite-dimensional then the Šilov decomposition for $A(K_1)$ must be finite, and hence the following result holds.

COROLLARY 4. *If K_1 is finite-dimensional then $A(K_1)$ and $A(K_2)$ are isometrically isomorphic if and only if $A_{\mathbf{C}}(K_1)$ and $A_{\mathbf{C}}(K_2)$ are isometrically isomorphic.*

As noted earlier, we cannot conclude in Corollary 4 that K_1 and K_2 are affinely homeomorphic. The next result gives special conditions under which such a conclusion can be drawn. The parts which refer to simplexes were first proved by Lazar [4] and Rao [5].

COROLLARY 5. *If K_1 is a simplex or the state space of a uniform algebra then the following are equivalent:*

(*a*) *K_1 is affinely homeomorphic to K_2;*

(*b*) *$A(K_1)$ and $A(K_2)$ are isometrically isomorphic;*

(*c*) *$A_{\mathbf{C}}(K_1)$ and $A_{\mathbf{C}}(K_2)$ are isometrically isomorphic.*

Proof. We need only prove the implication (*c*) $\Rightarrow$ (*a*). Using the notation of the proof of Theorem 3, if we can show that λ belongs to the centres of L and M then it will follow that $f \in L$ implies that $\lambda f \in L$, so that L is contained in M. Similarly we would see that M is contained in L, and hence that $L = M$. But then the state space K_2 of M would be affinely homeomorphic to the state space of L, which is affinely homeomorphic to K_1.

If K_1 is a simplex and if $f \in A_{\mathbf{C}}(K_1)$ has the property that there is some $h \in A_{\mathbf{C}}(K_1)$ such that $(f(x))^2 = h(x)$ for all $x \in X$, then f belongs to the centre of $A_{\mathbf{C}}(K_1)$ (see [1], p. 107). Now λ, λ^2 belong to L and so $\lambda = f \circ \tau$, some $f \in A_{\mathbf{C}}(K) \mid X$, and f^2 must belong to $A_{\mathbf{C}}(K) \mid X$. Consequently f is in the centre of $A_{\mathbf{C}}(K) \mid X$ and λ is in the centre of L as required.

If K_1 is the state space of a uniform algebra $\mathscr{A}$ on X, then consider a typical set E in the Bishop decomposition for $\mathscr{A}$. The function λ must be constant on E, since

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otherwise E splits into two disjoint closed sets and the Šilov idempotent theorem would show that the centre of $\mathscr{A}|E$ is non-trivial (cf. [3], corollary 2). Hence $f\in\mathscr{A}$ implies that $\lambda f|E\in\mathscr{A}|E$, and consequently $\lambda f\in L$. Therefore λ belongs to the centre of $\mathscr{A}$, and to the centre of L.

In general an antisymmetric complex function space $A_{\mathbb{C}}(K)$ may be isometrically isomorphic to a non-self-adjoint complex function space A_2 with $A(K)$ not isometrically isomorphic to $\mathrm{re}\,A_2$, as the following simple example shows.

Example. Let K be the square in $\mathbb{R}^2$ with vertices $w_1=0$, $w_2=e_1$, $w_3=e_1+e_2$, $w_4=e_2$, and let $\lambda\in A_{\mathbb{C}}(K)$ be defined by $\lambda(w_1)=1$, $\lambda(w_2)=i$, $\lambda(w_3)=-1$, $\lambda(w_4)=-i$. If $A_2=\{\lambda f: f\in A_{\mathbb{C}}(K)\,|\,\partial K\}$ then A_2 is a non-self-adjoint complex function space on ∂K. Writing $\lambda=\mu+i\nu$ where μ, ν are real-valued functions, we have

$$\mathrm{re}\,A_2=\{\mathrm{re}\,\lambda(u+iv): u,v\in A(K)\,|\,\partial K\}=\{\mu u-\nu v: u,v\in A(K)\,|\,\partial K\}.$$

Now

$$(\mu u-\nu v)(w_1)=u(w_1),\quad (\mu u-\nu v)(w_2)=-v(w_2),$$

$$(\mu u-\nu v)(w_3)=-u(w_3),\quad (\mu u-\nu v)(w_4)=v(w_4),$$

and it is evident that the functions in $\mathrm{re}\,A_2$ take arbitrary values on ∂K, i.e. $\mathrm{re}\,A_2=C_{\mathbb{R}}(\partial K)$. Therefore $A(K)$ and $\mathrm{re}\,A_2$ are not isometrically isomorphic, even though $A(K)$ is antisymmetric.

3. *Isometries between $A(Z_1)$ and $A(Z_2)$*

Let A_1, A_2 be complex function spaces with state spaces S_1, S_2 and complex state spaces Z_1, Z_2. In the case when A_1 and A_2 are uniform algebras it was shown [3] that $A(Z_1)$ and $A(Z_2)$ are isometrically isomorphic if and only if Z_1 and Z_2 are affinely homeomorphic. Even in the self-adjoint space when A_1, A_2 are $C_{\mathbb{C}}(X)$-spaces, this conclusion does not imply that A_1 and A_2 are isometrically isomorphic: see [2]. However, if A_1 has only finitely many distinct idempotents (which is the case if $A(Z_1)$ has a finite Šilov decomposition), then it is possible to conclude that $A(S_1)$ and $A(S_2)$ are isometrically isomorphic (see [3], corollary 6).

We now consider the case when A_1, A_2 are self-adjoint complex function spaces and study the relationship between isometric isomorphisms of $A(Z_1)$ with $A(Z_2)$ and isometric isomorphisms of $A(S_1)$ with $A(S_2)$. In the self-adjoint case S_j is a split face of Z_j and it is straightforward to verify that $A(Z_1)$ and $A(Z_2)$ are isometrically isomorphic whenever $A(S_1)$ and $A(S_2)$ are isometrically isomorphic.

THEOREM 6. *Let A_1, A_2 be self-adjoint complex function spaces and let the Šilov decomposition for A_1 be finite. If $A(Z_1)$ and $A(Z_2)$ are isometrically isomorphic then so are $A(S_1)$ and $A(S_2)$, and consequently A_1 and A_2 are isometrically isomorphic.*

Proof. Let $T: A(Z_2)\to A(Z_1)$ be an isometric isomorphism, so that $Tf=\lambda(f\circ\tau)$ for $f\in A(Z_2)$, where $\lambda\in A(Z_1)$ with $|\lambda|=1$ on ∂Z_1, and $\tau:\partial Z_1\to\partial Z_2$ extends to a homeomorphism between $\overline{\partial Z_1}$ and $\overline{\partial Z_2}$. Let $l\in A(Z_2)$ satisfy $l=1$ on S_2, $l=0$ on $-iS_2$. Then l belongs to the centre of $A(Z_2)$, and since for $f\in A(Z_2)$ we have

$$T(fl)=\lambda(f\circ\tau)(l\circ\tau)=Tf(l\circ\tau)$$

it follows that $l\circ\tau$ is in the centre of $A(Z_1)$.

In the case when A_1 is antisymmetric, the centre of $A(Z_1)$ consists of the linear span of the constant functions with l', where $l'=1$ on S_1, $l'=0$ on $-iS_1$. Hence either

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$l \circ \tau = l'$ or $l \circ \tau = 1 - l'$. If $l \circ \tau = l'$ then $T'g = \lambda(g \circ \tau)$ defines an isometric isomorphism between $A(S_2)$ and $A(S_1)$, since in this case τ maps ∂S_1 onto ∂S_2 and $-i\partial S_1$ onto $-i\partial S_2$. If $l \circ \tau = 1 - l'$ then τ maps ∂S_1 onto $-i\partial S_2$, and the mapping T' is an isometric isomorphism between $A(S_2)$ and $A(-iS_1)$. In either case $A(S_1)$ and $A(S_2)$ are isometrically isomorphic.

Suppose now that the Šilov decomposition for A_1 is finite, so that faces in the decomposition for Z_1 are of the form E_j, $-iE_j$ for $1 \leqslant j \leqslant n$. Since g belongs to the centre of $A(Z_2)$ if and only if $g \circ \tau$ belongs to the centre of $A(Z_1)$, we see that the faces in the Šilov decomposition for $A(Z_2)$ are of the form F_j, $-iF_j$ for $1 \leqslant j \leqslant n$, where either $F_j = \overline{\text{co}}\,(\tau(\partial E_k))$ or $F_j = \overline{\text{co}}\,(\tau(-i\partial E_k))$ for some k. Since the faces in the Šilov decompositions form split face decompositions of Z_1 and Z_2, it is evident that T is the direct sum of $2n$ isometric isomorphisms between $A\,(\overline{\text{co}}\,\tau(\partial E_j))$ or $A(\overline{\text{co}}\,\tau(-i\partial E_j))$ and $A(E_j)$ etc.

We may replace T by the direct sum of $2n$ isometric isomorphisms between $A(F_j)$ and $A(E_j)$ and between $A(-iF_j)$ and $A(-iE_j)$ for $1 \leqslant j \leqslant n$ (after some renumbering), in the following inductive manner. Suppose first that τ maps ∂E_1 into F_1. Then if τ maps $-i\partial E_1$ into $-iF_1$, the first step of the construction is complete. If τ maps $-i\partial E_1$ into F_2, say, then τ will map ∂E_2 (or $-i\partial E_2$) into $-iF_1$. But then $A(E_1)$, $A(F_1)$, $A(F_2)$ and $A(E_2)$ are all isometrically isomorphic, and so we may replace T by an isometric isomorphism mapping $A(F_1)$ onto $A(E_1)$ and $A(-iF_1)$ onto $A(-iE_1)$. In the case when τ maps ∂E_1 into $-iF_1$, a similar construction can be made, using the fact that $A(F_1)$ and $A(-iF_1)$ are isometrically isomorphic. Continuing in this way we obtain the required isometric isomorphism.

Since $A(S_1)$ is the direct sum of the $A(E_j)$ and $A(S_2)$ is the direct sum of the $A(F_j)$ we see that $A(S_1)$ and $A(S_2)$ are isometrically isomorphic. Applying Theorem 1 we obtain finally that A_1 and A_2 are isometrically isomorphic.

Example. There exist self-adjoint complex function spaces A_1 and A_2 which are isometrically isomorphic, but for which $A(Z_1)$ and $A(Z_2)$ are not isometrically isomorphic.

In fact, take $A_1 = L$ and $A_2 = M$, where L and M are as defined in the proof of Theorem 1. If $A(Z_1)$ and $A(Z_2)$ are isometrically isomorphic then there exists a homeomorphism $\tau : X \cup -iX \to X \cup -iX$ and a function $l \in A(Z_2)$ with $|l| = 1$ on X such that $A(Z_2)\,|\,(X \cup -iX) = \{l(f \circ \tau) : f \in A(Z_1)\,|\,(X \cup -iX)\}$.

The space $\{f \circ \tau : f \in A(Z_1)\,|\,X \cup -iX\}$ may be represented as an $A(K)$-space, where K is affinely homeomorphic to Z_1. Hence the faces of the Šilov decomposition for $A(K)$ are triangular prisms. Similar considerations to those in Theorem 1 show that Z_2 is affinely homeomorphic to the state space of $\{lf : f \in A(Z_1)\,|\,X \cup -iX\}$ and we obtain the required contradiction as before.

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