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Extremum principles for electrostatic and magnetostatic problems

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Abstract. Complementary variational principles are derived for a class of electrostatic and magnetostatic problems using the pairs of adjoint operators (grad, $-\mathrm{div}$) and (curl, curl). This theory unifies the principles of Dirichlet and Thomson in electrostatics and of Schrader in magnetostatics. The results are illustrated by deriving upper and lower bounds for the capacity of a surface, and it is shown how such bounds can be systematically improved by Ritz procedures.

1. *Introduction*

In a series of papers (1, 2, 3) complementary variational principles were derived for boundary value problems of the form

$$\nabla^2\Phi = f(\Phi) \quad \text{in} \quad V, \tag{1}$$

$$\Phi = \phi_0 \quad \text{on} \quad \partial V, \tag{2}$$

where V is some region with boundary ∂V, f is a function of Φ and ϕ_0 is a function of coordinates. The resulting formulae were used to derive upper and lower bounds associated with diffusion equations (1, 2), the Thomas–Fermi equation in quantum mechanics and the Liouville equation in equilibrium statistical mechanics (3).

The purpose of this paper is to derive complementary variational principles for electrostatic and magnetostatic problems. This enables us to generalize the work in (1, 2) to deal with boundary conditions of type (2), and to extend the basic analysis beyond the operator ∇^2 to include the operator curl curl which arises in the magnetostatic case. Our results therefore unify the Dirichlet and Thomson (4) and Schrader (5) principles and show how generalized forms of them arise in a complementary manner. As an example of the general theory we derive upper and lower bounds for the capacity of a surface, and finally we show how bounds of this general type can be systematically improved by Ritz procedures.

For the underlying theory and notation used here we refer to (3).

2. *Extremum principles*

2·1. *Electrostatic case.* We consider the Poisson equation which arises when Φ is the electrostatic potential, namely

$$\mathrm{div}\,(K\,\mathrm{grad}\,\Phi) = -4\pi\rho \quad \text{in} \quad V, \tag{3}$$

with

$$\Phi = \phi_0 \quad \text{on} \quad \partial V. \tag{4}$$

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Here K is the dielectric parameter and ρ is the charge density. Writing (3) in the form

$$\operatorname{grad}\Phi = K^{-1}\mathbf{U}, \tag{5}$$

$$-\operatorname{div}\mathbf{U} = 4\pi\rho, \tag{6}$$

and using the analysis of (3) with $T = \operatorname{grad}$, $T^* = -\operatorname{div}$, we obtain the extremum principles

$$G_1(\Phi) \leqslant I_1(\phi, \mathbf{u}) \leqslant J_1(\mathbf{U}), \tag{7}$$

where
$$G_1(\Phi) = \int_V \{-\tfrac{1}{2}K(\operatorname{grad}\Phi)^2 + 4\pi\rho\Phi\}\,d\tau, \quad \Phi = \phi_0 \quad \text{on} \quad \partial V, \tag{8}$$

$$I_1(\phi, \mathbf{u}) = \int_V (-\tfrac{1}{2}K(\operatorname{grad}\phi)^2 + 4\pi\rho\phi)\,d\tau$$

$$= \int_V 2\pi\rho\phi\,d\tau - \frac{1}{2}\int_{\partial V} K\phi_0 \operatorname{grad}\phi\,.\,d\mathbf{S}, \tag{9}$$

$$J_1(\mathbf{U}) = \int_V \tfrac{1}{2}K^{-1}\mathbf{U}\,.\,\mathbf{U}\,d\tau - \int_{\partial V}\phi_0\mathbf{U}\,.\,d\mathbf{S}, \quad \text{with} \quad \operatorname{div}\mathbf{U} = -4\pi\rho \quad \text{in} \quad V. \tag{10}$$

Here ϕ and $\mathbf{u}$ are the exact solutions of equations (4)–(6), and Φ and $\mathbf{U}$ are any trial functions satisfying the constraints in (8) and (10).

When $K^{-\frac{1}{2}}\nabla^2 K^{\frac{1}{2}}$ is positive, a more flexible upper bound $J_1'(\mathbf{U}')$ can be found (2) by means of the transformation $\Phi = K^{-\frac{1}{2}}\Phi'$, namely

$$J_1'(\mathbf{U}') = \frac{1}{2}\int_V \{\mathbf{U}'\,.\,\mathbf{U}' + K^{\frac{1}{2}}(\nabla^2 K^{\frac{1}{2}})^{-1}(\operatorname{div}\mathbf{U}')^2\}\,d\tau - \int_{\partial V}\phi_0 K^{\frac{1}{2}}(\mathbf{U}' - \tfrac{1}{2}\phi_0 \operatorname{grad} K^{\frac{1}{2}})\,.\,d\mathbf{S}. \tag{11}$$

In (11) there is no constraint on $\mathbf{U}'$.

2·2. *Magnetostatic case.* We next consider the problem of a steady current $\mathbf{j}$ in a medium of permeability μ. The field equation is

$$\operatorname{curl}(\mu^{-1}\operatorname{curl}\boldsymbol{\Phi}) = 4\pi\mathbf{j} \quad \text{in} \quad V, \tag{12}$$

with
$$\boldsymbol{\Phi} = \boldsymbol{\phi}_0 \quad \text{on} \quad \partial V, \tag{13}$$

and $\boldsymbol{\Phi}$ is now the vector potential. Writing (12) in the form

$$\operatorname{curl}\boldsymbol{\Phi} = \mu\mathbf{U}, \tag{14}$$

$$\operatorname{curl}\mathbf{U} = 4\pi\mathbf{j}, \tag{15}$$

and using the analysis of (3) with $T = \operatorname{curl}$, $T^* = \operatorname{curl}$, we obtain the extremum principles

$$G_2(\boldsymbol{\Phi}) \leqslant I_2(\boldsymbol{\phi}, \mathbf{u}) \leqslant J_2(\mathbf{U}), \tag{16}$$

where
$$G_2(\boldsymbol{\Phi}) = \int_V \{-\tfrac{1}{2}\mu^{-1}(\operatorname{curl}\boldsymbol{\Phi})^2 + 4\pi\mathbf{j}\,.\,\boldsymbol{\Phi}\}\,d\tau, \quad \boldsymbol{\Phi} = \boldsymbol{\phi}_0 \quad \text{on} \quad \partial V, \tag{17}$$

$$I_2(\boldsymbol{\phi}, \mathbf{u}) = \int_V \{-\tfrac{1}{2}\mu^{-1}(\operatorname{curl}\boldsymbol{\phi})^2 + 4\pi\mathbf{j}\,.\,\boldsymbol{\phi}\}\,d\tau$$

$$= \int_V 2\pi\mathbf{j}\,.\,\boldsymbol{\phi}\,d\tau - \frac{1}{2}\int_{\partial V}\mu^{-1}\boldsymbol{\phi}_0 \wedge \operatorname{curl}\boldsymbol{\phi}\,.\,d\mathbf{S}, \tag{18}$$

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$$J_2(\mathbf{U}) = \int_V \tfrac{1}{2}\mu \mathbf{U}.\mathbf{U}\,d\tau - \int_{\partial V} \boldsymbol{\phi}_0 \wedge \mathbf{U}.d\mathbf{S}, \quad \text{with} \quad \operatorname{curl}\mathbf{U} = 4\pi\mathbf{j} \quad \text{in} \quad V. \tag{19}$$

The principles in (16) reduce to those of Schrader (5) when

$$\boldsymbol{\phi}_0 = 0 \quad \text{and} \quad \mu = 1.$$

2·3. *Bounds for the capacity of a surface.* We return to the electrostatic field and consider the expression for the capacity C of any convex surface S, which is

$$C = \frac{1}{4\pi}\int_V K(\operatorname{grad}\phi)^2\,d\tau. \tag{20}$$

The integral in (20) is taken throught all space V external to S, and ϕ is the solution of the Dirichlet problem

$$\operatorname{div}(K\operatorname{grad}\Phi) = 0 \quad \text{in} \quad V, \quad \Phi = \phi_0 \quad \text{on} \quad \partial V, \tag{21}$$

with

$$\phi_0 = 1 \quad \text{on} \quad S; \quad \phi_0 = \text{order}\,(1/r) \quad \text{at} \quad \infty. \tag{22}$$

Using the results of section 2·1 with $\rho = 0$ we obtain the complementary variational principles

$$C_1(\mathbf{U}) \leqslant C \leqslant C_2(\Phi), \tag{23}$$

where

$$C_1(\mathbf{U}) = \frac{1}{4\pi}\left\{2\int_{\partial V}\phi_0\mathbf{U}.d\mathbf{S} - \int_V K^{-1}\mathbf{U}.\mathbf{U}\,d\tau\right\}, \tag{24}$$

and

$$C_2(\Phi) = \frac{1}{4\pi}\int_V K(\operatorname{grad}\Phi)^2\,d\tau. \tag{25}$$

Here Φ is any trial function satisfying $\Phi = \phi_0$ on ∂V, and $\mathbf{U}$ is any trial function satisfying $\operatorname{div}\mathbf{U} = 0$ in V and $\mathbf{U} = O(1/r^2)$ at infinity. Thus, equation (23) provides upper and lower bounds for the capacity and our analysis shows how they arise in a complementary manner. We note that the method of the hypercircle (6) also leads to the principles in (23).

3. *Improvement of bounds by Ritz procedures*

The bounds obtained in section 2 involve two kinds of functional which we can typify by

$$\text{(i)} \quad P(Y) = A(Y, Y) + 2L(Y), \tag{26}$$

$$\text{(ii)} \quad Q(\Psi) = B(\Psi, \Psi), \quad \Psi = \psi_0 \quad \text{on} \quad \partial V, \tag{27}$$

where A and B are quadratic forms and L is a linear form. Such bounds may be improved systematically by the Ritz method of linear combinations.

For type (i) we take a trial function Y of the form

$$Y_n = \sum_{i=1}^{n} \lambda_i \chi_i, \tag{28}$$

where the χ_i are linearly independent functions and where the λ_i are determined from the stationary conditions

$$\partial P/\partial\lambda_i = 0 \quad (i = 1, 2, \ldots, n). \tag{29}$$

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Then, setting (28) in (26) and using (29), we find that the stationary value of $P(Y)$ is given by

$$P_n = \frac{|A_0(n)|}{|A_1(n)|} \quad (n = 1, 2, \ldots), \tag{30}$$

where

$$|A_0(n)| = \begin{vmatrix} 0 & \alpha_1 & \alpha_2 & \ldots & \alpha_n \\ \alpha_1 & a_{11} & a_{12} & \ldots & a_{1n} \\ \alpha_2 & a_{21} & a_{22} & \ldots & a_{2n} \\ . & . & . & \ldots & . \\ \alpha_n & a_{n1} & a_{n2} & \ldots & a_{nn} \end{vmatrix}, \quad |A_1(n)| = \begin{vmatrix} a_{11} & a_{12} & \ldots & a_{1n} \\ a_{21} & a_{22} & \ldots & a_{2n} \\ . & . & \ldots & . \\ a_{n1} & a_{n2} & \ldots & a_{nn} \end{vmatrix}, \tag{31}$$

with

$$\alpha_i = L(\chi_i), \quad a_{ij} = A(\chi_i, \chi_j) = a_{ji}. \tag{32}$$

In a similar way, for type (ii) we take a trial function of the form

$$\Psi_n = \sum_{i=0}^{n} \mu_i \theta_i, \quad \mu_0 = 1, \tag{33}$$

where the θ_i are linearly independent functions such that

$$\theta_0 = \psi_0, \quad \theta_1 = \theta_2 = \ldots = \theta_n = 0 \quad \text{on} \quad \partial V. \tag{34}$$

Setting (33) in (27) and determining the $\mu_i (i = 1, 2, \ldots, n)$ from the stationary conditions

$$\partial Q / \partial \mu_i = 0 \quad (i = 1, 2, \ldots, n), \tag{35}$$

we find that the optimum value of $Q(\Psi)$ is given by

$$Q_n = \left\{ \begin{matrix} b_{00} & (n = 0), \\ \dfrac{|B_0(n)|}{|B_1(n)|} & (n = 1, 2, \ldots), \end{matrix} \right\} \tag{36}$$

where

$$|B_0(n)| = \begin{vmatrix} b_{00} & b_{01} & \ldots & b_{0n} \\ b_{10} & b_{11} & \ldots & b_{1n} \\ . & . & \ldots & . \\ b_{n0} & b_{n1} & \ldots & b_{nn} \end{vmatrix}, \quad |B_1(n)| = \begin{vmatrix} b_{11} & b_{12} & \ldots & b_{1n} \\ b_{21} & b_{22} & \ldots & b_{2n} \\ . & . & \ldots & . \\ b_{n1} & b_{n2} & \ldots & b_{nn} \end{vmatrix}, \tag{37}$$

with

$$b_{ij} = B(\theta_i, \theta_j) = b_{ji}. \tag{38}$$

According to the Ritz method, if $Q(\Psi)$ is a lower bound for I and if $P(Y)$ is an upper bound for I, then $\{\Psi_n\}$ is a maximizing sequence and $\{Y_n\}$ is a minimizing sequence giving

$$Q_0 \leqslant Q_1 \ldots \leqslant Q_n \leqslant I \leqslant P_n \leqslant \ldots \leqslant P_2 \leqslant P_1. \tag{39}$$

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