Mathematical Structures in Computer Science (2025), **35**, e39, pp. 1–22
doi:10.1017/S0960129525100340

CAMBRIDGE UNIVERSITY PRESS

PAPER

The Kock–Mikkelsen factorisation

J. M. E. Hyland

DPMMS, University of Cambridge, Cambridge, UK
Email: m.hyland@dpmms.cam.ac.uk

(Received 31 March 2025; revised 23 October 2025; accepted 28 October 2025)

Abstract

This paper responds to Rosolini's suggestion to use the ultracompletion of a category as a way to understand versions of conceptual completeness. Over 50 years ago, Kock and Mikkelsen observed in effect that one obtains ultracompletions of the category of sets by factorising ultrapower functors. They gave a concrete description of the factorisation under what they recognised were special conditions. In parallel work, Volger obtained a different description using categorical logic. Here, I revisit these ideas using Tripos Theory and show in particular that any left exact functor of toposes admits a Kock–Mikkelsen factorisation. In this reading, the ultracompletion appears amongst the various regular and exact completions which have been studied in particular by members of the Italian Category Theory School.

Keywords: Toposes; Tripos theory; factorization

1. Introduction

1.1 Background

I was very happy to be invited to speak at the workshop *Effectiveness and Continuity in Categorical Logic (ECCL 24)*, held in Genova to celebrate the 70th birthday of my long-time friend and colleague Pino Rosolini. But I also felt at a loss as to how to reflect his contributions to fundamental questions in category theory and logic. In many areas – Topos Theory generally, realisability toposes, polymorphism, the theory of partial maps, regular and exact completions – I along with many others have profited from Rosolini's characteristically imaginative research. What to choose? I discussed possibilities with Jacopo Emmenegger and eventually he encouraged me to say something relating to the idea of *conceptual completeness*.

Many years ago, Makkai had the idea that Gödel's Completeness Theorem should be augmented by a description of structure on the categories of models of first-order theories from which it was possible in essence to recover the theories themselves in the form of the corresponding pretopos. As it happens I learnt the idea from a short paper of Rosolini's explaining the essential points. It is a pity that this concise account was never published. Following the publication of Makkai's detailed account Makkai (1988), the topic was largely neglected. A notable exception is the work of Marmolejo (1995), Marmolejo (1998) and Marmolejo and Merino (2005) which takes inspiration from Makkai (1987). That provides some theoretical underpinnings for Stone Duality and points towards the reformulation of conceptual completeness undertaken more recently by Jacob Lurie (2018). Following Lurie, there was been a flurry of activity in particular from Garner (2020) and Di Liberti (2022); and recently Rosolini has himself returned to the topic. In Rosolini (2024), he makes the conceptually significant suggestion not simply to consider filter powers

© The Author(s), 2025. Published by Cambridge University Press. This is an Open Access article, distributed under the terms of the Creative Commons Attribution licence (https://creativecommons.org/licenses/by/4.0/), which permits unrestricted re-use, distribution and reproduction, provided the original article is properly cited.

https://doi.org/10.1017/S0960129525100340 Published online by Cambridge University Press

acting on individual structures but also to consider the action on categories of structures. This idea is proving influential. The referee for this paper drew my attention to developments since I spoke at the meeting for Rosolini: the papers Hamad (2024), Hamad (2025a), Hamad (2025b), Saadia (2025) and Tarantino and Wrigley (2025) develop theory which aims better to understand conceptual completeness from this point of view.

Though there is clearly much still to consider, I do not have anything to contribute to this ongoing research. Rather my aim in this paper is simply to provide some context by reformulating old, maybe forgotten, ideas of Kock and Mikkelsen (1974) about filter powers of toposes.

Filter powers of toposes were considered early in the development of Topos Theory with – I suppose – the idea of providing a clean alternative view of nonstandard analysis (NSA). For many years that line of thought remained unexplored. Recently, however, José Siqueira, a student of Peter Johnstone, gave it serious attention. In his PhD thesis, de Siqueira (2022) has given a topos-theoretic reformulation of Edward Nelson's set-theoretic foundations for NSA. (The published paper de Siqueira (2024) presents an extended account of the NSA perspective.) I had the good fortune to be internal examiner for Siqueira's thesis alongside Anders Kock as external. This led me to think about the old Kock–Mikkelsen paper. Its inspiration is the idea that what Rosolini calls an ultracompletion of the category of sets arises by factorising an ultrapower functor. That throws up a number of puzzles: how exactly to formulate the ultracompletion; and how to describe the factorisation?

The first of these is addressed by Garner (2020) and Rosolini (2024) and also variously in the very recent papers cited above. In this paper, I say something about the second – what I am calling the Kock–Mikkelsen factorisation. My account is based on the notion of Tripos as formulated by Pitts (1981) and Pitts (2002). While we need here only some very special triposes, their analysis does seem to shed interesting light on Tripos Theory as a whole and so more generally on the theory of doctrines as pioneered by Maietti and Rosolini (2013a), Maietti and Rosolini (2013b) and Maietti and Rosolini (2013c).

1.2 History

Let $\mathcal{U}$ be an ultrafilter on a set I. There is an evident functor

$$L : \mathbf{Set} \to \mathbf{Set}\,; A \mapsto A^I/\mathcal{U},$$

taking a set A to its ultrapower $A^I/\mathcal{U}$. The fundamental theorem of Łoś on ultraproducts – restricted to ultrapowers – implies that this functor preserves first-order logic. To get a hold on general ultraproducts, one can look at the category $\mathbf{Set}^I/\mathcal{U}$, a special case of a general filter power construction. As explained by Rosolini (2024), this is the image of **Set** under an ultracompletion pseudomonad. The data for an ultraproduct determine an object in $\mathbf{Set}^I/\mathcal{U}$, and the ultraproduct is its image under the global sections functor. This lies behind Rosolini's proposal to use the ultracompletion as a unifying theme in understanding the conceptual completeness constructions of Makkai (1988) and Lurie (2018).

The general factorisation of Kock and Mikkelsen (1974) was motivated by the following observation. The ultrapower functor $L : \mathbf{Set} \to \mathbf{Set}$ factors through Rosolini's ultracompletion as a composite

$$\mathbf{Set} \xrightarrow{\nabla} \mathbf{Set}^I/\mathcal{U} \xrightarrow{\Lambda} \mathbf{Set}.$$

Here, the constant functor ∇ is logical and Λ is the global sections functor which preserves first-order logic. At this point, the referee recommended making the standard terminology clear. A functor between elementary toposes is *logical* just when it preserves finite limits and power objects. Equivalently, a logical functor preserves finite limits function spaces and the subobject classifier. A functor between elementary toposes is *first order* just when it *preserves first-order*

https://doi.org/10.1017/S0960129525100340 Published online by Cambridge University Press

logic, which amounts to the following. It preserves finite limits, image factorisation (i.e., existential quantification), together with finite joins, Heyting implication and universal quantification of subobjects. Of course, a logical functor preserves first order logic. Details of these and related notions can be found in Johnstone (2002). Note however that (in contrast with, for example, Johnstone (1980)), I make no assumption that the functors in question are the inverse images of geometric morphisms.

Since the ultrapower functor $L : \mathbf{Set} \to \mathbf{Set}$ is itself first order, there are in principle many factorisations as above. What distinguishes this factorisation discovered by Kock and Mikkelsen seems to be the fact that Λ not only preserves first-order logic but also preserves elements or points.

With this leading example in mind, Kock and Mikkelsen studied the question when a functor $L : \mathbb{E} \to \mathbb{F}$ can be factorised as

$$\mathbb{E} \xrightarrow{\nabla} \mathbb{E}_* \xrightarrow{\Lambda} \mathbb{F}$$

in a fashion similar to that of the ultrapower functor, that is, with ∇ logical and Λ preserving elements. They assumed that the functor L preserves first-order logic and they showed how one can then find the topos $\mathbb{E}_*$ as a subcategory of $\mathbb{F}$. They raised but did not pursue the question whether a factorisation could be found under more general conditions.

Also in the early 1970s, Volger was developing forms of categorical logic. In Volger (1975), he considered what are in effect hyperdoctrines in the sense of Lawvere (2006, 1970) but of a restricted kind. He assumed that the base category $\mathbb{C}$ contains a suitably structured object Ω of propositions and that the slices of the hyperdoctrine over objects $X \in \mathbb{C}$ are the homsets $\mathbb{C}(X, \Omega)$. Notwithstanding these restrictions, one can see Volger's construction of a category of Ω-definable subobjects as a precursor of the regular completion of this paper. Volger scrupulously credits Lawvere with the suggestion to use his techniques to give an alternative approach to the construction of the Kock–Mikkelsen factorisation. Like Kock and Mikkelsen, Volger presents his construction as a factorisation of a functor preserving first-order logic. However, I think that he relies only on its preserving regular logic. In this paper, I recast Volger's categorical logic approach in the general setting of Tripos Theory as originally defined by Hyland et al. (1980) and Pitts (1981). The further analysis given by Pitts (2002) plays an essential role in the paper. At a technical level, ideas from Tripos Theory enable us to see that the Kock–Mikkelsen factorisation goes through so long as the functor in question is *left exact* – that is, preserves finite limits.

Aside Neither Kock and Mikkelsen (1974) nor Volger (1975) consider in any detail what it would be to show that the factorisation that they define does coincide in the leading example with the factorisation through the ultracompletion. That surely bears on the question of how best to treat ultrapowers in the formulation of conceptual completeness. In this connection, the referee drew my attention to old work of Blass (1976) and Adelman and Blass (1985). Setting aside the issue of measurable cardinals, the significant technical thought is that every finite-limit-preserving endofunctor on the category of sets is a reduced power. That surely is relevant here? My instinct is that we need some constructive understanding of the ultracompletion. That seems crazy, but I am inclined to see some of the work in the recent papers, Hamad (2024), Hamad (2025a), Hamad (2025b), Saadia (2025) and Tarantino and Wrigley (2025), as being in this spirit.

1.3 Outline

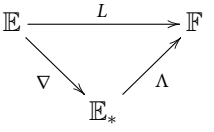
In this paper, I treat the Kock–Mikkelsen factorisation using the Theory of Triposes formulated by Hyland et al. (1980) and Pitts (1981). Triposes are higher-order hyperdoctrines. So I start with some background material on that indexed or fibrational approach to logic, what is called in the Italian Category Theory tradition the study of doctrines. I give the basic definitions and along the way refer in passing to important foundational research in the area.

https://doi.org/10.1017/S0960129525100340 Published online by Cambridge University Press

I then focus on the Theory of Triposes. The aspect which is important for here is that there are two distinct notions: the original one as in Hyland et al. (1980) and Pitts (1981) which I call *tripos in the external sense*; and the weaker alternative described in Pitts (2002) which I call *tripos in the internal sense*. Since both are due to Pitts, it is hard to distinguish them by attaching a name.

The distinction between the two notions comes into play as follows. Given a left exact functor $L : \mathbb{E} \to \mathbb{F}$ between toposes, one can obtain a hyperdoctrine $\mathcal{P}$ over $\mathbb{E}$ as follows. Each object $A \in \mathbb{E}$ has a power object PA; we map it to $\mathbb{F}$ using L and take global sections there. Structure transfers from the standard subobject hyperdoctrine Sub over $\mathbb{E}$ to $\mathcal{P}$ so that $\mathcal{P}$ is a first-order hyperdoctrine. Eventually, it will turn out that $\mathcal{P}$ is a tripos in the internal sense but as far as I can see that is very much after the event. It seems that to make progress, one has to replace $\mathbb{E}$ by a new category $\widehat{\mathbb{E}}$ (no longer a topos) with the same objects, over which there is a new fibration $\widehat{\mathcal{P}}$ with the same fibres as the old. The miracle is that $\widehat{\mathcal{P}}$ is then a tripos in the external sense. So its exact completion $\widehat{\mathbb{E}}[\widehat{\mathcal{P}}]$ is a topos. Since $\widehat{\mathbb{E}}[\widehat{\mathcal{P}}]$ can be identified with $\mathbb{E}[\mathcal{P}]$, it follows that it is also a topos and so by the Pitts Characterization Theorem of Pitts (2002) $\mathcal{P}$ is a tripos in the internal sense. My impression is that this convoluted path is necessary.

There is an evident logical map of triposes from the subobject tripos Sub on $\mathbb{E}$ to $\mathcal{P}$ on $\mathbb{E}$; and there is a map from the latter to the standard subobject tripos Sub on $\mathbb{F}$. The first of these readily induces a logical functor between the represented toposes. In the case of the second, things are not so straightforward. It seems better to consider instead the map of triposes from $\widehat{\mathcal{P}}$ over $\widehat{\mathbb{E}}$ to Sub on $\mathbb{F}$. From that, one can (with some effort) define a left exact functor with the property identified by Kock and Mikkelsen (1974) that it preserves elements or points. Writing $\mathbb{E}_*$ for $\mathbb{E}[\mathcal{P}]$ or $\widehat{\mathbb{E}}[\widehat{\mathcal{P}}]$, we get a factorisation



of the original left exact $L : \mathbb{E} \to \mathbb{F}$, with ∇ logical and Λ left exact and preserving points. That is what I want to call the general Kock–Mikkelsen factorisation.

The heart of this paper consists of filling in some details of this story. Towards the end, I shall briefly explain why this factorisation obtained from Tripos Theory coincides – in the cases they consider – with the factorisation obtained by Kock and Mikkelsen (1974).

2. Hyperdoctrines

Hyperdoctrines as envisaged by Lawvere are indexed models of logic given by a base category $\mathbb{C}$ equipped with $\mathcal{P} : \mathbf{C}^{\mathrm{op}} \to \mathcal{L}$, a contravariant pseudofunctor into some 2-category $\mathcal{L}$ of suitably structured logical categories. I usually think of this situation as a fibration so shall say that $\mathcal{P}$ is fibred or indexed over $\mathbb{E}$: or else that $\mathcal{P}$ is a hyperdoctrine over $\mathbb{E}$ or with base $\mathbb{E}$.

For each $X \in \mathbf{C}$, the fibre $\mathcal{P}(X)$ is regarded as a collection of (formal) predicates on X equipped with logical structure: generally, there is enough structure to interpret some logic. As is usual in categorical logic, I shall casually call this the internal logic. Here, it will be sufficient to consider what could be called entailment fibrations, those in which each $\mathcal{P}(X)$ is a preorder encapsulating proposional entailment. So our $\mathcal{L}$ is preorder-enriched. As suggested by the logical interpretation, I shall use the entailment symbol $\vdash$ for the preorder. In fact, $\mathcal{L}$ will always be some extension of meet preorders (Frequently, authors factor out by equivalence under entailment so would take meet semilattices, but I do not bother with that here.)

In the end, we will be concerned with triposes as defined by Pitts. These are hyperdoctrines with very rich and higher-order structure. However, partly to draw attention to a large and relatively neglected research area, I first survey hyperdoctrines with less structure. The various fragments

https://doi.org/10.1017/S0960129525100340 Published online by Cambridge University Press

of logic involved are described in Johnstone (2002) together with details of their interpretation in suitably structured categories. That is the standard perspective of categorical logic. Here, I concentrate on the fibrational or indexed point of view. That distinguishes between what happens in the fibres and what happens across fibres and the result especially when it comes to weak fragments is really quite subtle. What I shall say is at points tentative. I am conscious in particular that I have not adequately absorbed the important work on what they call doctrines, initiated by Maietti and Rosolini (see Maietti and Rosolini (2013a); Maietti and Rosolini (2013b); Maietti and Rosolini (2013c)). Equally I am not fully au fait with the more recent contributions in the same vein by Trotta (2021a) and Trotta (2021b). In this paper, I signal this lack of understanding by continuing for now to refer to hyperdoctrines.

2.1 Regular hyperdoctrines

Regular formulae are those with equality and closed under finitary conjunction and existential quantification; and regular logic is given by entailments between such regular formulae.

Definition 1. *A regular hyperdoctrine is given by a category $\mathbb{C}$ with products equipped with a fibred logic $\mathcal{P}$ interpreting regular logic. Explicitly that means:*

Finite Conjunction *For each object $X \in \mathbb{C}$, $\mathcal{P}(X)$ is equipped with binary meets $\phi \wedge \psi$, and a terminal or top element $\top$. Moreover, for each map $u : X \to Y$ of* **C**, *the reindexing $\mathcal{P}(u) : \mathcal{P}(Y) \to \mathcal{P}(X)$ preserves this structure.*

Equality *Each object $X \in \mathbb{C}$ is equipped with $\delta_X \in \mathcal{P}(X \times X)$ an equality predicate whose defining property in the internal logic is as follows. For $R \in \mathcal{P}(X \times Y)$ and $S \in \mathcal{P}(X \times Y \times Y)$, we have*

$$R(x, y_1) \wedge \delta(y_1, y_2) \vdash S(x, y_1, u_2) \quad \text{if and only if} \quad R(x, y) \vdash S(x, y, y).$$

(There is an evident reading of that as an adjunction.)

Existential Quantification *For each product projection $\pi : C \times X \to C$, there is a left adjoint*

$$\exists_X : \mathcal{P}(C \times X) \to \mathcal{P}(C)$$

to the reindexing $\mathcal{P}(\pi)$. Moreover, these satisfy the conditions:

Frobenius Reciprocity *For $R \in \mathcal{P}(A)$ and $S \in \mathcal{P}(A \times X)$, we have $\exists x.(R(a) \wedge S(a, x)) \cong R(a) = \exists x.S(a, x)$ in the internal logic. (Here I write $=$ for isomorphism in the preorder.)*

Beck-Chevalley *For $r : C \to D$ in* **C**, *the diagram*

$$\begin{array}{ccc} \mathcal{P}(D \times X) & \xrightarrow{\mathcal{P}(r \times X)} & \mathcal{P}(C \times X) \\ {\scriptstyle \exists_X}\downarrow & & \downarrow{\scriptstyle \exists_X} \\ \mathcal{P}(D) & \xrightarrow[\mathcal{P}(r)]{} & \mathcal{P}(C) \end{array}$$

commutes up to isomorphism.

Aside The condition for equality is taken from Maietti and Rosolini (2013a). They call a hyperdoctrine with conjunction and equality an *elementary doctrine*. They give a different but equivalent formulation in Maietti and Rosolini (2013b). There they call a hyperdoctrine with conjunction and existential quantification an *existential doctrine*. So I think that in their terms, a regular hyperdoctrine is both elementary and existential. But these matters are very delicate and I am not sure that the details which I give are exactly right. (In particular, I have not checked exactly how

https://doi.org/10.1017/S0960129525100340 Published online by Cambridge University Press

much Beck–Chevalley should be assumed.) The test is that the evident interpretation of (many-sorted) regular logic in regular hyperdoctrines is sound and complete. That is what determines just how strong the various conditions should be. Maietti and Rosolini have developed a wealth of quite delicate theory around their weak notions and that has been extended by Trotta. My impression is that there is still much to explore.

2.2 The free regular category

The notion of a regular hyperdoctrine lies at the heart of indexed categorical logic because a regular hyperdoctrine $\mathcal{P}$ over $\mathbb{C}$ freely generates a regular category $\mathbb{C}(\mathcal{P})$.

The objects of $\mathbb{C}(\mathcal{P})$ are of the form $R \in \mathcal{P}(X)$ with $X \in \mathbb{C}$. Write such as (X, R) and regard them as formal subobjects of the objects of $\mathbb{C}$. Maps from (X, R) to (Y, S) are equivalence classes of $F \in \mathcal{P}(X \times Y)$ which in the internal logic (the logic of $\mathcal{P}$) are graphs of functions from R to S. Composition is effected using the existential quantifier to compose the graphs. Restricted equalities give the graphs of the identities. The category $\mathbb{C}(\mathcal{P})$ comes equipped with a functor $\nabla : \mathbb{C} \to \mathbb{C}(\mathcal{P})$ taking an object X in $\mathbb{C}$ to $\top \in \mathcal{P}(X)$ and a map $u : X \to Y$ to $(u \times 1_Y)^*(\delta_Y)$. Objects of this kind are called *constant objects*.

A category is said to be regular (in the sense of Barr) just when it is cartesian or left exact (i.e., has finite limits) and has stable factorisation of maps as surjections followed by monics. (There are many equivalent formulations – see Johnstone (2002).)

Proposition 1. *The category $\mathbb{C}(\mathcal{P})$ is regular and the functor $\nabla : \mathbb{C} \to \mathbb{C}[\mathcal{P}]$ preserves products.*

The proof should be routine: the regular structure is given in the internal logic. Of course each regular category has an associated subobject regular hyperdoctrine and the construction $\mathbb{C}(\mathcal{P})$ has a universal property, but I leave that aside on this occasion. All the same I shall refer to $\mathbb{C}(\mathcal{P})$ as the regular completion of $\mathcal{P}$. Of course, regular completion refers to many other things, notably the regular completion of a left exact category. I hope that there will be no confusion.

2.3 The free exact category

In addition to the category $\mathbb{C}(\mathcal{P})$ of formal subobjects, one can consider the category $\mathbb{C}[\mathcal{P}]$ of formal quotients of subobjects (subquotients of objects) of $\mathbb{C}$. The objects of $\mathbb{C}[\mathcal{P}]$ are of the form $=_X \in \mathcal{P}(X \times X)$ where in the internal logic $=_X$ satisfies the statement that it is a partial equivalence relation (PER): in other words, it satisfies the standard conditions of symmetry and transitivity. Write such objects as $(X, =_X)$. Maps from $(X, =_X)$ to $(Y, =_Y)$ are equivalence classes of $F \in \mathcal{P}(X \times Y)$ which satisfy in the internal logic the conditions determining a function between the subquotients. Again composition is defined by existential quantification, and now identities are given by the PERs themselves. There is a functor $\nabla : \mathbb{C} \to \mathbb{C}[\mathcal{P}]$ taking an $X \in \mathbb{C}$ to $\mathrm{Id}_X \in \mathcal{P}(X \times X)$ and a map $u : X \to Y$ to $(u \times 1_Y)^*(\mathrm{Id}_Y)$. And again these are the *constant objects*.

In the logical setting, a category is said to be exact just when it is regular, it has quotients of equivalence relations and equivalence relations are effective (i.e., kernel pairs of the quotient). There are again many equivalent formulations; see Johnstone (2002).

Proposition 2. *The category $\mathbb{C}[\mathcal{P}]$ is exact and the functor $\nabla : \mathbb{C} \to \mathbb{C}[\mathcal{P}]$ preserves products.*

Again the proof should be routine. One could formulate a universal property. Again I let that go but I still refer to $\mathbb{C}[\mathcal{P}]$ as the exact completion of $\mathcal{P}$. Those new to the subject should be aware that there are other uses. At the end of this paper, I mention a different construction which is part

https://doi.org/10.1017/S0960129525100340 Published online by Cambridge University Press

of a wider picture. (A warning for the unwary is that particular care is needed with the free exact category over a regular one. The delicate issue is laid out in Carboni and Magno (1982).)

Obviously, there is a functor from the regular category $\mathbb{C}(\mathcal{P})$ to the exact category $\mathbb{C}[\mathcal{P}]$. It takes an object (X, R) of the first to an object $(X, =_R)$ of the second where the PER is defined in the internal logic by $x =_R y \leftrightarrow x \in R \wedge \delta(x, y)$. As a result, we have functors

$$\mathbb{C} \longrightarrow \mathbb{C}(\mathcal{P}) \longrightarrow \mathbb{C}[\mathcal{P}]$$

with the first preserving products and the second regular structure.

It is the exact category $\mathbb{C}[\mathcal{P}]$ which plays the central role in this paper. So for those new to the area, it may help to visualise the objects as follows. Given an object $(X, =_X)$, setting $E_X = \mathcal{P}(\Delta)(=_X) \in \mathcal{P}(X)$ gives an induced existence predicate. Officially (X, E_X) is an object of $\mathbb{E}(\mathcal{P})$, but it gives rise as just explained to an object of $\mathbb{E}[\mathcal{P}]$ which for simplicity I still write as (X, E_X). Then our original $(X, =_X)$ lies in a diagram

$$\begin{array}{ccc} (X, E_X) & \longrightarrow & \nabla X \\ \downarrow & & \\ (X, =_X) & & \end{array}$$

in $\mathbb{C}[\mathcal{P}]$, where the horizontal arrow is a monic and the vertical one a surjection. Thus, the diagram exhibits $(X, =_X)$ as a quotient of (X, E_X) which is itself a subobject of the constant object ∇X.

2.4 Coherent hyperdoctrines

Coherent formulae are obtained by adding closure under finite disjunctions to the closure properties of regular formulae. Coherent logic is given by entailments between such formulae. There is an extensive treatment in Johnstone (2002). The corresponding hyperdoctrines do not really feature in this paper, but in view of the central role of coherent toposes in Topos Theory I give a quick sketch.

Definition 2. *A coherent hyperdoctrine is given by a category* **C** *with products equipped with a fibred logic* $\mathcal{P}$ *interpreting coherent logic. That is, in addition to the conditions for a regular hyperdoctrine, we have the condition.*

Finite disjunction *Each* $\mathcal{P}(X)$ *is equipped with binary joins* $\phi \vee \psi$ *and an initial or bottom element* $\perp$ *in such a way that the standard distributive laws hold. (Of course generally up to isomorphism.) Moreover, for each map* $u : X \to Y$ *of* **C**, *the reindexing* $\mathcal{P}(u) : \mathcal{P}(Y) \to \mathcal{P}(X)$ *preserves this structure.*

Proposition 3. *Given a coherent hyperdoctrine* $\mathcal{P}$ *over* $\mathbb{C}$, *the regular completion* $\mathbb{C}(\mathcal{P})$ *and exact completion* $\mathbb{C}[\mathcal{P}]$ *are coherent categories. And the functor* $\mathbb{C}(\mathcal{P}) \to \mathbb{C}[\mathcal{P}]$ *preserves coherent structure.*

Aside A pretopos is a coherent category which is exact (here Johnstone says effective) and has disjoint finite coproducts (Johnstone calls this positive). Such categories are effectively invariants of coherent classifying toposes. The idea of conceptual completeness was formulated in these terms by Makkai and Lurie. Recent work – see in particular Hamad (2025a) and Saadia (2025) – extends the ideas to the general setting of classifying toposes for geometric theories. The geography of that setting, that is, of general Grothendieck toposes, is explored in detail in the recent Di Liberti and Ye (2025).

https://doi.org/10.1017/S0960129525100340 Published online by Cambridge University Press

2.5 First-order hyperdoctrines

The first treatments of the Kock–Mikkelsen factorisation concerned functors between toposes which preserved first-order intuitionistic logic. That does not play a big role in this paper, but it seems as well to have an account.

Definition 3. *A first-order hyperdoctrine is a hyperdoctrine which interprets first-order logic: in addition to the condition that it be coherent we have the following.*

Implication *Each $\mathcal{P}(X)$ is equipped with a Heyting implication $\phi \Rightarrow \psi$ such that*

$$\chi \vdash \phi \Rightarrow \psi \quad \text{if and only if} \quad \chi \wedge \phi \vdash \psi .$$

Moreover reindexing preserves this structure. (The result is that each $\mathcal{P}(X)$ is a pre-Heyting algebra with reindexing preserving that structure in the suitable sense.)

Universal quantification *For each product projection $\pi : C \times X \to C$, there is a right adjoint*

$$\forall_X : \mathcal{P}(C \times X) \to \mathcal{P}(C)$$

to the reindexing $\mathcal{P}(\pi)$. Moreover, these satisfy the evident naturality condition: for any $r : C \to D$,

$$\begin{array}{ccc} \mathcal{P}(D \times X) & \xrightarrow{\mathcal{P}(r \times X)} & \mathcal{P}(C \times X) \\ {\scriptstyle \forall_X}\downarrow & & \downarrow{\scriptstyle \forall_X} \\ \mathcal{P}(D) & \xrightarrow[\mathcal{P}(r)]{} & \mathcal{P}(C) \end{array}$$

Proposition 4. *Given a first-order hyperdoctrine $\mathcal{P}$ over $\mathbb{C}$, the regular completion $\mathbb{C}(\mathcal{P})$ and exact completion $\mathbb{C}[\mathcal{P}]$ are first-order categories. And the functor $\mathbb{C}(\mathcal{P}) \to \mathbb{C}[\mathcal{P}]$ preserves the first-order structure.*

Aside There is some redundancy in the definition. Frobenius Reciprocity follows from preservation of Heyting implication under reindexing.

3. Tripos Theory

3.1 Two definitions

As explained in Pitts (2002), the word *tripos* was coined for a general notion of an fibred logic or hyperdoctrine $\mathcal{P}$ over $\mathbb{C}$ for which the exact completion $\mathbb{C}[\mathcal{P}]$ is in fact a topos. The natural angle on this is to require that $\mathcal{P}$ interpret first-order logic and ask what more is needed to create the higher-order logic of a topos. Two distinct definitions have been considered. In both, we have a first-order hyperdoctrine $\mathcal{P}$ over $\mathbb{C}$ and equipped with *generic predicates* which is structure of the following shape. For each $A \in \mathbb{C}$, there is $PA \in \mathbb{C}$ and a predicate $\mathrm{In} \in \mathcal{P}(A \times PA)$. The idea is that the predicate *represents membership*.

The original definition given in Hyland et al. (1980) and Pitts (1981) was pragmatic; in that, it covered the leading examples and allowed the development of a substantial theory.

Definition 4. *A first-order hyperdoctrine $\mathcal{P}$ indexed over $\mathbb{C}$ is a tripos in the external sense just when it is equipped with generic predicates as above satisfying the following property. For any $R \in \mathcal{P}(A \times C)$, there is a map $f : C \to PA$ in $\mathbb{C}$ such that $R \cong (A \times f)^*\mathrm{In}$.*

https://doi.org/10.1017/S0960129525100340 Published online by Cambridge University Press

Of course, the label *in the external sense* did not appear in the original papers. I adopt it here because there is another possibly more compelling notion.

In Pitts (2002), a paper written for a retrospective volume on realisability organised by Jaap van Oosten, Andy Pitts looked more closely at the question of what was needed to obtain a topos. He observed that there is a weaker but quite natural internal version of the condition that the generic predicate represents membership which suffices to ensure that the represented category $\mathbb{C}[\mathcal{P}]$ be a topos.

Definition 5. *A first-order hyperdoctrine $\mathcal{P}$ indexed over $\mathbb{C}$ is a tripos in the internal sense just when it is equipped with generic predicates as above satisfying the following property. For every $R \in \mathcal{P}(A \times C)$, the condition*

$$\forall c \in C.\exists p \in PA.\forall a \in A.R(a, c) \iff \mathrm{In}(a, p)$$

holds in the logic of the hyperdoctrine.

Following Pitts (2002), I shall call the condition the Comprehension Axiom, shortening that to CA.

Both the definitions in the external and the internal sense are due to Pitts. The merit of his second more relaxed definition is apparent in the following theorem from Pitts (2002).

Theorem 1. *Let $\mathcal{P}$ be a first-order hyperdoctrine over $\mathbb{C}$. Then $\mathcal{C}[\mathcal{P}]$ is a topos just when it is equipped with generic predicates making it a tripos in the internal sense.*

I shall call this the Pitts Caracterisation Theorem. Pitts (2002) suggests that had he conducted the analysis earlier then he might have considered making what I am calling a tripos in the internal sense the official definition. That is tempting to a purist but as we shall see in this paper there is also something to be said in favour of the original definition.

To see what that is we need to understand what one gets from the external Skolemised definition of tripos. Obviously, it cannot be a property of the represented topos $\mathbb{C}[\mathcal{P}]$. (After all every topos is represented over itself by its subobject tripos.) It has to involve its relationship with the base category $\mathbb{C}$, that is, the functor $\nabla : \mathbb{C} \to \mathbb{C}[\mathcal{P}]$. To understand that, it seems good to start with the notion of tripos in the internal sense and the Pitts Characterisation Theorem.

Aside Our two notions of tripos start with a first-order hyperdoctrine. One might wonder about other formulations. If the aim is to obtain a topos, one should consider that long ago Robinson and Rosolini (1990) gave a construction of the effective topos via a delicate process of adding recursively indexed coproducts and then freely adding quotients as in Carboni and Magno (1982). In a related direction of work, Carboni and Rosolini (2000) considered what was required to ensure that an exact completion was locally cartesian closed; and in Menni (2003), Menni characterised the left exact categories whose exact completions are toposes. There are many unexplored directions of this kind.

3.2 Triposes in the internal sense

Let us try to understand the CA. In it, we are given some $R \in \mathcal{P}(A \times C)$. That corresponds to a subobject R of $\nabla A \times \nabla C$ in $\mathbb{C}[\mathcal{P}]$. What then does the condition

$$\forall c \in C.\exists p \in PA.\forall a \in A.R(a, c) \iff \mathrm{In}(a, p)$$

give? Well our given structure $\mathrm{In} \in \mathcal{P}(A \times PA)$ enables us to define an equivalence relation $\Leftrightarrow \in \mathcal{P}(PA \times PA)$ by the formula

$$\forall a \in A.\mathrm{In}(a, p) \iff \mathrm{In}(a, q).$$

https://doi.org/10.1017/S0960129525100340 Published online by Cambridge University Press

That gives us a quotient $\widetilde{P}(\nabla A) = (PA, \Leftrightarrow)$ of $\nabla(PA)$ and the generic predicate induces a generic subobject of $\nabla A \times \widetilde{P}(\nabla A)$ which I continue to call In. Now consider the relation defined in the internal logic by

$$\forall a \in A.R(a, c) \iff \mathrm{In}(a, p)$$

It is evidently a total functional relation from C to $(PA, \Leftrightarrow)$, so it represents a map $h : \nabla C \to \widetilde{P}\nabla A$ in $\mathbb{C}[\mathcal{P}]$. It is easy to see that pulling back the subobject $\in$ along id $\times\, h$ gives the original R.

The upshot is that the Pitts CA says exactly that the generic subobject $\mathrm{In} \in \nabla A \times \widetilde{P}(\nabla A)$ satisfies the condition for being the power object of ∇A for subobjects indexed over objects themselves of the form ∇C. Thus, the force of the Characterisation Theorem is that this restricted condition implies that we have a full power object for all objects of the category. One can derive that fact in two steps. First using the internal logic extend from ∇C to an arbitrary indexing object $(C, =)$ so that $\mathrm{In} \in \nabla A \times \widetilde{P}(\nabla A)$ is indeed the power object of ∇A. Then observe that each object $(A, =)$ of $\mathbb{C}[\mathcal{P}]$ is a quotient of a subobject of an object ∇A in the image of ∇ as in the diagram

$$\begin{array}{ccc} (A, E) & \longrightarrow & \nabla A \\ \downarrow & & \\ (A, =) & & \end{array}$$

We obtain the power object of (A, E) as a subobject of that of ∇A by direct image and the power object of $(A, =)$ as a subobject of that via inverse image. (That is easy: both the monics involved are split.)

What I have just presented is a sketch proof of the Pitts Characterisation Theorem. With that explanation in place, I turn to the question of the force of the definition of tripos in the external sense.

Aside The definition of tripos in the original paper of Hyland et al. (1980) was unnecessarily demanding as regards the structure of the base category. The point that it suffices to consider $\mathcal{P}$ over categories with products is already made in Pitts (1981). Note that van Oosten (2008) uses a seemingly different setting from Pitts (2002). I do not believe there is really much difference between the two and certainly both cover the substantial examples.

3.3 Triposes in the external sense

In the discussion above, we got a generic subobject In of $\nabla A \times \widetilde{P}A$. Now given a subobject ϕ of $\nabla A \times \nabla C$, the internal condition simply gave us a classifying map $\nabla C \to \widetilde{P}(A)$ in $\mathbb{C}[\mathcal{P}]$. By contrast, the external condition gives us a classifying map $r : C \to PA$ in the base category $\mathbb{C}$ of the tripos. That gives a map $\nabla r : \nabla C \to \nabla PA$ in $\mathbb{C}[\mathcal{P}]$ and composing that with the quotient $\nabla PA \to (PA, \Leftrightarrow)$ gives the classifying map $\nabla C \to \widetilde{P}(A)$. Thus, the additional force of the external condition amounts to this. Any map $\nabla C \to \widetilde{P}(A)$ factors through the quotient via a map of the form $\nabla r : \nabla C \to \nabla PA$. As in our sketch of a proof of the Pitts Characterization Theorem, this condition extends from constant to all objects. This has an important consequence. I give a brief outline using some temporary homespun notation.

Any object in a topos is isomorphic to its object of singletons which is itself a subobject of its power object. Take an object $(A, =)$ of $\mathbb{E}[\mathcal{P}]$. We observed above that it can be regarded as a quotient of a subobject of ∇A. But we can find another representation as a quotient of a subobject of a constant object as follows. In any topos, each object X is a subobject of its power object PX as the subobject SX of singletons. But in $\mathbb{C}[\mathcal{P}]$, the power object

$$\widetilde{P}(A, =) = (PA, \Leftrightarrow)$$

https://doi.org/10.1017/S0960129525100340 Published online by Cambridge University Press

is a quotient of a subobject of ∇PA. I write (PA, E) for the subobject of ∇PA. Then I write $(SA, \Leftrightarrow)$ for the object of singletons and clumsily (PA, S) for the result of pulling back that singleton subobject along the quotient map to $(PA, \Leftrightarrow)$. That gives a diagram as follows.

$$\begin{array}{ccccc} (PA,S) & \longrightarrow & (PA,E) & \longrightarrow & \nabla PA \\ \downarrow & & \downarrow & & \\ (SA,\Leftrightarrow) & \longrightarrow & (PA,\Leftrightarrow) & & \end{array}$$

which exhibits the object of singletons $(SA, \Leftrightarrow)$ and so the original object $(A, =)$ as the quotient of a subobject of ∇PA.

The value of the external definition of tripos seems to be that the presentation

$$\begin{array}{ccc} (PA,S) & \longrightarrow & \nabla(PA) \\ \downarrow & & \\ (SA,\Leftrightarrow) & & \end{array}$$

enjoys a kind of lifting and extension property involving the functor ∇. Suppose that we have an object $(C, =)$ with some presentation

$$\begin{array}{ccc} (C,E) & \longrightarrow & \nabla C \\ \downarrow & & \\ (C,=) & & \end{array}$$

and suppose that in $\mathbb{C}[\mathcal{P}]$ we have a map from $f : (C, =) \to (SA, \Leftrightarrow)$. Then there is a map $h : C \to PA$ in $\mathbb{C}$ with the following property. In $\mathbb{C}[\mathcal{P}]$, the map $\nabla h : \nabla C \to \nabla(PA)$ first restricts (obviously uniquely) along the subobjects to give an intermediate map $(C, E) \to (PA, S)$; and that map then induces the given map $f : (C, =) \to (SA, \Leftrightarrow)$ between the quotients. Essentially this is what was called in Hyland et al. (1980) and Pitts (1981) and then in van Oosten (2008) the property that $(SA, \Leftrightarrow)$ is *weakly complete*. The terminology comes from sheaf theory and does not seem brilliant, but I do not have a good alternative suggestion. (At Rosolini's birthday meeting I just said that we have a *good presentation* which is no better.)

Proposition 5. *If $\mathcal{P}$ is a tripos over $\mathbb{C}$ in the external sense, then any object of the topos $\mathbb{C}[\mathcal{P}]$ is isomorphic to a weakly complete object $(SA, \Leftrightarrow)$ presented in a diagram as above.*

For the proof assume that $f : (C, =) \to (SA, \Leftrightarrow)$ is represented by a functional relation $F \in \mathcal{P}(C \times PA)$ and consider $H \in \mathcal{P}(C \times A)$ defined by

$$H(c, a) = \exists p.(F(c, p) \wedge \mathrm{In}(a, p)).$$

By the tripos condition, we obtain from H a map $h : C \to PA$ with the property that

$$\forall a.H(c, a) \iff \mathrm{In}(a, h(c))$$

holds in the internal logic. It is then routine to check that h satisfies the property explained above. The essential point – and this was the original formulation of weak completeness – is that

$$\exists p \in PA.F(c, p) \vdash F(c, h(c))$$

holds in $\mathcal{P}(C)$.

https://doi.org/10.1017/S0960129525100340 Published online by Cambridge University Press

3.4 Left exact maps of triposes

The property of weak completeness may seem arcane, but its use is essential for the theory of geometric morphisms of triposes. It is used in Hyland et al. (1980), Pitts (1981) and then in van Oosten (2008) to show that these do indeed induce geometric morphisms at the level of the constructed toposes. More generally – as already remarked in passing in Hyland et al. (1980) – the idea applies to left exact or cartesian maps of triposes. We shall need this so I give a brief sketch.

Suppose that we have hyperdoctrines $\mathcal{P}$ over $\mathbb{C}$ and $\mathcal{Q}$ over $\mathbb{D}$. A map of hyperdoctrines is given by a product-preserving $F : \mathbb{C} \to \mathbb{D}$ together with a map $\Phi : \mathcal{P} \to F^*\mathcal{Q}$ of fibrations over $\mathbb{C}$. Explicitly the latter gives for each $A \in \mathbb{C}$ a map $\Phi_A : \mathcal{P}(A) \to \mathcal{Q}(FA)$ with the Φ_A commuting with reindexing up to isomorphism. Such a map is left exact just when Φ preserves finite conjunctions and equalities.

Proposition 6. *Let $\mathcal{P}$ over $\mathbb{C}$ and $\mathcal{Q}$ over $\mathbb{D}$ be triposes in the external sense. Then a left exact map F, Φ from $\mathcal{P}$ to $\mathcal{Q}$ induces a left exact functor from $\mathbb{C}[\mathcal{P}]$ to $\mathbb{D}[\mathcal{Q}]$.*

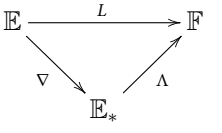
Almost all the proof is routine with one crucial exception which is this. Suppose that we have a map $r : (A, =_A) \to (B, =_B)$ in $\mathbb{C}[\mathcal{P}]$ represented by a functional relation $R \in \mathcal{P}(A \times B)$, we would like to take that to a map $(FA, \Phi(=_A)) \to (FB, \Phi(=_B))$ represented by $\Phi(R) \in \mathcal{Q}(FA \times FB)$. Certainly, the objects $(FA, \Phi(=_A))$ and $(FB, \Phi(=_B))$ lie in $\mathbb{D}[\mathcal{Q}]$: being a PER is preserved. And $\Phi(R)$ is certainly a single-valued relation in the internal $\mathcal{Q}$-logic. But if existential quantification is not preserved, it need not to be total.

That looks disastrous, but there is a way round the difficulty. Replace $(A, =_A)$ and $(B, =_B)$ by their isomorphic weakly complete versions $(SA, \Leftrightarrow)$ and $(SB, \Leftrightarrow)$. So as not to overload notation, write again r for the replacement map and $R \in \mathcal{P}(PA \times PB)$ for a representing functional relation. Now R is total, in other words, $S(p) \vdash \exists q.R(p, q)$ holds in $\mathcal{P}(PA)$. But by weak completeness of $(SB, \Leftrightarrow)$, we have a map $h : PA \to PB$ in $\mathbb{C}$ with $\exists q.R(p, q) \vdash R(p, h(p))$ holding. So $S(p) \vdash R(p, h(p)$ holds in $\mathcal{P}(PA)$. Applying the map of triposes, we see that $\Phi S(p) \vdash \Phi R(p, Fh(p))$ holds in $\mathcal{Q}(FPA)$. But a fortiori that means that $\Phi S(p) \vdash \exists q.\Phi R(p, q)$ holds which gives the seemingly problematic totality of ΦR. The upshot is that we do get a map in the topos $\mathbb{D}[\mathcal{Q}]$ so long as we remember to replace objects by weakly complete versions of themselves.

Aside The problem which we handle above does not arise for maps of triposes which preserve regular structure. But without that condition, it simply is not the case that all total relations will be carried to total relations. The problem was recognised early in the development of Tripos Theory. The solution is adapted from the Theory of Sheaves and can now be regarded as standard. I sketch a fresh view on the topic at the end of this paper.

4. The Kock–Mikkelsen Factorisation

The situation considered by Kock and Mikkelsen (1974) is that of a left exact functor $L : \mathbb{E} \to \mathbb{F}$ where both $\mathbb{E}$ and $\mathbb{F}$ are toposes. Their idea is that such a functor should factorise as



where $\mathbb{E}_*$ is a topos, ∇ is logical and Λ is preserving points. Assuming L preserves first-order logic, Kock and Mikkelsen identify $\mathbb{E}_*$ as a subcategory of $\mathbb{F}$. Volger (1975) sketches an approach based on his take on categorical logic. Again assuming L preserves first-order logic, he essentially identifies $\mathbb{E}_*$ as a regular completion of the form $\mathbb{E}(\mathcal{P})$ for the tripos we are going to define.

https://doi.org/10.1017/S0960129525100340 Published online by Cambridge University Press

Here I recast Volger's categorical logic approach using the notion of Tripos. We shall see that the factorisation goes through when L is merely left exact.

4.1 The basic hyperdoctrine

Given a left exact functor $L : \mathbb{E} \to \mathbb{F}$ between toposes, we can define a hyperdoctrine $\mathcal{P}$ indexed over $\mathbb{E}$ and a map of hyperdoctrines from the standard subobject hyperdoctrine Sub over $\mathbb{E}$ to $\mathcal{P}$. Every $A \in \mathbb{E}$ has a power object PA, and we take $\mathcal{P}(A) = \mathbb{F}(1, L(PA))$ the points of the image of PA in $\mathbb{F}$. We have evident maps

$$L : \mathrm{Sub}(A) \cong \mathbb{E}(1, PA) \to \mathbb{F}(1, L(PA)) = \mathcal{P}(A)$$

and the hyperdoctrine structure is obtained from the internal structure in $\mathbb{E}$ by transferring along these maps. So to give reindexing along $u : A \to B$ in $\mathbb{E}$, we have the internal pullback $u^* : PB \to PA$ in $\mathbb{E}$ and so we have $L(u^*) : L(PB) \to L(PA)$ in $\mathbb{F}$. Composing with $L(u^*)$ gives the reindexing

$$\mathcal{P}(u) : \mathcal{P}(B) = \mathbb{F}(1, L(PB)) \to \mathbb{F}(1, L(PA)) = \mathcal{P}(A)$$

Since PA is an internal Heyting algebra in $\mathbb{E}$, it is immediate that $\mathcal{P}(A)$ is Heyting algebra. Moreover, u^* preserves the Heyting algebra structure in $\mathbb{E}$; so $L(u^*)$ does so in $\mathbb{F}$; and so $\mathcal{P}(u)$ does so externally. Finally, u^* has left and right adjoints $PA \to PB$ in $\mathbb{E}$; these are mapped by L to left and right adjoints in $\mathbb{F}$ and then as before we simply compose to get

$$\exists u : \mathcal{P}(A) \to \mathcal{P}(B) \quad \text{and} \quad \forall u : \mathcal{P}(A) \to \mathcal{P}(B).$$

The Beck–Chevalley conditions for pullbacks are themselves equational and so transfer without fuss. This makes $\mathcal{P}$ a first-order hyperdoctrine. Finally, we have the standard membership relation on $A \times PA$ in $\mathbb{E}$ is classified by a map $A \times PA \to \Omega$ so gives a point $1 \to P(A \times PA)$. Applying L, we obtain $\mathrm{In} \in \mathcal{P}(A \times PA)$. Pretty clearly $\mathcal{P}$ cannot in general be a tripos in the external sense, but we might hope to show that it is one in the internal sense. The natural thing to do is to use the fact that the topos $\mathbb{E}$ satisfies a fully internal form of the Comprehension Axiom:

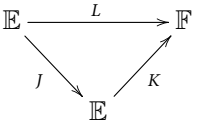
$$\forall \phi \in P(A \times C). \forall c \in C. \exists p \in PA. \forall a \in A. \phi(a, c) \iff a \in p$$

And indeed this does extend. The same holds in the logic of $\mathcal{P}$ (obviously with In replacing $\in$). The trouble, however, that there is no way in the hyperdoctrine to instantiate the variable ϕ at any given $R \in \mathcal{P}(C \times A)$ as is required for CA. One cannot see R in the base $\mathbb{E}$. To address that issue, I replace the hyperdoctrine by one over a richer category. I find the solution attractive and do not see any other way to proceed.

4.2 The extended base

The key idea is to replace the category $\mathbb{E}$ by one with maps more attuned to the hyperdoctrine structure. I first sketch some background.

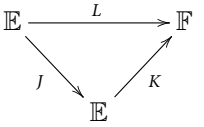
It is an easy and familiar fact that any functor $L : \mathbb{E} \to \mathbb{F}$ between categories can be factored



as a bijective-on-objects functor $J : \mathbb{E} \to \widehat{\mathbb{E}}$ followed by a full and faithful functor $K : \widehat{\mathbb{E}} \to \mathbb{F}$. This is a basic factorisation of category theory and unsurprisingly there are many versions of it. We need the following version sketched in passing many years ago by Kelly (1969).

Suppose that $\mathbb{E}$ is a symmetric monoidal closed category, that $\mathbb{F}$ is a symmetric monoidal category and that $L : \mathbb{E} \to \mathbb{F}$ is monoidal in the usual lax sense. Then L factorises as

https://doi.org/10.1017/S0960129525100340 Published online by Cambridge University Press



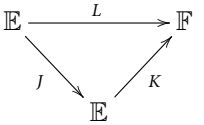
where $\widehat{\mathbb{E}}$ is symmetric monoidal, the functor $J : \mathbb{E} \to \widehat{\mathbb{E}}$ is not only bijective on objects but also strong monoidal closed and the functor $K : \widehat{\mathbb{E}} \to \mathbb{F}$ is a normal monoidal functor. Here, *normal* means that the functor preserves points in the sense of monoidal category theory: that is, the evident maps $\widehat{\mathbb{E}}(I, A) \to \mathbb{F}(I.KA)$ are isomorphisms.

The construction of $\widehat{\mathbb{E}}$ is straightforward. It has the same objects as $\mathbb{E}$, and its maps or hom-sets are given in terms of the closed structure $[A, B]$ in $\mathbb{E}$ by the formula

$$\widehat{\mathbb{E}}(A, B) = \mathbb{F}(I, L([A, B])).$$

Even in the late 1960s, Kelly did not feel the need to give further details and they can safely be left to the reader.

As a special case of the above factorisation, one can consider the situation where the closed structure on $\mathbb{E}$ is cartesian and $\mathbb{F}$ is a category with products. Suppose that the functor $L : \mathbb{E} \to \mathbb{F}$ preserves products. Then both functors in the factorisation

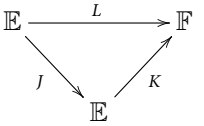


preserve products. In particular, $J : \mathbb{E} \to \widehat{\mathbb{E}}$ preserves the cartesian closed structure, while now K preserves points.

We are going to extend the hyperdoctrine $\mathcal{P}$ over $\mathbb{E}$ to a hyperdoctrine $\widehat{\mathcal{P}}$ over $\widehat{\mathbb{E}}$ with the same fibres. Before doing that, let us review what we know about our intermediate category. First, $\widehat{\mathbb{E}}$ is cartesian closed and the bijective on objects functor $J : \mathbb{E} \to \widehat{\mathbb{E}}$ preserves that structure. It is also the case that J preserves the pullbacks (finite limits) of $\mathbb{E}$. On the other hand, $\widehat{\mathbb{E}}$ cannot be expected to have all pullbacks. Fortunately (see in particular Pitts (2002)) to give a topos of the form $\widehat{\mathbb{E}}[\mathcal{P}]$, we only need pullbacks arising from the product structure on $\widehat{\mathbb{E}}$ and these we do have.

4.3 The extended hyperdoctrine

The idea now is to consider the factorisation



of the previous section and to construct a tripos $\widehat{\mathcal{P}}$ in the external sense over the intermediate category $\widehat{\mathbb{E}}$ with the same fibres as $\mathcal{P}$. Obviously, there will be maps of hyperdoctrines which we will be able to exploit.

When working with the hyperdoctrine $\widehat{\mathcal{P}}$ on $\widehat{\mathbb{E}}$, we make constant use of the composite

$$\mathbb{E} \xrightarrow{L} \mathbb{F} \xrightarrow{\Gamma} \mathbf{Set}$$

of L followed by the global sections functor Γ on $\mathbb{F}$. I introduce the notation $L_0 : \mathbb{E} \to \mathbf{Sets}$ for this left exact composite.

https://doi.org/10.1017/S0960129525100340 Published online by Cambridge University Press

We have already been making use of L_0. We defined $\mathcal{P}$ on $\mathbb{E}$ by setting $\mathcal{P}(A) = L_0(PA)$. We defined $\widehat{E}$ by $\widehat{E}(A, B) = L_0(B^A)$. And now we define $\widehat{\mathcal{P}}$ on $\widehat{\mathbb{E}}$ by again setting $\widehat{\mathcal{P}}(A) = L_0(PA)$. So as promised $\widehat{\mathcal{P}}(A) = \mathcal{P}(A)$. I now sketch the structure of a hyperdoctrine on $\widehat{\mathcal{P}}$ over $\widehat{\mathbb{E}}$.

Reindexing Given $f \in \widehat{\mathbb{E}}(A, B) = L_0(B^A)$, we want $\widehat{\mathcal{P}}(f) : \widehat{\mathcal{P}}(B) \to \widehat{\mathcal{P}}(A)$. We give it uniformly in f. There is an internal reindexing $PB \times B^A \to PA$ which is the evident internal composition $\Omega^A \times A^B \to \Omega^B$ in $\mathbb{E}$. Applying the product preserving L_0 gives the required reindexing

$$\widehat{\mathcal{P}}(A) \times \widehat{\mathbb{E}}(B, A) \to \widehat{\mathcal{P}}(B).$$

This simple procedure is a model for everything we do. All the tripos structure on $\widehat{\mathcal{P}}$ will be obtained in the same way from the internal topos structure in $\mathbb{E}$ by applying L_0.

Propositional Logic We want to show that each $\widehat{\mathcal{P}}(A)$ has the structure of a Heyting algebra and the reindexing functors preserve that structure.

The first claim is clear. Ω and so each PA is an internal Heyting algebra in $\mathbb{E}$. As L_0 is left exact it preserves such structure. For the reindexing, we take $f : A \to B$ in $\widehat{\mathbb{E}}$ and want to show that reindexing $\widehat{\mathcal{P}}(f) : \widehat{\mathcal{P}}(B) \to \widehat{\mathcal{P}}(A)$ is a map of Heyting algebras. Again that goes through uniformly. To show that

$$\widehat{\mathcal{P}}(B) \times \widehat{\mathbb{E}}(A, B) \to \widehat{\mathcal{P}}(A)$$

is a map of Heyting algebras indexed by $\widehat{\mathbb{E}}(A, B)$ we show that

$$\widehat{\mathcal{P}}(B) \to (\widehat{\mathbb{E}}(A, B) \Rightarrow \widehat{\mathcal{P}}(A))$$

is a map of Heyting algebra, where on the right-hand side I use a temporary arrow notation for the function space. (Of course the function space is given the pointwise Heyting algebra structure.) To see this note that internally to $\mathbb{E}$, the map from $P(B) \to (B^A \Rightarrow P(A)$ is a map of Heyting algebras. Applying L_0 we see that

$$L_0(P(B)) \to L_0((B^A) \Rightarrow P(A))$$

is also a map of Heyting algebras. But the evident map

$$L_0((B^A) \Rightarrow P(A)) \to L_0((B^A)) \Rightarrow L_0(P(B))$$

also preserves the Heyting algebra structure. Composing gives the desired result.

Entailment It is good to be clear about the preorder $\vdash$ on the hyperdoctrine $\mathcal{P}$. The internal order on the subobject classifier Ω of $\mathbb{E}$ is given by the subobject $\Omega_1 \to \Omega^2$ which is classified by the Heyting implication $\Rightarrow$. So we have a pullback

$$\begin{array}{ccc} \Omega^2 & \xrightarrow{\Rightarrow} & \Omega \\ \uparrow & & \uparrow{\scriptstyle \top} \\ \Omega_1 & \longrightarrow & 1 \end{array}$$

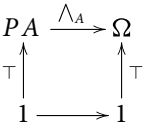
and since exponentiation preserve pullbacks and $\Omega^A \cong PA$, we have for every $A \in \mathbb{E}$ a pullback diagram

$$\begin{array}{ccc} PA^2 & \xrightarrow{\Rightarrow_A} & PA \\ \uparrow & & \uparrow{\scriptstyle \top} \\ PA_1 & \longrightarrow & 1 \end{array}$$

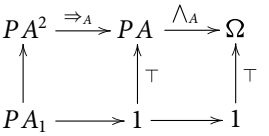
where $\Rightarrow_A$ is the Heyting implication on PA and PA_1 is the internal order relation on PA.

Also in $\mathbb{E}$, we have for every A an infimum map $\bigwedge_A : PA \to \Omega$ which classifies the top element of PA as in the pullback

https://doi.org/10.1017/S0960129525100340 Published online by Cambridge University Press



Composing this with the previous pullback gives a big pullback



showing that the internal order PA_1 on PA is classified by the top composite. That demonstrates that for $R, S \in \mathcal{P}(A) = \widehat{\mathbb{E}}(A, \Omega)$ we have

$$R \vdash S \quad \text{if and only if} \quad \bigwedge_A (R \Rightarrow S) = \top$$

in the internal logic of $\mathbb{E}$.

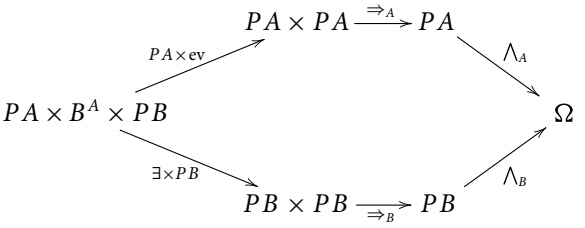
Adjoints Reindexing $\widehat{\mathcal{P}}(B) \times \widehat{\mathbb{E}}(A, B) \to \widehat{\mathcal{P}}(A)$ was obtained by applying the functor L_0 to the internal reindexing map $PA \times A^B \to PB$ of $\mathbb{E}$. We give the left and right adjoints in the same way. Within $\mathbb{E}$ we have existential quantification and universal quantification maps

$$\exists : B^A \times PA \to PB \quad \text{and} \quad \forall : B^A \times PA \to PB,$$

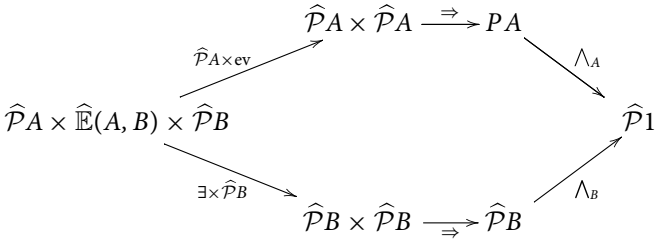
and applying L_0 gives maps

$$\exists : \widehat{\mathbb{E}}(A, B) \times \mathcal{P}A \to \mathcal{P}B \quad \text{and} \quad \forall : \widehat{\mathbb{E}}(A, B) \times \mathcal{P}A \to \mathcal{P}B.$$

So for $f \in \widehat{\mathbb{E}}(A, B)$, we have $\exists f : \mathcal{P}(A) \to \mathcal{P}(B)$ and $\forall f : \mathcal{P}A \to \mathcal{P}B$. That these give adjoints is again uniform in f. Those are expressed in $\mathbb{E}$ by commuting diagrams. For $\exists$ it is of shape



and similarly for $\forall$. (In the diagram, ev is the evaluation map giving reindexing.) Applying L_0 gives a commuting diagram of shape



Input $R \in \widehat{\mathcal{P}}A$, $f \in \widehat{\mathbb{E}}(A, B)$ and $S \in \widehat{\mathcal{P}}B$ on the left-hand side. The top composite takes them to $\top$ just when $R \vdash \widehat{\mathcal{P}}fS$; and the bottom composite takes them to $\top$ just when $\exists_f R \vdash S$. So the

https://doi.org/10.1017/S0960129525100340 Published online by Cambridge University Press

two conditions are equivalent. Of course mutatis mutandis the same argument goes through for universal quantification. We have both left and right adjoints to reindexing.

Aside Weirdly these general adjoints are not of much interest or use to us. That is because good adjoints should satisfy some kind of Beck–Chevalley condition. We cannot have this in general because we have no handle on pullbacks in $\widehat{\mathbb{E}}$. Fortunately for a tripos, we only need quantification along product projections and Beck–Chevalley for pullbacks of them.

Beck–Chevalley The Beck–Chevalley condition refers to a commutation of substitution and quantification. For a tripos, we need this only for quantification along product projections, that is, for quantification in the usual restricted sense of standard logic. For $g \in \widehat{\mathbb{E}}(C, D)$, we have an evident $g \times A \in \widehat{\mathbb{E}}(C \times A, D \times A)$ and a pullback diagram of the form

$$\begin{array}{ccc} C \times A & \xrightarrow{\pi} & C \\ {\scriptstyle g\times A}\downarrow & & \downarrow{\scriptstyle g} \\ D \times A & \xrightarrow[\pi]{} & D \end{array}$$

For existential quantification, there are two induced maps $\exists_\pi.\widehat{\mathcal{P}}(g \times A)$ and $\widehat{\mathcal{P}}g.\exists_\pi$ from $\widehat{\mathcal{P}}(D \times A)$ to $\widehat{\mathcal{P}}(C)$. Adjunction induces a natural comparison from the first to the second and the Beck–Chevalley condition states that this is an isomorphism. More concretely for $R \in \widehat{\mathcal{P}}(D \times A)$, we have an isomorphism

$$\exists_\pi.(g \times A)^* R \xrightarrow{\cong} g^*.\exists_\pi R.$$

Again we find an internal version of this condition in $\mathbb{E}$. The diagram

$$\begin{array}{ccccc} P(D \times A) \times D^C & \xrightarrow{\exists_A \times D^C} & P(D) \times D^C & \xrightarrow{\text{ev}} & PC \\ \downarrow & & & & \| \\ P(D \times A) \times (D \times A)^{(C \times A)} & \xrightarrow[\text{ev}]{} & P(C \times A) & \xrightarrow[\exists_A]{} & PC \end{array}$$

commutes. Applying L_0 gives a commuting diagram of the same shape and as was the case for the adjoints we can read off from it the required Beck–Chevalley condition.

Equality I put off saying anything about equality because we can now deal with it quickly. We want $\delta_A \in \widehat{\mathcal{P}}A$ satisfying the axioms for equality. We get exactly what we had for $\mathcal{P}$. We simply apply L_0 to the map $1 \to P(A \times A)$ picking out the internal equality on $A \in \mathbb{E}$. With the structure, we now have it is sufficient to show that for $R \in \widehat{\mathcal{P}}(A \times A)$ we have

$$\delta_A \vdash R \quad \text{if and only if} \quad \top_A \vdash \widehat{\mathcal{P}}(\Delta_A)(R).$$

That is easy as we have essentially set $\delta_A = \exists_{\Delta_A}(\top_A)$.

Generic predicates At this point, we know that we have a first-order hyperdoctrine $\widehat{\mathcal{P}}$ over our category $\widehat{\mathbb{E}}$ and it remains to show that it has generic predicates. In a sense that is the easiest property. $\widehat{\mathbb{E}}$ is exactly constructed to have maps making $\widehat{\mathcal{P}}$ a tripos in the external sense. We just transfer structure from $\mathbb{E}$.

In $\mathbb{E}$, we have the membership predicate as a subobject of each $A \times PA$. It is classified by an evaluation map $A \times PA \to \Omega$ and that corresponds by transpose to a point $P(A \times PA)$. Applying L gives a point of $LP(A \times PA)$ and that gives the membership relation $\text{In} \in \mathcal{P}(A \times PA)$.

Suppose that we are given $R \in \widehat{\mathcal{P}}(A \times C)$. That is a point of $LP(A \times C) \cong L(PA^C)$ and that gives a map $r : C \to PA$ in $\widehat{\mathbb{E}}$. That has to do the trick.

https://doi.org/10.1017/S0960129525100340 Published online by Cambridge University Press

Why? Well in $\mathbb{E}$ we have $P(A \times C) \cong PA^C$ and there is an evident map $PA^C \to (A \times PA)^{(A \times C)}$ which is taking the product with the identity on A. So we have $P(A \times C) \to (A \times PA)^{(A \times C)}$. Take the product of that with the point of $P(A \times PA)$ above to give

$$P(A \times C) \to (A \times PA)^{(A \times C)} \times P(A \times PA)$$

Composing with the internal compositon (which we use to define reindexing)

$$(A \times PA)^{(A \times C)} \times P(A \times PA) \to P(A \times C)$$

returns the identity on $P(A \times C)$. Applying L_0 to that unravels to show that for each $R \in \widehat{\mathcal{P}}(A \times C)$ with corresponding $r : C \to PA$ in $\widehat{\mathbb{E}}$ as above, the reindexing $\widehat{\mathcal{P}}(A \times r)(\mathrm{In})$ is R as required.

In the discussion above, we have sketched the proof of our main technical theorem.

Theorem 2. *With the structure described above $\widehat{\mathcal{P}}$ is a tripos over $\widehat{\mathbb{E}}$ in the external sense.*

It is evident from the definitions that we can identify the categories $\widehat{\mathbb{E}}[\widehat{\mathcal{P}}]$ and $\mathbb{E}[\mathcal{P}]$. In line with the notation of Kock and Mikkelsen (1974) I write $\mathbb{E}_*$ for either version. Now $\widehat{\mathcal{P}}$ is a tripos in the external sense so we can deduce the following.

Theorem 3. *The category $\widehat{\mathbb{E}}[\widehat{\mathcal{P}}]$ and hence the category $\mathbb{E}[\mathcal{P}]$ is a topos.*

By the Pitts Characterisation Theorem, we can now deduce that $\mathcal{P}$ is a tripos over $\mathbb{E}$ in the internal sense.

4.4 The logical functor

We have a map of triposes from Sub over $\mathbb{E}$ to $\mathcal{P}$ over $\mathbb{E}$. It is the identity on $\mathbb{E}$. On fibres it is the following map $\mathrm{Sub}(A) \to \mathcal{P}(A) = L_0(PA)$. A subobject R of A is classified by a map $A \to \Omega$ and taking the transpose we get a point $r : 1 \to PA$; then R is taken to $L(r) \in L_0(PA)$. It is clear from the definitions that this is what one could call a logical map of triposes. In fact, the structure here is precisely preserved. So we can deduce at once the following.

Proposition 7. *The induced functor $\nabla : \mathbb{E} \to \mathbb{E}_*$ is a logical functor.*

4.5 The left exact functor

We have a map of hyperdoctrines from $\widehat{\mathcal{P}}$ over $\widehat{\mathbb{E}}$ to the subobject hyperdoctrine Sub over $\mathbb{F}$. The details need a little care.

At the category level, we take an object A in $\widehat{\mathbb{E}}$, equivalently in $\mathbb{E}$ to the object $L(A)$ of $\mathbb{F}$. But we also need to give the action on maps. Suppose we have $f \in \widehat{\mathbb{E}}(A, B) = L_0(B^A)$. Since L preserves products, there is an evident standard comparison $L(B^A) \to (LB)^{LA}$ and composing with it gives a point $1 \to (LB)^{(LA)}$. The exponential transpose is then the required map $LA \to LB$ in $\mathbb{F}$.

At the level of the fibres, we have $\widehat{\mathcal{P}}A = L_0(PA)$. Again as L preserves products we have the comparison $L(PA) = L(\Omega^A) \to (L\Omega)^{LA}$. And as the functor L is left exact, we get a classifier $L\Omega \to \Omega$ of the subobject $L\top \to L\Omega$. So we have $L(PA) \to \Omega^{LA} = P(LA)$. Given $R : 1 \to L(PA)$ in $\widehat{\mathcal{P}}A$ we compose to get a point $1 \to P(LA)$ and hence a subobject of LA as required.

While the structure of this map of hyperdoctrines is natural enough, it is not quite straightforward to see that it induces even a functor between the generated toposes as generally the map will preserve left exact structure but not regular structure. Fortunately, we dealt with that problem in a general setting in Proposition 6. We can apply that here and deduce the following.

https://doi.org/10.1017/S0960129525100340 Published online by Cambridge University Press

Proposition 8. *The map of hyperdoctrines above induces a left exact map of toposes* $\Lambda : \mathbb{E}_* = \widehat{\mathbb{E}}[\widehat{\mathcal{P}}] \to \mathbb{F}$.

4.6 The factorisation

Given that the definition of the functor Λ involves replacing objects by an isomorphic weakly complete one, it is a bit fiddly to check properties of it. All the same it is fairly clear that it does preserve elements or points, the condition identified by Kock and Mikkelsen (1974). Moreover with a little more attention to detail one sees that the composite

$$\mathbb{E} \xrightarrow{\nabla} \mathbb{E}_* \xrightarrow{\Lambda} \mathbb{F}$$

is isomorphic to the original $L : \mathbb{E} \to \mathbb{F}$.

4.7 The regular completion

We have paid a good deal of attention to the power objects $\widetilde{P}(\nabla A)$ of constant objects ∇A of our toposes. For a general tripos $\mathcal{P}$ over $\mathbb{C}$, the power object is given by $(PA, \Leftrightarrow)$ where $p \Leftrightarrow q$ is defined in the internal logic by the formula $\forall a.\mathrm{In}(a, p) \Leftrightarrow \mathrm{In}(a, q)$. Evidently $\Leftrightarrow$ is an equivalence relation and the power object is the quotient of ∇PA which it determines.

That is the typical situation but when we consider the triposes $\mathcal{P}$ over $\mathbb{E}$ or $\widehat{\mathcal{P}}$ over $\widehat{\mathbb{E}}$ arising from a left exact $L : \mathbb{E} \to \mathbb{F}$ something unusual happens. For in $\mathbb{E}$ the relation $\Leftrightarrow$ on PA is just equality. It follows that the quotient $\nabla(PA) \to \widetilde{P}(\nabla A)$ is an isomorphism. (In one reading of the situation it is literally the identity.) But any object of our toposes is (isomorphic to) a subobject of the power object $\widetilde{P}(\nabla)A)$. Hence in the case of the (equivalent) toposes $\mathbb{E}[\mathcal{P}]$ and $\widehat{\mathbb{E}}[\widehat{\mathcal{P}}]$ every object is isomorphic to a subobject of a constant object. It follows that the standard embeddings $\mathbb{E}(\mathcal{P}) \to \mathbb{E}[\mathcal{P}]$ and $\widehat{\mathbb{E}}(\widehat{\mathcal{P}}) \to \widehat{\mathbb{E}}[\widehat{\mathcal{P}}]$ of the regular completion in the exact completion are in fact equivalences. That explains the fact that no quotients are to be found in the old treatments of Kock and Mikkelsen (1974) and Volger (1975). In a sense that reflects very well on those involved. The natural perspective is surely via the Tripos Theory of Pitts and avoiding the quotients along the way made the issues they faced just a bit harder.

4.8 First-order functors

In Kock and Mikkelsen (1974), the assumption that their functor $L : \mathbb{E} \to \mathbb{F}$ is first-order logic preserving. At one point they appear to say that this means that the subobject classifier is preserved. While that is true in their leading example, it is stronger than is needed and the paper suggests that they intended simply that it preserves first-order logic in what is now the standard sense mentioned in section 1.2. In other words, they assume that L is first-order. Now that implies that the various comparisons

$$L\Omega \to \Omega, \quad L(B^A) \to (LB)^{(LA)}, \quad L(PA) \to P(LA),$$

which we have seen above are all monics. (It seems that considerations of this kind were first studied by Mikkelsen (2022) in his thesis. Readers are likely to be familiar with the condition for inverse images of geometric morphisms. A geometric morphism of toposes is open (see Johnstone (1980) for example) just when the inverse image functor is first-order.)

Suppose that $L : \mathbb{E} \to \mathbb{F}$ is first order and return to the description of the map of triposes from $\widehat{\mathcal{P}}$ over $\widehat{\mathbb{E}}$ to Sub over $\mathbb{F}$. Since L is first-order the comparison $L(B^A) \to (LB)^{(LA)}$ is monic and that implies that the map of triposes is faithful at the level of categories. But as $L(PA) \to P(LA)$ is monic, it is also faithful at the level of fibres. Since in particular L preserves existential

https://doi.org/10.1017/S0960129525100340 Published online by Cambridge University Press

quantification, we have no need to consider replacement by weakly complete objects so we immediately deduce the following.

Proposition 9. *Let $L : \mathbb{E} \to \mathbb{F}$ be first-order. Then in the Kock–Mikkelsen factorisation, the functor $\Lambda : \mathbb{E}_* \to \mathbb{F}$ is faithful.*

This proposition explains why Kock and Mikkelsen were able to carve out $\mathbb{E}_*$ as a subcategory of the ambient category $\mathbb{F}$.

5. Afterthoughts

5.1 Artin Gluing

Clearly, the Kock–Mikkelsen factorisation of a left exact functor between toposes into a logical one followed by a left exact one preserving points belongs in the part of Topos Theory concerned with logical functors. What should the place of the factorisation be in the Topos Theory story? The referee suggested saying more about one aspect.

Artin glueing takes a finite-limit-preserving functor $L : \mathbb{E} \to \mathbb{F}$ between toposes and gives the glued topos $\mathbb{G}$ with an open inclusion $u : \mathbb{E} \to \mathbb{G}$ and corresponding closed inclusion $v : \mathbb{F} \to \mathbb{G}$. The functor L is then the composite of the direct image $u_* : \mathbb{E} \to \mathbb{G}$ and the inverse image $v^*\mathbb{G} \to \mathbb{F}$. There is a detailed discussion in Johnstone (2002).

It is a perhaps confusing feature of Artin gluing that the inverse image $u^* : \mathbb{G} \to \mathbb{E}$ is logical. By contrast, this paper takes a finite-limit-preserving functor $L : \mathbb{E} \to \mathbb{F}$ and provides a logical functor $\mathbb{E} \to \mathbb{E}_*$. It feels as if there should be something to say here, but I have not thought it through. Quite specifically, the referee notes that Artin gluing can be thought of as a tabulation and asks whether there is a notion of cotabulation for toposes. That is a good question. I have no idea about it, but that certainly seems worth pursuing.

5.2 Methodology

In a different direction, one might wonder about the methodology of this paper, specifically the use of the tripos $\widehat{\mathcal{P}}$ on the intermediate category $\widehat{\mathbb{E}}$. Is all that really necessary? As to that I would remark first that both Kock and Mikkelsen (1974) and Volger (1975) are led in their treatments to consider a category corresponding to $\widehat{\mathbb{E}}$. Indeed Kock and Mikkelsen explicitly draw attention to it in their leading example as a category of standard objects and non-standard maps. More than that however I feel – and regret not having conveyed better – that the theory of the tripos $\widehat{\mathcal{P}}$ over $\widehat{\mathbb{E}}$ is incredibly smooth. It exploits systematically the way in which the structure of an elementary topos is internalised within the topos itself and carries that over to the new tripos using the functor L_0 as explained above. That leads me to think that the construction may be a fundamental one.

5.3 Weak completeness

In the first version of this paper, I observed that the need for the seemingly technical notion of weak completeness in the development of the theory told against the ideas of this paper. I hoped to clarify that to make it all seem completely natural.

Since writing that I have had the great pleasure, at her invitation, to spend a week in June 2025 with Milly Maietti in Padova. For my visit, she arranged what was an ideal mini-conference: it had just a few talks and ample time for discussion and reflection. In the course of it, I learnt much of current research in the general area of doctrines. The meeting began with a wonderful survey of the field by Rosolini and prompted by that I came to see that the story of weak completeness has a natural home within the theory of doctrines. Here, I give just a glimpse of what is involved.

https://doi.org/10.1017/S0960129525100340 Published online by Cambridge University Press

First from the Maietti–Rosolini perspective, it is natural to consider for suitable doctrines $\mathcal{P}$ over $\mathbb{C}$ not only the categories $\mathbb{C}(\mathcal{P})$ of section 2.2 and $\mathbb{C}[\mathcal{P}]$ of section 2.3 but also versions of these where one restricts to maps which are tracked by maps in $\mathbb{C}$. Put otherwise, we have in each case the same objects, but maps are equivalence classes of maps in $\mathbb{C}$ satisfying evident properties in the internal logic. Just for now I shall refer to these as the tracked maps in the larger categories. The simple but crucial point exploited in section 3.3 is that a left exact map of doctrines automatically acts on the (categories of) tracked maps.

The notion of weak completeness makes sense in this context. An object B is weakly complete just when for all objects A all maps $A \to B$ are tracked. So evidently left exact maps of doctrines act on the subcategory of weakly complete objects.

The story is completed by showing that for a tripos $\mathcal{P}$ over $\mathbb{C}$ in the external sense all objects in $\mathbb{C}[\mathcal{P}]$ are isomorphic to weakly complete ones. I discussed that in section 3.3, but perhaps made too much fuss. The genericity condition pretty much says that power objects of constant objects are weakly complete. That easily extends to all power objects. But in a topos, every object is isomorphic to the subobject of singletons in the power object and the usual presentation makes these weakly complete.

Of course the proper context for the story just sketched would be an extensive development of the theory of doctrines. As I learnt during my recent visit to Padova that is a very lively area. When a definitive account is written, one might hope that better understanding will suggest a better term than 'weakly complete'.

5.4 Thanks

Given what I have just said I obviously owe an intellectual debt as regards this paper to Pino Rosolini for whom it is written. Thanks are also due to Milly Maietti for her determined promotion of research in the area of doctrines and more generally for her insistence on precision in research on categorical logic. Alongside that I would like to thank all who came to the meeting in June. I learnt a great deal from the presentations and from the discussions. The area of doctrines has proved to be (perhaps surprisingly) subtle and there remains much still worth pursuing.

Finally, any reader of this paper will recognise that it has been substantially enriched by suggestions made by a sympathetic, thoughtful and well-informed referee. It is probably as well that such contributions to research remain anonymous. However, I would like the referee and the editor who chose that referee to know that I appreciate their efforts. As ever, it remains the case that the inevitable errors and infelicities in the paper are my responsibility.

References

Adelman, M. and Blass, A. (1985). Exact functors, local connectedness and measurable cardinals. *Rendiconti del Seminario Matematico e Fisico di Milano* **54** 9–28.

Blass, A. (1976). Exact functors and measurable cardinals. *Pacific Journal of Mathematics* **63** 335–346.

Carboni, A. and Magno, R. C. (1982). The free exact category on a left exact one. *Journal of the Australian Mathematical Society. Series A. Pure Mathematics and Statistics* **33** 295–301.

Carboni, A. and Rosolini, G. (2000). Locally cartesian closed exact completions. *Journal of Pure and Applied Algebra* **154** 103–116.

Di Liberti, I. (2022). The geometry of coherent topoi and ultrastructures. arXiv preprint arXiv: 2211.03104.

Di Liberti, I. and Ye, L. (2025). Logic and concepts in the 2-category of topoi. arXiv preprint arXiv: 2504.16690.

Garner, R. (2020). Ultrafilters, finite coproducts and locally connected classifying toposes. *Annals of Pure and Applied Logic* **171** 1–29.

Hamad, A. (2024). Bundles of metric structures as left ultrafunctors, arXiv preprint arXiv: 2406.11076.

Hamad, A. (2025a). Generalised ultracategories and conceptual completeness of geometric logic. arXiv preprint arXiv: 2507.07922.

Hamad, A. (2025b). Ultracategories as colax algebras for a pseudo-monad on cat. arXiv preprint arXiv: 2502.20597.

https://doi.org/10.1017/S0960129525100340 Published online by Cambridge University Press

Hyland, J. M. E., Johnstone, P. T. and Pitts, A. M. (1980). Tripos theory. *Mathematical Proceedings of the Cambridge Philosophical Society* **88** 205–232.
Johnstone, P. T. (1980). Open maps of toposes. *Manuscripta Mathematica* **31** 217–247.
Johnstone, P. T. (2002). *Sketches of an Elephant: A Topos Theory Compendium*. Oxford Logic Guides, vol. 43, 44, Oxford University Press.
Kelly, G. M. (1969). Adjunctions for enriched categories. In: Reports of the Midwest Category Seminar III, vol. 106, springer LNM, 166–177.
Kock, A. and Mikkelsen, C. J. (1974).Topos theoretic factorization of non-standard extensions. In: Hurd, A. and Loeb, P. (eds.) Victoria Symposium on Nonstandard Analysis, vol. 369, Springer LNM, 122–143.
Lawvere, F. W. (1970). Equality in hyperdoctrines and comprehension schema as an adjoint functor. In: Proceedings of the AMS Symposium on Pure Mathematics XVII, 1–14.
Lawvere, F. W. (2006). Adjointness in foundations. Dialectica 23, 1969, 281-296. Available as Reprints in Theory and Applications of Categories, 16, 2006, 1-16, 1969.
Lurie, J. (2018) Ultracategories. Available at https://www.math.ias.edu/∼lurie/.
Maietti, M. E. and Rosolini, G. (2013a). Elementary quotient completion . *Theory and Applications of Categories* **27** 445–463.
Maietti, M. E. and Rosolini, G. (2013b). Quotient completion for the foundations of constructive mathematics. *Logica Universalis* **7** 371–402.
Maietti, M. E. and Rosolini, G. (2013c). Unifying exact completions. *Applied Categorical Structures* **23** 43–52.
Makkai, M. (1987). Stone duality for first order logic. *Advances in Mathematics* **65** 97–170.
Makkai, M. (1988). Strong conceptual completeness for first-order logic. *Annals of Pure and Applied Logic* **40** 167–215.
Marmolejo, F. (1995). Ultraproducts and continuous families of models. PhD Thesis, Dalhousie University.
Marmolejo, F. (1998). Continuous families of coalgebras. *Journal of Pure and Applied Algebra* **130** 197–215.
Marmolejo, F. and Merino, A. (2005). Locale morphisms with exact direct image functor in sheaves. *Journal of Pure and Applied Algebra* **202** 195–200.
Menni, M. (2003). A characterization of the left exact categories whose exact completions are toposes. *Journal of Pure and Applied Algebra* **177** 287–301.
Mikkelsen, C. J. (2022). Lattice theoretic and logical aspects of elementary topoi. PhD Thesis, Aarhus Universitet, 1976. First published in Aarhus Universitet Various Publications Series no 25, 1976; then in Reprints In Theory and Applications of Categories No. 29, Aarhus Universitet.
Pitts, A. M. (1981). The theory of triposes. Phd Thesis, University of Cambridge.
Pitts, A. M. (2002). Tripos theory in retrospect. *Mathematical Structures in Computer Science* **12** 265–279.
Robinson, E. P. and Rosolini, G. (1990). Colimit completions and the effective topos. *Journal of Symbolic Logic* **55** 678–699.
Rosolini, G. About Makkai's theorem. Unpublished paper.
Rosolini, G. (2024). Ultracompletions. Abstract for talk given at the International Category Theory Conference CT2024, Santiago de Compostela.
Saadia, G. (2025). Extending conceptual completeness via virtual ultracategories, arXiv preprint arXiv: 2506.23935.
de Siqueira, J. V. P. M. (2022). Tripos models of internal set theory. PhD Dissertation, University of Cambridge.
de Siqueira, J. V. P. M. (2024). Nonstandard proof methods in toposes. *Annals of Pure and Applied Logic* **175** 103424.
Tarantino, U. and Wrigley, J. (2025). Ultracategories via Kan extensions of relative monads, arXiv preprint arXiv: 2506.09788.
Trotta, D. (2021a). The existential completion, arXiv preprint arXiv: 2108.03416.
Trotta, D. (2021b). An algebraic approach to the completions of elementary doctrines, arXiv preprint arXiv: 2108.03415.
van Oosten, J. (2008). *Realizability: An Introduction to its Categorical Side*. Studies in Logic and Foundations of Mathematics, vol. 152, Elsevier.
Volger, H. (1975). Logical categories, semantic categories and topoi. In: Lawvere, F. W., Maurer, C. and Wraith, G. C. (eds.) *Model Theory and Topoi*, vol. 445, Springer LNM, 87–100.

Cite this article: Hyland JME (2025). The Kock–Mikkelsen factorisation. *Mathematical Structures in Computer Science* **35**(e39), 1–22. https://doi.org/10.1017/S0960129525100340

https://doi.org/10.1017/S0960129525100340 Published online by Cambridge University Press