Math. Proc. Camb. Phil. Soc. (1999), **125**, 511
Printed in the United Kingdom © 1999 *Cambridge Philosophical Society*

Möbius-invariant metrics

By PASI SEITTENRANTA

University of Helsinki, Department of Mathematics, Nokia Research Center, P.O. Box 407, *FIN*-00045 *Nokia Group, Finland*
e-mail: pasi.seittenranta@research.nokia.com

(*Received* 16 *April* 1997*; revised* 29 *August* 1997)

Abstract

A new Möbius-invariant metric δ_G defined on an open set $G \subset \bar{\mathbf{R}}^n$ with at least two boundary points is introduced. This metric coincides with the hyperbolic metric if G is the unit ball. Some inequalities between this and other metrics are proved.

The behaviour of δ_G under K-quasiconformal maps is also studied. In particular, a generalization of the Schwarz lemma is obtained for K-quasiconformal maps defined on quasiballs.

1. *Introduction*

Conformal invariants play an important role in geometric function theory. One of the most elementary conformal invariants is the modulus of a curve family, which is used to define quasiconformal maps [**Vä1**]. In Euclidean spaces of dimension greater than 2, conformal maps are restrictions of Möbius maps (see [**Vu2**, p. 19] for references). Therefore, it is natural to use Möbius-invariant quantities to study quasiconformality.

Some metrics have been useful in the study of quasiconformal maps; one is the conformally invariant metric defined by Ferrand in [**LF**] (see Remark 2·13 (3)) and another is the quasihyperbolic metric, introduced by Gehring and Palka [**GP**]. The purpose of this paper is to introduce a new metric, the absolute ratio metric δ_G, which is Möbius-invariant and simpler than the aforementioned metrics. It is defined as follows.

1·1. *Definition.* Let G be an open subset of $\bar{\mathbf{R}}^n$ with $\operatorname{card} \partial G \geqslant 2$. Set

$$m_G(x,y) = \sup_{a,b \in \partial G} |a,x,b,y| \tag{1·1}$$

and

$$\delta_G(x,y) = \log(1 + m_G(x,y)) \tag{1·2}$$

for all $x, y \in G$.

Here $\bar{\mathbf{R}}^n$ is the Möbius space $\mathbf{R}^n \cup \{\infty\}$, ∂G is the boundary of G, and the absolute (cross) ratio $|a,x,b,y|$ is defined in (2·1). For a proof that δ_G is a metric see Theorem 3·3.

https://doi.org/10.1017/S0305004198002904 Published online by Cambridge University Press

In Section 2 there are some preliminary facts about various metrics and quasiconformal maps. In Section 3 the metric δ_G is compared with some other metrics such as j_G (see Lemma 2·2) and Ferrand's metric σ_G (see Definition 2·7). For example, we obtain the inequality $j_G \leqslant \delta_G \leqslant 2j_G$ in Theorem 3·4. For some other related results, see Corollary 3·10 and Theorem 3·12. We also compare δ_G with the Apollonian metric α_G, introduced in [**KM**, pp. 319–325] and [**Be2**]. Note that the inequalities $\alpha_G \leqslant 2\rho_G$ and $\alpha_G \leqslant 2j_G$, proved in [**Be2**], also follow from Theorems 3·11, 3·4, 3·12, and 3·13 of this paper.

Section 4 gives a new definition for uniform domains. For a survey of this class of domains see [**Vä2**]. Our definition has the following advantages over the usual definition: (1) the uniformity constant in our definition is 1 for Euclidean balls; (2) the image of a c-uniform domain under a Möbius map is c-uniform.

The main results of this paper are given in Section 5. The first main result establishes the following quasi-invariance property of δ_G (see also Corollary 5·8 for the case $n = 2$):

1·2. THEOREM. *Let $f: \bar{\mathbf{R}}^n \to \bar{\mathbf{R}}^n$ be a K-qc map, G and $G' = fG$ open sets of $\bar{\mathbf{R}}^n$ with* $\operatorname{card} \partial G \geqslant 2$, *and $x, y \in G$. Then*

$$\delta_{G'}(f(x), f(y)) \leqslant c(K,n) \max\{\delta_G(x,y), \delta_G(x,y)^\alpha\},$$

where $c(K,n) = \lambda_n^{\beta-1}\beta\,\eta_{K,n}(1)$ and $\alpha = K^{1/(1-n)} = 1/\beta$. Here $c(K,n)$ tends to 1 as K tends to 1.

In this theorem, λ_n is the Grötzsch ring constant with $\lambda_n \in [4, 2e^{n-1})$ and $\lambda_2 = 4$ (see [**AVV2**, chapter 12], [**Vu2**, p. 89]). For $\eta_{K,n}$ see Theorem 5·2.

We shall also study quasiconformal maps between quasiballs. A domain $D \subset \bar{\mathbf{R}}^n$ is a K-quasiball if there exists a K-qc map $f: \bar{\mathbf{R}}^n \to \bar{\mathbf{R}}^n$ such that $f(\mathbf{B}^n) = D$. Note that quasiballs constitute a very general class of domains whose boundaries may be complicated nonrectifiable curves or surfaces. For instance, in [**CG**] it is shown that some Julia sets are boundaries of quasiballs in the plane. In our second main result, Theorem 5·12, we prove an inequality for the absolute ratio metric under K-quasiconformal maps defined on a K_1-quasiball. This inequality gives the Schwarz lemma, when K_1 tends to 1.

In the last section we study connections between the absolute ratio metric and some conformally invariant quantities. Finally, in Table 1 there is a summary of known inequalities between the Möbius-invariant quantities in this paper.

2. *Definitions and preliminary results*

This section contains some preliminary material that will be needed later. For more information, see, for example, [**AVV2**], [**Vä1**] or [**Vu2**].

2·1. *Notation.* We shall write

$$B^n(x,r) = \{z \in \mathbf{R}^n : |z-x| < r\},$$
$$B^n(r) = B^n(0,r), \quad \mathbf{B}^n = B^n(0,1),$$
$$S^{n-1}(x,r) = \{z \in \mathbf{R}^n : |z-x| = r\},$$
$$S^{n-1}(r) = S^{n-1}(0,r), \quad \mathbf{S}^{n-1} = S^{n-1}(0,1)$$

https://doi.org/10.1017/S0305004198002904 Published online by Cambridge University Press

and

$$\mathbf{H}^n = \{(x_1, \ldots, x_n) \in \mathbf{R}^n : x_n > 0\}.$$

The standard unit vectors of $\mathbf{R}^n$ are denoted by $e_1, e_2, \ldots, e_n$. In addition, ω_{n-1} is the surface area of the unit sphere $\mathbf{S}^{n-1}$ and Ω_n is the volume of the unit ball $\mathbf{B}^n$. We always assume that $n \geqslant 2$. The complex plane is denoted by $\mathbf{C}$.

If (X, d) is a metric space and G is a subset of X with nonempty boundary, then $d(x) = d(x, \partial G)$ is the distance between a point $x \in G$ and the boundary ∂G. If $G \subset \bar{\mathbf{R}}^n$, and the metric space (X, d) is not given explicitly, then ∂G means the boundary of G with respect to the topology of $\bar{\mathbf{R}}^n$ and $d(x)$ is the Euclidean distance between x and ∂G, when $x \in G \setminus \{\infty\}$ and $\partial G \neq \{\infty\}$.

For distinct points $a, b, c, d \in \bar{\mathbf{R}}^n$, the absolute (cross) ratio is defined by

$$|a, b, c, d| = \frac{q(a, c)q(b, d)}{q(a, b)q(c, d)}, \tag{2·1}$$

where $q(x, y)$ is the spherical (chordal) metric, defined by the formulas

$$\begin{cases} q(x, y) = \dfrac{|x - y|}{\sqrt{1 + |x|^2}\sqrt{1 + |y|^2}}, & x, y \neq \infty, \\ q(x, \infty) = q(\infty, x) = \dfrac{1}{\sqrt{1 + |x|^2}}, & x \neq \infty. \end{cases} \tag{2·2}$$

It is not difficult to prove [**Be1**, p. 32] that a function from $\bar{\mathbf{R}}^n$ into $\bar{\mathbf{R}}^n$ is a Möbius transformation if and only if it preserves absolute ratios. Note also that if a, b, c, and d are distinct points in $\mathbf{R}^n$, then

$$|a, b, c, d| = \frac{|a - c||b - d|}{|a - b||c - d|}.$$

The metrics j_G and $\tilde{j}_G$ in Lemmas 2·2 and 2·3 have appeared in the literature in the case where G is a proper subset of the Euclidean space $\mathbf{R}^n$ (see, for example [**GO**]). The proofs of Lemmas 2·2 and 2·3 are included because, as far as I know, they have not been published before, and because Lemma 2·2 is an essential part of the proof of Theorem 3·3.

2·2. Lemma. *Let (X, d) be a metric space and $G \subset X$ an open set with nonempty boundary. The formula*

$$j_G(x, y) = \log\left(1 + \frac{d(x, y)}{\min\{d(x), d(y)\}}\right), \quad x, y \in G,$$

defines a metric on G.

Proof. We need only prove the triangle inequality

$$j_G(x, y) + j_G(y, z) \geqslant j_G(x, z) \quad \text{for all } x, y, z \in G.$$

By symmetry, we may assume that $d(x) \leqslant d(z)$. Then we need to prove that

$$\frac{d(x, y)}{\min\{d(x), d(y)\}} + \frac{d(y, z)}{\min\{d(y), d(z)\}} + \frac{d(x, y)d(y, z)}{\min\{d(x), d(y)\}\min\{d(y), d(z)\}} \geqslant \frac{d(x, z)}{d(x)}. \tag{$*$}$$

If $d(y) \leqslant d(x)$, then $(*)$ follows immediately. Next, suppose that $d(x) < d(y)$. Then

https://doi.org/10.1017/S0305004198002904 Published online by Cambridge University Press

the left-hand side of $(*)$ is equal to

$$\frac{1}{d(x)}\left(d(x,y)+d(y,z)\left(\frac{d(x)+d(x,y)}{\min\{d(y),d(z)\}}\right)\right),$$

which is greater than or equal to $d(x,z)/d(x)$, since $d(x)+d(x,y)\geqslant d(y)$. □

We shall use this metric only when X is the Euclidean space $\mathbf{R}^n$.

2·3. LEMMA. *Let (X,d) be a metric space and $G\subset X$ be an open set with nonempty boundary. The formula*

$$\tilde{j}_G(x,y)=\log\left(1+\frac{d(x,y)}{d(x)}\right)+\log\left(1+\frac{d(x,y)}{d(y)}\right),\quad x,y\in G,$$

defines a metric on G.

Proof. We need only prove the triangle inequality

$$\tilde{j}_G(x,y)+\tilde{j}_G(y,z)\geqslant\tilde{j}_G(x,z)\quad\text{for all } x,y,z\in G.$$

This is equivalent to the inequality

$$\begin{aligned}\log(d(x)+d(x,y))+\log(d(y)+d(x,y))+\log(d(y)+d(y,z))+\log(d(z)+d(y,z))\\ \geqslant\log(d(x)+d(x,z))+\log(d(z)+d(x,z))+2\log d(y).\end{aligned}\tag{$**$}$$

Now we have

$$\begin{aligned}\log(d(x)+d(x,y))-\log(d(x)+d(x,z))&\\ \geqslant\log(d(x)+d(x,y))-\log(d(x)+d(x,y)+d(y,z))&\\ \geqslant\log d(y)-\log(d(y)+d(y,z))&\end{aligned}$$

and, similarly,

$$\log(d(z)+d(y,z))-\log(d(z)+d(x,z))\geqslant\log d(y)-\log(d(y)+d(x,y)).$$

We obtain $(**)$ by combining these inequalities. □

In what follows, we sometimes omit the subindex G of the metrics j_G and $\tilde{j}_G$, when the set is clear from the context.

2·4. *Remark.* Since

$$\frac{|x-y|}{\max\{d(x),d(y)\}}\geqslant\frac{|x-y|}{\min\{d(x),d(y)\}+|x-y|}=1-e^{-j(x,y)},$$

we always have

$$j(x,y)+\log(2-e^{-j(x,y)})=\log(2e^{j(x,y)}-1)\leqslant\tilde{j}(x,y)\leqslant 2j(x,y).$$

2·5. *Hyperbolic metric.* The hyperbolic metric $\rho_{\mathbf{B}^n}$ in $\mathbf{B}^n$ is defined by

$$\rho_{\mathbf{B}^n}(x,y)=\inf_{\gamma\in\Gamma}\int_\gamma\frac{2|dt|}{1-|t|^2},\quad x,y\in\mathbf{B}^n,$$

and the hyperbolic metric $\rho_{\mathbf{H}^n}$ in $\mathbf{H}^n$ is defined by

$$\rho_{\mathbf{H}^n}(x,y)=\inf_{\gamma\in\Gamma}\int_\gamma\frac{|dt|}{d(t,\partial\mathbf{H}^n)},\quad x,y\in\mathbf{H}^n,$$

https://doi.org/10.1017/S0305004198002904 Published online by Cambridge University Press

where Γ is the family of all rectifiable curves in $\mathbf{B}^n$ or $\mathbf{H}^n$, respectively, joining x and y.

If D is a simply connected domain of the extended complex plane $\bar{\mathbf{C}}$ and $\operatorname{card}\partial D > 1$, then by the Riemann mapping theorem there exists a conformal mapping f from D onto the unit disk $\{z \in \mathbf{C}: |z| < 1\}$. The hyperbolic metric in D is defined by the formula

$$\rho_D(x,y) = \inf_{\gamma\in\Gamma} \int_\gamma \eta_D(t)|dt|, \quad \eta_D(t) = \frac{2|f'(t)|}{1-|f(t)|^2},$$

where Γ is the family of all rectifiable curves in D joining x and y. It is well known that ρ_D is independent of f. In addition, if $D \subset \mathbf{C}$ then the weight function η_D satisfies the inequalities

$$\frac{1}{2d(t,\partial D)} \leqslant \eta_D(t) \leqslant \frac{2}{d(t,\partial D)} \tag{2·3}$$

for every $t \in D$ [**Le**, p. 6].

2·6. *Quasihyperbolic metric.* Let $D \subsetneq \mathbf{R}^n$ be a domain. The quasihyperbolic metric k_D is defined by

$$k_D(x,y) = \inf_{\gamma\in\Gamma} \int_\gamma \frac{|dt|}{d(t,\partial D)}, \quad x,y \in D,$$

where Γ is the family of all rectifiable curves in D joining x and y [**GP**].

Ferrand [**Fe1**, p. 122] has defined the following metric:

2·7. *Definition* (Ferrand's metric). Let $D \subset \bar{\mathbf{R}}^n$ be a domain with $\operatorname{card}\partial D \geqslant 2$. We define a density function

$$w_D(x) = \sup_{a,b\in\partial D} \frac{|a-b|}{|x-a||x-b|}, \quad x \in D \setminus \{\infty\},$$

and a metric σ in D,

$$\sigma_D(x,y) = \inf_{\gamma\in\Gamma} \int_\gamma w_D(t)|dt|,$$

where Γ is the family of all rectifiable curves joining x and y in D.

2·8. Theorem ([**Fel**, p. 122]).
(1) *σ_D is Möbius-invariant.*
(2) *$\sigma_{\mathbf{B}^n}$ and $\sigma_{\mathbf{H}^n}$ coincide with the hyperbolic metrics $\rho_{\mathbf{B}^n}$ and $\rho_{\mathbf{H}^n}$, respectively.*
(3) *$k_D \leqslant \sigma_D \leqslant 2k_D$ for every domain $D \subsetneq \mathbf{R}^n$.*

See also Theorem 3·13 and Remarks 3·14.

2·9. *Apollonian metric.* Beardon [**Be2**] has studied the pseudo-metric α_G defined in open proper subsets G of $\bar{\mathbf{R}}^n$ by

$$\alpha_G(x,y) = \sup_{a,b\in\partial G} \log|a,x,y,b| \quad \text{for all } x,y \in G$$

(see also [**KM**, pp. 319–325] and [**AVV2**, 7.28(2)]). This formula defines a metric if and only if $\bar{\mathbf{R}}^n \setminus G$ is not contained in an $(n-1)$-dimensional sphere in $\bar{\mathbf{R}}^n$, and it is called the Apollonian metric.

https://doi.org/10.1017/S0305004198002904 Published online by Cambridge University Press

2·10. *Rings.* If D is a domain in $\bar{\mathbf{R}}^n$ and if $\bar{\mathbf{R}}^n \setminus D$ has exactly two components E and F, then D is called a ring and we write $D = R(E,F)$. In particular, $R_{G,n}(s) \equiv R(\bar{\mathbf{B}}^n, [se_1, \infty])$, $s > 1$, is called the Grötzsch ring and $R_{T,n}(s) \equiv R([-e_1, 0], [se_1, \infty])$, $s > 0$, is called the Teichmüller ring.

For sets $E, F, G \subset \bar{\mathbf{R}}^n$, let $\Delta(E, F; G)$ denote the curve family of all curves joining the sets E and F in G, and let $\Delta(E,F) = \Delta(E, F; \bar{\mathbf{R}}^n)$.

2·11. *Modulus of a curve family.* Let Γ be a family of curves in $\bar{\mathbf{R}}^n$. We say that a non-negative Borel-measurable function $\rho\colon \mathbf{R}^n \to \mathbf{R} \cup \{+\infty\}$ is an admissible function, if

$$\int_\gamma \rho\, ds \geqslant 1$$

for each locally rectifiable curve γ in Γ. The modulus of Γ is

$$\mathsf{M}(\Gamma) = \inf_{\rho \in \mathscr{F}(\Gamma)} \int_{\mathbf{R}^n} \rho^n\, dm,$$

where $\mathscr{F}(\Gamma)$ is the family of admissible functions and m stands for the n-dimensional Lebesgue measure. The capacity of a ring $R(E,F)$ is

$$\operatorname{cap} R(E,F) = \mathsf{M}(\Delta(E,F)).$$

The capacities of $R_{T,n}(s)$ and $R_{G,n}(s)$ are denoted by $\tau_n(s)$ and $\gamma_n(s)$, respectively. By [**Vu2**, lemma 5.53], $\tau_n\colon (0,\infty) \to (0,\infty)$ and $\gamma_n\colon (1,\infty) \to (0,\infty)$ are decreasing homeomorphisms and they satisfy the functional identity

$$\gamma_n(t) = 2^{n-1}\tau_n(t^2 - 1), \quad t > 1. \tag{2·4}$$

2·12. *Conformal invariants μ_D and λ_D.* Let D be a proper subdomain of $\bar{\mathbf{R}}^n$ and $x, y \in D$. Then

$$\mu_D(x,y) = \inf_{C_{xy}} \mathsf{M}(\Delta(C_{xy}, \partial D; D)),$$

where C_{xy} is a curve in D joining x and y. If x and y are distinct points and $\operatorname{card} \partial D \geqslant 2$, then

$$\lambda_D(x,y) = \inf_{C_x, C_y} \mathsf{M}(\Delta(C_x, C_y; D)),$$

where C_z is a curve in D joining z and ∂D [**Vu2**].

2·13. *Remarks.*

(1) It is easy to observe from the definitions that if D_1 and D_2 are proper subdomains of $\bar{\mathbf{R}}^n$, $D_1 \subset D_2$, and $x, y \in D_1$ are distinct points, then $\lambda_{D_1}(x,y) \leqslant \lambda_{D_2}(x,y)$ and $\mu_{D_1}(x,y) \geqslant \mu_{D_2}(x,y)$.

(2) If $\operatorname{cap} \partial D = 0$ [**Vu2**, definition 7.12], then $\mu_D \equiv 0$; otherwise it follows easily from the definition that μ_D is a conformally invariant metric.

(3) Ferrand [**LF**] has shown that $\lambda_D^{-1/n}$ is a metric. In [**Vu2**, p. 193] it was asked whether the exponent can be improved to $1/(n-1)$, that is, whether $\lambda_D^{1/(1-n)}$ is a metric. This was subsequently answered in the affirmative in [**So1**, **Je**, **Fe2**].

2·14. *Special functions.* The function $\mu\colon (0,1] \to [0,\infty)$ is defined by the formula

$$\mu(r) = \frac{\pi}{2}\frac{\mathscr{K}(\sqrt{1-r^2})}{\mathscr{K}(r)}, \quad r \in (0,1), \quad \mu(1) = 0,$$

https://doi.org/10.1017/S0305004198002904 Published online by Cambridge University Press

where

$$\mathscr{K}(r) = \int_0^{\pi/2} \frac{dt}{\sqrt{1 - r^2 \sin^2 t}}.$$

For all $K > 0$ and $n \geqslant 2$, the distortion function $\varphi_{K,n}\colon [0,1] \to [0,1]$ is defined by

$$\varphi_{K,n}(t) = \frac{1}{\gamma_n^{-1}(K\gamma_n(1/t))}, \quad t \in (0,1), \tag{2·5}$$

$\varphi_{K,n}(0) = 0$, and $\varphi_{K,n}(1) = 1$.

The function μ is a decreasing and $\varphi_{K,n}$ is an increasing homeomorphism. It can be shown that $\varphi_K(r) \equiv \varphi_{K,2}(r) = \mu^{-1}(\mu(r)/K)$ for all $r \in (0,1)$ and $K > 0$.

Let D and D' be domains in $\bar{\mathbf{R}}^n$, $K \geqslant 1$, and let $f\colon D \to D'$ be a homeomorphism. Then f is K-quasiconformal (qc) if

$$\mathsf{M}(\Gamma)/K \leqslant \mathsf{M}(f\Gamma) \leqslant K\mathsf{M}(\Gamma)$$

for every curve family Γ in D. The following quasiconformal Schwarz lemma is proved in [**MRV**].

2·15. Theorem (Schwarz lemma). *Let $f\colon \mathbf{B}^n \to f\mathbf{B}^n$ be a K-qc map with $f\mathbf{B}^n \subset \mathbf{B}^n$. Then*

$$\tanh \frac{\rho(f(x), f(y))}{2} \leqslant \varphi_{K,n}\left(\tanh \frac{\rho(x,y)}{2}\right) \quad \textit{for all } x, y \in \mathbf{B}^n.$$

In particular, if $f(0) = 0$, then

$$|f(x)| \leqslant \varphi_{K,n}(|x|) \quad \textit{for all } x \in \mathbf{B}^n.$$

2·16. *Notation.* Let $f\colon G \to \mathbf{R}^n$ be a continuous injective function, where $G \subset \mathbf{R}^n$. The linear dilatation of f at a point $x \in G \setminus \partial G$ is

$$H(x,f) = \limsup_{r \to 0+} \frac{L(x,f,r)}{l(x,f,r)},$$

where

$$L(x,f,r) = \max_z \{|f(z) - f(x)|\colon |z - x| = r\},$$
$$l(x,f,r) = \min_z \{|f(z) - f(x)|\colon |z - x| = r\}$$

for $r \in (0, d(x))$.

2·17. *Remark.* Let D and D' be domains in $\mathbf{R}^n$ and let $f\colon D \to D'$ be an L-bilipschitz map. Then $H(x,f) \leqslant L^2$ for all $x \in D$, and it can be proved [**Vä1**, remark 34·2] that f is K-qc with $K \leqslant L^{2(n-1)}$.

2·18. *Some elementary inequalities.* The following Bernoulli inequalities will be used frequently:

$$\log(1 + sx) \leqslant s \log(1 + x), \quad x > 0,\ s \geqslant 1, \tag{2·6}$$
$$\log(1 + sx) \geqslant s \log(1 + x), \quad x > 0,\ 0 \leqslant s \leqslant 1. \tag{2·7}$$

2·19. Lemma. *For all $0 < s \leqslant 1$ and $0 \leqslant a \leqslant 1$,*

$$\log(1 + s^a) \leqslant (\log(1 + s))^a.$$

https://doi.org/10.1017/S0305004198002904 Published online by Cambridge University Press

Proof. By the Bernoulli inequality (2·6),

$$\log(1+s^a) \leqslant s^{a-1}\log(1+s) \leqslant (\log(1+s))^{a-1}\log(1+s) = (\log(1+s))^a. \quad \square$$

3. *Absolute ratio metric* δ_G

Recall that δ_G was defined in (1·1).

3·1. THEOREM ([**Vu2**, lemma 8·39]). *The functions* $\delta_{\mathbf{B}^n}$ *and* $\delta_{\mathbf{H}^n}$ *coincide with the hyperbolic metrics* $\rho_{\mathbf{B}^n}$ *and* $\rho_{\mathbf{H}^n}$, *respectively.*

3·2. *Remarks.*

(1) The supremum in Definition 1·1 is also the maximum.
(2) The function m_G has the following monotonicity property [**Vu2**, p. 116]: if G_1 and G_2 are sets for which m_{G_1} and m_{G_2} are defined, and $G_1 \subset G_2$, then $m_{G_1} \geqslant m_{G_2}$.
(3) It follows from the definitions that $\delta_{\mathbf{R}^n\setminus\{a\}} = j_{\mathbf{R}^n\setminus\{a\}}$ for all $a \in \mathbf{R}^n$.

3·3. THEOREM. *The function* δ_G *is a metric in* G.

Proof. Clearly $\delta_G(x,y) = 0$ if and only if $x = y$. Also δ_G is symmetric, because $|a,x,b,y| = |b,y,a,x|$. For the triangle inequality, let x, y, and z be points in G. Then there exist distinct points u and v in ∂G such that

$$m_G(x,z) = |u,x,v,z|.$$

Since

$$\delta_G(x,y) + \delta_G(y,z) \geqslant \delta_{\bar{\mathbf{R}}^n\setminus\{u,v\}}(x,y) + \delta_{\bar{\mathbf{R}}^n\setminus\{u,v\}}(y,z)$$

and

$$\delta_G(x,z) = \delta_{\bar{\mathbf{R}}^n\setminus\{u,v\}}(x,z),$$

we need to prove the triangle inequality only when $G = \bar{\mathbf{R}}^n\setminus\{u,v\}$. In addition, since δ_G is a Möbius-invariant, we can assume that $\{u,v\} = \{0,\infty\}$. But then $\delta_G = j_G$, and the triangle inequality follows, because j_G is a metric by Lemma 2·2. $\square$

3·4. THEOREM. *The inequalities* $j_G \leqslant \delta_G \leqslant \tilde{j}_G \leqslant 2j_G$ *hold for every open set* $G \subsetneq \mathbf{R}^n$.

Proof. Let x and y be distinct points in G, with $d(x) \leqslant d(y)$. Choose $a, b \in \partial G$ such that $d(x) = |x-a|$ and $|b-a| \geqslant |b-y|$ (possibly $b = \infty$). Then

$$m_G(x,y) \geqslant |a,x,b,y| \geqslant \frac{|x-y|}{d(x)},$$

and we obtain the first inequality.

For the second inequality, let a and b be in ∂G. If $a, b \neq \infty$, then

$$\begin{aligned} |a,x,b,y| &\leqslant \frac{|x-y|}{|x-a|}\left(\frac{|a-x|}{|y-b|} + \frac{|x-y|}{|y-b|} + 1\right) \\ &\leqslant \frac{|x-y|}{d(y)} + \frac{|x-y|^2}{d(x)d(y)} + \frac{|x-y|}{d(x)} \end{aligned}$$

and we get

$$\delta_G(x,y) \leqslant \log\left(\left(1+\frac{|x-y|}{d(x)}\right)\left(1+\frac{|x-y|}{d(y)}\right)\right) = \tilde{j}_G(x,y).$$

https://doi.org/10.1017/S0305004198002904 Published online by Cambridge University Press

If $b = \infty$, then $|a, x, b, y| = |x - y|/d(x)$ and similarly if $a = \infty$. □

3·5. *Remark.* The equality $\delta_G(x, y) = \tilde{j}_G(x, y)$ holds if and only if $x = y$ or there exist points $a, b \in \partial G$ such that $d(x) = |x - a|$, $d(y) = |y - b|$, and a, x, y, and b lie on the same line in this order. In particular, $\delta_{\mathbf{B}^n}(x, -x) = 2j_{\mathbf{B}^n}(x, -x)$ for all $x \in \mathbf{B}^n$. Also, $\delta_{\mathbf{H}^n}(re_n, se_n) = j_{\mathbf{H}^n}(re_n, se_n) = |\log(r/s)|$ for all $r, s > 0$.

3·6. Theorem. *If G is an open subset of $\bar{\mathbf{R}}^n$ with* $\operatorname{card} \partial G \geqslant 2$ *and the metric j_G^* is defined by*

$$j_G^*(x, y) = \log\left(1 + \frac{q(x, y)}{q(x, \partial G)}\right) + \log\left(1 + \frac{q(x, y)}{q(y, \partial G)}\right), \quad x, y \in G,$$

then $\delta_G \leqslant j_G^*$.

Proof. The function j_G^* is a metric by Lemma 2·3. The inequality $\delta_G \leqslant j_G^*$ is proved in the same way as the second inequality in Theorem 3·4.

3·7. *Notation.* Let G be a set for which j_G is defined. We define the open ball $D_j(x, t)$, with centre x and radius t, in this metric as the set

$$D_j(x, t) = \{z \in G\colon j_G(x, z) < t\},$$

for all $x \in G$ and $t > 0$. The balls $D_{\tilde{j}}(x, t)$ and $D_\delta(x, t)$ are defined similarly.

3·8. Theorem. *If $G \subsetneq \mathbf{R}^n$ is open, $x \in G$ and $t > 0$ then*

$$B^n(x, r) \subset D_j(x, t) \subset B^n(x, R),$$

where $r = (1 - e^{-t})d(x)$ and $R = (e^t - 1)d(x)$. The formulas for r and R are the best possible expressed in terms of t and $d(x)$ only.

Proof. Let $y \in G$ and write $j = j_G(x, y)$. Then

$$|x - y| = (1 - e^{-j}).(\min\{d(x), d(y)\} + |x - y|) \geqslant (1 - e^{-j})d(x)$$

and the first inclusion follows. The second inclusion follows from the inequality

$$|x - y| = (e^j - 1)\min\{d(x), d(y)\} \leqslant (e^j - 1)d(x).$$

To show that r and R are the best possible, let $G = \mathbf{R}^n \setminus \{0\}$, $x \in G$, and $t > 0$. Now, choose $y = e^t x$ and $z = e^{-t}x$. Then $j_G(x, y) = j_G(x, z) = t$, $|x - y| = R$, and $|x - z| = r$.

3·9. Theorem. *If $G \subsetneq \mathbf{R}^n$ is open, $x \in G$, and $t > 0$, then*

$$B^n(x, r) \subset D_{\tilde{j}}(x, t) \subset B^n(x, R),$$

where $r = (\tanh(t/2))d(x)$ and $R = ((e^t - 1)/2)d(x)$. The formulas for r and R are the best possible expressed in terms of t and $d(x)$ only.

Proof. By Remark 2·4, $j \leqslant \log((e^{\tilde{j}} + 1)/2)$. Then

$$D_{\tilde{j}}(x, t) \subset D_j\left(x, \log\left(\frac{e^t + 1}{2}\right)\right) \subset B^n\left(x, \frac{e^t - 1}{2}d(x)\right)$$

by Theorem 3·8.

https://doi.org/10.1017/S0305004198002904 Published online by Cambridge University Press

For the first inclusion, let $y \in G$ and denote $\tilde{j} = \tilde{j}_G(x,y)$. If $|x-y| < d(x)$, then

$$\begin{aligned}\tilde{j} &\leqslant \log\left(\left(1+\frac{|x-y|}{d(x)}\right)\left(1+\frac{|x-y|}{d(x)-|x-y|}\right)\right)\\ &= \log\frac{d(x)+|x-y|}{d(x)-|x-y|} = 2\operatorname{artanh}\frac{|x-y|}{d(x)},\end{aligned}$$

and $|x-y| \geqslant (\tanh(\tilde{j}/2))d(x)$. The case $|x-y| \geqslant d(x)$ is clear.

To show that r and R are the best possible, let $G = \mathbf{R}^n \setminus \{0\}$, $x \in G$, and $t > 0$. Now, choose $y = \big((e^t+1)/2\big)\,x$ and $z = \big(2/(e^t+1)\big)\,x$. Then $\tilde{j}_G(x,y) = \tilde{j}_G(x,z) = t$, $|x-y| = R$ and $|x-z| = r$.

3·10. COROLLARY. *If $G \subsetneq \mathbf{R}^n$ is an open set, $x \in G$ and $t > 0$ then*

$$B^n(x,r) \subset D_\delta(x,t) \subset B^n(x,R),$$

where $r = (\tanh(t/2))d(x)$ and $R = (e^t-1)d(x)$. If, in addition, $G \subset \mathbf{R}^n$ is a half-space, then there exists a point $y \in G$ for which $\delta_G(x,y) = t$ and $|x-y| = R$. If $G = B^n(x,d)$, $d > 0$, then $D_\delta(x,t) = B^n(x,r)$.

Proof. The inequalities $j_G \leqslant \delta_G \leqslant \tilde{j}_G$ imply that

$$D_{\tilde{j}}(x,t) \subset D_\delta(x,t) \subset D_j(x,t).$$

The inclusions follow now from Theorems 3·8 and 3·9. For the cases where G is a half-space or a ball, see [**Vu2**, pp. 22,25]. □

The following theorem gives a connection between the Apollonian metric α_G and the absolute ratio metric δ_G.

3·11. THEOREM. *Let $G \subset \bar{\mathbf{R}}^n$ be an open set with* card $\partial G \geqslant 2$. *Then*

$$\alpha_G \leqslant \delta_G \leqslant \log(e^{\alpha_G}+2) \leqslant \alpha_G + \log 3.$$

The first two inequalities give the best possible bounds for δ_G expressed in terms of α_G only.

Proof. Let $x, y \in G$. The first two inequalities follow if

$$|a,x,y,b| \leqslant 1 + |a,x,b,y| \leqslant |a,x,y,b| + 2 \qquad (*)$$

for all $a, b \in \partial G$. Since the absolute ratio is invariant under Möbius maps, we may assume that $b = \infty$. But then $(*)$ takes the form

$$\frac{|a-y|}{|a-x|} \leqslant 1 + \frac{|x-y|}{|a-x|} \leqslant \frac{|a-y|}{|a-x|} + 2,$$

and the result follows from the triangle inequality.

The third inequality is obvious. □

For curves $\gamma\colon [0,1] \to D$, we write $d(\gamma, \partial D) = \inf_{t\in[0,1]} d(\gamma(t), \partial D)$.

3·12. THEOREM. *The inequality $\sigma_D \geqslant \delta_D$ holds for any domain $D \subset \bar{\mathbf{R}}^n$ with* card $\partial D \geqslant 2$.

https://doi.org/10.1017/S0305004198002904 Published online by Cambridge University Press

Proof. Since both metrics are Möbius-invariant, we may assume that $D \subset \mathbf{R}^n$. Let $\gamma\colon [0,1] \to D$ be a rectifiable curve with end points $\gamma(0) = x$ and $\gamma(1) = y$. Set $d = d(\gamma, \partial D)$ and choose $\varepsilon \in (0, d)$. Now divide $[0,1]$ into m subintervals $[t_i, t_{i+1}]$, $i = 0, \ldots, m-1$, such that the diameters of the sets $\gamma([t_i, t_{i+1}])$ are less than ε. Set $x_i = \gamma(t_i)$ and $y_i = \gamma(t_{i+1})$ for all $i = 0, 1, \ldots, m-1$. Then

$$w_D(\gamma(t)) \geqslant \left(1 - \frac{\varepsilon}{d}\right)^2 \left(\sup_{a,b \in \partial D} \frac{|a-b|}{|x_i - a||y_i - b|} \right) = \left(1 - \frac{\varepsilon}{d}\right)^2 \frac{m_D(x_i, y_i)}{|x_i - y_i|}$$

for every $t \in [t_i, t_{i+1}]$. Now

$$\begin{aligned} \int_\gamma w_D(\gamma(t))|d\gamma(t)| &\geqslant \sum_{i=0}^{m-1} |x_i - y_i| \,.\, \left(1 - \frac{\varepsilon}{d}\right)^2 \,.\, \frac{m_D(x_i, y_i)}{|x_i - y_i|} \\ &\geqslant \left(1 - \frac{\varepsilon}{d}\right)^2 \sum_{i=0}^{m-1} \delta_D(x_i, y_i) \geqslant \left(1 - \frac{\varepsilon}{d}\right)^2 \delta_D(x, y), \end{aligned}$$

and we obtain $\sigma_D(x,y) \geqslant \delta_D(x,y)$ as ε tends to 0. □

In general, there need not be a constant $c \geqslant 1$ such that $\sigma_D \leqslant c\delta_D$. In Section 4 we shall study domains for which such a constant exists.

The first inequality in Theorem 3·13 below has been proved by Solynin [**So2**] recently. The second inequality follows from the inequality $w_D \leqslant 2\eta_D$ between the density functions of σ_D and ρ_D [**Le**, (1·5)].

3·13. Theorem. *If D is a simply connected domain of $\bar{\mathbf{C}}$ and* card $\partial D > 1$, *then* $\rho_D \leqslant \sigma_D \leqslant 2\rho_D$.

3·14. *Remarks.*

(1) In [**Be2**] there is an example of a domain D such that $\alpha_D(0,\infty) = 2\rho_D(0,\infty)$. This implies that the constant 2 in the second inequality of Theorem 3·13 is the best possible, because $\alpha_D \leqslant \sigma_D$.

(2) The definition of the hyperbolic metric can be extended to multiply connected domains of $\bar{\mathbf{C}}$ with at least 3 boundary points. The inequalities in (2·3) and in Theorem 3·13 also hold for these domains.

Next we shall study the behaviour of δ_G and σ_G under Euclidean bilipschitz maps. Using the definition of the absolute ratio, one can easily prove the following lemma.

3·15. Lemma. *Let $G, G' \subsetneq \mathbf{R}^n$ be open sets and $f\colon G \to G'$ an L-bilipschitz map. Then*

$$m_G(x,y) \leqslant L^4 m_{G'}(f(x), f(y)) \quad \textit{for all } x, y \in G. \tag{3·1}$$

The inequality (3·1) *also holds if G and G' are open subsets of $\bar{\mathbf{R}}^n$ with at least two boundary points and $f\colon G \to G'$ is an L-bilipschitz map with respect to the spherical metric.*

3·16. Theorem. *Let $G, G' \subsetneq \mathbf{R}^n$ be open sets and $f\colon G \to G'$ an L-bilipschitz map. Then*

$$\frac{1}{L^4}\delta_{G'}(f(x), f(y)) \leqslant \delta_G(x,y) \leqslant L^4 \delta_{G'}(f(x), f(y)) \quad \textit{for all } x, y \in G. \tag{3·2}$$

https://doi.org/10.1017/S0305004198002904 Published online by Cambridge University Press

The inequalities (3·2) *also hold if* G *and* G' *are open subsets of* $\bar{\mathbf{R}}^n$ *with at least two boundary points and* $f\colon G \to G'$ *is an* L*-bilipschitz map with respect to the spherical metric.*

Proof. The theorem follows from Lemma 3·15 and the Bernoulli inequality (2·6).

3·17. LEMMA. *Let* $D \subset \bar{\mathbf{R}}^n$ *be a domain with* card $\partial D \geqslant 2$. *For every* $\varepsilon \in (0, \frac{1}{2})$, *if* $x, y \in D$ *and* $\delta_D(x,y) < \varepsilon$, *then* $\sigma_D(x,y) \leqslant c(\varepsilon)\delta_D(x,y)$, *where* $c(\varepsilon) = (1-\varepsilon)/(1-2\varepsilon)^2$. *Also,* $c(0+) = 1$.

Proof. We can assume that $D \subset \mathbf{R}^n$ and $d(x) \leqslant d(y)$. Let $\varepsilon \in (0, \frac{1}{2})$ and $j_D(x,y) \leqslant \delta_D(x,y) < \varepsilon$. Then $|x-y| < d(x)$ so that the segment $[x,y]$ is contained in D, and

$$\begin{aligned}\sigma_D(x,y) &\leqslant |x-y| \sup_{a,b\in\partial D} \frac{|a-b|}{|x-a||y-b|}\,\frac{|x-a|}{|x-a|-|x-y|}\,\frac{|y-b|}{|y-b|-|x-y|}\\ &\leqslant m_D(x,y)\left(\frac{d(x)}{d(x)-|x-y|}\right)^2 \leqslant \frac{e^\varepsilon - 1}{\varepsilon}\,\delta_D(x,y)\cdot\frac{1}{(2-e^\varepsilon)^2}.\end{aligned}$$

The formula for $c(\varepsilon)$ follows now from the inequality $e^\varepsilon < 1/(1-\varepsilon)$.

3·18. THEOREM. *Let* $D, D' \subsetneq \mathbf{R}^n$ *be domains and let* $f\colon D \to D'$ *be an* L*-bilipschitz map. Then*

$$\frac{1}{L^4}\sigma_{D'}(f(x),f(y)) \leqslant \sigma_D(x,y) \leqslant L^4\sigma_{D'}(f(x),f(y)) \quad \textit{for all } x,y \in D. \tag{3·3}$$

The inequalities (3·3) *also hold if* D *and* D' *are subdomains of* $\bar{\mathbf{R}}^n$ *with at least two boundary points and* $f\colon D \to D'$ *is an* L*-bilipschitz map with respect to the spherical metric.*

Proof. Since σ is Möbius-invariant, we may assume that $D, D' \subset \mathbf{R}^n$. Let $\gamma\colon [0,1] \to D$ be a rectifiable curve such that $\gamma(0) = x$ and $\gamma(1) = y$. Set $d = d(\gamma, \partial D)$. For $\varepsilon \in (0, \frac{1}{2})$, divide $[0,1]$ into subintervals $[t_i, t_{i+1}]$, $0 = t_0 < t_1 < \ldots < t_m = 1$, so that $|\gamma(t_i) - \gamma(t_{i+1})| < \varepsilon d/2$ for all $i = 0, 1, \ldots, m-1$. Let $x_i = \gamma(t_i)$ and $y_i = \gamma(t_{i+1})$. Then $\delta_D(x_i, y_i) \leqslant 2j_D(x_i, y_i) < \varepsilon$, and, by Lemma 3·17,

$$\sigma_D(x_i, y_i) \leqslant c(\varepsilon)\delta_D(x_i, y_i)$$

for all $i = 0, 1, \ldots, m-1$. Now

$$\begin{aligned}\sigma_D(x,y) &\leqslant \sum_{i=0}^{m-1}\sigma_D(x_i,y_i) \leqslant L^4c(\varepsilon)\sum_{i=0}^{m-1}\delta_{D'}(f(x_i),f(y_i))\\ &\leqslant L^4c(\varepsilon)\sum_{i=0}^{m-1}\sigma_{D'}(f(x_i),f(y_i)) \leqslant L^4c(\varepsilon)\int_{f(\gamma)} w_{D'}(t)|dt|.\end{aligned}$$

Next, we can choose γ so that $\int_{f(\gamma)} w_{D'}(t)|dt|$ is arbitrarily close to $\sigma_{D'}(f(x), f(y))$. Then, letting ε tend to 0, we obtain

$$\sigma_D(x,y) \leqslant L^4\sigma_{D'}(f(x),f(y)).$$

The first inequality in (3·3) follows from the second one applied to f^{-1}.

3·19. *Lipschitz maps with respect to* δ_G. Next we shall study maps $f\colon G \to G' = fG$

https://doi.org/10.1017/S0305004198002904 Published online by Cambridge University Press

that are bilipschitz with respect to the absolute ratio metric, namely there exists a constant $L \geqslant 1$ such that

$$\frac{1}{L}\delta_G(x,y) \leqslant \delta_{G'}(f(x),f(y)) \leqslant L\delta_G(x,y) \quad \text{for all } x,y \in G.$$

It follows from Corollary 3·10 that these maps are homeomorphisms.

Now we shall find an upper bound for the linear dilatation $H(x,f)$, defined in 2·16, when f is a bilipschitz map with respect to δ.

3·20. Lemma. *Let $G \subsetneq \mathbf{R}^n$ be an open set and $x_0 \in G$. Then*

$$\lim_{r\to 0+} \frac{\delta_G(x,x_0)}{\delta_G(y,x_0)} = 1,$$

where $|x-x_0| = |y-x_0| = r \in (0, d(x_0))$.

Proof. Let $a,b,c,d \in \partial G$ be points for which $m_G(x,x_0) = |a,x,b,x_0|$ and $m_G(y,x_0) = |c,y,d,x_0|$. Then

$$m_G(x,x_0) = |a,y,b,x_0|\,.\,\frac{|a-y|}{|a-x|} \leqslant m_G(y,x_0)\,.\,\left(1+\frac{2r}{d(x_0)-r}\right).$$

Similarly,

$$m_G(x,x_0) \geqslant m_G(y,x_0)\,.\,\left(1-\frac{2r}{d(x_0)-r}\right).$$

Using the Bernoulli inequalities (2·6) and (2·7), we have

$$1-\frac{2r}{d(x_0)-r} \leqslant \frac{\delta_G(x,x_0)}{\delta_G(y,x_0)} \leqslant 1+\frac{2r}{d(x_0)-r}.$$

3·21. Theorem. *Let $G, G' \subsetneq \mathbf{R}^n$ be open sets and $f: G \to G'$ an L-bilipschitz map with respect to the metric δ_G. Then $H(x,f) \leqslant L^2$ for all $x \in G$.*

Proof. Let $x_0 \in G$ and $r \in (0, d(x_0))$. Choose $x, y \in S^{n-1}(x_0, r)$ such that

$$m \equiv l(x_0,f,r) = |f(x)-f(x_0)|$$

and

$$M \equiv L(x_0,f,r) = |f(y)-f(x_0)|,$$

and let $a,b \in \partial G'$ be points for which $m_{G'}(f(x),f(x_0)) = |a,f(x),b,f(x_0)|$. Then

$$\begin{aligned}\frac{M}{m} &= \frac{|a,f(y),b,f(x_0)|}{|a,f(x),b,f(x_0)|}\cdot\frac{|a-f(y)|}{|a-f(x)|}\\ &\leqslant \frac{m_{G'}(f(y),f(x_0))}{m_{G'}(f(x),f(x_0))}\cdot\left(1+\frac{m+M}{d(f(x_0))-m}\right),\end{aligned}$$

and

$$\lim_{r\to 0+}\frac{M}{m} \leqslant \limsup_{r\to 0+}\frac{\exp\left(\delta_{G'}(f(y),f(x_0))\right)-1}{\exp\left(\delta_{G'}(f(x),f(x_0))\right)-1} \leqslant \lim_{r\to 0+}\frac{L\delta_G(y,x_0)}{\delta_G(x,x_0)/L} = L^2$$

by Lemma 3·20. □

https://doi.org/10.1017/S0305004198002904 Published online by Cambridge University Press

It follows from the previous theorem and [**Vä1**, theorem 34.1] that bilipschitz maps with respect to δ_G are quasiconformal and that isometries in δ_G are conformal.

In Theorem 3·16 it has been shown that bilipschitz maps with respect to the Euclidean metric are also bilipschitz in the metric δ_G. However, the converse is not true even in the case of bounded domains. For example, the function $f\colon \mathbf{B}^2 \setminus \{0\} \to \mathbf{B}^2 \setminus \{0\}$, $f(x) = |x|\,.\,x$ is not bilipschitz with respect to the Euclidean metric, but it can be shown that it is bilipschitz with respect to the metric $\delta_{\mathbf{B}^2\setminus\{0\}}$.

4. *Bounded, convex and uniform domains*

In this section we study the metrics δ_G, α_G, and σ_G when G has some additional properties.

4·1. THEOREM. *If G is a bounded open subset of $\mathbf{R}^n$, then*

$$\delta_G(x,y) \geqslant \log\left(1 + \frac{d(\partial G)}{d(\partial G) - \max\{d(x),d(y)\}}\,\frac{|x-y|}{\min\{d(x),d(y)\}}\right) \geqslant j_G(x,y)$$

for all $x, y \in G$, where $d(\partial G)$ is the diameter of the set ∂G.

Proof. Assume that $d(x) \leqslant d(y)$ and let $a \in \partial G$ be a point for which $d(x) = |x-a|$. Then there exists a point $b \in \partial G$ on the line determined by a and y, such that $|a-b| > |a-y|$. Now

$$|a,x,b,y| = \frac{|x-y|}{d(x)}\left(\frac{|a-b|}{|a-b|-|y-a|}\right) \geqslant \frac{|x-y|}{d(x)}\,\frac{d(\partial G)}{d(\partial G)-d(y)}.$$

Thus

$$\delta_G(x,y) \geqslant \log\left(1 + \frac{d(\partial G)}{d(\partial G)-d(y)}\,\frac{|x-y|}{d(x)}\right) \geqslant j_G(x,y). \quad \square$$

If $G = \mathbf{B}^n$, then Theorem 4·1 gives an inequality

$$\delta_{\mathbf{B}^n}(x,y) \geqslant \log\left(1 + \frac{2|x-y|}{(1+|y|)(1-|x|)}\right).$$

This lower bound has the property

$$\lim_{y\to x} \frac{\log\,(1 + 2|x-y|/((1+|y|)(1-|x|)))}{\delta_{\mathbf{B}^n}(x,y)} = 1$$

for every fixed $x \in \mathbf{B}^n$.

In [**Be2**] it has been proved that if D is a simply connected bounded plane domain such that the inequality $\alpha_D \leqslant \rho_D$ or $j_D \leqslant \rho_D$ holds, then D is convex. The next theorem shows that the former inequality implies the latter inequality.

4·2. THEOREM. *Let $D \subsetneq \mathbf{R}^n$ be a convex domain; then $j_D \leqslant \alpha_D$.*

Proof. Fix $x, y \in D$ and write

$$q_1 = \sup_{u\in\partial D}\frac{|u-y|}{|u-x|}, \quad q_2 = \sup_{u\in\partial D}\frac{|u-x|}{|u-y|}.$$

Then $\alpha_D(x,y) = \log(q_1 q_2)$. We may also assume that $|x-y| = 1$ and $q_1 \geqslant q_2 \geqslant 1$. Define the sets

$$B_1 = \{u \in \mathbf{R}^n\colon |u-y| = q_1\,.\,|u-x|\}$$

https://doi.org/10.1017/S0305004198002904 Published online by Cambridge University Press

and

$$B_2 = \{u \in \mathbf{R}^n \colon |u - x| = q_2 \,.\, |u - y|\}.$$

Now $B_1 = B^n(x_0, r_1)$, where $r_1 = q_1/(q_1^2 - 1)$ and $|x - x_0| = 1/(q_1^2 - 1)$. Similarly $B_2 = B^n(y_0, r_2)$, where $r_2 = q_2/(q_2^2 - 1)$ and $|y - y_0| = 1/(q_2^2 - 1)$, if $q_2 > 1$. Now

$$\min\{d(x), d(y)\} \geqslant r_1 + \frac{r_2 - r_1}{|x_0 - y_0|} \,.\, |x - x_0| = \frac{q_2}{q_1 q_2 - 1}$$

if $q_2 > 1$, and $\min\{d(x), d(y)\} \geqslant 1/(q_1 - 1)$ if $q_2 = 1$. Thus

$$\frac{j_D(x, y)}{\alpha_D(x, y)} \leqslant \frac{\log\,(1 + (q_1 q_2 - 1)/q_2)}{\log\,(q_1 q_2)} \leqslant 1. \quad \square$$

Equality $j_D(x, y) = \alpha_D(x, y)$ holds if $D = \mathbf{H}^n$, $x = re_n$, and $y = se_n$, where $r, s > 0$.

4·3. *Definition.* A domain $D \subsetneq \mathbf{R}^n$ is said to be uniform if there exists a constant $c \geqslant 1$ such that

$$k_D(x, y) \leqslant c j_D(x, y) \quad \text{for all } x, y \in D$$

(see, for example, [**Vä2**]).

If $D, D' \subsetneq \mathbf{R}^n$ are domains and $f \colon D \to D'$ is L-bilipschitz , then it is easy to prove that

$$\frac{1}{L^2} k_D(x, y) \leqslant k_{D'}(f(x), f(y)) \leqslant L^2 k_D(x, y)$$

and, by the Bernoulli inequalities (2·6) and (2·7),

$$\frac{1}{L^2} j_D(x, y) \leqslant j_{D'}(f(x), f(y)) \leqslant L^2 j_D(x, y).$$

Thus D is uniform if and only if D' is. Since $j_D \leqslant \delta_D \leqslant 2j_D$ and $k_D \leqslant \sigma_D \leqslant 2k_D$, we can define the uniformity using the metrics σ_D and δ_D as well.

4·4. *Definition.* A domain $D \subset \bar{\mathbf{R}}^n$ with card $\partial D \geqslant 2$ is said to be c-uniform, $c \geqslant 1$, if

$$\sigma_D(x, y) \leqslant c \delta_D(x, y) \quad \text{for all } x, y \in D.$$

4·5. *Remarks.*

(1) Definitions 4·3 and 4·4 have the following connection: if a domain $D \subset \mathbf{R}^n$ is uniform in the sense of Definition 4·3 with constant c, then it is $2c$-uniform in the sense of Definition 4·4. If a domain $D \subset \mathbf{R}^n$ is c-uniform in the sense of Definition 4·4, then it is uniform in the sense of Definition 4·3 with constant $2c$.

(2) The sets $\mathbf{B}^n$ and $\mathbf{H}^n$ are 1-uniform, whereas the best possible constant c for $D = \mathbf{B}^n$ and $D = \mathbf{H}^n$ in Definition 4·3 is 2.

Combining Theorems 3·16 and 3·18 we obtain the following result.

4·6. Theorem. *Let $D, D' \subsetneq \mathbf{R}^n$ be domains. If $f \colon D \to D'$ is an L-bilipschitz map, then*

$$\frac{\sigma_{D'}(f(x), f(y))}{\delta_{D'}(f(x), f(y))} \leqslant L^8 \frac{\sigma_D(x, y)}{\delta_D(x, y)}$$

https://doi.org/10.1017/S0305004198002904 Published online by Cambridge University Press

for all $x, y \in D$, $x \neq y$. The same is also true if D and D' are domains of $\bar{\mathbf{R}}^n$ with card ∂D, card $\partial D' \geqslant 2$, *and if $f\colon D \to D'$ is an L-bilipschitz map with respect to the spherical metric.*

4·7. COROLLARY. *The image of a c-uniform domain (in the sense of Definition* 4·4*) under an L-bilipschitz map with respect to the Euclidean or spherical metric is cL^8-uniform.*

5. *Quasi-invariance of δ*

In this section we study the behaviour of the absolute ratio metric δ_G under quasiconformal maps of $\bar{\mathbf{R}}^n$. For this, we need to define a distortion function $\eta_{K,n}$.

5·1. *Definition.* Let D be a domain of $\mathbf{R}^n$. An injective mapping $f\colon D \to \mathbf{R}^n$ is η-quasisymmetric, if there exists an increasing homeomorphism $\eta\colon [0,\infty) \to [0,\infty)$ such that

$$\frac{|f(a)-f(b)|}{|f(a)-f(c)|} \leqslant \eta\left(\frac{|a-b|}{|a-c|}\right) \tag{5·1}$$

for all $a, b, c \in D$ with $a \neq c$ [**TV**].

5·2. THEOREM ([**Vu3**, theorem 1.8]). *For all $K \geqslant 1$ and $n \geqslant 2$ there exists a homeomorphism $\eta_{K,n}\colon [0,\infty) \to [0,\infty)$, such that every K-*qc *map of $\mathbf{R}^n$ is $\eta_{K,n}$-quasisymmetric and*

$$\eta_{K,n}(1) \leqslant \exp\left(6(K+1)^2\sqrt{K-1}\right), \tag{5·2}$$

$$\eta_{K,n}(t) \leqslant \eta_{K,n}(1)\varphi_{K,n}(t), \quad 0 \leqslant t < 1, \tag{5·3}$$

$$\eta_{K,n}(t) \leqslant \eta_{K,n}(1)/\varphi_{1/K,n}(1/t), \quad t > 1. \tag{5·4}$$

Hereafter, $\eta_{K,n}$ will be a function that satisfies the conditions in Theorem 5·2, when $n \geqslant 3$ and $K \geqslant 1$. When $n = 2$, we define

$$\eta_K(t) \equiv \eta_{K,2}(t) = \frac{s^2}{1-s^2}, \quad s = \varphi_K\left(\sqrt{\frac{t}{1+t}}\right)$$

for all $t > 0$ and $K > 0$. Agard [**Ag**] has shown that this function gives the smallest possible value of $\eta_{K,2}(t)$ in Theorem 5·2 at every point $t > 0$.

The following theorem is easy to prove [**AVV2**, theorem 14·18].

5·3. THEOREM. *Let $f\colon \bar{\mathbf{R}}^n \to \bar{\mathbf{R}}^n$ be K-*qc. *Then*

$$|f(a), f(b), f(c), f(d)| \leqslant \eta_{K,n}(|a, b, c, d|) \tag{5·5}$$

for all distinct points $a, b, c, d \in \bar{\mathbf{R}}^n$.

Next we prove Theorem 1·2 concerning the quasi-invariance of δ_G. See also [**GO**, pp. 62–65] for quasi-invariance properties of the metrics k_G and j_G. We need the following inequalities:

5·4. THEOREM ([**Vu2**, theorem 7·47]). *For $n \geqslant 2$, $K \geqslant 1$, and $0 \leqslant r \leqslant 1$,*

$$\varphi_{K,n}(r) \leqslant \lambda_n^{1-\alpha} r^\alpha, \quad \alpha = K^{1/(1-n)}, \tag{5·6}$$

$$\varphi_{1/K,n}(r) \geqslant \lambda_n^{1-\beta} r^\beta, \quad \beta = K^{1/(n-1)}. \tag{5·7}$$

https://doi.org/10.1017/S0305004198002904 Published online by Cambridge University Press

5·5. *Proof of Theorem* 1·2. By Theorem 5·3 and the definition of δ,

$$\delta_{G'}(f(x), f(y)) \leqslant \log(1 + \eta_{K,n}(e^{\delta_G(x,y)} - 1)). \tag{5·8}$$

Case 1. Suppose that $\delta_G(x, y) > \log 2$. Using the inequalities (5·4), (5·7), and the Bernoulli inequality (2·6), we get

$$\begin{aligned}\delta_{G'} &\leqslant \log(1 + \eta_{K,n}(1)\lambda_n^{\beta-1}(e^{\delta_G} - 1)^\beta) \\ &\leqslant \eta_{K,n}(1)\lambda_n^{\beta-1}\log(e^{\delta_G \cdot \beta}) = \lambda_n^{\beta-1}\beta\,\eta_{K,n}(1)\delta_G.\end{aligned}$$

Case 2. Suppose that $\delta_G(x, y) \leqslant \log 2$. Using the inequalities (5·3), (5·6), the Bernoulli inequality (2·6), and Lemma 2·19, we obtain

$$\delta_{G'} \leqslant \log(1 + \eta_{K,n}(1)\lambda_n^{1-\alpha}(e^{\delta_G} - 1)^\alpha) \leqslant \lambda_n^{1-\alpha}\eta_{K,n}(1)\delta_G^\alpha.$$

5·6. Corollary. *If f, G, and G' are as in Theorem* 1·2, *then*

$$\delta_{G'}(f(x), f(y)) \geqslant \frac{\lambda_n^{1-\beta}\alpha}{\eta_{K,n}(1)^\beta}\min\{\delta_G(x, y), \delta_G(x, y)^\beta\}$$

for all $x, y \in G$.

Proof. If we replace f by f^{-1} in the proof of Theorem 1·2, it follows that for each x and y either

$$\delta_G(x, y) \leqslant \lambda_n^{\beta-1}\beta\eta_{K,n}(1)\delta_{G'}(f(x), f(y))$$

or

$$\delta_G(x, y) \leqslant \lambda_n^{1-\alpha}\eta_{K,n}(1)\delta_{G'}(f(x), f(y))^\alpha.$$

The statement follows when we combine these inequalities. □

The following result has recently appeared (see [**AQV**]).

5·7. Theorem. *For $K > 1$, the function $f(t) = \eta_K(t) - \log(1 + \eta_K(e^t - 1))$ is strictly increasing on $(0, \infty)$. In particular,*

$$\eta_K(t) > \log(1 + \eta_K(e^t - 1)) \tag{5·9}$$

for all $K > 1$ and $t > 0$.

Using (5·8) and (5·9) we obtain the following corollary.

5·8. Corollary. *If $f\colon \bar{\mathbf{R}}^2 \to \bar{\mathbf{R}}^2$ is a K-qc map, G and $G' = fG$ open sets of $\bar{\mathbf{R}}^2$ with* card $\partial G \geqslant 2$, *and $x, y \in G$, then*

$$\delta_{G'}(f(x), f(y)) \leqslant \eta_K(\delta_G(x, y)).$$

Next we shall study the conformal invariants λ_D and μ_D, defined in 2·12, especially when D is a quasiball. Recall that quasiballs were defined in Section 1 after Theorem 1·2.

For the proof of Theorem 5·11, we need some preliminary results.

5·9. Theorem ([**Vu2**, theorem 8·6]). *For all distinct $x, y \in \mathbf{B}^n$,*

$$\mu_{\mathbf{B}^n}(x, y) = \gamma_n\left(\frac{1}{\tanh(\rho(x, y)/2)}\right), \tag{5·10}$$

https://doi.org/10.1017/S0305004198002904 Published online by Cambridge University Press

$$\lambda_{\mathbf{B}^n}(x,y) = \tfrac{1}{2}\tau_n\left(\sinh^2\frac{\rho(x,y)}{2}\right). \tag{5·11}$$

The next theorem follows from the definitions of μ_D, λ_D, and quasiconformality.

5·10. THEOREM. *If $f\colon D \to D'$ is a K-qc map, where D and D' are proper subdomains of $\bar{\mathbf{R}}^n$, then*

$$\frac{\mu_D(x,y)}{K} \leqslant \mu_{D'}(f(x),f(y)) \leqslant K\mu_D(x,y) \quad \textit{for all } x,y \in D.$$

If also card $\partial D \geqslant 2$, *then*

$$\frac{\lambda_D(x,y)}{K} \leqslant \lambda_{D'}(f(x),f(y)) \leqslant K\lambda_D(x,y)$$

for all distinct $x,y \in D$.

5·11. THEOREM. *Let $D \subset \bar{\mathbf{R}}^n$ be a K-quasiball and let $x,y \in D$ be distinct points. Then*

$$\frac{1}{K}\gamma_n\left(\frac{1}{\tanh(r/2)}\right) \leqslant \mu_D(x,y) \leqslant K\gamma_n\left(\frac{1}{\tanh(s/2)}\right) \tag{5·12}$$

and

$$\frac{1}{2K}\tau_n\left(\sinh^2\frac{s}{2}\right) \leqslant \lambda_D(x,y) \leqslant \frac{K}{2}\tau_n\left(\sinh^2\frac{r}{2}\right), \tag{5·13}$$

where

$$r = \log(1+\eta_{K,n}^{-1}(e^{\delta_D(x,y)}-1)),$$
$$s = \log(1+\eta_{K,n}(e^{\delta_D(x,y)}-1)).$$

If $n = 2$, then also

$$\frac{1}{K}\gamma_2\left(1/\tanh\frac{\eta_K^{-1}(\delta_D)}{2}\right) \leqslant \mu_D \leqslant K\gamma_2\left(1/\tanh\frac{\eta_K(\delta_D)}{2}\right) \tag{5·14}$$

and

$$\frac{1}{2K}\tau_2\left(\sinh^2\frac{\eta_K(\delta_D)}{2}\right) \leqslant \lambda_D \leqslant \frac{K}{2}\tau_2\left(\sinh^2\frac{\eta_K^{-1}(\delta_D)}{2}\right). \tag{5·15}$$

Proof. Let $f\colon \bar{\mathbf{R}}^n \to \bar{\mathbf{R}}^n$ be a K-qc map for which $f(D) = \mathbf{B}^n$. Theorem 5·3 implies that

$$\rho(f(x),f(y)) \leqslant \log(1+\eta_{K,n}(e^{\delta_D(x,y)}-1)),$$
$$\rho(f(x),f(y)) \geqslant \log(1+\eta_{K,n}^{-1}(e^{\delta_D(x,y)}-1)).$$

The inequalities (5·12) and (5·13) now follow from Theorems 5·9 and 5·10. The inequalities in the case $n = 2$ follow from (5·12) and (5·13), since $r \geqslant \eta_K^{-1}(\delta_D(x,y))$ and $s \leqslant \eta_K(\delta_D(x,y))$ by (5·9). □

Note that Theorem 5·11 reduces to Theorem 5·9 when $K = 1$.

5·12. THEOREM (Generalized Schwarz lemma). *Let $D_i \subset \bar{\mathbf{R}}^n$ be a K_i-quasiball, $i = 1,2$ and $f\colon D_1 \to fD_1$ a K-qc map such that $fD_1 \subset D_2$. Then*

$$\tanh\frac{r}{2} \leqslant \varphi_{K',n}\left(\tanh\frac{s}{2}\right), \tag{5·16}$$

https://doi.org/10.1017/S0305004198002904 Published online by Cambridge University Press

where

$$r = \log\left(1 + \eta_{K_2,n}^{-1}(\exp(\delta_{D_2}(f(x), f(y))) - 1)\right),$$
$$s = \log\left(1 + \eta_{K_1,n}(\exp(\delta_{D_1}(x, y)) - 1)\right),$$

$K' = KK_1K_2$, *and* $x, y \in D_1$. *In particular, if* $K_1 = K_2 = 1$, *this reduces to the Schwarz lemma* 2·15. *If* $n = 2$, *then also*

$$\tanh\frac{\eta_{K_2}^{-1}(\delta_{D_2}(f(x), f(y)))}{2} \leqslant \varphi_{K'}\left(\tanh\frac{\eta_{K_1}(\delta_{D_1}(x, y))}{2}\right). \tag{5·17}$$

Proof. By (5·12), Remark 2·13(1) and Theorem 5·10,

$$\frac{1}{K_2}\gamma_n\left(\frac{1}{\tanh(r/2)}\right) \leqslant \mu_{D_2}(f(x), f(y)) \leqslant K\mu_{D_1}(x, y) \leqslant KK_1\gamma_n\left(\frac{1}{\tanh(s/2)}\right).$$

Then

$$\tanh\frac{r}{2} \leqslant \frac{1}{\gamma_n^{-1}(KK_1K_2\gamma_n(1/\tanh(s/2)))} = \varphi_{K',n}\left(\tanh\frac{s}{2}\right)$$

by formula (2·5). For $n = 2$, the inequality (5·17) follows, since $r \geqslant \eta_{K_2}^{-1}(\delta_{D_2}(f(x), f(y)))$ and $s \leqslant \eta_{K_1}(\delta_{D_1}(x, y))$ by (5·9).

6. *Inequalities between Möbius-invariants*

Next we shall give some inequalities between the conformal invariants μ_D and λ_D and the absolute ratio metric δ_D. For this we need the following lemma.

6·1. Lemma ([**Vu2**, lemma 7.35]). *Let* $R = R(E, F)$ *be a ring in* $\bar{\mathbf{R}}^n$ *and let* $a, b \in E$ *and* $c, d \in F$ *be distinct points. Then*

$$\operatorname{cap} R \geqslant \tau_n(|a, b, c, d|).$$

Here equality holds if $E = [a, b]$ *and* $F = [c, d]$, *where* $a = s_2e_1$, $b = s_1e_1$, $c = s_3e_1$, $d = s_4e_1$, *and* $s_1 < s_2 < s_3 < s_4$.

The inequality (6·2) in the next theorem is due to S.-L. Qiu.

6·2. Theorem. *Let* D *be a domain in* $\bar{\mathbf{R}}^n$, *for which* $\operatorname{card}\partial D \geqslant 2$ *and* ∂D *is connected. Then, for distinct points* $x, y \in D$,

$$\mu_D(x, y) \geqslant \tau_n\left(\frac{1}{e^{\delta_D(x,y)} - 1}\right). \tag{6·1}$$

Equality holds if $x = s_1e_1$, $y = s_2e_1$ *and* $D = \bar{\mathbf{R}}^n \setminus [s_3e_1, s_4e_1]$, *where* $s_1 < s_2 < s_3 < s_4$. *If* $n = 2$, *then*

$$\delta_D(x, y) \leqslant \frac{\pi}{2}\mu_D(x, y) \quad \textit{for all } x, y \in D. \tag{6·2}$$

Proof. By Lemma 6·1,

$$\mu_D(x, y) \geqslant \sup_{a,b\in\partial D} \tau_n(|x, y, a, b|) = \tau_n\left(\frac{1}{m_D(x, y)}\right),$$

thus proving (6·1).

For the equality in (6·1), we may assume that $s_1 = -1$, $s_2 = 0$, and $s_4 = \infty$, since

https://doi.org/10.1017/S0305004198002904 Published online by Cambridge University Press

μ_D and m_D are Möbius-invariant. Then we observe by spherical symmetrization that $\mu_D(x,y) = \tau_n(s_3)$. Also

$$m_D(x,y) = \sup_{a,b\in[s_3e_1,\infty]} \frac{|a-b|}{|a||b+e_1|} = \frac{1}{s_3},$$

and equality holds.

If $n = 2$, then

$$\tau_n\left(\frac{1}{e^{\delta_D(x,y)}-1}\right) = \frac{\pi}{\mu\left(\sqrt{1-\exp\left(-\delta_D(x,y)\right)}\right)} = \frac{4}{\pi}\mu\left(\exp\left(-\frac{\delta_D(x,y)}{2}\right)\right),$$

and (6·2) follows from the well-known inequality $\mu(r) \geqslant \log(1/r)$ (see e.g. [**Vu2**, pp. 66–67].

6·3. LEMMA ([**Vu4**, theorem 1·7]). *For distinct points $x, y \in \mathbf{R}^n \setminus \{0\}$,*

$$\tau_n\left(\frac{|x-y|}{\min\{|x|,|y|\}}\right) \leqslant \lambda_{\mathbf{R}^n\setminus\{0\}}(x,y) \leqslant \tau_n\left(\frac{|x-y|+||x|-|y||}{2\min\{|x|,|y|\}}\right).$$

6·4. THEOREM. *Let $D = \mathbf{R}^n \setminus \{0\}$. For distinct points $x, y \in D$,*

$$\tau_n(m_D(x,y)) \leqslant \lambda_D(x,y) \leqslant \tau_n\left(\tfrac{1}{2}m_D(x,y)\right) \leqslant \sqrt{2}\tau_n(m_D(x,y)).$$

Proof. Since $m_D(x,y) = |x-y|/\min\{|x|,|y|\}$, the first two inequalities follow from Lemma 6·3. The last inequality follows from [**AVV1**, theorem 1·8(3)]. □

6·5. THEOREM. *Let $D \subset \bar{\mathbf{R}}^n$ be a domain with* card $\partial D \geqslant 2$. *Then*

$$\lambda_D(x,y) \leqslant \tau_n\left(\tfrac{1}{2}m_D(x,y)\right) \leqslant \sqrt{2}\tau_n(m_D(x,y)).$$

Proof. Choose $a, b \in \partial D$ such that $m_D(x,y) = |a,x,b,y|$. Then, by Remark 2·13(1) and Theorem 6·4,

$$\lambda_D(x,y) \leqslant \lambda_{\mathbf{R}^n\setminus\{a,b\}}(x,y) \leqslant \tau_n\left(\tfrac{1}{2}m_D(x,y)\right).$$

6·6. *Notation.* If E and F are distinct sets in $\bar{\mathbf{R}}^n$, then we see from the definition of the absolute ratio that

$$\sup_{\substack{a,b\in E\\ c,d\in F}} |a,c,b,d| = \sup_{\substack{a,b\in \partial E\\ c,d\in \partial F}} |a,c,b,d|.$$

We shall write

$$\delta_{E,F} = \delta_{\bar{\mathbf{R}}^n\setminus E}(F) = \delta_{\bar{\mathbf{R}}^n\setminus F}(E) = \log\left(1 + \sup_{\substack{a,b\in \partial E\\ c,d\in \partial F}} |a,c,b,d|\right).$$

In the next three theorems we shall find lower and upper bounds for the capacity of a ring $R(E,F)$ expressed in terms of $\delta_{E,F}$. See also [**Vu2**, pp. 96–97] for the corresponding bounds in terms of $\min\{j_{\mathbf{R}^n\setminus E}(F), j_{\mathbf{R}^n\setminus F}(E)\}$.

6·7. THEOREM. *If $R(E,F)$ is a ring with* card E, card $F \geqslant 2$, *then*

$$\operatorname{cap} R(E,F) \geqslant \tau_n\left(\frac{1}{e^{\delta_{E,F}}-1}\right).$$

Equality holds if $E = [-e_1, 0]$, $F = [te_1, \infty]$, and $t > 0$.

https://doi.org/10.1017/S0305004198002904 Published online by Cambridge University Press

Proof. By Lemma 6·1,

$$\operatorname{cap} R(E,F) \geqslant \sup_{\substack{a,b\in E\\ c,d\in F}} \tau_n(|a,b,c,d|) = \tau_n\left(\frac{1}{\sup\limits_{\substack{a,b\in E\\ c,d\in F}} |a,c,b,d|}\right) = \tau_n\left(\frac{1}{e^{\delta_{E,F}}-1}\right).$$

The equality cases also follow from Lemma 6·1. □

We need the following lemma in the proof of Theorem 6·9.

6·8. Lemma ([**Vä1**, p. 22]). *If* $0 < a < b < \infty$, *then*

$$\mathsf{M}(\Delta(S^{n-1}(a), S^{n-1}(b); B^n(b) \setminus \bar{B}^n(a))) = \omega_{n-1}\left(\log\frac{b}{a}\right)^{1-n}.$$

6·9. Theorem. *Let* $E, F \subset \bar{\mathbf{R}}^n$, $\bar{E} \cap \bar{F} = \varnothing$, $\operatorname{card} E, \operatorname{card} F \geqslant 2$, *and* $\delta_{E,F} < \log 2$. *Then*

$$\mathsf{M}(\Delta(E,F)) \leqslant \omega_{n-1}\left(\log\frac{1}{e^{\delta_{E,F}}-1}\right)^{1-n}.$$

Proof. Let $a, b \in \partial E$ and $c, d \in \partial F$ be points, such that

$$|a,c,b,d| = e^{\delta_{E,F}} - 1.$$

By the Möbius-invariance of $\mathsf{M}(\Delta(E,F))$ and $\delta_{E,F}$, we can assume that $a = e_1$, $b = 0$ and $c = \infty$. Now

$$|d| = \frac{1}{|e_1,\infty,0,d|} = \frac{1}{e^{\delta_{E,F}}-1} > 1$$

and

$$\frac{1}{|x|} = |e_1,\infty,0,x| \leqslant e^{\delta_{E,F}} - 1 \quad \text{for all } x \in F,$$

and therefore $F \subset \bar{\mathbf{R}}^n \setminus B^n(0, 1/(e^{\delta_{E,F}}-1))$. Also

$$\frac{|y|}{|d|} = |y,\infty,0,d| \leqslant e^{\delta_{E,F}} - 1 \quad \text{for all } y \in E,$$

and we have $E \subset \bar{B}^n(1)$. Then, by Lemma 6·8,

$$\begin{aligned}\mathsf{M}(\Delta(E,F)) &\leqslant \mathsf{M}\left(\Delta(S^{n-1}(1), S^{n-1}(1/(e^{\delta_{E,F}}-1)))\right)\\ &= \omega_{n-1}\left(\log\frac{1}{e^{\delta_{E,F}}-1}\right)^{1-n}.\end{aligned}$$

6·10. Theorem. *Let* E *and* F *be as in Theorem* 6·9. *Then*

$$\mathsf{M}(\Delta(E,F)) \leqslant \Omega_n e^{n\delta_{E,F}}.$$

Proof. Because of the Möbius-invariance we can assume that the set E is bounded. Now, let $a \in \bar{E}$ be a point such that $t \equiv d(E,F) = d(a,F)$, and let $b \in \bar{E}$ be a point such that $r \equiv \max_{x\in E} |a-x| = |a-b|$. Then $\frac{1}{t}\chi_{B^n(a,r+t)}$ is an admissible function for the curve family $\Delta(E,F)$, and

$$\mathsf{M}(\Delta(E,F)) \leqslant \Omega_n\left(\frac{r+t}{t}\right)^n.$$

https://doi.org/10.1017/S0305004198002904 Published online by Cambridge University Press

Table 1. *Inequalities between Möbius-invariants.*

	j_G, δ_G	k_G, σ_G	μ_G, λ_G
j_G, δ_G	$j_G \leqslant \delta_G \leqslant 2j_G$ (Theorem 3·4)	$\delta_G \leqslant \sigma_G$, $j_G \leqslant k_G$ (Theorem 3·12, [**GP** (2·2)])	∂G connected: $\tau_n\left(\frac{1}{e^{\delta_G}-1}\right) \leqslant \mu_G$ (Theorem 6·2)
k_G, σ_G	G c-uniform: $\sigma_G \leqslant c\delta_G$ (Definition 4·4)	$k_G \leqslant \sigma_G \leqslant 2k_G$ [**Fe1**, p. 122]	
μ_G, λ_G	$\lambda_G \leqslant \tau_n\left(\frac{e^{\delta_G}-1}{2}\right)$ (Theorem 6·5)	$\mu_G \leqslant h(3k_G)$ [**Vu1**, lemma 2·39]	

Here

$$\frac{r+t}{t} = \frac{|a-b|}{d(a,F)} + 1 \leqslant e^{j_{\mathbf{R}^n \setminus F}(E)} \leqslant e^{\delta_{E,F}},$$

and we have $\mathsf{M}(\Delta(E,F)) \leqslant \Omega_n e^{n\delta_{E,F}}$.

6·11. *Remark.* Theorems 6·7, 6·9 and 6·10 say that if $R(E,F)$ is a ring, where the sets $E, F \subset \bar{\mathbf{R}}^n$ have at least two points, then there exist increasing homeomorphisms $h_i\colon [0,\infty) \to [0,\infty)$, $i = 1,2$, such that

$$h_1(\delta_{E,F}) \leqslant \operatorname{cap} R(E,F) \leqslant h_2(\delta_{E,F}).$$

Table 1 summarizes some inequalities between pairs of metrics $\delta_G, \sigma_G, \mu_G$, and λ_G. The function $h(T)$ is defined in [**Vu1**, lemma 2.39] as

$$\begin{aligned} h(T) &= \begin{cases} \max\{1,\gamma\}\,.\,d_1(T), & e^T \leqslant \frac{5}{4}, \\ \max\{1,1/\gamma\}\,.\,d_2(T), & e^T > \frac{5}{4}, \end{cases} \\ \gamma &= d_2\left(\log\tfrac{5}{4}\right)/d_1\left(\log\tfrac{5}{4}\right), \\ d_1(T) &= \omega_{n-1}(-\log(2(e^T-1)))^{1-n}, \\ d_2(T) &= 6^n \Omega_n (e^T-1)^n. \end{aligned}$$

Acknowledgements. I wish to express my gratitude to Matti Vuorinen whose suggestions and ideas were invaluable during my work. I want to thank also Glen D. Anderson, Song-Liang Qiu, Mavina K. Vamanamurthy, the referee, and all others who have read this paper for their valuable comments.

REFERENCES

[**Ag**] S. AGARD. Distortion theorems for quasiconformal mappings. *Ann. Acad. Sci. Fenn. Ser. A I Math.* **413** (1968), 1–12.

[**AQV**] G. D. ANDERSON, S.-L. QIU, and M. K. VAMANAMURTHY. Inequalities for distortion functions and plane quasiconformal mappings. *New Zealand J. Math.* **26** (1997), 1–19.

[**AV**] G. D. ANDERSON and M. K. VAMANAMURTHY. Some properties of quasiconformal distortion functions. *New Zealand J. Math.* **24** (1995), 1–15.

[**AVV1**] G. D. ANDERSON, M. K. VAMANAMURTHY and M. VUORINEN. Functional inequalities for the Teichmüller ring capacity. *Israel J. Math.* **88** (1994), 367–380.

[**AVV2**] G. D. ANDERSON, M. K. VAMANAMURTHY and M. VUORINEN. *Conformal invariants, inequalities, and quasiconformal maps* (Wiley, 1997).

https://doi.org/10.1017/S0305004198002904 Published online by Cambridge University Press

[**Be1**] A. F. Beardon. *The geometry of discrete groups*. Graduate Texts in Math., Vol. **91**, (Springer-Verlag, 1983).

[**Be2**] A. F. Beardon. The Apollonian metric of a domain in $\mathbf{R}^n$; in *Quasiconformal mappings and analysis – a collection of papers honoring F. W. Gehring*, ed. by P. L. Duren, J. M. Heinonen, B. G. Osgood and B. P. Palka (Springer-Verlag, 1998), 91–108.

[**CG**] L. Carleson and T. W. Gamelin. *Complex dynamics*. (Springer-Verlag, 1993).

[**Fe1**] J. Ferrand. A characterization of quasiconformal mappings by the behavior of a function of three points; in *Proc. 13th Rolf Nevanlinna Colloquium* (Joensuu, 1987), ed. by I. Laine, S. Rickman and T. Sorvali, Lecture Notes in Math., Vol. 1351 (Springer-Verlag, 1988), 110–123.

[**Fe2**] J. Ferrand. Conformal capacity and conformally invariant metrics. *Pacific J. Math.* **180** (1997), 41–49.

[**GO**] F. W. Gehring and B. G. Osgood. Uniform domains and the quasi-hyperbolic metric. *J. Anal. Math.* **36** (1979), 50–74.

[**GP**] F. W. Gehring and B. P. Palka. Quasiconformally homogeneous domains. *J. Anal. Math.* **30** (1976), 172–199.

[**Je**] J. A. Jenkins. On metrics defined by modules. *Pacific J. Math.* **167** (1995), 289–292.

[**KM**] P. Kelly and G. Matthews. *The non-Euclidean hyperbolic plane* (Springer-Verlag, 1981).

[**Le**] O. Lehto. *Univalent functions and Teichmüller spaces*. Graduate Texts in Math., Vol. 109 (Springer-Verlag, 1987).

[**LF**] J. Lelong-Ferrand. Invariants conformes globaux sur les variétés riemanniennes. *J. Differential Geom.* **8** (1973), 487–510.

[**MRV**] O. Martio, S. Rickman and J. Väisälä. Distortion and singularities of quasiregular mappings. *Ann. Acad. Sci. Fenn. Ser. A I Math.* **465** (1970), 1–13.

[**So1**] A. Yu. Solynin. Moduli of doubly-connected domains and conformally invariant metrics (Russian). *Zap. Nautsh. Semin. LOMI* **196** (1991), 122–131.

[**So2**] A. Yu. Solynin. Ordering of sets and the hyperbolic metric. *Zap. Nautsh. Semin. POMI* **237** (1997), 129–147.

[**TV**] P. Tukia and J. Väisälä. Quasisymmetric embeddings of metric spaces. *Ann. Acad. Sci. Fenn. Ser. A I Math.* **5** (1980), 97–114.

[**Vä1**] J. Väisälä. *Lectures on n-dimensional quasiconformal mappings*. Lecture Notes in Math., Vol. 229 (Springer-Verlag, 1971).

[**Vä2**] J. Väisälä. Domains and maps; in *Quasiconformal space mappings – a collection of surveys* 1960–1990, ed. by M. Vuorinen, Lecture Notes in Math., Vol. 1508 (Springer-Verlag, 1992), 119–131.

[**Vu1**] M. Vuorinen. Conformal invariants and quasiregular mappings. *J. Anal. Math.* **45** (1985), 69–115.

[**Vu2**] M. Vuorinen. *Conformal geometry and quasiregular mappings*. Lecture Notes in Math., Vol. 1319 (Springer-Verlag, 1988).

[**Vu3**] M. Vuorinen. Quadruples and spatial quasiconformal mappings. *Math. Z.* **205** (1990), 617–628.

[**Vu4**] M. Vuorinen. Conformally invariant extremal problems and quasiconformal maps. *Quart. J. Math. Oxford Ser.* (2) **43** (1992), 501–514.

https://doi.org/10.1017/S0305004198002904 Published online by Cambridge University Press