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# Quadratic stochastic operators and zero-sum game dynamics

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*Abstract*. In this paper we consider the set of all extremal Volterra quadratic stochastic operators defined on a unit simplex $S^4$ and show that such operators can be reinterpreted in terms of zero-sum games. We show that an extremal Volterra operator is non-ergodic and an appropriate zero-sum game is a rock-paper-scissors game if either the Volterra operator is a uniform operator or for a non-uniform Volterra operator $V$ there exists a subset $I \subset \{1, 2, 3, 4, 5\}$ with $|I| \leq 2$ such that $\sum_{i \in I}(V^n \mathbf{x})_i \to 0$, and the restriction of $V$ on an invariant face $\Gamma_I = \{\mathbf{x} \in S^{m-1} : x_i = 0, i \in I\}$ is a uniform Volterra operator.

## 1. *Introduction*

Let

$$S^{m-1} = \left\{ \mathbf{x} = (x_1, x_2, \ldots, x_m) \in R^m : \text{ for any } i \ x_i \geq 0 \text{ and } \sum_{i=1}^{m} x_i = 1 \right\} \tag{1.1}$$

be the $(m-1)$-dimensional simplex. As shown by Jenks [**15**], the homogeneous differential system on $S^{m-1}$ given by

$$\frac{dx_i}{dt} = \sum_{j,k} a^i_{jk} x_j x_k, \tag{1.2}$$

with (i) $a^i_{jk} = a^i_{kj}$ for all $i, j, k \in \{1, \ldots, m\}$, (ii) $\sum_i a^i_{jk} = 0$ for all $j, k \in \{1, \ldots, m\}$ and (iii) $a^i_{jk} \geq 0$ for all $j \neq i, k \neq i \in \{1, \ldots, m\}$, governs mathematical models for large interacting populations of $m$ constituents. Here the numbers $x_i$ represent a fraction of the constituents of type $i$, $i = 1, \ldots, m$, and satisfy the conservation law $\sum_i x_i = 1$.

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The equations for Volterra's treatise on the biological struggle for life [**23**] are distinguished by the form

$$\frac{dx_i}{dt} = x_i \sum_j a_{ij} x_j, \quad i = 1, \ldots, m, \tag{1.3}$$

that is, the special case where relation (iii) in (1.2) holds with equality. Here the $a_{ij}$ are biological constants satisfying $a_{ij} = -a_{ji}$, that is, the $m \times m$ matrix $A = (a_{ij})$ is skew-symmetric. We will call the system of equations (1.3) a *Volterra dynamical system*.

In recent decades in game theory, evolutionary and dynamical aspects have dramatically increased in popularity. Hofbauer and Sigmund's book *Evolutionary Games and Population Dynamics* [**14**] serves as a very good introduction to this theory. Zero-sum games and their evolutionary dynamics were studied by Akin and Losert [**1**] (see also [**14**] and references therein). We recall the definition of zero-sum games following [**1**] and show their connection with Volterra dynamical systems. A two-player symmetric game consists of a finite set of strategies indexed by $\Phi = \{1, \ldots, m\}$ and an $m \times m$ payoff matrix $(a_{ij})$. When a player $i$ meets a player $j$ their payoffs are $a_{ij}$ and $a_{ji}$, respectively. In evolutionary game dynamics a large population of game players is assumed, each with a fixed strategy. The state of the population is a vector in $\mathbb{R}^m_+ = \{\mathbf{P} \in \mathbb{R}^m : p_i \geq 0\}$ where $p_i$ measures the subpopulation of $i$ strategists. So the total population size is $|p| = \sum_i p_i$. The associated distribution vector $\mathbf{x} = (x_1, \ldots, x_m)$ lies in the simplex $S^{m-1}$, where $x_i = p_i/|p|$ is the ratio of $i$ strategists to the total population. The system of differential equations describing the evolution of the distribution vector has the following form:

$$\frac{dx_i}{dt} = \sum_j x_j a_{ij} - \sum_{j,k} x_j x_k a_{kj}, \tag{1.4}$$

where $\sum_j x_j a_{ij}$ is the average payoff to the $i$th strategist because his opponent is a $j$ strategist with probability $x_j$. If a payoff matrix is antisymmetric, then such games are called *zero-sum*, since $a_{ij} + a_{ji} = 0$. This implies $\sum_{i,j} x_i x_j a_{ij} = 0$ for any point $\mathbf{x}$, $|a_{ij}| \leq 1$, and the associated system of differential equations (1.4) becomes

$$\frac{dx_i}{dt} = x_i \sum_j a_{ij} x_j. \tag{1.5}$$

Thus the Volterra dynamical system yields a zero-sum evolutionary game dynamics.

PROPOSITION 1.1. [**12**] *The discrete time system corresponding to (1.2) is defined by the mapping $V : S^{m-1} \to S^{m-1}$ with*

$$(V\mathbf{x})_k = \sum_{i,j=1}^{m} p_{ij,k} x_i x_j, \tag{1.6}$$

*where $p_{ij,k} = a^i_{jk} + (\delta_{ik} + \delta_{jk})/2$, and $\delta_{ij}$ is Kronecker delta. It is evident that*

$$\begin{aligned} &p_{ij,k} \geq 0, \\ &p_{ij,k} = p_{ji,k} \quad \textit{for all } i, j, k, \\ &\textstyle\sum_{k=1}^{m} p_{ij,k} = 1. \end{aligned} \tag{1.7}$$

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A mapping (1.6) with conditions (1.7), called a *quadratic stochastic operator* (qso), was first introduced by Bernstein [**3**]. Such operators frequently arise in many models of mathematical genetics, namely, the theory of heredity [**3**, **11**, **15**, **16**, **18**, **19**]. Consider a biological population, that is, a community of organisms closed with respect to reproduction. Assume that each individual in this population belongs to precisely one species (genotype) $\Phi = \{1, \ldots, m\}$. The scale of the species is such that the species of the parents $i$ and $j$ unambiguously determines the probability $p_{ij,k}$ of every species $k$ for the first generation of direct descendants of $i$ and $j$. This probability is called the heredity coefficient. It is then obvious that $p_{ij,k} \geq 0$ for all $i, j, k$ and, for any pair $i, j$,

$$\sum_{k=1}^{m} p_{ij,k} = 1.$$

The state of the population can be described by the $m$-tuple $(x_1, x_2, \ldots, x_m)$ of species probabilities, that is, $x_k$ is the fraction of the species $k$ in the total population. In the case of panmixia (random interbreeding) the parent pairs $i$ and $j$ arise for a fixed state $\mathbf{x} = (x_1, x_2, \ldots, x_m)$ with probability $x_i x_j$. Hence the total probability of the species $k$ in the first generation of direct descendants is defined by

$$\sum_{i,j=1}^{m} p_{ij,k} x_i x_j \quad (k = 1, \ldots, m).$$

Note that the linear operators are a special case of quadratic operators. If $\Pi = (r_{ij})$ is a stochastic matrix of size $m \times m$, that is, $r_{ij} \geq 0$ and $\sum_j r_{ij} = 1$, then with $p_{ij,k} = (r_{ik} + r_{jk})/2$ we get

$$(V\mathbf{x})_k = \sum_{i=1}^{m} r_{ik} x_i. \tag{1.8}$$

For example, if $p_{ij,k} = (\delta_{ik} + \delta_{jk})/2$, then the corresponding qso is an identity transformation.

*Volterra quadratic stochastic operators.* Let $\Phi = \{1, \ldots, m\}$ be the scale of genotypes or collection of strategies in a zero-sum game and $V$ be a qso on $S^{m-1}$.

*Definition 1.2.* The quadratic stochastic operator $V$ is called Volterra, if $p_{ij,k} = 0$ for any $k \notin \{i, j\}$.

The biological treatment of such operators is rather clear: the offspring repeats one of its parents. Evidently, for any Volterra qso,

$$p_{ii,i} = 1, \quad p_{ik,k} + p_{ki,i} = p_{ik,k} + p_{ik,i} = 1 \quad \text{for all } i, k = 1, \ldots, m, \ i \neq k. \tag{1.9}$$

A Volterra qso $V$ defined on $S^{m-1}$ has the form

$$(V\mathbf{x})_k = x_k^2 + 2 \sum_{i=1, i\neq k}^{m} p_{ik,k} x_i x_k, \tag{1.10}$$

where $k = 1, \ldots, m$.

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PROPOSITION 1.3. [**8**, **15**] *A qso $V$ is Volterra if and only if*

$$(V\mathbf{x})_k = x_k\left(1 + \sum_{i=1}^{m} a_{ki}x_i\right) \tag{1.11}$$

*where $A = (a_{ij})_1^m$ is a skew-symmetric matrix with $a_{ki} = 2p_{ik,k} - 1$ for $i \neq k$, $a_{ii} = 0$ and $|a_{ij}| \leq 1$. Here $i, j \in \{1, 2, \ldots, m\}$.*

*Proof.* The proof follows easily from (1.9). □

PROPOSITION 1.4. *The discrete time system corresponding to a Volterra dynamical system (1.3) is defined by the Volterra operator $V : S^{m-1} \to S^{m-1}$ with*

$$(V\mathbf{x})_k = x_k\left(1 + \sum_{i=1}^{m} a_{ki}x_i\right) \tag{1.12}$$

*where $A = (a_{ij})_1^m$ is a skew-symmetric matrix with $|a_{ij}| \leq 1$. Here $i, j \in \{1, 2, \ldots, m\}$.*

The proof follows immediately from Proposition 1.3. This proposition is also proven independently in [**1**].

The nonlinear dynamical system (1.2) can be reinterpreted in terms of evolutionary games and a gene conversion and in those forms it has a fair history [**1**, **14**, **17**, **20**, **21**]. Hofbauer and Sigmund's book [**14**] reflects well the state of evolutionary games up to 1998.

In the discrete case, the Volterra operator yields the discrete time zero-sum evolutionary game dynamics [**1**]. Note also that Nagylaki [**20**, **21**] has introduced (1.12) with antisymmetric $a_{ij}$ as a model for a gene conversion, and one can interpret the models considered below as gene conversion models.

In this paper we will study the trajectory behavior of Volterra operators and the evolutionary dynamics of discrete zero-sum games. As noted in [**1**], there is a radical difference between discrete and continuous models in the antisymmetric case. The continuous case will be the subject of another paper.

For a Volterra operator $V$ defined on $S^{m-1}$, we introduce a conjugacy invariant $\mathrm{Id}(V) = (C(V), D(V))$ as follows. Let $(\Phi, V)$ be a tournament generated by a Volterra operator $V$, where $\Phi$ is a set of vertices of the simplex $S^{m-1}$, and let $N_k(V)$ be a number of $k$-cycles in the tournament $(\Phi, V)$, where $k = 3, \ldots, m$. Then we put $C(V) = (N_3(V), N_4(V), \ldots, N_m(V))$. For any element $k \in \Phi$ we denote by $I_k(V)$ and $O_k(V)$ the indegree and outdegree of the vertex $k$ respectively, where $I_k(V) > 0$ and $O_k(V) < 0$. Then we put $D(V) = \{(I_1(V), O_1(V)); (I_2(V), O_2(V)); \ldots; (I_m(V), O_m(V))\}$.

The space of zero-sum game matrices, that is, skew-symmetric matrices generating Volterra operators, is parameterized by the cube $[-1, 1]^{m(m-1)/2}$. The extremal points of the cube are its vertices. The qso $V$ is called extremal Volterra if the corresponding skew-symmetric is a vertex of the cube, that is, $a_{ij} = -1$ or 1 for any $i \neq j$.

Let $\mathcal{E}$ be the set of all extremal Volterra qsos on $S^{m-1}$. It is evident that the total number of extremal Volterra qsos is equal to $2^{m(m-1)/2}$.

In this paper we consider the set of all extremal Volterra qsos defined on the unit simplex $S^{m-1}$ with $m = 5$. We prove that a zero-sum game is the paper-rock-scissors

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game if either the corresponding Volterra operator is a uniform operator or for non-uniform Volterra operator $V$ there exists a subset $I \subset \{1, 2, 3, 4, 5\}$ with $|I| \leq 2$ such that $\sum_{i \in I}(V^n \mathbf{x})_i \to 0$, and the restriction of $V$ on the invariant face $\Gamma_I = \{\mathbf{x} \in S^{m-1} : x_i = 0, i \in I\}$ is a uniform Volterra operator. In §2 we discuss the ergodicity of a qso and paper-rock-scissors games. For $m = 3, 4$ the asymptotic behavior of Volterra qsos has been studied in [**5**–**7**]. In §3 we describe the paper-rock-scissors games for $m = 3, 4$ using the results of these papers. In §§4 and 5 we present our main results, that is, we describe the asymptotic behavior of the extremal Volterra qsos for $m = 5$. Note that one can reformulate these results in terms of their corresponding zero-sum games, as was done in §3. Finally, in §6 we discuss how one can generalize these results to an arbitrary Volterra operator and an arbitrary $m$.

## 2. *Ergodicity and paper-rock-scissors games*

In statistical mechanics an ergodic hypothesis proposes a connection between dynamics and statistics. In the classical theory the assumption was made that the average time spent in any region of phase space is proportional to the volume of the region in terms of the invariant measure, or, more generally, that time averages may be replaced by space averages. For nonlinear dynamical system (1.6), Ulam [**22**] suggested an analogue of a measure-theoretic ergodicity in the form of the following ergodic hypothesis: a nonlinear operator $V$ defined on the unit simplex $S^{m-1}$ is called ergodic if the limit

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} V^k(x) \tag{2.1}$$

exists for any $\mathbf{x} \in S^{m-1}$. Note that a continuous time analogue of an ergodicity has the form [**14**]

$$\lim_{T\to\infty} \frac{1}{T} \int_0^T x_i(t)\, dt = p_i, \quad i = 1, \ldots, m. \tag{2.2}$$

On the basis of numerical calculations, Ulam [**22**] conjectured that an ergodic theorem holds for any qso $V$. In 1977, Zakharevich [**24**] proved that in general this conjecture is false. He considered the Volterra operator on $S^2$,

$$\begin{aligned} x_1' &= x_1(1 + x_2 - x_3), \\ x_2' &= x_2(1 - x_1 + x_3), \\ x_3' &= x_3(1 + x_1 - x_2), \end{aligned} \tag{2.3}$$

and proved that it is a non-ergodic transformation. Later, in [**5**], the authors established a necessary and sufficient condition for a qso defined on $S^2$ to be a non-ergodic transformation. Recently, in [**6**, **7**], the authors studied special classes of a Volterra qso defined on $S^3$.

Note that in [**2**] the author investigated the dynamics of the classical paper-rock-scissors games and studied the limit sets for the Cesàro sums (2.1).

Let $\Phi = \{1, 2, \ldots, m\}$ and $V$ be a Volterra operator defined on $S^{m-1}$,

$$(V\mathbf{x})_k = x_k\left(1 + \sum_{i=1}^{m} a_{ki} x_i\right), \tag{2.4}$$

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where $A = (a_{ij})_1^m$ is a skew-symmetric matrix with $|a_{ij}| \leq 1$. The set $\Phi$ can be interpreted as a collection of genotypes (constituents) or a collection of strategies. In the first case the Volterra operator describes a model of heredity and in the second case a zero-sum game.

The following notation will be used throughout this paper. We let int $S^{m-1}$ denote the interior and $\partial S^{m-1}$ denote the boundary of $S^{m-1}$:

$$\text{int } S^{m-1} = \{\mathbf{x} \in S^{m-1} : x_1 x_2 \cdots x_m > 0\}, \quad \partial S^{m-1} = S^{m-1} \backslash \text{int } S^{m-1}.$$

Also, for $i = 1, 2, \ldots, m$, we let $\mathbf{e}_i = (\delta_{1i}, \delta_{2i}, \ldots, \delta_{mi})$ denote the $i$th vertex of the simplex $S^{m-1}$, where $\delta_{ij}$ is the Kronecker symbol. $\Gamma_I = \{\mathbf{x} \in S^{m-1} : x_i = 0, i \in I\}$ denotes a face of $S^{m-1}$, where $I \subset \{1, 2, \ldots, m\}$. Assume that $\{V^k(\mathbf{x}) \in S^{m-1} : k = 0, 1, 2, \ldots\}$ is a trajectory of the initial point $\mathbf{x} \in S^{m-1}$, where $V^{k+1}(\mathbf{x}) = V(V^k(\mathbf{x}))$ for all $k = 0, 1, 2, \ldots$, with $V^0(\mathbf{x}) = \mathbf{x}$.

*Definition 2.1.* A point $\mathbf{a} \in S^{m-1}$ is called a fixed point of a qso $V$ if $V(\mathbf{a}) = \mathbf{a}$.

Let Fix($V$) be a set of all fixed points of qso $V$. It is evident that for any Volterra qso $\{\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_m\} \subset \text{Fix}(V)$, that is, all the vertices of $S^{m-1}$ are fixed points of $V$.

*Definition 2.2.* A qso $V$ is called regular if, for any initial point $\mathbf{x} \in S^{m-1}$, a limit

$$\lim_{n\to\infty} V^n(\mathbf{x}) \tag{2.5}$$

exists.

Note that the limit point is a fixed point of a qso $V$. Thus the fixed points of qso describe a limit or long-run behavior of the trajectories at any initial point. A limit behavior of trajectories and the fixed points of a qso play an important role in many applied problems [**1**, **8**–**10**, **14**–**16**, **18**].

Let $\omega(\mathbf{x}^0)$ be the set of limit points of a trajectory $\{V^k(\mathbf{x}^0) \in S^{m-1} : k = 0, 1, 2, \ldots\}$. One can control the $\omega$-limit points. Recall the definition of a Lyapunov function.

*Definition 2.3.* A continuous function $\varphi$ : int $S^{m-1} \to R$ is called a Lyapunov function for a qso $V$ if $\varphi(V(\mathbf{x})) \geq \varphi(\mathbf{x})$ for all $\mathbf{x}$ (or $\varphi(V(\mathbf{x})) \leq \varphi(\mathbf{x})$ for all $\mathbf{x}$).

A Lyapunov function is very helpful in describing the upper estimate $\omega(\mathbf{x}^0)$. However, there is no general recipe for finding such a Lyapunov function. In [**1**, **5**, **8**] the authors presented examples of Lyapunov functions for dynamical systems.

It is evident that a regular qso $V$ is ergodic; however, regularity does not follow from the ergodicity. For example, the operator appearing in (1.8) is an ergodic Markov chain and if $r_{ik} = \delta_{\pi(i)k}$, we get an operator $T_\pi$ which is ergodic but not regular.

Below we recall some of the well-known properties of Volterra operators and we also consider some of the new properties of Volterra operators.

*Some basic properties of Volterra operators.* As shown above, we can consider two forms of Volterra operators:

$$(V\mathbf{x})_k = x_k^2 + 2 \sum_{i=1, i\neq k}^{m} p_{ik,k} x_i x_k, \tag{2.6}$$

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where $p_{ij,k} = 0$ for any $k \notin \{i, j\}$; and

$$(V\mathbf{x})_k = x_k\left(1 + \sum_{i=1}^{m} a_{ki}x_i\right), \tag{2.7}$$

where $A = (a_{ij})$ is a skew-symmetric matrix with $a_{ki} = 2p_{ik,k} - 1$ for $i \neq k$, $a_{ii} = 0$ and $|a_{ij}| \leq 1$. The form we use below depends on the problem being considered.

PROPOSITION 2.4. *Let $V$ be a Volterra qso. Then for any permutation $\pi \in S_m$ the transformation $T_\pi^{-1} V T_\pi$ is a Volterra qso.*

*Proof.* For a qso $V$ (2.6) and a permutation $\pi \in S_m$, simple algebra gives

$$(T_\pi^{-1} V T_\pi \mathbf{x})_k = x_k^2 + 2 \sum_{i=1, i\neq k}^{m} p_{\pi^{-1}(i)\pi^{-1}(k),\pi^{-1}(k)} x_i x_k, \tag{2.8}$$

where $k = 1, \ldots, m$. It is evident that the transformation (2.8) is a Volterra qso. □

Note that, for any permutation $\pi$, a qso $T_\pi^{-1} V T_\pi$ is regular (respectively, ergodic) if and only if a qso $V$ is regular (respectively, ergodic).

PROPOSITION 2.5. *Let $V$ be an extremal Volterra qso. Then for any permutation $\pi \in S_m$ the transformation $T_\pi^{-1} V T_\pi$ is an extremal Volterra qso.*

*Proof.* The proof follows immediately from Proposition 2.4. □

Let $\mathcal{V}$ be the set of all Volterra qsos on $S^{m-1}$. Let us introduce the following binary relation on $\mathcal{V}$.

*Definition 2.6.* Let $V_1, V_2 \in \mathcal{V}$ be two Volterra qsos. We will say that a qso $V_1$ is equivalent to $V_2$ and write $V_1 \sim V_2$ if there exists a permutation $\pi \in S_m$ such that $V_2 = T_\pi^{-1} V_1 T_\pi$.

Since this relation is an equivalence relation, we can split the set $\mathcal{V}$ into equivalence classes. Note that considering this relation on a finite set $\mathcal{E}$ of all extremal Volterra qsos on $S^{m-1}$ we can explicitly list all elements of each equivalence class.

Below we consider Volterra qsos such that $a_{ij} \neq 0$ for any $i, j = 1, \ldots, m$. Let $V$ be a Volterra qso. Then on the set $\Phi = \{1, \ldots, m\}$ of genotypes one can introduce the following binary relation: if $a_{ij} > 0$, we say that genotype $i$ *dominates* genotype $j$ and write $i \succ j$. Any two genotypes $i$ and $j$ are comparable, that is, $i \succ j$ or $j \succ i$. If we assume that $\Phi = \{1, \ldots, m\}$ is a set of strategies in a zero-sum game, then if $a_{ij} > 0$ we say that a strategy $j$ *is beaten* by a strategy $i$ [**1**]. Along with a Volterra operator $V$ we consider a complete graph $(\Phi, V)$, with vertices $\Phi = \{1, \ldots, m\}$ as follows: the edge joining vertices $k$ and $i$ is directed to a dominant genotype (a winning strategy), that is, directed from the $k$th to the $i$th vertex if $a_{ki} < 0$ and in the opposite direction if $a_{ki} > 0$. Since $a_{ki} = -a_{ik}$ and $a_{ki} \neq 0$, the directions of all the edges of $(\Phi, V)$ are uniquely determined by the specification of the Volterra operator. The directed graph obtained is called a tournament [**13**].

A path of $(\Phi, V)$ is defined as an alternating sequence $i_1, e_1, i_2, e_2, \ldots, e_{n-1}, i_n$ of distinct vertices $i$ and edges $e_k = \langle i_k, i_{k+1}\rangle$. Such a path has length $n - 1$ and is said to connect $i_1$ to $i_n$.

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*Definition 2.7.* A cycle of a graph $(\Phi, V)$ is an alternating sequence $i_1, e_1, i_2, e_2, \ldots, e_{n-1}, i_n, e_n, i_1$ of vertices and edges such that $i_1, e_1, i_2, e_2, \ldots, e_{n-1}, i_n$ is a path and $e_n = \langle i_n, i_1 \rangle$; such a cycle has length $n$ and is called a $n$-cycle.

If the vertices $i_1, i_2, \ldots, i_n$ form a $n$-cycle, then we say that the strategies $i_1, i_2, \ldots, i_n$ define a paper-rock-scissors game.

*Definition 2.8.* A cycle $i_1, e_1, i_2, e_2, \ldots, e_{n-1}, i_n, e_n, i_1$ is called a Hamiltonian cycle if $\{i_1, i_2, \ldots, i_n\} = \Phi$. A graph containing no cycles of any length is known as an acyclic graph, whereas a graph containing at least one cycle is called a cyclic graph.

It is evident that a zero-sum game generated by a Volterra operator $V$ is a paper-rock-scissors game if and only if the graph $(\Phi, V)$ contains a Hamiltonian cycle.

Let the indegree of a vertex be the number of edges leading to that vertex, and the outdegree of a vertex be the number of edges leading away from that vertex.

*Definition 2.9.* A vertex with an indegree of 0 is called a *source* (since one can only leave it) and a vertex with an outdegree of 0 is called a *sink* (since one cannot leave it).

It is easy to verify that these graph-theoretic definitions of source and sink coincide with the well-known definitions for a discrete dynamical system. Let $D_x V(\mathbf{a}) = (\partial V_i / \partial x_j(\mathbf{x}))_{\mathbf{x}=\mathbf{a}}$ be a Jacobian of $V$ at the point $\mathbf{a} \in S^{m-1}$.

*Definition 2.10.* A fixed point $\mathbf{a} \in S^{m-1}$ is called a sink (respectively, source) if the spectrum $D_x V(\mathbf{a})$ is contained inside (respectively, outside) the unit disk on a complex plane.

Let $V$ be a Volterra operator. Without loss of generality, one can consider a vertex $\mathbf{e}_1 = (1, 0, \ldots, 0)$. Then

$$D_x V = \begin{pmatrix} 1 + \sum a_{1i}x_i & a_{12}x_1 & a_{13}x_1 & \cdots & a_{1m}x_1 \\ a_{21}x_2 & 1 + \sum a_{2i}x_i & a_{23}x_2 & \cdots & a_{2m}x_2 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{m1}x_m & a_{m2}x_m & a_{m3}x_m & \cdots & 1 + \sum a_{mi}x_i \end{pmatrix} \tag{2.9}$$

and

$$D_x V(\mathbf{e}_1) = \begin{pmatrix} 1 & a_{12} & a_{13} & \cdots & a_{1m} \\ 0 & 1 + a_{21} & 0 & \cdots & 0 \\ 0 & 0 & 1 + a_{31} & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 + a_{m1} \end{pmatrix}. \tag{2.10}$$

It is evident that a spectrum of $D_x V(\mathbf{e}_1)$ is a set $\{1, 1 + a_{21}, 1 + a_{31}, \ldots, 1 + a_{m1}\}$. The spectrum is contained inside (respectively, outside) a unit disk on a complex plane if and only if $a_{1k} > 0$ for all $k = 2, \ldots, m$, that is, $\mathbf{e}_1$ is a sink (respectively, $a_{1k} < 0$ for all $k = 2, \ldots, m$, that is, $\mathbf{e}_1$ is a source).

For any Volterra operator $V$ we define a conjugacy invariant $\mathrm{Id}(V) = (C(V), D(V))$ as follows. Let $N_k(V)$ be a number of $k$ cycles in a tournament $(\Phi, V)$, where $k = 3, \ldots, m$. Then we put $C(V) = (N_3(V), N_4(V), \ldots, N_m(V))$. For

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any element $k \in \Phi$ denote by $I_k(V)$ and $O_k(V)$ the indegree and outdegree, respectively, of the vertex $k$, where $I_k(V) > 0$ and $O_k(V) < 0$. Then we put $D(V) = \{(I_1(V), O_1(V)); (I_2(V), O_2(V)); \ldots; (I_m(V), O_m(V))\}$.

Now we select Volterra qsos such that $I_k(V) + O_k(V) = 0$ for any $k = 1, \ldots, m$.

*Definition 2.11.* A Volterra qso $V$ is called uniform if in any row of a skew-symmetric matrix $A = (a_{ij})$ the number of positive entries is equal to the number of negative ones.

It is evident that there are uniform Volterra qsos if and only if $m$ is odd. The qso (2.3) considered by Zakharevich is a uniform Volterra operator.

We conclude this section by generalizing the notions of ergodicity and regularity.

*Definition 2.12.* Let $V$ be a nonlinear transformation on $S^{m-1}$ and let $A$ be a maximal measurable subset of $S^{m-1}$ such that for any $x \in A$ the limit (2.5) (respectively, (2.1)) does not exist. If $\mu(A) > 0$ with the usual Lebesgue measure $\mu$ on $S^{m-1}$, then the transformation $V$ is called non-regular (respectively, non-ergodic); if $\mu(A) = 0$, then the transformation $V$ is called regular by mod 0 (ergodic by mod 0 respectively).

## 3. *Non-ergodicity and paper-rock-scissors game*

In this section we show the correlation between non-ergodicity of Volterra operators and paper-rock-scissors games. According to Definition 2.12, a Volterra operator $V$ is non-ergodic if there exists a subset $A \subset S^{m-1}$ with a positive measure such that the limit

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} V^k(x) \tag{3.1}$$

does not exist for any $\mathbf{x} \in A$. In game theory $(V^k(\mathbf{x}))_i$ denotes the ratio of $i$ strategists to the total population at the moment of time $k$ with $i = 1, \ldots, m$ and $k = 0, 1, 2, \ldots$, and (3.1) denotes that their time averages have chaotic behavior.

First, consider the case $m = 3$. The general paper-rock-scissors zero-sum game is characterized by having three pure strategies such that $R_1$ is beaten by $R_2$, which is beaten by $R_3$, which is beaten by $R_1$ (see [**14**]). Then the corresponding payoff matrix has the form

$$A = \begin{pmatrix} 0 & a & -b \\ -a & 0 & c \\ b & -c & 0 \end{pmatrix}. \tag{3.2}$$

In [**5**], the authors proved the following theorem.

THEOREM 3.1. *Any Volterra operator*

$$V : (x, y, z) \to (x(1 + ax - bz), y(1 - ax + cz), z(1 + bx - cy)), \tag{3.3}$$

*where* $a, b, c \in [-1, 1]$, *is non-ergodic if and only if the three parameters* $a, b, c$ *have the same signs.*

Thus any Volterra operator (3.3), such that the parameters $a, b, c$ have the same signs, describes a paper-rock-scissors game. The point $(c/(a + b + c), b/(a + b + c), a/(a + b + c))$ is the fixed point; the function $f(x, y, z) = x^c y^b z^a$ is a Lyapunov function and the boundary $\partial S^2$ is an attractor [**5**].

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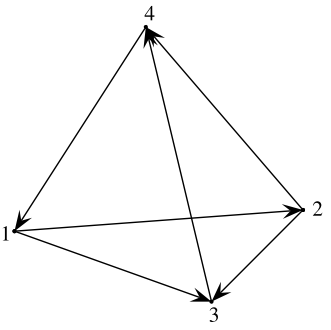



FIGURE 1. Tournament generated by operators in the first class, $m = 4$.

As a corollary, we have a non-ergodicity Zakharewich's example (2.3) with a skew-symmetric matrix

$$A = \begin{pmatrix} 0 & 1 & -1 \\ -1 & 0 & 1 \\ 1 & -1 & 0 \end{pmatrix} \tag{3.4}$$

and its corresponding game is a paper-rock-scissors game.

Let $m = 4$. Then the set $\mathcal{E}$ consists of 64 extremal Volterra operators

$$\begin{aligned} x_1' &= x_1^2 + 2\alpha x_1x_2 + 2\beta x_1x_3 + 2\gamma x_1x_4, \\ x_2' &= x_2^2 + 2(1-\alpha)x_1x_2 + 2\delta x_2x_3 + 2\varepsilon x_2x_4, \\ x_3' &= x_3^2 + 2(1-\beta)x_1x_3 + 2(1-\delta)x_2x_3 + 2\lambda x_3x_4, \\ x_4' &= x_4^2 + 2(1-\gamma)x_1x_4 + 2(1-\varepsilon)x_2x_4 + 2(1-\lambda)x_3x_4, \end{aligned} \tag{3.5}$$

where $\alpha, \beta, \gamma, \delta, \varepsilon, \lambda \in \{0, 1\}$. Simple but tedious algebra gives that the set $\mathcal{E}$ is split into four equivalence classes.

*First equivalent class.* If $\alpha = \beta = \delta = \varepsilon = \lambda = 1$ and $\gamma = 0$ then the corresponding qso (3.5) with the payoff matrix $A$ has the form

$$A = \begin{pmatrix} 0 & 1 & 1 & -1 \\ -1 & 0 & 1 & 1 \\ -1 & -1 & 0 & 1 \\ 1 & -1 & -1 & 0 \end{pmatrix} \qquad \begin{aligned} x_1' &= x_1^2 + 2x_1x_2 + 2x_1x_3, \\ x_2' &= x_2^2 + 2x_2x_3 + 2x_2x_4, \\ x_3' &= x_3^2 + 2x_3x_4, \\ x_4' &= x_4^2 + 2x_1x_4, \end{aligned} \tag{3.6}$$

and a tournament generated by this operator contains the Hamiltonian cycle $1 \succ 2 \succ 3 \succ 4 \succ 1$ (see Figure 1).

The equivalent class containing this operator consists of 24 extremal Volterra qsos, and each of them generates a tournament containing a Hamiltonian cycle. It is easy to see that a set of fixed points contains the points $\mathbf{e}_1$, $\mathbf{e}_2$, $\mathbf{e}_3$, $\mathbf{e}_4$, $M_5 = (1/3, 0, 1/3, 1/3)$, $M_6 = (1/3, 1/3, 0, 1/3)$. Introducing new variables $y_1 = x_1$, $y_2 = x_2 + x_3$, $y_4 = x_4$, one can see that

$$y_1' = x_1', \quad y_2' = x_2' + x_3', \quad y_4' = x_4',$$

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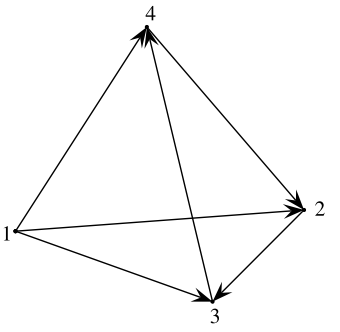



FIGURE 2. Tournament generated by operators in the second class, $m = 4$.

and qso (3.6) is reduced to a Zakharevich example (2.3),

$$\begin{aligned} y_1' &= y_1^2 + 2y_1y_2, \\ y_2' &= y_2^2 + 2y_2y_3, \\ y_3' &= y_3^2 + 2y_1y_3. \end{aligned} \tag{3.7}$$

Note that qso (3.6) has the invariant subset $\{\tau M_5 + (1-\tau)M_6, \tau \in [0, 1]\}$ [**7**]. Since a Lebesgue measure of an invariant subset is equal to 0, qso (3.6) is non-ergodic. Thus we have proved the following proposition.

PROPOSITION 3.2. *Any qso from the first class is a non-ergodic operator.*

Thus a zero-sum game corresponding to a Volterra operator (3.6) is a paper-rock-scissors game.

*Second equivalent class*. If $\alpha = \beta = \gamma = \delta = \lambda = 1$ and $\varepsilon = 0$, then the corresponding qso (3.5) with the payoff matrix $A$ has the form

$$A = \begin{pmatrix} 0 & 1 & 1 & 1 \\ -1 & 0 & 1 & -1 \\ -1 & -1 & 0 & 1 \\ -1 & 1 & -1 & 0 \end{pmatrix} \qquad \begin{aligned} x_1' &= x_1^2 + 2x_1x_2 + 2x_1x_3 + 2x_1x_4, \\ x_2' &= x_2^2 + 2x_2x_3, \\ x_3' &= x_3^2 + 2x_3x_4, \\ x_4' &= x_4^2 + 2x_2x_4, \end{aligned} \tag{3.8}$$

and in the tournament generated by this operator (see Figure 2), the vertex $\mathbf{e}_1$ is a sink, and the other three vertices form a 3-cycle $2 \succ 3 \succ 4 \succ 2$.

The equivalent class containing this operator consists of eight extremal Volterra qsos, and each of them generates a tournament containing a 3-cycle. It is easy to see that a set of fixed points consists of the points $\mathbf{e}_1$, $\mathbf{e}_2$, $\mathbf{e}_3$, $\mathbf{e}_4$, $M_5(0, 1/3, 1/3, 1/3)$. It is easy to verify that for any initial $\mathbf{x}^0 \in S^3$ with $x_1^0 \neq 0$ its trajectory converges to a vertex $\mathbf{e}_1$ and the restriction of qso (3.8) on the face $\Gamma_1$ is reduced to Zakharevich's example (2.3). Since the Lebesgue measure of this face is equal to 0, we have proved the following proposition.

PROPOSITION 3.3. *Any extremal Volterra qso $V$ in the second class is regular by* mod 0.

Thus a zero-sum game corresponding to a Volterra operator in the second class is not a paper-rock-scissors game.

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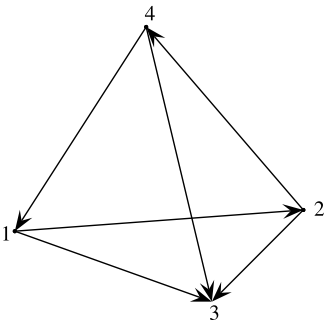



FIGURE 3. Tournament generated by operators in the third class, $m = 4$.

*Third equivalent class*. If $\alpha = \beta = \delta = \varepsilon = 1$ and $\gamma = \lambda = 0$, the corresponding qso (3.5) with the payoff matrix $A$ has the form

$$A = \begin{pmatrix} 0 & 1 & 1 & -1 \\ -1 & 0 & 1 & 1 \\ -1 & -1 & 0 & -1 \\ 1 & -1 & 1 & 0 \end{pmatrix} \quad \begin{aligned} x_1' &= x_1^2 + 2x_1x_2 + 2x_1x_3, \\ x_2' &= x_2^2 + 2x_2x_3 + 2x_2x_4, \\ x_3' &= x_3^2, \\ x_4' &= x_4^2 + 2x_1x_4 + 2x_3x_4, \end{aligned} \tag{3.9}$$

and in the tournament generated by this operator (see Figure 3) the vertex $\mathbf{e}_3$ is a source, and the other three vertices form a 3-cycle $1 \succ 2 \succ 4 \succ 1$.

The equivalent class containing this operator also consists of eight extremal Volterra qsos, and each of them generates a tournament containing a 3-cycle. It is easy to see that a set of fixed points consists of the points $\mathbf{e}_1$, $\mathbf{e}_2$, $\mathbf{e}_3$, $\mathbf{e}_4$, $M_5(1/3, 1/3, 0, 1/3)$. It is evident that $\lim_{n\to\infty} x_3^{(n)} = 0$. If $\mathbf{x}^0 \in S^3$ is an initial point with $x_1^0 \neq 0$ then $\omega(\mathbf{x}^0) \subset \Gamma_3$. It is evident that the restriction of qso (3.9) on $\Gamma_3$ is reduced to Zakharevich's example (2.3). Thus we have proved the following proposition.

PROPOSITION 3.4. *If an extremal Volterra qso $V$ belongs to the third class, then it is a non-ergodic transformation.*

Thus a zero-sum game corresponding to a Volterra operator (3.9) is a paper-rock-scissors game. Note that in [**1**] the authors considered an example similar to (3.9).

*Fourth equivalent class*, If $\alpha = \beta = \gamma = \delta = \varepsilon = \lambda = 1$, then the corresponding qso (3.5) with the payoff matrix $A$ has the form

$$A = \begin{pmatrix} 0 & 1 & 1 & 1 \\ -1 & 0 & 1 & 1 \\ -1 & -1 & 0 & 1 \\ -1 & -1 & -1 & 0 \end{pmatrix} \quad \begin{aligned} x_1' &= x_1^2 + 2x_1x_2 + 2x_1x_3 + 2x_1x_4, \\ x_2' &= x_2^2 + 2x_2x_3 + 2x_2x_4, \\ x_3' &= x_3^2 + 2x_3x_4, \\ x_4' &= x_4^2, \end{aligned} \tag{3.10}$$

and a tournament generated by this operator (see Figure 4) is acyclic.

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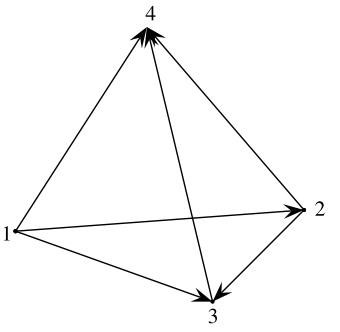



FIGURE 4. Tournament generated by operators in the fourth class, $m = 4$.

The equivalent class containing this operator consists of 24 extremal Volterra qsos. In [**7**] the authors proved the following proposition.

PROPOSITION 3.5. *Any extremal Volterra qso $V$ belonging to the fourth class is a regular transformation.*

## 4. *Classification of extremal Volterra operators on $S^4$*

In this and the next sections we discuss the trajectory behavior of extremal Volterra operators on $S^4$. In these sections we omit the reinterpretation of evolutionary dynamics of zero-sum games. For $m = 5$ the set $\mathcal{E}$ consists of 1024 extremal Volterra operators

$$\begin{aligned}
x_1' &= x_1^2 + 2\alpha x_1x_2 + 2\beta x_1x_3 + 2\gamma x_1x_4 + 2\delta x_1x_5,\\
x_2' &= x_2^2 + 2(1-\alpha)x_1x_2 + 2\varepsilon x_2x_3 + 2\theta x_2x_4 + 2\lambda x_2x_5,\\
x_3' &= x_3^2 + 2(1-\beta)x_1x_3 + 2(1-\varepsilon)x_2x_3 + 2\mu x_3x_4 + 2\nu x_3x_5,\\
x_4' &= x_4^2 + 2(1-\gamma)x_1x_4 + 2(1-\theta)x_2x_4 + 2(1-\mu)x_3x_4 + 2\eta x_4x_5,\\
x_5' &= x_5^2 + 2(1-\delta)x_1x_5 + 2(1-\lambda)x_2x_5 + 2(1-\nu)x_3x_5 + 2(1-\eta)x_4x_5,
\end{aligned} \tag{4.1}$$

where $\alpha, \beta, \gamma, \delta, \varepsilon, \theta, \lambda, \mu, \nu, \eta \in \{0, 1\}$.

Let us consider a partition of the set $\mathcal{E}$ into equivalence classes with respect to the equivalence relation defined above (see Definition 2.6). A simple but tedious algebraic manipulation gives that $\mathcal{E}$ is split into 12 equivalent classes. It is evident that any two extremal Volterra operators in the same class have the same Id. Below we consider the trajectory behavior of the extremal Volterra operators in each class. It is convenient to formulate the following two evident statements.

LEMMA 4.1. *Let $\varphi(x) = x(2-x)$ be a one-dimensional transformation on the segment $[0, 1]$. Then it has two fixed points $x^* = 0$, $x^{**} = 1$, where $x^* = 0$ is repelling, $x^{**} = 1$ is attracting, and, for any $x \in (0, 1)$, $\omega(x) = \{x^{**}\}$.*

LEMMA 4.2. *Let $V$ be a Volterra qso on $S^{m-1}$ and $\Gamma_I = \{\mathbf{x} \in S^{m-1} : x_i = 0, i \in I\}$ be a face of $S^{m-1}$, where $I$ is a subset of $\{1, 2, \ldots, m\}$. If the restriction of $V$ on the invariant face $\Gamma_I$ is non-ergodic and $\sum_{i\in I}(V^n\mathbf{x})_i \to 0$, then the operator $V$ itself is non-ergodic.*

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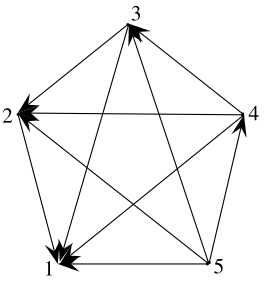



FIGURE 5. Tournament generated by operators in the first class, $m = 5$.

*Class of operators generating acyclic graphs*. The first class consists of 120 extremal Volterra operators such that the corresponding graphs are acyclic. As a representative of this class we consider the extremal Volterra operator with $\alpha = \beta = \gamma = \delta = \varepsilon = \theta = \lambda = \mu = \nu = \eta = 0$; the corresponding qso with the payoff matrix $A$ has the form

$$A = \begin{pmatrix} 0 & -1 & -1 & -1 & -1 \\ 1 & 0 & -1 & -1 & -1 \\ 1 & 1 & 0 & -1 & -1 \\ 1 & 1 & 1 & 0 & -1 \\ 1 & 1 & 1 & 1 & 0 \end{pmatrix} \qquad \begin{aligned} x_1' &= x_1^2, \\ x_2' &= x_2^2 + 2x_1x_2, \\ x_3' &= x_3^2 + 2x_1x_3 + 2x_2x_3, \\ x_4' &= x_4^2 + 2x_1x_4 + 2x_2x_4 + 2x_3x_4, \\ x_5' &= x_5^2 + 2x_1x_5 + 2x_2x_5 + 2x_3x_5 + 2x_4x_5, \end{aligned} \tag{4.2}$$

and a tournament generated by this operator (see Figure 5) is acyclic.

It is evident that $\mathrm{Id}(V) = ((0, 0, 0); \{(+4, 0), (+3, -1)(+2, -2), (+1, -3), (0, -4)\})$ and $\mathrm{Fix}(V) = \{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3, \mathbf{e}_4, \mathbf{e}_5\}$, where the vertex $\mathbf{e}_1$ is repelling, $\mathbf{e}_5$ is attracting and other fixed points are saddles.

One can rewrite the fifth equality of (4.2) as $x_5' = x_5(2 - x_5)$ and, using Lemma 4.1, one can show that for an initial point with $x_5^0 > 0$ a trajectory of the qso (4.2) converges to the vertex $\mathbf{e}_5$.

Similarly one can show that:

for $\mathbf{x} \in \Gamma_5$, with $x_4^0 > 0$, the trajectory of qso (4.2) converges to the vertex $\mathbf{e}_4$;

for $\mathbf{x} \in \Gamma_{45}$, with $x_3^0 > 0$, the trajectory of qso (4.2) converges to the vertex $\mathbf{e}_3$;

for $\mathbf{x} \in \Gamma_{345}$, with $x_2^0 > 0$, the trajectory of qso (4.2) converges to the vertex $\mathbf{e}_2$.

Thus we have proved the following proposition.

PROPOSITION 4.3. *Any extremal Volterra qso $V$ belonging to the first class is a regular operator.*

*Three classes of operators generating graphs with a single 3-cycle*. For any extremal Volterra qso $V$ in these three classes $C(V) = (1, 0, 0)$, that is, a graph generated by $V$ contains a single 3-cycle. However, for operator $V$ in the second (respectively, third, fourth) class we have $D(V) = \{(+4, 0); (+3, -1); (+1, -3); (+1, -3); (+1, -3)\}$

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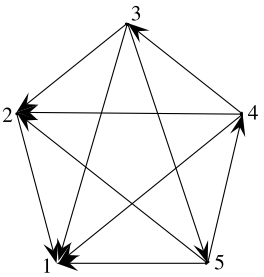



FIGURE 6. Tournament generated by operators in the second class, $m = 5$.

(respectively, $D(V) = \{(+4, 0); (+2, -2); (+2, -2); (0, -4); (+2, -2)\}$, $D(V) = \{(+3, -1); (+3, -1); (+1, -3); (0, -4); (+3, -1)\}$), that is, the operators in different classes have different Ids.

*Second class with* $D(V) = \{(+4, 0); (+3, -1); (+1, -3); (+1, -3); (+1, -3)\}$. The second class consists of 40 operators. As a representative of this class one can choose the operator with $\alpha = \beta = \gamma = \delta = \varepsilon = \theta = \lambda = \mu = \eta = 0$, $\nu = 1$; the corresponding qso with the payoff matrix $A$ has the form

$$A = \begin{pmatrix} 0 & -1 & -1 & -1 & -1 \\ 1 & 0 & -1 & -1 & -1 \\ 1 & 1 & 0 & -1 & 1 \\ 1 & 1 & 1 & 0 & -1 \\ 1 & 1 & -1 & 1 & 0 \end{pmatrix} \qquad \begin{aligned} x_1' &= x_1^2, \\ x_2' &= x_2^2 + 2x_1x_2, \\ x_3' &= x_3^2 + 2x_1x_3 + 2x_2x_3 + 2x_3x_5, \\ x_4' &= x_4^2 + 2x_1x_4 + 2x_2x_4 + 2x_3x_4, \\ x_5' &= x_5^2 + 2x_1x_5 + 2x_2x_5 + 2x_4x_5, \end{aligned} \tag{4.3}$$

and a tournament generated by this operator (see Figure 6) contains a single 3-cycle $3 \succ 5 \succ 4 \succ 3$.

It is easy to show that $\mathrm{Fix}(V) = \{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3, \mathbf{e}_4, \mathbf{e}_5, M_6(0, 0, 1/3, 1/3, 1/3)\}$, where the vertex $\mathbf{e}_1$ is repelling, and all others are saddle. Note that the restriction of qso (4.3) on an invariant face $\Gamma_{12}$ is reduced to Zakharevich's example (2.3). Introducing new variables $y_1 = x_1 + x_2$, $y_2 = x_3$, $y_3 = x_4$, $y_4 = x_5$, one can see that

$$y_1' = x_1' + x_2', \quad y_2' = x_3', \quad y_3' = x_4', \quad y_4' = x_5'$$

and qso (4.3) is reduced to the following extremal Volterra operator on $S^3$:

$$\begin{aligned} y_1' &= y_1^2, \\ y_2' &= y_2^2 + 2y_1y_2 + 2y_2y_4, \\ y_3' &= y_3^2 + 2y_1y_3 + 2y_2y_3, \\ y_4' &= y_4^2 + 2y_1y_4 + 2y_3y_4. \end{aligned} \tag{4.4}$$

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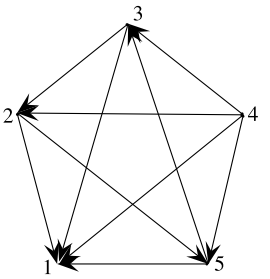



FIGURE 7. Tournament generated by operators in the third class, $m = 5$.

This operator has been studied in [**7**] and its non-ergodicity proven. In fact, for this operator, according to [**8**, Theorem 7], we have $(V^n\mathbf{x})_1 + (V^n\mathbf{x})_2 \to 0$ when $n \to \infty$ and from Lemma 4.2 the non-ergodicity of qso (4.3) follows. Thus the following proposition is valid.

PROPOSITION 4.4. *Any extremal Volterra qso in the second class is non-ergodic.*

*Third class with* $D(V) = \{(+4, 0); (+2, -2); (+2, -2); (0, -4); (+2, -2)\}$. The third class also consists of 40 operators. As a representative of this class one can choose the operator with $\alpha = \beta = \gamma = \delta = \varepsilon = \theta = \mu = \nu = 0$, $\lambda = \eta = 1$; the corresponding qso with the payoff matrix $A$ has the form

$$A = \begin{pmatrix} 0 & -1 & -1 & -1 & -1 \\ 1 & 0 & -1 & -1 & 1 \\ 1 & 1 & 0 & -1 & -1 \\ 1 & 1 & 1 & 0 & 1 \\ 1 & -1 & 1 & -1 & 0 \end{pmatrix} \qquad \begin{aligned} x_1' &= x_1^2, \\ x_2' &= x_2^2 + 2x_1x_2 + 2x_2x_5, \\ x_3' &= x_3^2 + 2x_1x_3 + 2x_2x_3, \\ x_4' &= x_4^2 + 2x_1x_4 + 2x_2x_4 + 2x_3x_4 + 2x_4x_5, \\ x_5' &= x_5^2 + 2x_1x_5 + 2x_3x_5, \end{aligned} \tag{4.5}$$

and a tournament generated by this operator (see Figure 7) contains a single 3-cycle $2 \succ 5 \succ 3 \succ 2$. One can see that the vertex $\mathbf{e}_1$ is a sink, and the vertex $\mathbf{e}_4$ is a source.

It is easy to see that $\mathrm{Fix}(V) = \{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3, \mathbf{e}_4, \mathbf{e}_5, M_6(0, 1/3, 1/3, 0, 1/3)\}$, where the fixed point $\mathbf{e}_1$ is repelling, $\mathbf{e}_4$ is attracting, and $\mathbf{e}_2$, $\mathbf{e}_3$, $\mathbf{e}_5$, $M_6$ are saddles.

One can rewrite the fourth equality of (4.5) as $x_4' = x_4(2 - x_4)$ and, using Lemma 4.1, we have that for any initial point with $x_4^0 > 0$, the trajectory converges to $\{\mathbf{e}_4\}$.

The restriction of qso (4.5) on an invariant face $\Gamma_4$ is reduced to the qso

$$\begin{aligned} x_1' &= x_1^2, \\ x_2' &= x_2^2 + 2x_1x_2 + 2x_2x_5, \\ x_3' &= x_3^2 + 2x_1x_3 + 2x_2x_3, \\ x_5' &= x_5^2 + 2x_1x_5 + 2x_3x_5. \end{aligned} \tag{4.6}$$

The non-ergodicity of this operator was proven in [**7**].

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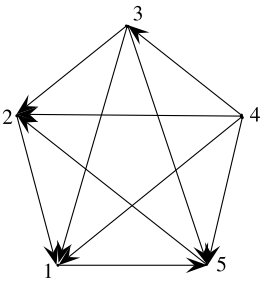



FIGURE 8. Tournament generated by operators in the fourth class, $m = 5$.

PROPOSITION 4.5. *Any extremal Volterra qso $V$ in the third class is regular by* mod 0.

*Fourth class with* $D(V) = \{(+3, -1); (+3, -1); (+1, -3); (0, -4); (+3, -1)\}$. The fourth class also consists of 40 operators. As a representative of this class one can choose the operator with coefficients $\alpha = \beta = \gamma = \varepsilon = \theta = \lambda = \mu = 0$, $\delta = \eta = \nu = 1$; the corresponding qso with the payoff matrix $A$ has the form

$$A = \begin{pmatrix} 0 & -1 & -1 & -1 & 1 \\ 1 & 0 & -1 & -1 & -1 \\ 1 & 1 & 0 & -1 & 1 \\ 1 & 1 & 1 & 0 & 1 \\ -1 & 1 & -1 & -1 & 0 \end{pmatrix} \qquad \begin{aligned} x_1' &= x_1^2 + 2x_1x_5, \\ x_2' &= x_2^2 + 2x_1x_2, \\ x_3' &= x_3^2 + 2x_1x_3 + 2x_2x_3 + 2x_3x_5, \\ x_4' &= x_4^2 + 2x_1x_4 + 2x_2x_4 + 2x_3x_4 + 2x_4x_5, \\ x_5' &= x_5^2 + 2x_2x_5, \end{aligned} \tag{4.7}$$

and a tournament generated by this operator (see Figure 8) contains a single 3-cycle $1 \succ 5 \succ 2 \succ 1$, and the vertex $\mathbf{e}_4$ is a source.

It is easy to show that $\mathrm{Fix}(V) = \{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3, \mathbf{e}_4, \mathbf{e}_5, M_6(1/3, 1/3, 0, 0, 1/3)\}$, where the fixed points $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3, \mathbf{e}_5$ are saddles, $\mathbf{e}_4$ is attracting and $M_6$ is repelling. It is evident that the restriction of qso (4.7) on an invariant face $\Gamma_{34}$ is reduced to Zakharevich's example (2.3). One can rewrite the fourth equality of (4.7) as $x_4' = x_4(2 - x_4)$ and, using Lemma 4.1, we have:

(1) for any initial point with $x_4^0 > 0$ its trajectory converges to the vertex $\mathbf{e}_4$;

(2) for any initial point $\mathbf{x} \in \Gamma_4$ with $x_3^0 > 0$ its trajectory converges to the vertex $\mathbf{e}_3$.

Therefore the following proposition is valid.

PROPOSITION 4.6. *Any extremal Volterra qso $V$ that belongs to the fourth class is regular by* mod 0.

*Two classes of operators that generate graphs with 3- and 4-cycles.* Below we consider two classes of operators that generate graphs with two 3-cycles and one 4-cycle, that is, $C(V) = (2, 1, 0)$.

*Fifth class of operators with* $D(V) = \{(+4, 0); (+2, -2); (+2, -2); (+1, -3); (+1, -3)\}$. The fifth class consists of 120 operators. As a representative of this class one can

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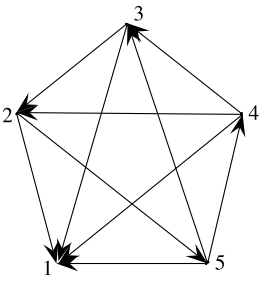



FIGURE 9. Tournament generated by operators in the fifth class, $m = 5$.

choose the operator with parameters $\alpha = \beta = \gamma = \delta = \varepsilon = \theta = \mu = \nu = \eta = 0$, $\lambda = 1$; the corresponding qso with the payoff matrix $A$ has the form

$$A = \begin{pmatrix} 0 & -1 & -1 & -1 & -1 \\ 1 & 0 & -1 & -1 & 1 \\ 1 & 1 & 0 & -1 & -1 \\ 1 & 1 & 1 & 0 & -1 \\ 1 & -1 & 1 & 1 & 0 \end{pmatrix} \qquad \begin{aligned} x_1' &= x_1^2, \\ x_2' &= x_2^2 + 2x_1x_2 + 2x_2x_5, \\ x_3' &= x_3^2 + 2x_1x_3 + 2x_2x_3, \\ x_4' &= x_4^2 + 2x_1x_4 + 2x_2x_4 + 2x_3x_4, \\ x_5' &= x_5^2 + 2x_1x_5 + 2x_3x_5 + 2x_4x_5, \end{aligned} \tag{4.8}$$

and a tournament generated by this operator has the following form shown in Figure 9.

One can see that the vertex $\mathbf{e}_1$ is a sink; the four vertices $\mathbf{e}_2$, $\mathbf{e}_3$, $\mathbf{e}_4$, $\mathbf{e}_5$ form two 3-cycles $2 \succ 5 \succ 3 \succ 2$ and $2 \succ 5 \succ 4 \succ 2$, and a one 4-cycle $4 \succ 2 \succ 5 \succ 3 \succ 4$.

It is easy to show that $\mathrm{Fix}(V) = \{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3, \mathbf{e}_4, \mathbf{e}_5, M_6(0, 1/3, 1/3, 0, 1/3), M_7(0, 1/3, 0, 1/3, 1/3)\}$, where fixed point $\mathbf{e}_1$ is repelling and $\mathbf{e}_2$, $\mathbf{e}_3$, $\mathbf{e}_4$, $\mathbf{e}_5$, $M_6(0, 1/3, 1/3, 0, 1/3)$, $M_7(0, 1/3, 0, 1/3, 1/3)$ are saddles. Note that the restrictions of qso (4.8) on the invariant faces $\Gamma_{14}$ and $\Gamma_{13}$ are reduced to a non-ergodic Zakhrevich example (2.3). It is evident that the segment $[M_6, M_7]$ is the invariant subset of qso (4.8), and if $\mathbf{x} \in \{\tau M_6 + (1-\tau)M_7\}$ then $\omega(\mathbf{x}) = \{M_7\}$. It is also easy to check that the restriction of qso (4.8) on the face $\Gamma_1$ is reduced to a permutation of

$$\begin{aligned} x_1' &= x_1^2 + 2x_1x_2 + 2x_1x_3, \\ x_2' &= x_2^2 + 2x_2x_3 + 2x_2x_4, \\ x_3' &= x_3^2 + 2x_3x_4, \\ x_4' &= x_4^2 + 2x_1x_4, \end{aligned} \tag{4.9}$$

which is non-ergodic [**7**]. From the first equality of (4.8) one can see that $\lim_{n\to\infty}(V^n\mathbf{x})_1 = 0$, such that, using Lemma 4.2, we have the following proposition.

PROPOSITION 4.7. *Any extremal Volterra qso $V$ that belongs to the fifth class is non-ergodic.*

*Sixth class of operators with* $D(V) = \{(+3, -1); (+3, -1); (+2, -2); (0, -4); (+2, -2)\}$. The sixth class also consists of 120 operators. As a representative of this class one can

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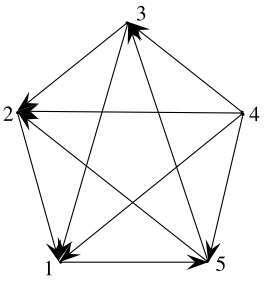



FIGURE 10. Tournament generated by operators in the sixth class, $m = 5$.

choose the operator with coefficients $\alpha = \beta = \gamma = \varepsilon = \theta = \lambda = \mu = \nu = 0$, $\delta = \eta = 1$; the corresponding qso with the payoff matrix $A$ has the form

$$A = \begin{pmatrix} 0 & -1 & -1 & -1 & 1 \\ 1 & 0 & -1 & -1 & -1 \\ 1 & 1 & 0 & -1 & -1 \\ 1 & 1 & 1 & 0 & 1 \\ -1 & 1 & 1 & -1 & 0 \end{pmatrix} \qquad \begin{aligned} x_1' &= x_1^2 + 2x_1x_5, \\ x_2' &= x_2^2 + 2x_1x_2, \\ x_3' &= x_3^2 + 2x_1x_3 + 2x_2x_3, \\ x_4' &= x_4^2 + 2x_1x_4 + 2x_2x_4 + 2x_3x_4 + 2x_4x_5, \\ x_5' &= x_5^2 + 2x_2x_5 + 2x_3x_5, \end{aligned} \tag{4.10}$$

and a tournament generated by this operator has the form shown in Figure 10.

It is evident that the vertex $\mathbf{e}_4$ is a source and the vertices $\mathbf{e}_1$, $\mathbf{e}_2$, $\mathbf{e}_3$, $\mathbf{e}_5$ form two 3-cycles $1 \succ 5 \succ 2 \succ 1$ and $1 \succ 5 \succ 3 \succ 1$, and a single 4-cycle $1 \succ 5 \succ 3 \succ 2 \succ 1$.

It is easy to verify that $\mathrm{Fix}(V) = \{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3, \mathbf{e}_4, \mathbf{e}_5, M_6(1/3, 1/3, 0, 0, 1/3), M_7(1/3, 0, 1/3, 0, 1/3)\}$, where fixed point $\mathbf{e}_4$ is attracting, $M_6$ is repelling and $\mathbf{e}_1$, $\mathbf{e}_2$, $\mathbf{e}_3$, $\mathbf{e}_5$, $M_7(1/3, 0, 1/3, 0, 1/3)$ are saddles. It is evident that the restriction of qso (4.10) on invariant faces $\Gamma_{34}$ and $\Gamma_{24}$ are reduced to Zakharevich's example (2.3). Note that the qso (4.10) has the invariant subset $\{\tau M_6 + (1-\tau)M_7, \tau \in [0, 1]\}$ and the trajectory of (4.10) which starts from this invariant subset converges to the fixed point $M_7$. One can rewrite the fourth equality in (4.10) as $x_4' = x_4(2 - x_4)$ and, using Lemma 4.1, we have:

(1) for any initial point with $x_4^0 > 0$ its trajectory converges to the vertex $\mathbf{e}_4$;
(2) the restriction of qso (4.10) on $\Gamma_4$, that is, $x_4^0 = 0$, reduces to qso (4.9).

Thus we have the following proposition.

PROPOSITION 4.8. *Any extremal Volterra qso $V$ that belongs to the sixth class is regular by* mod 0.

*Six classes of operators generating graphs with 3-, 4- and 5-cycles.* Below we consider the following six classes of operators that generate graphs with 3, 4 and 5-cycles.

*Class of operators with $C(V) = (3, 2, 1)$.* The seventh class consists of 120 operators. As a representative of this class one can choose the operator with coefficients $\alpha = \beta = \gamma =$

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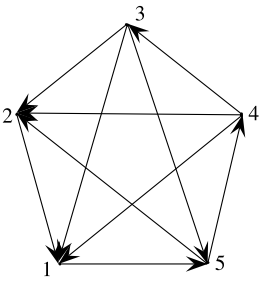



FIGURE 11. Tournament generated by operators in the seventh class, $m = 5$.

$\varepsilon = \theta = \lambda = \mu = \eta = 0$, $\delta = \nu = 1$; the corresponding qso with the payoff matrix $A$ has the form

$$A = \begin{pmatrix} 0 & -1 & -1 & -1 & 1 \\ 1 & 0 & -1 & -1 & -1 \\ 1 & 1 & 0 & -1 & 1 \\ 1 & 1 & 1 & 0 & -1 \\ -1 & 1 & -1 & 1 & 0 \end{pmatrix} \qquad \begin{aligned} x_1' &= x_1^2 + 2x_1x_5, \\ x_2' &= x_2^2 + 2x_1x_2, \\ x_3' &= x_3^2 + 2x_1x_3 + 2x_2x_3 + 2x_3x_5, \\ x_4' &= x_4^2 + 2x_1x_4 + 2x_2x_4 + 2x_3x_4, \\ x_5' &= x_5^2 + 2x_2x_5 + 2x_4x_5, \end{aligned} \tag{4.11}$$

and a tournament generated by this operator has the form shown in Figure 11.

The corresponding graph contains three 3-cycles $1 \succ 5 \succ 2 \succ 1$, $1 \succ 5 \succ 4 \succ 1$ and $3 \succ 5 \succ 4 \succ 3$, two 4-cycles $1 \succ 5 \succ 4 \succ 2 \succ 1$ and $1 \succ 5 \succ 4 \succ 3 \succ 1$, and one Hamiltonian cycle $1 \succ 5 \succ 4 \succ 3 \succ 2 \succ 1$.

It is easy to verify that $\mathrm{Fix}(V) = \{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3, \mathbf{e}_4, \mathbf{e}_5, M_6(1/3, 0, 0, 1/3, 1/3), M_7(1/3, 1/3, 0, 0, 1/3), M_8(0, 0, 1/3, 1/3, 1/3)\}$, where the fixed point $M_7$ is repelling and all other fixed points are saddles. It is evident that the restriction of qso (4.11) on the invariant faces $\Gamma_{34}$, $\Gamma_{23}$, $\Gamma_{12}$ is reduced to Zakharevich's example (2.3). Note that the qso (4.11) has the invariant subset $\{\tau M_6 + (1-\tau)M_7, \tau \in [0, 1]\}$ and for any initial point in this set its trajectory converges to the fixed point $M_6$. Let $\mathbf{x} \in \mathrm{int}\, S^4 \backslash \{\tau M_6 + (1-\tau)M_7, \tau \in [0, 1]\}$ be the initial point; then from (4.11) we have

$$\frac{x_1'}{x_3'} = \frac{x_1}{x_3} \cdot \frac{1 - x_2 - x_3 - x_4 + x_5}{1 + x_1 + x_2 - x_4 + x_5} < \frac{x_1}{x_3},$$
$$\frac{x_2'}{x_4'} = \frac{x_2}{x_4} \cdot \frac{1 + x_1 - x_3 - x_4 - x_5}{1 + x_1 + x_2 + x_3 - x_5} < \frac{x_2}{x_4}.$$

According to [**8**] we have $\lim_{n\to\infty}(V^n\mathbf{x})_1 = \lim_{n\to\infty}(V^n\mathbf{x})_2 = 0$. Thus, $\lim_{n\to\infty}((V^n\mathbf{x})_1 + (V^n\mathbf{x})_2) = 0$ and since the restriction of qso (4.11) on the face $\Gamma_{12}$ is a non-ergodic qso, we have, using Lemma 4.2, the following proposition.

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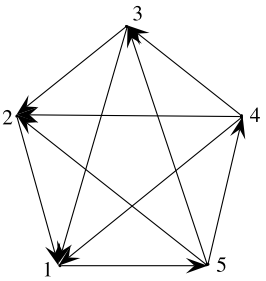



FIGURE 12. Tournament generated by operators in the eighth class, $m = 5$.

PROPOSITION 4.9. *Any extremal Volterra qso $V$ in the seventh class is a non-ergodic transformation.*

*Class of operators with $C(V) = (3, 3, 1)$.* The eighth class also consists of 120 operators. As a representative of this class one can choose the operator with $\alpha = \beta = \gamma = \theta = \lambda = \mu = \nu = \eta = 0$, $\delta = \varepsilon = 1$; the corresponding qso with the payoff matrix $A$ has the form

$$A = \begin{pmatrix} 0 & -1 & -1 & -1 & 1 \\ 1 & 0 & -1 & -1 & -1 \\ 1 & 1 & 0 & -1 & -1 \\ 1 & 1 & 1 & 0 & -1 \\ -1 & 1 & 1 & 1 & 0 \end{pmatrix} \qquad \begin{aligned} x_1' &= x_1^2 + 2x_1x_5, \\ x_2' &= x_2^2 + 2x_1x_2, \\ x_3' &= x_3^2 + 2x_1x_3 + 2x_2x_3, \\ x_4' &= x_4^2 + 2x_1x_4 + 2x_2x_4 + 2x_3x_4, \\ x_5' &= x_5^2 + 2x_2x_5 + 2x_3x_5 + 2x_4x_5, \end{aligned} \tag{4.12}$$

and a tournament generated by this operator has the form shown in Figure 12.

Then the vertices $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3, \mathbf{e}_4, \mathbf{e}_5$ form three 3-cycles $1 \succ 5 \succ 2 \succ 1$, $1 \succ 5 \succ 3 \succ 1$ and $1 \succ 5 \succ 4 \succ 1$, three 4-cycles $1 \succ 5 \succ 3 \succ 2 \succ 1$, $1 \succ 5 \succ 4 \succ 2 \succ 1$ and $1 \succ 5 \succ 4 \succ 3 \succ 1$, and a single Hamilton cycle $1 \succ 5 \succ 4 \succ 3 \succ 2 \succ 1$.

It is easy to verify that $\mathrm{Fix}(V) = \{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3, \mathbf{e}_4, \mathbf{e}_5, M_6(1/3, 1/3, 0, 0, 1/3), M_7(1/3, 0, 1/3, 0, 1/3), M_8(1/3, 0, 0, 1/3, 1/3)\}$, where the fixed point $M_6$ is repelling and all other fixed points are saddles. It is evident that the restriction of qso (4.12) on faces $\Gamma_{34}, \Gamma_{24}, \Gamma_{23}$ are reduced to Zakharevich's example (2.3). Note that the triangle $\Delta = \{\mathbf{x} \in S^4 : x_1 = x_5 = 1/3, x_2 + x_3 + x_4 = 1/3\}$ is an invariant subset and if $\mathbf{x} \in \Delta \backslash \{\tau M_6 + (1-\tau)M_7, \tau \in [0, 1]\}$ then $\omega(\mathbf{x}) = \{M_8\}$, and if $\mathbf{x} \in \{\tau M_6 + (1-\tau)M_7, \tau \in [0, 1]\}$ then $\omega(\mathbf{x}) = \{M_7\}$. Let $\mathbf{x} \in \mathrm{int}\, S^4 \backslash \Delta$ be an any initial point. Then introducing new variables $y_1 = x_1$, $y_2 = x_2 + x_3 + x_4$, $y_3 = x_5$, we have

$$y_1' = x_1', \quad y_2' = x_2' + x_3' + x_4', \quad y_3' = x_5'$$

and the qso (4.12) reduces to (2.3). Therefore we obtain the following proposition.

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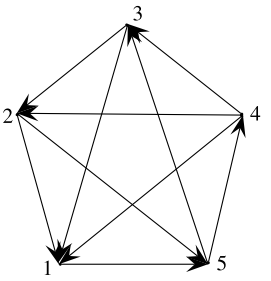



FIGURE 13. Tournament generated by operators in the ninth class, $m = 5$.

PROPOSITION 4.10. *Any extremal Volterra qso $V$ in the eighth class is a non-ergodic transformation.*

*Class of operators with $C(V) = (4, 3, 1)$.* The ninth class also consists of 120 operators. As a representative of this class one can choose the operator with $\alpha = \beta = \gamma = \varepsilon = \theta = \mu = \nu = \eta = 0$, $\delta = \lambda = 1$; the corresponding qso with the payoff matrix $A$ has the form

$$A = \begin{pmatrix} 0 & -1 & -1 & -1 & 1 \\ 1 & 0 & -1 & -1 & 1 \\ 1 & 1 & 0 & -1 & -1 \\ 1 & 1 & 1 & 0 & -1 \\ -1 & -1 & 1 & 1 & 0 \end{pmatrix} \qquad \begin{aligned} x_1' &= x_1^2 + 2x_1x_5, \\ x_2' &= x_2^2 + 2x_1x_2 + 2x_2x_5, \\ x_3' &= x_3^2 + 2x_1x_3 + 2x_2x_3, \\ x_4' &= x_4^2 + 2x_1x_4 + 2x_2x_4 + 2x_3x_4, \\ x_5' &= x_5^2 + 2x_3x_5 + 2x_4x_5, \end{aligned} \tag{4.13}$$

and a tournament generated by this operator has the form shown in Figure 13.

Then the vertices $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3, \mathbf{e}_4, \mathbf{e}_5$ form four 3-cycles $1 \succ 5 \succ 3 \succ 1$, $1 \succ 5 \succ 4 \succ 1$, $5 \succ 4 \succ 2 \succ 5$ and $5 \succ 3 \succ 2 \succ 5$, three 4-cycles $1 \succ 5 \succ 3 \succ 2 \succ 1$, $1 \succ 5 \succ 4 \succ 2 \succ 1$ and $1 \succ 5 \succ 4 \succ 3 \succ 1$, and a single Hamiltonian cycle $1 \succ 5 \succ 4 \succ 3 \succ 2 \succ 1$.

It is easy to verify that $\mathrm{Fix}(V) = \{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3, \mathbf{e}_4, \mathbf{e}_5, M_6(1/3, 0, 0, 1/3, 1/3), M_7(1/3, 0, 1/3, 0, 1/3), M_8(0, 1/3, 0, 1/3, 1/3), M_9(0, 1/3, 1/3, 0, 1/3)\}$, where the fixed point $M_7$ is repelling and all the other fixed points are saddles. It is evident that the restriction of qso (4.13) on faces $\Gamma_{24}$, $\Gamma_{23}$, $\Gamma_{14}$, $\Gamma_{13}$ reduces to Zakharevich's example (2.3). Moreover, the subsets $M = \{\tau M_6 + (1 - \tau)M_7, \tau \in [0, 1]\}$ and $N = \{\xi M_8 + (1 - \xi)M_9, \xi \in [0, 1]\}$ are invariant subsets. If $\mathbf{x} \in M$ then $\omega(\mathbf{x}) = \{M_6\}$; similarly, if $\mathbf{x} \in N$ then $\omega(\mathbf{x}) = \{M_8\}$. Let $\mathbf{x} \in \mathrm{int}\, S^4 \backslash (M \cup N)$; then, introducing new variables $y_1 = x_1 + x_2$, $y_2 = x_3 + x_4$, $y_3 = x_5$, one can see that $y_1' = x_1' + x_2'$, $y_2' = x_3' + x_4'$, $y_3' = x_5'$ and the qso (4.13) reduces to (2.3). Then the following proposition is valid.

PROPOSITION 4.11. *If extremal Volterra qso $V$ belongs to the ninth class, then it is a non-ergodic transformation.*

*Class of operators with $C(V) = (4, 4, 2)$.* The tenth class also consists of 120 operators. As a representative of this class one can choose the operator with $\alpha = \beta = \gamma = \varepsilon = \lambda =$

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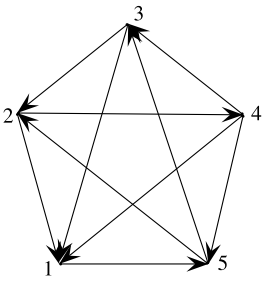



FIGURE 14. Tournament generated by operators in the tenth class, $m = 5$.

$\mu = \nu = 0$, $\delta = \theta = \eta = 1$; the corresponding qso with the payoff matrix $A$ has the form

$$A = \begin{pmatrix} 0 & -1 & -1 & -1 & 1 \\ 1 & 0 & -1 & 1 & -1 \\ 1 & 1 & 0 & -1 & -1 \\ 1 & -1 & 1 & 0 & 1 \\ -1 & 1 & 1 & -1 & 0 \end{pmatrix} \quad \begin{aligned} x_1' &= x_1^2 + 2x_1x_5, \\ x_2' &= x_2^2 + 2x_1x_2 + 2x_2x_4, \\ x_3' &= x_3^2 + 2x_1x_3 + 2x_2x_3, \\ x_4' &= x_4^2 + 2x_1x_4 + 2x_3x_4 + 2x_4x_5, \\ x_5' &= x_5^2 + 2x_2x_5 + 2x_3x_5, \end{aligned} \tag{4.14}$$

and a tournament generated by this operator has the form shown in Figure 14.

Then the vertices $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3, \mathbf{e}_4, \mathbf{e}_5$ form four 3-cycles $1 \succ 5 \succ 2 \succ 1$, $1 \succ 5 \succ 3 \succ 1$, $2 \succ 4 \succ 3 \succ 2$ and $2 \succ 4 \succ 5 \succ 2$, three 4-cycles $1 \succ 5 \succ 2 \succ 4 \succ 1$, $1 \succ 5 \succ 3 \succ 2 \succ 1$ and $5 \succ 3 \succ 2 \succ 4 \succ 5$, and two Hamiltonian cycles $1 \succ 5 \succ 2 \succ 4 \succ 3 \succ 1$ and $1 \succ 5 \succ 3 \succ 2 \succ 4 \succ 1$.

It is easy to verify that $\mathrm{Fix}(V) = \{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3, \mathbf{e}_4, \mathbf{e}_5, M_6(1/3, 1/3, 0, 0, 1/3), M_7(1/3, 0, 1/3, 0, 1/3), M_8(0, 1/3, 1/3, 1/3, 0), M_9(0, 1/3, 0, 1/3, 1/3)\}$, where the fixed point $M_6$ is repelling and all the other fixed points are saddles. It is evident that the restriction of qso (4.14) on faces $\Gamma_{34}$, $\Gamma_{24}$, $\Gamma_{15}$, $\Gamma_{12}$ reduces to Zakharevich's example (2.3). The subsets $M = \{\tau M_6 + (1-\tau)M_7, \tau \in [0, 1]\}$ and $N = \{\xi M_8 + (1-\xi)M_9, \xi \in [0, 1]\}$ are invariant subsets. If $\mathbf{x} \in M$ then $\omega(\mathbf{x}) = \{M_7\}$; similarly, if $\mathbf{x} \in N$ then $\omega(\mathbf{x}) = \{M_9\}$. Let $\mathbf{x} \in \mathrm{int}\, S^4 \backslash (M \cup N)$; then from (4.14) we have

$$\frac{x_1'}{x_4'} = \frac{x_1}{x_4} \cdot \frac{1 - x_2 - x_3 - x_4 + x_5}{1 + x_1 - x_2 - x_4 + x_5} < \frac{x_1}{x_4}.$$

According to [**8**] we have that $\lim_{n\to\infty}(V^n\mathbf{x})_1 = 0$ and consequently $\omega(\mathbf{x}) \subset \Gamma_1$. The restriction of qso (4.14) on this invariant subset $\Gamma_1$ reduces to a non-ergodic qso (4.9). Then, using Lemma 4.2, we prove the following statement.

PROPOSITION 4.12. *If an extremal Volterra qso $V$ belongs to the tenth class, then it is a non-ergodic transformation.*

*Class of operators with $C(V) = (4, 4, 3)$.* The eleventh class consists of 40 operators. As a representative of this class one can choose the operator with $\alpha = \beta = \gamma = \varepsilon = \lambda = \mu =$

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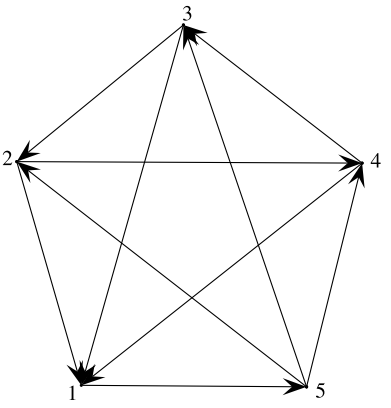



FIGURE 15. Tournament generated by operators in the eleventh class, $m = 5$.

$\nu = \eta = 0$, $\delta = \theta = 1$; the corresponding qso with the payoff matrix $A$ has the form

$$A = \begin{pmatrix} 0 & -1 & -1 & -1 & 1 \\ 1 & 0 & -1 & 1 & -1 \\ 1 & 1 & 0 & -1 & -1 \\ 1 & -1 & 1 & 0 & -1 \\ -1 & 1 & 1 & 1 & 0 \end{pmatrix} \qquad \begin{aligned} x_1' &= x_1^2 + 2x_1x_5, \\ x_2' &= x_2^2 + 2x_1x_2 + 2x_2x_4, \\ x_3' &= x_3^2 + 2x_1x_3 + 2x_2x_3, \\ x_4' &= x_4^2 + 2x_1x_4 + 2x_3x_4, \\ x_5' &= x_5^2 + 2x_2x_5 + 2x_3x_5 + 2x_4x_5, \end{aligned} \tag{4.15}$$

and a tournament generated by this operator has the form shown in Figure 15.

Then the corresponding graph contains four 3-cycles $1 \succ 5 \succ 2 \succ 1$, $1 \succ 5 \succ 3 \succ 1$, $2 \succ 4 \succ 3 \succ 2$ and $1 \succ 5 \succ 4 \succ 1$, three 4-cycles $1 \succ 5 \succ 2 \succ 4 \succ 1$, $1 \succ 5 \succ 3 \succ 2 \succ 1$ and $1 \succ 5 \succ 4 \succ 3 \succ 1$, and three Hamiltonian cycles $1 \succ 5 \succ 2 \succ 4 \succ 3 \succ 1$, $1 \succ 5 \succ 3 \succ 2 \succ 4 \succ 1$ and $1 \succ 5 \succ 4 \succ 3 \succ 2 \succ 1$.

It is easy to verify that $\mathrm{Fix}(V) = \{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3, \mathbf{e}_4, \mathbf{e}_5, M_6(1/3, 1/3, 0, 0, 1/3), M_7(1/3, 0, 1/3, 0, 1/3), M_8(1/3, 0, 0, 1/3, 1/3), M_9(0, 1/3, 1/3, 1/3, 0), M_{10}(1/3, 1/9, 1/9, 1/9, 1/3)\}$, where the fixed points $M_7$, $M_{10}$ are repelling and all other fixed points are saddles. It is evident that the restriction of qso (4.15) on faces $\Gamma_{34}$, $\Gamma_{24}$, $\Gamma_{23}$, $\Gamma_{15}$ reduces to Zakharevich's example (2.3). Moreover, a triangle $\Delta = \{\mathbf{x} \in S^4 : x_1 = x_5 = 1/3, x_2 + x_3 + x_4 = 1/3\}$ is an invariant subset. The restriction of qso (4.15) on this invariant triangle has the form

$$\begin{aligned} x_1' &= 1/3, \\ x_2' &= x_2(1 - x_3 + x_4), \\ x_3' &= x_3(1 + x_2 - x_4), \\ x_4' &= x_4(1 - x_2 + x_3), \\ x_5' &= 1/3, \end{aligned} \tag{4.16}$$

and this operator is non-ergodic.

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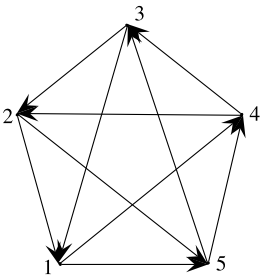



FIGURE 16. Tournament generated by operators in the twelfth class, $m = 5$.

Let $\mathbf{x} \in \operatorname{int} S^4 \backslash \Delta$; then introducing new variables $y_1 = x_1$, $y_2 = x_2 + x_3 + x_4$, $y_3 = x_5$ we have $y_1' = x_1'$, $y_2' = x_2' + x_3' + x_4'$, $y_3' = x_5'$ and qso (4.15) reduces to a non-ergodic qso (2.3). Therefore we obtain the following proposition.

PROPOSITION 4.13. *Any extremal Volterra qso $V$ in the eleventh class is a non-ergodic transformation.*

Note that in all the classes considered we can reduce the corresponding operator to Zakharevich's qso by introducing some new variables. The twelfth and final class consists of uniform extremal Volterra operators. This class is essentially complex and we need to develop a new approach and a new technique to study it. We therefore consider this class in a separate section.

## 5. *Class of uniform extremal operators*

The twelfth class consists of 24 operators. The Id of any operator $V$ in this class has the form $\mathrm{Id}(V) = ((5, 5, 2); \{(+2, -2), (+2, -2), (+2, -2), (+2, -2), (+2, -2)\})$. As a representative of this class one can choose the operator with coefficients $\alpha = \beta = \varepsilon = \theta = \mu = \nu = \eta = 0$, $\gamma = \delta = \lambda = 1$; the corresponding qso with the payoff matrix $A$ has the form

$$A = \begin{pmatrix} 0 & -1 & -1 & 1 & 1 \\ 1 & 0 & -1 & -1 & 1 \\ 1 & 1 & 0 & -1 & -1 \\ -1 & 1 & 1 & 0 & -1 \\ -1 & -1 & 1 & 1 & 0 \end{pmatrix} \qquad \begin{aligned} x_1' &= x_1^2 + 2x_1x_4 + 2x_1x_5, \\ x_2' &= x_2^2 + 2x_1x_2 + 2x_2x_5, \\ x_3' &= x_3^2 + 2x_1x_3 + 2x_2x_3, \\ x_4' &= x_4^2 + 2x_2x_4 + 2x_3x_4, \\ x_5' &= x_5^2 + 2x_3x_5 + 2x_4x_5, \end{aligned} \tag{5.1}$$

and a tournament generated by this operator has the form shown in Figure 16.

Then the corresponding graph contains five 3-cycles $1 \succ 4 \succ 2 \succ 1$, $1 \succ 4 \succ 3 \succ 1$, $1 \succ 5 \succ 3 \succ 2$, $2 \succ 5 \succ 3 \succ 2$ and $4 \succ 2 \succ 5 \succ 4$, five 4-cycles $1 \succ 4 \succ 3 \succ 2 \succ 1$, $1 \succ 5 \succ 3 \succ 2 \succ 1$, $1 \succ 5 \succ 4 \succ 3 \succ 1$ and $4 \succ 3 \succ 2 \succ 5 \succ 4$, and two Hamiltonian cycles $1 \succ 4 \succ 2 \succ 5 \succ 3 \succ 1$ and $1 \succ 5 \succ 4 \succ 3 \succ 2 \succ 1$.

It is easy to verify that $\mathrm{Fix}(V) = \{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3, \mathbf{e}_4, \mathbf{e}_5, M_6(1/3, 1/3, 0, 1/3, 0), M_7(1/3, 0, 1/3, 1/3, 0), M_8(1/3, 0, 1/3, 0, 1/3), M_9(0, 1/3, 1/3, 0, 1/3), M_{10}(0, 1/3, 0, 1/3, 1/3), M_{11}(1/5, 1/5, 1/5, 1/5, 1/5)\}$, where the fixed point $M_{11}$ is repelling and the

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other fixed points are saddles. It is evident that the restriction of qso (5.1) on faces $\Gamma_{35}, \Gamma_{25}, \Gamma_{24}, \Gamma_{14}, \Gamma_{13}$ reduces to Zakharevich's example (2.3). Let $\langle i, j\rangle$ denote an edge connecting vertices $i$ and $j$. According to $D(V) = \{(+2, -2), (+2, -2), (+2, -2), (+2, -2), (+2, -2)\}$, one can introduce the following sequence of edges, which forms a 'cycle':

$$\langle 45\rangle \to \langle 12\rangle \to \langle 34\rangle \to \langle 15\rangle \to \langle 23\rangle \to \langle 45\rangle,$$

or, in a more detailed form,

$$\langle 45\rangle \rightrightarrows 3 \rightrightarrows \langle 12\rangle \rightrightarrows 5 \rightrightarrows \langle 34\rangle \rightrightarrows 2 \rightrightarrows \langle 15\rangle \rightrightarrows 4 \rightrightarrows \langle 23\rangle \rightrightarrows 1 \rightrightarrows \langle 45\rangle$$

where, for example, $\langle 23\rangle \rightrightarrows 1 \rightrightarrows \langle 45\rangle$ means that there are two edges from $\mathbf{e}_2, \mathbf{e}_3$ to $\mathbf{e}_1$ and there are two edges from $\mathbf{e}_1$ to $\mathbf{e}_4, \mathbf{e}_5$. Below we will use the following theorem.

THEOREM 5.1. [**1**, **8**, **11**] *Let* $\mathbf{p} = (p_1, \ldots, p_m)$ *be the repelling fixed point of the Volterra qso (2.7). Then there exists a global decreasing Lyapunov function of the form* $\varphi_p(\mathbf{x}) = x_1^{p_1} \cdots x_m^{p_m}$, *where* $p_i \geq 0$, $\sum_{i=1}^m p_i = 1$, $\mathbf{p} \in S^{m-1}$.

Let us denote by $\phi_i(\mathbf{x}) \equiv \phi_{M_{i+5}}(\mathbf{x})$, $i = \overline{1, 6}$, a Lyapunov function that corresponds to a fixed point $M_{i+5}$, $i = \overline{1, 6}$. According to Theorem 5.1 these Lyapunov functions have the form

$$\phi_1(\mathbf{x}) = (x_1x_2x_4)^{1/3}, \quad \phi_2(\mathbf{x}) = (x_1x_3x_4)^{1/3}, \quad \phi_3(\mathbf{x}) = (x_1x_3x_5)^{1/3},$$
$$\phi_4(\mathbf{x}) = (x_2x_3x_5)^{1/3}, \quad \phi_5(\mathbf{x}) = (x_2x_4x_5)^{1/3}, \quad \phi_6(\mathbf{x}) = (x_1x_2x_3x_4x_5)^{1/5}.$$

It is easy to verify that the Jacobian of qso (5.1) at fixed point $M_{i+5}$, $i = \overline{1, 5}$, has eigenvalues $\lambda_1 < 1$ and $\lambda_2 > 1$, $\lambda_3 > 1$, $\lambda_4 > 1$. Then according to [**4**, Theorem 6.5] there exists an invariant smooth curve $\gamma_i$, $i = \overline{1, 5}$, and for any initial point from this curve its trajectory converges to a fixed point $M_{i+5}$, $i = \overline{1, 5}$. Therefore $\lim_{n\to\infty} \phi_i(V^n\mathbf{x}) = 1/3$ or $\lim_{n\to\infty} \phi_i(V^n\mathbf{x}) = 0$, $i = \overline{1, 5}$.

The case $\lim_{n\to\infty} \phi_i(V^n\mathbf{x}) = 1/3$, $i = \overline{1, 5}$, is not interesting and we consider the case where

$$\lim_{n\to\infty} \phi_i(V^n\mathbf{x}) = 0, \quad i = \overline{1, 5}. \tag{5.2}$$

These relations will be very useful. For example, if $i = 1$ with a Lyapunov function $\phi_1(\mathbf{x}) = (x_1x_2x_4)^{1/3}$, then relation (5.2) means that

$$\lim_{n\to\infty} \min\{x_1^{(n)}, x_2^{(n)}, x_4^{(n)}\} = 0.$$

Consider the functions

$$\varphi_1(\mathbf{x}) = x_1x_2x_4, \quad \varphi_2(\mathbf{x}) = x_1x_3x_4, \quad \varphi_3(\mathbf{x}) = x_1x_3x_5,$$
$$\varphi_4(\mathbf{x}) = x_2x_3x_5, \quad \varphi_5(\mathbf{x}) = x_2x_4x_5, \quad \varphi(\mathbf{x}) = \max_{1\leq k\leq 5} \varphi_k(x).$$

Thus we can conclude that for any $\mathbf{x} \in \operatorname{int} S^4 \backslash \{M_{11} \cup \bigcup_i \gamma_i\}$, $\lim_{n\to\infty} \varphi(V^n\mathbf{x}) = 0$, that is,

$$\omega(\mathbf{x}) \subset \Gamma_{123} \cup \Gamma_{125} \cup \Gamma_{145} \cup \Gamma_{234} \cup \Gamma_{345}. \tag{5.3}$$

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We introduce the following notation:

$$G_1 = \{\mathbf{x} \in S^4 : x_1 \geq x_2 \geq x_3 \geq x_4 \geq x_5\}, \ldots,$$
$$G_{120} = \{\mathbf{x} \in S^4 : x_5 \geq x_4 \geq x_3 \geq x_2 \geq x_1\}.$$

Let $W$ be a 'four-dimensional tetrahedronal' neighborhood of the point $M_{11}$ and $G_k^i = \{G_k : x_i = \max_{1\leq j\leq 5} x_j, \forall \mathbf{x} \in G_k\}$. We define the neighborhood $U_i$ of the vertex $\mathbf{e}_i$ as follows: $U_i = (\bigcup_k G_k^i)\backslash W$, that is, $U_i = \{\mathbf{x} : x_i = \max_{1\leq j\leq 5} x_j\}\backslash W$.

Let us consider a part of the above-mentioned 'cycle', say, $\langle 23\rangle \rightrightarrows 1 \rightrightarrows \langle 45\rangle$. Then for any point $M \in U_4 \cup U_5$ its route is given in the opposite direction $U_4 \cup U_5 \to U_1 \to U_2 \cup U_3$.

LEMMA 5.2. *If* $\mathbf{x} \in U_{45}$, $\mathbf{x}^{(k)} \in U_1$, $k = \overline{1, n}$, *then there is some* $n \in N$ *such that*

$$n > A \log(B/\varphi(\mathbf{x}')), \tag{5.4}$$

*and* $\mathbf{x}^{(n+1)} \in U_{23}$, *where* $U_{45} = U_4 \cup U_5$, $U_{23} = U_2 \cup U_3$, $\mathbf{x}' = V(\mathbf{x})$ *and* $A, B$ *are absolute constants.*

*Proof.* (a) Let $\mathbf{x} \in U_4$, $\mathbf{x}^{(k)} \in U_1$, $k = \overline{1, n}$, and $\mathbf{x}^{(n+1)} \in U_3$; then from (5.1) for $\mathbf{x} \in \operatorname{int} S^4$ we obtain

$$\frac{x_3^{(n+1)}}{x_3'} = \prod_{k=1}^{n}(1 + x_1^{(k)} + x_2^{(k)} - x_4^{(k)} - x_5^{(k)}) < 2^n.$$

For $\mathbf{x} \in U_4$, $\mathbf{x}' \in U_1$ and $\mathbf{x}^{(n+1)} \in U_3$ one has

$$2^n > \frac{x_3^{(n+1)}}{x_3'} \geq \frac{1}{5x_3'} = \frac{x_1'x_4'}{5\varphi_2(\mathbf{x}')} \geq \frac{x_1'x_4'}{5\varphi(\mathbf{x}')}.$$

From (5.1) it follows that

$$x_4' = x_4(x_4 + 2x_2 + 2x_3) \geq x_1x_4 = \frac{x_1(x_4 + 4x_4)}{5} \geq \frac{x_1'}{5},$$

and since $\varphi_1(\mathbf{x}') \leq 1$ we have

$$2^n\varphi(\mathbf{x}') > \frac{x_1'x_4'}{5} \geq \frac{(x_1')^2}{25} \geq \frac{1}{625} \geq \frac{\tau}{625},$$

where $\tau = \min\{\varphi_i(\mathbf{x}'), i = 1, \ldots, 5\}$ and it is clear that $\tau > 0$ for all $\mathbf{x} \in \operatorname{int} S^4$.

(b) Let $\mathbf{x} \in U_4$, $\mathbf{x}^{(k)} \in U_1$, $k = \overline{1, n}$ and $\mathbf{x}^{(n+1)} \in U_2$; then from (5.1) for $\mathbf{x} \in \operatorname{int} S^4$ we have

$$\frac{x_2^{(n+1)}}{x_2'} = \prod_{k=1}^{n}(1 + x_1^{(k)} - x_3^{(k)} - x_4^{(k)} + x_5^{(k)}) < 2^n.$$

Hence for $\mathbf{x} \in U_4$, $\mathbf{x}' \in U_1$ and $\mathbf{x}^{(n+1)} \in U_2$ as above, we have

$$2^n > \frac{x_2^{(n+1)}}{x_2'} \geq \frac{1}{5x_2'} = \frac{x_1'x_4'}{5\varphi_1(\mathbf{x}')} \geq \frac{x_1'x_4'}{5\varphi(\mathbf{x}')}.$$

From (5.1) it follows that

$$x_4' = x_4(x_4 + 2x_2 + 2x_3) \geq x_1(x_4 + 2x_2 + 2x_3) \geq x_1x_4 = \frac{x_1(x_4 + 4x_4)}{5} \geq \frac{x_1'}{5},$$

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and since $\varphi_1(\mathbf{x}') \le 1$ we have

$$2^n\varphi(\mathbf{x}') > \frac{x_1'x_4'}{5} \ge \frac{(x_1')^2}{25} \ge \frac{1}{625} \ge \frac{\tau}{625}.$$

(c) Let $\mathbf{x} \in U_5$, $\mathbf{x}^{(k)} \in U_1$, $k = \overline{1,n}$, and $\mathbf{x}^{(n+1)} \in U_3$; then from (5.1) for $\mathbf{x} \in \operatorname{int} S^4$ we obtain

$$\frac{x_3^{(n+1)}}{x_3'} = \prod_{k=1}^{n}(1 + x_1^{(k)} + x_2^{(k)} - x_4^{(k)} - x_5^{(k)}) < 2^n.$$

Hence for $\mathbf{x} \in U_5$, $\mathbf{x}' \in U_1$ and $\mathbf{x}^{(n+1)} \in U_3$ we have

$$2^n > \frac{x_3^{(n+1)}}{x_3'} \ge \frac{1}{5x_3'} = \frac{x_1'x_5'}{5\varphi_3(\mathbf{x}')} \ge \frac{x_1'x_5'}{5\varphi(\mathbf{x}')}.$$

From (5.1) as above one can show that

$$x_5' = x_5(x_5 + 2x_3 + 2x_4) \ge x_1x_5 = \frac{x_1(x_5 + 4x_5)}{5} \ge \frac{x_1'}{5},$$

and, using $\varphi_1(\mathbf{x}') \le 1$, we get

$$2^n\varphi(\mathbf{x}') > \frac{x_1'x_5'}{5} \ge \frac{(x_1')^2}{25} \ge \frac{1}{625} \ge \frac{\tau}{625}.$$

(d) Let $\mathbf{x} \in U_5$, $\mathbf{x}^{(k)} \in U_1$, $k = \overline{1,n}$, and $\mathbf{x}^{(n+1)} \in U_2$; then from (5.1) for $\mathbf{x} \in \operatorname{int} S^4$ we obtain

$$\frac{x_2^{(n+1)}}{x_2'} = \prod_{k=1}^{n}(1 + x_1^{(k)} - x_3^{(k)} - x_4^{(k)} + x_5^{(k)}) < 2^n.$$

Hence for $\mathbf{x} \in U_5$ and $\mathbf{x}' \in U_1$ as above, we have

$$2^n > \frac{x_2^{(n+1)}}{x_2'} \ge \frac{1}{5x_2'} = \frac{x_1'x_4'}{5\varphi_1(\mathbf{x}')} \ge \frac{x_1'x_4'}{5\varphi(\mathbf{x}')} \ge \frac{x_1'x_4'x_2'}{5\varphi(\mathbf{x}')} = \frac{\varphi_1(x')}{5\varphi(\mathbf{x}')}$$

and

$$2^n\varphi(x') > \frac{\varphi_1(\mathbf{x}')}{5} \ge \frac{\varphi_1(\mathbf{x}')}{625} \ge \frac{\tau}{625}.$$

Therefore

$$n > \log_2 \frac{\tau}{625\varphi(\mathbf{x}')} > A\log(B/\varphi(\mathbf{x}')).$$ □

*Remark 5.3.* The other parts of the cycle can be considered in a similar manner.

Lemma 5.2 shows that there are neighborhoods of vertices such that the trajectory moves from the neighborhood of a vertex to the neighborhood of another vertex, staying for some time in this neighborhood and then moving to the neighborhood of a third vertex, .... For any starting point from $\mathbf{x} \in \operatorname{int} S^4$, the problem of describing the exact sequence of visits to these neighborhoods turns out to be rather difficult.

From this lemma, it follows that the time spent in any neighborhood of any vertex tends to infinity and the average time spent in the exterior of the union of sufficiently small neighborhoods of all vertices is equal to 0.

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We now consider the degree of convergence of $\varphi(\mathbf{x})$. Evidently,

$$\varphi(\mathbf{x}') = \varphi(\mathbf{x}) \cdot Q(\mathbf{x}) = \varphi(\mathbf{x}) \cdot \max\{Q_1(\mathbf{x}),\, Q_2(\mathbf{x}),\, Q_3(\mathbf{x}),\, Q_4(\mathbf{x}),\, Q_5(\mathbf{x})\},$$

where

$$\begin{aligned}
Q_1(\mathbf{x}) &= (x_1 + 2x_4 + 2x_5)(x_2 + 2x_1 + 2x_5)(x_4 + 2x_2 + 2x_3),\\
Q_2(\mathbf{x}) &= (x_1 + 2x_4 + 2x_5)(x_3 + 2x_1 + 2x_2)(x_4 + 2x_2 + 2x_3),\\
Q_3(\mathbf{x}) &= (x_1 + 2x_4 + 2x_5)(x_3 + 2x_1 + 2x_2)(x_5 + 2x_3 + 2x_4),\\
Q_4(\mathbf{x}) &= (x_2 + 2x_1 + 2x_5)(x_3 + 2x_1 + 2x_2)(x_5 + 2x_3 + 2x_4),\\
Q_5(\mathbf{x}) &= (x_2 + 2x_1 + 2x_5)(x_4 + 2x_2 + 2x_3)(x_5 + 2x_3 + 2x_4).
\end{aligned} \tag{5.5}$$

(i) Let $x_3 = 0$ and without loss the generality assume that $x_1 \neq x_2$, because $V$: $(0.5, 0.5, 0, 0, 0) \to (0.25, 0.75, 0, 0, 0)$. On the face $\Gamma_{123}$ we get $Q(\mathbf{x}) < 1$ by the relations

$$\begin{aligned}
Q_1(\mathbf{x}) &= 2x_1(1 + x_1)(1 - x_1) < 1,\\
Q_2(\mathbf{x}) &= 4x_1(1 - x_1) < 1,\\
Q_3(\mathbf{x}) &= Q_4(\mathbf{x}) = Q_5(\mathbf{x}) = 0.
\end{aligned} \tag{5.6}$$

(ii) Let $x_3 > 0$; then the restriction of qso (5.1) on the invariant face has the form

$$\begin{aligned}
x_1' &= x_1^2,\\
x_2' &= x_2^2 + 2x_1x_2,\\
x_3' &= x_3^2 + 2x_1x_3 + 2x_2x_3,\\
x_4' &= x_5' = 0.
\end{aligned} \tag{5.7}$$

It is easy to see that, after a finite number of iterations, $\min\{x_1^{(n+1)}, x_2^{(n+1)}, x_3^{(n+1)}\} \leq (x_1^{(n)} + x_2^{(n)})^2$. For $x_3 > 0$, we have $\varphi(\mathbf{x}^{(n+1)}) \leq \min\{x_1^{(n+1)}, x_2^{(n+1)}, x_3^{(n+1)}\} \leq (x_1^{(n)} + x_2^{(n)})^2$ and so $\varphi(\mathbf{x}^{(n+1)}) \leq (x_1 + x_2)^{2^n}$.

Hence from (i) and (ii) it follows that, for any $\mathbf{x} \in \operatorname{int} S^4$, the function $\varphi(\mathbf{x})$ rapidly decreases exponentially. The other two-dimensional faces without cycles can be considered in a similar manner.

Thus for any $\theta > 0$ there exists a neighborhood $U$ of a vertex set such that $\varphi(\mathbf{x}') \leq \theta\varphi(\mathbf{x})$ for $\mathbf{x} \in U$. For any $n$ let $n = n_1 + n_2$, where $n_1$ is the time spent by $\mathbf{x}^n$ in $U$ and $n_2$ is the remaining time; then $\varphi(\mathbf{x}^n) \leq \theta^{n_1}\varphi(\mathbf{x})$. Therefore $\lim_{n\to\infty} \sup \sqrt[n]{\varphi(\mathbf{x}^n)} \leq \theta$ since $\lim_{n\to\infty} n_1/n = 1$ is shown above. It is clear that $\lim_{n\to\infty} \sup \sqrt[n]{\varphi(\mathbf{x}^n)} = 0$, because $\theta$ is arbitrary. Using (5.4), we have $2^{t(n_1)}\varphi(\mathbf{x}^{(n_1+1)}) > \tau/625$ and $t(n_1)/n \to \infty$ as $n \to \infty$. However, the average $(\mathbf{x} + \mathbf{x}' + \cdots + \mathbf{x}^n)/(n + 1)$ is arbitrarily close to the closure $\overline{U}_1$. Hence if the limit of averages exists and it lies in $\overline{U}_1$, then by the same argument one can show that this limit lies in $\overline{U}_2$, $\overline{U}_3$, $\overline{U}_4$ and $\overline{U}_5$ respectively, which contradicts with $\overline{U}_1 \cap \overline{U}_2 \cap \overline{U}_3 \cap \overline{U}_4 \cap \overline{U}_5 = \emptyset$.

So we get the following proposition.

PROPOSITION 5.4. *If an extremal Volterra qso $V$ belongs to the twelfth class, then it is a non-ergodic transformation.*

Thus we have proved that there are 12 classes of extremal Volterra operators on $S^4$ with different Id, described the trajectory behavior of operators in each class, and shown that

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operators with different Id have different trajectory behaviors. Note that the operators in the twelfth class have essentially different trajectory behaviors.

In the next section we discuss briefly how to generalize these results for arbitrary, not necessarily extremal Volterra operators and for an arbitrary $m$.

## 6. *Conclusion*

We have considered the set of all extremal Volterra qsos on $S^4$ and described their trajectory behaviors. It is proven that any uniform extremal Volterra operator is non-ergodic. A non-uniform extremal Volterra operator $V$ is non-ergodic if and only if there exists a subset $I \subset \{1, 2, 3, 4, 5\}$ with $|I| \leq 2$ such that $\sum_{i \in I}(V^n \mathbf{x})_i \to 0$, and the restriction of $V$ on an invariant face $\Gamma_I = \{\mathbf{x} \in S^{m-1} : x_i = 0, i \in I\}$ is a uniform Volterra operator. An extremal Volterra operator $V$ is non-ergodic if and only if the graph generated by this operator contains a Hamiltonian cycle or a Hamiltonian component with the property described above.

It is evident that for a generic Volterra qso with $a_{ij} \neq 0$, $i, j = 1, \ldots, 5$, a tournament generated by this operator coincides with the tournament generated by some extremal Volterra operator, so that one can split the set of all Volterra operators into 12 classes. Then, following [**5**], we can describe the trajectory behavior of any Volterra operator.

One can prove that if $m$ is odd, then a uniform extremal Volterra operator is non-ergodic and any non-uniform extremal Volterra operator can be reduced to a uniform Volterra operator on $S^{m-3}$; and if $m$ is even, a uniform Volterra operator on $S^{m-1}$ does not exist and any extremal Volterra operator can be reduced to a uniform Volterra operator on $S^{m-2}$. Proofs of these statements will be considered in another paper.

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