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Two notes on ideal-transforms

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Abstract

The first note gives two new characterization of the ideal-transform $T(I)$ of a finitely generated regular ideal I in a large class of rings. Specifically, if b is a regular element in I, then there exists a regular element $c \in I$ and a multiplicatively closed set S of regular elements in R such that $T(I) = T((b,c)R) = R_b \cap R_c = R_b \cap R_S$, so $T(I)$ is the ideal-transform of an ideal generated by two elements, and every ring of the form $R_b \cap R_S$ is an ideal-transform. The second theorem shows that if $T(I)$ is integrally closed, then it is a Krull ring. As an application of these results we strengthen some known results concerning when certain ideal-transforms of the Rees ring $\mathbf{R}(R,I)$ are finite or integral extension rings of $\mathbf{R}(R,I)$.

1. *Introduction*

In [**11**], M. Nagata introduced the ideal-transform $T(I)$ of an ideal I in a ring R, and these transforms proved very useful in his series of papers on the Fourteenth Problem of Hilbert. Also, D. Rees used ideal-transforms in his paper [**20**] on this problem, and Nagata noted in [**12**] that they could be used in the study of the Catenary Chain Conjectures. Since then, ideal-transforms have proved useful in the study of over-rings of a ring [**1**, **2**], and Brodmann [**3**], Kiyek [**5**], Matijevic [**6**], McAdam and Ratliff [**10**] and Nishimura [**14**, **15**] have studied when these transforms are finite or integral extensions, and McAdam [**7**, **8**] and Schenzel [**22**] have noted their usefulness in the study of asymptotic prime divisors. Thus, ideal-transforms have proved to be very useful in several research areas in commutative algebra, and they are quite interesting in their own right.

In this paper we prove two new results concerning ideal-transforms for finitely generated regular ideals in a large class of rings (including all Krull domains and all Noetherian rings). The first of these shows that if b is a regular element in I, then there exists a regular element $c \in I$ such that $T(I) = T((b,c)R) = R_b \cap R_c$ and every ring of the form $R_b \cap R_S$ (where S is a multiplicatively closed set of regular elements of R) is the ideal-transform $T((b,c)R)$ for some regular element c in R. The second main result shows that if $T(I)$ is integrally closed, then $T(I)$ is a Krull ring. Finally, as applications of these results we strengthen some known results concerning when certain ideal-transforms of the Rees ring $\mathbf{R}(R,I)$ of R with respect to I are either a finite extension ring or an integral extension ring of $\mathbf{R}(R,I)$.

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2. *First theorem*

We begin with four definitions and a remark concerning them.

(2·1) *Definition.* Let I be an ideal in a ring R. Then:

(2·1·1) If F is the total quotient ring of R, then the *I-transform $T(I)$ of R* is defined by $T(I) = \{r/s \in F;\ r, s \in R,\ (0):sR = (0), \text{ and } (r/s)I^n \subseteq R \text{ for some } n \geqslant 1\}$.

(2·1·2) I is said to be *regular* if it contains some regular element of R.

(2·1·3) $\mathbf{P}(R) = \{P \in \operatorname{Spec}(R);$ there exist $b, c \in R$ such that $(0):bR = (0)$ and P is minimal with respect to being prime and containing $bR:cR\}$.

(2·1·4) $\mathbf{C}^*$ denotes the class of rings R such that (i) R has only finitely many maximal prime divisors of zero, and (ii) each regular non-unit in R is contained in only finitely many $P \in \mathbf{P}(R)$.

(2·2) *Remark.* (2·2·1) It is readily checked that $T(I)$ is an over-ring of R (that is, a ring between R and F), and if I is finitely generated, then $T(I) = \{r/s \in F;\ r, s \in R,\ (0):sR = (0), \text{ and } I \subseteq \operatorname{Rad}(sR:rR)\}$.

(2·2·2) It follows from Zorn's Lemma that if $bR:cR \neq R$, then there exists a prime ideal in R that is minimal with respect to containing $bR:cR$.

(2·2·3) It follows immediately from (2·2·1) and (2·2·2) that if I is finitely generated, then $T(I) = R$ if and only if $I \not\subseteq P$ for all $P \in \mathbf{P}(R)$.

(2·2·4) It readily follows from (2·2·2) and the definition of $\mathbf{P}(R)$ that if b is a regular non-unit in R and $c \in R$, then $R = \bigcap\{R_{(P)};\ P \in \mathbf{P}(R)\}$ and $bR:cR = \bigcap\{(bR_{(P)}:cR_{(P)}) \cap R;\ bR:cR \subseteq P \in \mathbf{P}(R)\}$, where (P) denotes the set of regular elements in $R - P$.

(2·2·5) It is clear that each Noetherian ring and each Krull domain is in $\mathbf{C}^*$, and it is shown in ([16], proposition 2·13) that the integral closure of a Noetherian ring is in $\mathbf{C}^*$. And it is readily seen that if $R \in \mathbf{C}^*$ and S is a multiplicatively closed set of regular elements in R, then $R_S \in \mathbf{C}^*$ and its set $\mathbf{P}(R_S)$ consists of the ideals PR_S with $P \in \mathbf{P}(R)$ such that $P \cap S = \varnothing$.

To prove the new characterizations of $T(I)$ we need the following lemma.

(2·3) LEMMA. *If $R \in \mathbf{C}^*$ and if I and J are finitely generated ideals in R, then $T(I) \subseteq T(J)$ if and only if $J \subseteq P$ for every $P \in \mathbf{P}(R)$ that contains I.*

Proof. Assume first that $T(I) \subseteq T(J)$ and let $P \in \mathbf{P}(R)$ contain I. (By (2·2·3) it may be assumed that $I \subseteq P$ for some $P \in \mathbf{P}(R)$.) Let $b,\ c \in R$ such that $(0):bR = (0)$ and P is minimal over $bR:cR$. Now only finitely many $P \in \mathbf{P}(R)$ contain $bR:cR$, by (2·1·4) (ii) (since $R \in \mathbf{C}^*$), so it follows that there exists $d \in R$ such that $P = \operatorname{Rad}(bR:cdR)$. Therefore $I \subseteq P = \operatorname{Rad}(bR:cdR)$, so $cd/b \in T(I) \subseteq T(J)$, by (2·2·1), so $J \subseteq \operatorname{Rad}(bR:cdR) = P$, by (2·2·1).

Conversely, let $r/s \in T(I)$, so $I \subseteq \operatorname{Rad}(sR:rR)$, by (2·2·1). It may clearly be assumed that $r/s \notin R$, so by (2·2·2) let $P \in \mathbf{P}(R)$ be minimal with respect to containing $sR:rR$. Then $I \subseteq P$, so the hypothesis implies that $J \subseteq P$, so it follows that $J \subseteq \operatorname{Rad}(sR:rR)$. Therefore $r/s \in T(J)$, by (2·2·1), so $T(I) \subseteq T(J)$. ∎

(2·4) COROLLARY. *If $R \in \mathbf{C}^*$ and if I and J are finitely generated ideals in R, then $T(I) = T(J)$ if and only if I and J are contained in exactly the same $P \in \mathbf{P}(R)$.*

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Proof. This is clear by (2·3). ∎

The following corollary of (2·3) is well known.

(2·5) COROLLARY. *If* $R \in \mathbf{C}^*$, *if* I *and* J *are finitely generated ideals in* R, *and if* $\mathrm{Rad}(I) = \mathrm{Rad}(J)$, *then* $T(I) = T(J)$.

Proof. This is also clear by (2·3). ∎

Nagata showed in ([**11**], corollary, p. 59) and in ([**13**], corollary, p. 113) that if I is an ideal in a Krull domain K, then $T(I) = T((b, c)K)$ for some $b, c \in K$. Our first theorem, (2·6), sharpens this result and shows that it continues to hold for finitely generated regular ideals in a considerably larger class of rings.

(2·6) THEOREM. *If* $R \in \mathbf{C}^*$, *and if* T *is an over-ring of* R, *then the following are equivalent*:

(2·6·1) $T = T(I)$ *for some finitely generated regular ideal* I *in* R.

(2·6·2) $T = T((b, c)R)$ *for some regular elements* b, c *in* R.

(2·6·3) $T = R_b \cap R_c$ *for some regular elements* b, c *in* R.

(2·6·4) $T = R_b \cap R_S$ *for some regular element* b *in* R *and for some multiplicatively closed subset* S *of regular elements in* R.

Proof. Assume that (2·6·1) holds. If $P \in \mathbf{P}(R)$ and $I \subseteq P$, then P contains each regular principal ideal contained in I, so since $R \in \mathbf{C}^*$ there exist only finitely many ideals in $\mathbf{P}(R)$ that contain I, say $P_1, \ldots, P_g$ (possibly $g = 0$; that is, no $P \in \mathbf{P}(R)$ contains I). Let b be a regular non-unit in I, so $b \in \bigcap\{P_i; i = 1, \ldots, g\}$. Let $P_{g+1}, \ldots, P_h$ be the other prime ideals in $\mathbf{P}(R)$ that contain bR. (If $g > 0$, then possibly $h = g$; that is, $P_1, \ldots, P_g$ are the only ideals in $\mathbf{P}(R)$ that contain b.) Then $I \nsubseteq \bigcup\{P_j; j = g+1, \ldots, h\}$, and there exist only finitely many maximal prime divisors of zero in R, by (2·1·4) (i) (since $R \in \mathbf{C}^*$), so there exists $c \in I$ such that

$$c \notin \bigcup\{P_j; j = g+1, \ldots, h\} \cup \bigcup\{z; z \text{ is a maximal prime divisor of zero in } R\},$$

so c is regular. (If $h = g$, then we may take $c = b$.) Therefore if $P \in \mathbf{P}(R)$, then $(b, c)R \subseteq P$ if and only if $I \subseteq P$, so (2·6·2) holds, by (2·4). Thus (2·6·1) $\Rightarrow$ (2·6·2), and it is clear that (2·6·2) $\Rightarrow$ (2·6·1).

To show that (2·6·2) $\Leftrightarrow$ (2·6·3) it will be shown that $T((b, c)R) = R_b \cap R_c$. For this, let $r/s \in T((b, c)R)$, so $(r/s)(b, c)^n R \subseteq R$ for some $n \geqslant 1$. Therefore there exists r', $r'' \in R$ such that $(r/s)b^n = r'$ and $(r/s)c^n = r''$, so $r/s = r'/b^n = r''/c^n \in R_b \cap R_c$, so $T((b, c)R) \subseteq R_b \cap R_c$. For the opposite inclusion, if $r/s \in R_b \cap R_c$, then there exist r', $r'' \in R$ and positive integers n and m such that $r/s = r'/b^n = r''/c^m$. Therefore $(r/s)(b^n, c^m) \subseteq R$, so $r/s \in T((b^n, c^m)R) = T((b, c)R)$, by (2·5). Thus $T((b, c)R) = R_b \cap R_c$, so (2·6·2) $\Leftrightarrow$ (2·6·3).

It is clear that (2·6·3) $\Rightarrow$ (2·6·4), so it remains to show that (2·6·4) $\Rightarrow$ (2·6·3). For this, let c be a regular element in R which is in a prime ideal $P \in \mathbf{P}(R)$ that contains bR if and only if $P \cap S \neq \varnothing$ and let $V = \{c^n; n \geqslant 0\}$. Then the choice of c and the definition of V imply: (*a*) if $b \in P \in \mathbf{P}(R)$, then $P \cap V = \varnothing$ if and only if $P \cap S = \varnothing$. Now $R_S \in \mathbf{C}^*$ and $\mathbf{P}(R_S)$ consists of the ideals PR_S with $P \in \mathbf{P}(R)$ and $P \cap S = \varnothing$, by (2·2·5),

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and a similar statement holds for R_V, so: (b) $b^nR_S \cap R = b^nR_V \cap R$ for all $n \geqslant 1$, since it follows from (2·2·4), (2·2·5), and (a) that

$$b^nR_V \cap R = \bigcap\{b^n(R_V)_{(pR_V)} \cap R_V \cap R;\ b \in P \in \mathbf{P}(R) \text{ and } P \cap V = \varnothing\}$$
$$= \bigcap\{b^n(R_S)_{(pR_S)} \cap R_S \cap R;\ b \in P \in \mathbf{P}(R) \text{ and } P \cap S = \varnothing\} = b^nR_S \cap R.$$

Finally, $x \in R_b \cap R_V$ (resp., $R_b \cap R_S$) if and only if there exists $n \geqslant 1$ and $r \in b^nR_V \cap R$ (resp., $b^nR_S \cap R$) such that $x = r/b^n$, so it follows from (b) that $R_b \cap R_V = R_b \cap R_S$, so (2·6·4) $\Rightarrow$ (2·6·3), since $R_V = R_c$. ∎

It should be noted that if condition (i) if the definition of $\mathbf{C}^*$ is omitted, then essentially the same proof as for (2·6·1) $\Rightarrow$ (2·6·2) shows that for each regular element $b \in I$ there exists $c \in I$ such that $T(I) = T((b, c)R)$, but c may not be regular. Also, if c is regular, then $T((b, c)R) = R_b \cap R_c$, by the proof of (2·6·2) $\Rightarrow$ (2·6·3), and it can be shown that $R_b \cap R_c = R_b \cap R_S$, where S is the set of regular elements in $R - \bigcup\{P \in \mathbf{P}(R);\ b \in P \text{ and } c \notin P\}$.

The following remark contains some results related to (2·6).

(2·7) *Remark.* Let I be a finitely generated regular ideal in a ring $R \in \mathbf{C}^*$. Then:

(2·7·1) If b is a regular element in I, then for each regular element $c \in I$, $\notin \bigcup\{P \in \mathbf{P}(R);\ b \in P \text{ and } I \nsubseteq P\}$ it holds that $T(I) = T((b, c)R) = R_b \cap R_c$, and $T(I) = R_b \cap R_W$, where W is the set of regular elements in $R - \bigcup\{P \in \mathbf{P}(R);\ b \in P \text{ and } c \notin P\}$.

(2·7·2) $T(I) = \bigcap\{R_{(P)};\ P \in \operatorname{Spec}(R) \text{ and } I \nsubseteq P\} = \bigcap\{R_{(P)};\ P \in \mathbf{P}(R) \text{ and } I \nsubseteq P\}$, where $(P) = \{r \in R - P;\ r \text{ is regular}\}$.

(2·7·3) $T(I) \in \mathbf{C}^*$.

(2·7·4) If $b_1, \ldots, b_k$ are regular non-units in R, then there exist regular non-units $c_1, \ldots, c_k$ in R such that $R_{b_1} \cap \ldots \cap R_{b_k} = R_{b_i} \cap R_{c_i}$ for $i = 1, \ldots, k$.

(2·7·5) Given a regular non-unit $b \in R$ there exists a regular non-unit $c \in R$ such that $R_b \cap R_c = R$ if and only if there exists a regular non-unit $c \in R$ such that $(b, c)R \nsubseteq P$ for all $P \in \mathbf{P}(R)$.

Proof. For (2·7·1), $T(I) = T((b, c)R) = R_b \cap R_c$, by the proof of (2·6·1) $\Leftrightarrow$ (2·6·2) $\Leftrightarrow$ (2·6·3). Also, by the definition of W it follows, as in the proof of (2·6·4) $\Rightarrow$ (2·6·3), that $b^nR_W \cap R = b^nR_c \cap R$ for all $n \geqslant 1$, so it follows as in the last sentence of the proof of (2·6) that $R_b \cap R_W = R_b \cap R_c$, and it was just noted that $R_b \cap R_c = T(I)$.

For (2·7·2), let $A = \bigcap\{R_{(P)};\ P \in \operatorname{Spec}(R) \text{ and } I \nsubseteq P\}$, and $B = \bigcap\{R_{(P)};\ P \in \mathbf{P}(R) \text{ and } I \nsubseteq P\}$. To see that $T(I) \subseteq A$ let $r/s \in T(I)$, so $I \subseteq \operatorname{Rad}(sR : rR)$, by (2·2·1). Then if $P \in \operatorname{Spec}(R)$ and $I \nsubseteq P$, then it is clear that $sR : rR \nsubseteq P$, so $r/s \in R_{(P)}$, hence $T(I) \subseteq A$. It is clear that $A \subseteq B$. Finally, let $r/s \in B$. Let $P \in \mathbf{P}(R)$ such that P is minimal with respect to containing $sR : rR$. Then $r/s \notin R_{(P)}$, so it follows from the definition of B that $I \subseteq P$. Therefore $I \subseteq \operatorname{Rad}(sR : rR)$, so $r/s \in T(I)$, by (2·2·1), and so $B \subseteq T(I)$.

For (2·7·3), condition (i) in the definition of $\mathbf{C}^*$ holds, since $T(I)$ and R have the same total quotient ring. Also, by (2·6·1) $\Leftrightarrow$ (2·6·2) there exist regular elements b, $c \in I$ such that $T(I) = T((b, c)R)$. Let $T = T((b, c)R)$, so $T((b, c)T) = T((b, c)R)$, by [**13**, lemma 2·5], hence $(b, c)T \nsubseteq P'$ for all $P' \in \mathbf{P}(T)$, by (2·2·3). Also, $T = R_b \cap R_c$, by (2·6·1) $\Rightarrow$ (2·6·3), so $T_b = R_b$ and $T_c = R_c$. Further, if $P' \in \mathbf{P}(T)$, then since $(b, c)T \nsubseteq P'$ it follows that either $P'T_b \in \mathbf{P}(T_b)$ or $P'T_c \in \mathbf{P}(T_c)$, so it follows from (2·2·5) that there exists $P \in \mathbf{P}(R)$ such that either $P'T_b = PR_b$ or $P'T_c = PR_c$. Therefore, since $R \in \mathbf{C}^*$, it

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readily follows from (2·2·5) that each regular non-unit in T is in only finitely many $P' \in \mathbf{P}(T)$, so $T \in \mathbf{C}^*$.

For (2·7·4), it follows as in the proof of (2·6·2) $\Leftrightarrow$ (2·6·3) that $R_{b_1} \cap \ldots \cap R_{b_k} = T((b_1, \ldots, b_k) R)$, so this follows from (2·7·1).

Finally, (2·7·5) follows from (2·2·3), since, by (2·6·2) $\Leftrightarrow$ (2·6·3), $T((b, c) R) = R_b \cap R_c$. ∎

3. *Second theorem*

The main result in this section, (3·3), shows that if $T(I)$ is integrally closed, then $T(I)$ is a Krull ring. To prove this result we need the following two definitions and a remark concerning them.

(3·1) *Definition.* (3·1·1) $\mathbf{C}$ is the set of rings $R \in \mathbf{C}^*$ that have the following properties: (i) R has only finitely many minimal prime ideals, and (ii) for each minimal prime ideal z in R the integral closure $(R/z)'$ of R/z is a Krull domain.

(3·1·2) A *Krull ring* is a ring $R \in \mathbf{C}^*$ such that $(R_P)/(\mathrm{Rad}\,(R_P))$ is a discrete valuation ring for all $P \in \mathbf{P}(R)$.

(3·2) *Remark.* (3·2·1) It is clear that each Noetherian ring and each Krull domain is in $\mathbf{C}$, and it is noted in ([**16**], remark, p. 212) that every ring between a Noetherian ring and its integral closure is in $\mathbf{C}$.

(3·2·2) If $R \in \mathbf{C}$ and b is a regular non-unit in R, then it is readily seen that $R_b \in \mathbf{C}$.

(3·2·3) Concerning (3·1·2), if R is a Krull ring, then each regular non-unit is in only finitely many $P \in \mathbf{P}(R)$ (since $R \in \mathbf{C}^*$), it is readily seen that each $P \in \mathbf{P}(R)$ has height one, and it follows from (2·2·4) and (3·1·2) that regular principal ideals in R have a finite primary decomposition (since height $(P) = 1$ implies $bR_{(P)} \cap R$ is P-primary) and that R is integrally closed. Also, it was shown in ([**16**], propositions 2·13 and 2·7) that if $R \in \mathbf{C}$ and R is integrally closed, then R is a Krull ring.

We can now state and prove the main result in this section.

(3·3) Theorem. *If $R \in \mathbf{C}$, if I is a finitely generated regular ideal in R, and if $T(I)$ is integrally closed, then $T(I)$ is a Krull ring.*

Proof. $T = T(I) \in \mathbf{C}^*$, by (2·7·3), so it suffices to show that $(R_P)/(\mathrm{Rad}\,(R_P))$ is a discrete valuation ring for all $P \in \mathbf{P}(R)$. For this, by (2·6·1) $\Leftrightarrow$ (2·6·2) $\Leftrightarrow$ (2·6·3) let b, c be regular elements in I such that $T = T((b, c) R) = R_b \cap R_c$. Then it is clear that $T_b = R_b$ and $T_c = R_c$, so R_b and R_c are integrally closed (by the hypothesis on T) and in $\mathbf{C}$ (by (3·2·2)), so T_b and T_c are Krull rings, by (3·2·3). Also, it is shown in ([**13**], lemma 2·5) that $T((b, c) T) = T((b, c) R) = T$, so it follows from (2·2·3) that $(b, c) T \nsubseteq P'$ for all $P' \in \mathbf{P}(T)$. Therefore, since T_b and T_c are Krull rings, it readily follows that T is a Krull ring. ∎

(3·4) Corollary. *If I is a regular ideal in a Noetherian ring R such that $T(I)$ is integrally closed, then $T(I)$ is a Krull ring.*

Proof. This is clear by (3·2·1) and (3·3). ∎

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4. *Some applications*

In this section we use (2·6) to strengthen the known result that the Rees rings $\mathbf{R}(R,f)$ of certain filtrations f associated with an ideal I in a Noetherian ring R are ideal-transforms of the Rees ring $\mathbf{R}(R,I)$. Then, using this, we give new proofs of some known results concerning when $\mathbf{R}(R,f)$ is either finite or integral over $\mathbf{R}(R,I)$. Finally, two additional corollaries of (3·3) are proved. We begin with several definitions.

(4·1) *Definition.* If I is an ideal in a Noetherian ring R, then:

(4·1·1) The *Rees ring* $\mathbf{R} = \mathbf{R}(R,I)$ *of* R *with respect to* I is the graded subring $\mathbf{R} = R[u, tI]$ of $R[u, t]$, where t is an indeterminate and $u = 1/t$.

(4·1·2) $\hat{A}^*(I) = \{P \in \operatorname{Spec}(R);\ P \in \operatorname{Ass}(R/(I^n)_a)$ for all large $n\}$, where $(I^n)_a$ denotes the integral closure in R of I^n. Members of $\hat{A}^*(I)$ are called the *asymptotic prime divisors* of I.

(4·1·3) $E(I) = \{P \in \operatorname{Spec}(R);$ there exists $z \in \operatorname{Ass}((R_P)^*)$ such that $(I+z)(R_P)^*$ is primary for $P(R_P)^*\}$, where $(R_P)^*$ is the completion of R_P. Members of $E(I)$ are called the *essential prime divisors* of I.

(4·1·4) $U(I) = \{P \in \operatorname{Spec}(R);\ P = p \cap R$ for some $p \in E(u\mathbf{R}(R,I))\}$. Members of $U(I)$ are called the *u-essential prime divisors* of I.

(4·1·5) A *filtration* $f = \{I_n\}_{n \geqslant 0}$ on R is a decreasing sequence of ideals I_n of R such that $I_0 = R$ and $I_j I_k \subseteq I_{j+k}$ for all positive integers j, k.

(4·1·6) If $f = \{I_n\}_{n \geqslant 0}$ is a filtration on R, then the *Rees ring* $\mathbf{R} = \mathbf{R}(R,f)$ *of* R *with respect to* f is the graded subring $\mathbf{R} = R[u, tI_1, t^2I_2, \ldots]$ of $R[u,t]$.

Concerning these definitions, it is clear that u is regular in $\mathbf{R}(R, f)$ and that $u^n\mathbf{R}(R,f) \cap R = I_n$ for all $n \geqslant 1$. Also, it is shown in [17] (resp., [9], [4]) that $\hat{A}^*(I)$, (resp., $E(I)$, $U(I)$) are well-defined finite sets, and it is shown in [4] that

$$\hat{A}^*(I) \cup E(I) \subseteq U(I).$$

The main result in [8] showed that if I and J are ideals in a Noetherian ring R and if $f = \{I_n\}_{n \geqslant 0}$, where $I_n = \cup\{I^n\colon J^m;\ m \geqslant 1\}$, then $\mathbf{R}(R, f) = T((J,u)\mathbf{R})$, where $\mathbf{R} = \mathbf{R}(R,I)$. Now $\cup\{P \in \operatorname{Spec}(R);\ P \in \operatorname{Ass}(R/I^n)$ for some $n \geqslant 1\}$ is a finite set, by ([7], corollary 1·5), so it is readily seen that there exists a multiplicatively closed set S in R such that $I_n = (I^n)_S = \{r \in R;\ rs \in I^n$ for some $s \in S\}$, for all $n \geqslant 1$, and it follows from (2·6) that $T((J, u)\mathbf{R}) = \mathbf{R}[1/u] \cap \mathbf{R}_{S^*}$ for some multiplicatively closed set S^* of regular elements in $\mathbf{R}$. Therefore (4·2) strengthens and extends this result from [8].

(4·2) THEOREM. *Let I be an ideal in a Noetherian ring R, let $\mathbf{R} = \mathbf{R}(R, I)$, let S be a multiplicatively closed set of regular elements in $\mathbf{R}$, let $f = \{I_n\}_{n \geqslant 0}$, where $I_n = u^n\mathbf{R}_S \cap R$, and let $\mathbf{T} = \mathbf{R}(R, f)$. Then $\mathbf{T} = R[u, t] \cap \mathbf{R}_S = \mathbf{R}[1/u] \cap \mathbf{R}[1/c] = T((u, c)\mathbf{R})$, for some regular element c in $\mathbf{R}$.*

Proof. If $p \cap S \neq \varnothing$ for all prime divisors p of $u\mathbf{R}$, then $I_n = R$ for all $n \geqslant 0$ and $\mathbf{T} = R[u,t] \subseteq \mathbf{R}_S$, so the conclusion follows with $c = u$ (since $\mathbf{R}[1/u] = R[u,t]$). If $p \cap S = \varnothing$ for some prime divisor p of $u\mathbf{R}$, then let c be a regular element in $\mathbf{R}$ which is in a prime divisor q of $u\mathbf{R}$ if and only if $q \cap S \neq \varnothing$ and let W be the set of regular

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elements in $\mathbf{R}-\bigcup\{p;\ p \text{ is a prime divisor of } u\mathbf{R} \text{ and } c \notin p\}$. Then it follows as in the proof of (2·7·1) and the proof of (2·6·4) $\Rightarrow$ (2·6·3) that

$$\mathbf{R}[1/u] \cap \mathbf{R}_S = \mathbf{R}[1/u] \cap \mathbf{R}_W = \mathbf{R}[1/u] \cap \mathbf{R}[1/c] = T((u,c)\mathbf{R}).$$

Finally, $\mathbf{R}[1/u] = R[u,t]$, and it is shown in ([**18**], section 5) that $\mathbf{T} = R[u,t] \cap \mathbf{R}_W$. ∎

It is of some importance to know when $\mathbf{R}(R,f)$ is a finite $\mathbf{R}(R,I)$-module, since this implies that some rather nice relations hold between I and the ideals I_n (for example, see [**19**]), and it is always useful to know when a given ring is integrally dependent on its subring. (Also, the finiteness of $T(I)$ over R was of importance in the work on Hilbert's Fourteenth Problem.) (4·3) gives some results in this area. Of these, (4·3·1) and (4·3·3) are new, and (4·3·2) and (4·3·4) were first proved by McAdam in [**8**] (and in somewhat different form in [**19**] and [**22**]). In the proof of (4·3·1) and (4·3·3) we use the following result proved in ([**7**], propositions 10·9, 10·11 (i $\Leftrightarrow$ ii), and corollary 10·4): if I is an ideal in a Noetherian ring R, then $T(I)$ is a finite R-module (resp., $T(I)$ is integral over R) if and only if I is not contained in any essential (resp., asymptotic) prime divisor of a regular principal ideal in R. (In ([**19**], (3·3) and (3·4)) it was proved that if R is Noetherian, then an over-ring of the form $R_b \cap R_S$ is a finite R-module if and only if $S \subseteq R-\bigcup\{P;\ P \in U(bR)\}$. Thus, because of (2·6), this gives another equivalence of the finiteness of $T(I)$. At the time [**19**] was written, (2·6) above was unknown.)

(4·3) Corollary. *With* $\mathbf{T} = T((u,c)\mathbf{R}) = R[u,t] \cap \mathbf{R}_S$ *as in* (4·2), *the following statements hold*:

(4·3·1) $\mathbf{T}$ *is a finite* $\mathbf{R}$*-module if and only if* $c \notin \bigcup\{p;\ p \in E(u\mathbf{R})\}$ *if and only if* $p \cap S = \varnothing$ *for all* $p \in E(u\mathbf{R})$.

(4·3·2) *If* $S \subseteq R$, *then* $\mathbf{T}$ *is a finite* $\mathbf{R}$*-module if and only if* $p \cap S = \varnothing$ *for all* $p \in U(I)$.

(4·3·3) $\mathbf{T}$ *is integral over* $\mathbf{R}$ *if and only if* $c \notin \bigcup\{p;\ p \in \hat{A}^*(u\mathbf{R})\}$ *if and only if* $p \cap S = \varnothing$ *for all* $p \in \hat{A}^*(u\mathbf{R})$.

(4·3·4) *If* $S \subseteq R$, *then* $\mathbf{T}$ *is integral over* $\mathbf{R}$ *if and only if* $p \cap S = \varnothing$ *for all* $p \in \hat{A}^*(I)$.

Proof. $\mathbf{T}$ is an ideal-transform of $\mathbf{R}$, by (4·2). Therefore the first equivalence in (4·3·1) and in (4·3·3) readily follows from the result in [**7**] noted just before this corollary. Also, $U(I) = \{p \cap R;\ p \in E(u\mathbf{R})\}$ and $\hat{A}^*(I) = \{p \cap R;\ p \in \hat{A}^*(u\mathbf{R})\}$, so (4·3·2) and (4·3·4) follow from the second equivalence in (4·3·1) and (4·3·3), respectively. Therefore it remains to prove the second equivalence in (4·3·1) and (4·3·3).

For this, the proof of (4·2) shows that if c is a regular element in $\mathbf{R}$ which is in a prime divisor p of $u\mathbf{R}$ if and only if $p \cap S \neq \varnothing$, then $\mathbf{R}[1/u] \cap \mathbf{R}[1/c] = \mathbf{R}[1/u] \cap \mathbf{R}_S$. Therefore, as in the proof of (2·6·4) $\Rightarrow$ (2·6·3) it follows that $u^n\mathbf{R}[1/c] \cap \mathbf{R} = u^n\mathbf{R}_S \cap \mathbf{R}$, for all $n \geqslant 1$, so if $p \in \mathrm{Ass}(\mathbf{R}/u\mathbf{R})$, then $p \cap S = \varnothing$ if and only if $p \cap \{c^n;\ n \geqslant 1\} = \varnothing$ if and only if $c \notin p$. Therefore the last equivalence in (4·3·1) and in (4·3·3) holds. ∎

The following result is another corollary of (3·3). In its statement we use the following definition: if $f = \{I_n\}_{n \geqslant 0}$ is a filtration on a ring R, then the *weak integral closure* $(I_n)_a$ of I_n is $\{x \in R;\ x \text{ is a root of a polynomial of the form } X^k + i_1 X^{k-1} + \ldots + i_k, \text{ where } i_j \in I_{jn} \text{ for } j = 1, \ldots, k\}$. This concept was studied by Ribenboim in [**21**].

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(4·4) COROLLARY. *With the notation of* (4·2), *consider the following statements*:
(4·4·1) $\mathbf{T}$ *is integrally closed.*
(4·4·2) $\mathbf{T}$ *is a Krull ring.*
(4·4·3) R *is integrally closed and* $I_n = (I_n)_a$ *for all* $n \geqslant 1$.
(4·4·4) R *is integrally closed and* $I_n = (I_n)_a$ *for all* $n \geqslant 1$.
Then (4·4·1) $\Leftrightarrow$ (4·4·2) $\Leftrightarrow$ (4·4·3) $\Rightarrow$ (4·4·4), *and if* $S \subseteq R$, *then* (4·4·4) $\Rightarrow$ (4·4·1).

Proof. $\mathbf{R} \in \mathbf{C}$, since $\mathbf{R}$ is Noetherian, so (4·4·1) $\Rightarrow$ (4·4·2), by (4·2) and (3·3). Also, (4·4·2) $\Rightarrow$ (4·4·1), by (3·2·3), and by considering equations of integral dependence it is readily seen that (4·4·1) $\Leftrightarrow$ (4·4·3).

Finally, $(I_n)^k \subseteq I_{kn}$, for all positive integers k, n, so it is clear that (4·4·3) $\Rightarrow$ (4·4·4). And if (4·4·4) holds and $S \subseteq R$, then $\mathbf{R}_S \cong \mathbf{R}(R_S, IR_S)$, so $I^n R_S = I_n R_S = (I_n)_a R_S = (I_n R_S)_a = (I^n R_S)_a$ for all $n \geqslant 1$, so it follows by considering equations of integral dependence that $\mathbf{R}_S$ is integrally closed. Therefore (4·4·4) implies that $\mathbf{R}_S$ and $R[u, t]$ are integrally closed, so (4·4·4) $\Rightarrow$ (4·4·1), since $\mathbf{T} = R[u,t] \cap \mathbf{R}_S$, by (4·2). ∎

An important special case of our final corollary shows that $\mathbf{T}$ is always a Krull domain when I is a radical ideal in a regular local ring.

(4·5) COROLLARY. *Let* R *be an integrally closed Noetherian ring, let* $P_1, \ldots, P_k$ *be incomparable prime ideals in* R, *let* $I = \bigcap\{P_i;\ i = 1, \ldots, k\}$, *and let* S *be the set of regular elements in* $R - \bigcup\{P_i;\ i = 1, \ldots, k\}$. *If* R_{P_i} *is a regular local ring for* $i = 1, \ldots, k$ *and if* $f = \{I^n R_S \cap R\}_{n \geqslant 0}$, *then* $\mathbf{R}(R, f)$ *is a Krull ring.*

Proof. $R \in \mathbf{C}$, by (3·2·1), and $I^n R_S$ is integrally closed for all $n \geqslant 1$, by the hypothesis on the R_{P_i}. Let $\mathbf{R} = \mathbf{R}(R, I)$, so $\mathbf{R}_S \cong \mathbf{R}(R_S, IR_S)$, and so

$$f = \{I^n R_S \cap R\}_{n \geqslant 0} = \{(u^n \mathbf{R}_S \cap R_S) \cap R\}_{n \geqslant 0}.$$

Therefore $\mathbf{R}(R,f) = \mathbf{R}[1/u] \cap \mathbf{R}_S = T((u, c)\mathbf{R})$ for some regular element $c \in \mathbf{R}$, by (4·2), so the conclusion follows from (4·4·4) $\Rightarrow$ (4·4·2). ∎

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