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Pseudo-orbits, pseudoleaves and geometric entropy of foliations

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Abstract. We show that the entropy of a finitely generated pseudogroup (respectively, of a foliation on a compact Riemannian manifold) can be calculated by suitable counting separated pseudo-orbits (respectively, pseudoleaves).

0. *Introduction*

For some years, pseudo-orbits have played an important role in the theory of classical dynamical systems. In particular, it is shown [**Mi**, **BS**] that pseudo-orbits can be used to calculate the topological entropy of transformations. More recently, a similar result was obtained [**Hur**] for the inverse-image entropy introduced earlier by Remi Langevin and the second author [**LW1**].

On the other hand, geometric entropy $h(\mathcal{F})$ of a foliation $\mathcal{F}$ of a compact Riemannian manifold has been introduced [**GLW**] and shown to be a useful tool for studying the topology and dynamics of foliated manifolds [**Hu1–3**, **GLW**, **LW2**, **Eg**, **IT**, etc.]. The entropy $h(\mathcal{F})$ can be calculated either by counting the number of points separated along the leaves or by counting the number of separated orbits of holonomy pseudogroups generated by nice coverings by charts distinguished by $\mathcal{F}$. Also, pseudoleaves of foliations have been defined by Takashi Inaba [**In**] who has shown (among the other results) that expansive C^1-foliations (in the sense of [**IT**]) of codimension-one which have the pseudoleaf tracing property are topologically stable.

In this paper, we show that the geometric entropy $h(\mathcal{F})$ of any foliation $\mathcal{F}$ of a compact Riemannian manifold M coincides with that calculated by suitable counting separated pseudoleaves (Theorem 2 in §4). To this end, we study the entropy of finitely generated pseudogroups of local transformations of compact metric spaces and we show that it can be calculated by counting (again, in a suitable way) separated pseudo-orbits (Theorem 1 in §2). In §5, we sketch an easy proof of Theorem 2 for foliated bundles. Finally, in §6 we provide an example of a group acting on S^1 for which the 'usual' formula for the entropy in terms of pseudo-orbits does not work: it gives a number strictly bigger than the entropy calculated in terms of separated orbits.

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The significance of these results can be observed by trying to calculate (or to estimate) entropies of pseudogroups or foliations with the aid of computers.

1. *Pseudo-orbits of pseudogroups*

Let (X, d) be a compact metric space and let G be a finitely generated pseudogroup of local transformations of X. For any $g \in G$, $U_g \subset X$ will denote the domain of g. Let us fix a finite symmetric generating set $G_1 \subset G$. Recall that 'symmetric' means that $e = \mathrm{id}_X \in G_1$ and $G_1^{-1} \subset G_1$. For any $k \in \mathbb{N}$ let $G_k = (G_1)^k = \{(h_1, \dots, h_k); h_i \in G_1\}$. We shall identify G_k with a subset of G_{k+1} consisting of all the sequences of the form $(e, h_1, \dots, h_k)$, $h_i \in G_1$. Also, given two sequences $h = (h_1, \dots, h_k) \in G_k$ and $g = (g_1, \dots, g_l) \in G_l$ we shall denote by (h, g) the element $(h_1, \dots, h_k, g_1, \dots, g_l) \in G_{k+l}$. Finally, for any $h = (h_1, \dots, k_k) \in G_k$ we shall denote by the same symbol h the map $h_1 \circ \cdots \circ h_k \in G$. Therefore, we may write that $\{e\} = G_0 \subset G_1 \subset G_2 \subset \cdots \subset G_\infty = \cup_k G_k$ and define the sets $G(p)$, $p \in X$, by $G(p) = \{g \in G_\infty; p \in U_g\}$.

The entropy of the system (G, G_1) can be defined whenever the maps of G are homeomorphisms between open sets. However, in general the entropy can be infinite and the most natural setting ensuring its finiteness is the category of locally Lipschitz maps. Therefore, throughout the paper we shall assume that the maps generating pseudogroups are Lipschitz.

Let us fix $\alpha \geq 0$. An *α-pseudo-orbit* of G is a map $x : D_x \to X$ such that $e \in D_x$, $D_x \subset G(x(e))$, for any $h \in G_1$ and $g \in D_x$, $(h, g) \in D_x$ iff $x(g) \in U_h$, U_h being the domain of h, and then

$$d(h(x(g)), x(h, g)) \leq \alpha. \tag{1}$$

If G is a group of global transformations, then $D_x = G_\infty$ for any pseudo-orbit x and inequality (1) should be satisfied for all $g \in G_\infty$ and $h \in G_1$. In particular, if $G = \mathbb{Z}$ acts on X and f is a generator, then $x : \mathbb{Z} \to X$ is just a sequence of points and (1) reduces to

$$d(f(x_n), x_{n+1}) \leq \alpha, \quad n \in \mathbb{Z},$$

i.e. x becomes a pseudo-orbit of f in the classical sense.

Let us denote by Y_α the space of all the α-pseudo-orbits. Clearly, $Y_\alpha \subset Y_\beta$ whenever $\alpha \leq \beta$ and $Y_0 = \cap_{\alpha \geq 0} Y_\alpha$ is the space of 'real' G-orbits. Recall that $x \in Y_0$ iff $x = x_p$ for some $p \in X$, $D_{x_p} = \{g \in G_\infty; p \in U_g\}$ and $x_p(g) = g(p)$ for all $g \in D_{x_p}$.

For any pseudo-orbits x and y in Y_α put

$$d_0(x, y) = \sum_{k=0}^{\infty} \frac{1}{2^k} \max\{d(x(g), y(g)); g \in G_k \cap D_x \cap D_y\} \tag{2}$$

and

$$d_1(x, y) = \inf\left\{ \sum_{i=0}^{m-1} d_0(z_i, z_{i+1}); z_i \in Y_\alpha, z_0 = x, z_m = y \text{ and } m \in \mathbb{N} \right\}. \tag{3}$$

LEMMA 1. *For any α, (Y_α, d_1) is a metric space.*

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Proof. The symmetry and triangle inequality for d_1 as well as the relation $d_1(x, x) = 0$ for all $x \in Y_\alpha$ follow immediately from (2) and (3). It remains to show that the condition $d_1(x, y) = 0$ implies

$$D_x \cap G_k = D_y \cap G_k \quad \text{and} \quad x = y \quad \text{on } D_x \cap D_y \cap G_k \tag{4}$$

for $k = 0, 1, 2, \ldots$.

First, observe that

$$\sum_{i=0}^{m-1} d_0(z_i, z_{i+1}) \geq \sum_{i=0}^{m-1} d(z_i(e), z_{i+1}(e)) \geq d(x(e), y(e))$$

whenever $z_0 = x$ and $z_m = y$. Therefore, the condition $d_1(x, y) = 0$ implies that $x(e) = y(e)$ and then (4) is satisfied with $k = 0$.

Next, observe that the equality $x(e) = y(e) = p$ implies that $D_x \cap G_1 = D_y \cap G_1 = G(p) \cap G_1$. Find $\epsilon > 0$ such that $B(p, \epsilon) \subset U_h$ for all $h \in G(p) \cap G_1$. If $z_0, z_1, \ldots, z_m \in Y_\alpha$, $z_0 = x$, $z_m = y$ and $\sum_i d_0(z_i, z_{i+1}) < \epsilon/2$, then

$$d(p, z_j(e)) \leq \sum_{i \leq j} d(z_i(e), z_{i+1}(e)) \leq \sum_{i \leq j} d_0(z_i, z_{i+1}) < \frac{\epsilon}{2}$$

and $z_j(e) \in U_h$ for any $h \in G(p) \cap G_1$ and $j \leq m$. Therefore,

$$d(x(h), y(h)) \leq \sum_i d(z_i(h), z_{i+1}(h)) \leq 2 \cdot \sum_i d_0(z_i, z_{i+1}) < \epsilon$$

for all $h \in G(p) \cap G_1$. This shows that the condition $d_1(x, y) = 0$ implies (4) with $k = 1$.

Proceed by induction to complete the proof. □

LEMMA 2. *The space (Y_α, d_1) is compact.*

Proof. Take any sequence (x_n) of α-pseudo-orbits. Choose a subsequence (x_n^0) of (x_n) such that $x_n^0(e) \to p \in X$. Put $x(e) = p$ and $D_x \cap G_1 = G(p) \cap G_1$. Find a subsequence (x_n^1) of (x_n^0) such that $x_n^1(h) \to p_h \in X$ for any $h \in D_x \cap G_1$. Put $x(h) = p_h$ and $D_x \cap G_2 = \{(h', h); h \in D_x \cap G_1, h' \in G_1 \text{ and } p_h \in U_{h'}\}$. Proceed by induction to define $D_x \cap G_k$, to find subsequences (x_n^k) such that $x_n^k(g) \to p_g \in X$ for all $g \in D_x \cap G_k$, $k \in \mathbb{N}$, and put $x(g) = p_g$ for all such g.

If $h \in D_x \cap G_1$, then $h \in D_{x_n^0}$ and $d(x_n^0(h), h(x_n^0(e))) \leq \alpha$ for all n large enough. Therefore, $d(x(h), h(x(e))) \leq \alpha$. Also, if $(h', h) \in D_x \cap G_2$, then $h' \in D_{x_n^1}$ and $d(x_n^1(h', h), h'(x_n^1(h))) \leq \alpha$ for all n large enough. Therefore $d(x(h', h), h'(x(h))) \leq \alpha$. Proceed by induction to show that $x \in Y_\alpha$.

Obviously, $x_n^n \to x$. □

Now, take $g_0 \in G_\infty$ and $x \in Y_\alpha$ such that $g_0 \in D_x$. Define $y = \sigma_{g_0}(x) \in Y_\alpha$ by

$$y(g) = x(g, g_0)$$

on $D_y = \{g \in G_\infty; (g, g_0) \in D_x\}$. It is easy to see that this definition is correct and that all the maps σ_h, $h \in G_1$, generate a pseudogroup, denoted again by G, of local transformations of Y_α.

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Note that whenever $g_0 \in G_n$,

$$\begin{aligned} d_0(\sigma_{g_0}x, \sigma_{g_0}x') &= \sum_k \frac{1}{2^k} \max\{d(x(g, g_0), x'(g, g_0)); g \in G_k \cap D_x \cap D_{x'}\} \\ &\leq \sum_k \frac{1}{2^k} \max\{d(x(g), x'(g)); g \in G_{k+n} \cap D_x \cap D_{x'}\} \leq 2^n d_0(x, x') \end{aligned}$$

and

$$\begin{aligned} d_1(\sigma_{g_0}x, \sigma_{g_0}x') &= \inf\left\{\sum_{i\leq m-1} d_0(z_i, z_{i+1}); z_0 = \sigma_{g_0}x \text{ and } z_m = \sigma_{g_0}x'\right\} \\ &\leq \inf\left\{\sum_{i\leq m-1} d_0(\sigma_{g_0}z_i, \sigma_{g_0}z_{i+1}); z_0 = x \text{ and } z_m = x'\right\} \leq 2^n d_1(x, x'). \end{aligned}$$

Therefore, G acts on Y_α also *via* locally Lipschitz transformations.

2. *Entropy of pseudogroups*

As before, let G be a pseudogroup on X equipped with a finite symmetric generating set G_1. Following [**GLW**] we shall say that two points p and q of X are (n, ϵ)-*separated* w.r.t. G ($n \in \mathbb{N}$, $\epsilon > 0$) if there exists $g \in G_n$ such that $\{p, q\} \subset U_g$ and $d(g(p), g(q)) \geq \epsilon$. A set $A \subset X$ is (n, ϵ)-separated when any two points $p, q \in A$, $p \neq q$, have this property. Since X is compact, separated sets are always finite and we may put

$$N(n, \epsilon, X) = \max\{\#A; A \subset X \text{ is } (n, \epsilon)\text{-separated}\}, \tag{5}$$

$$N(\epsilon, X) = \limsup_{n\to\infty} \frac{1}{n} \log N(n, \epsilon, X) \tag{6}$$

and

$$h(G, G_1, X) = \lim_{\epsilon\to 0^+} N(\epsilon, X). \tag{7}$$

The number $h(G, G_1, X)$ is the *entropy* of G (w.r.t. G_1) on X.

LEMMA 3. *$h(G, G_1, X) = h(G, G_1, Y_0)$, Y_0 being the space of G-orbits.*

Proof. The inequality '$\leq$' follows immediately from the estimate

$$d_1(\sigma_g x, \sigma_g y) \geq d(x(g), y(g)) = d(g(p), g(q))$$

which holds for the orbits x and y of any points p and q of X and any element $g \in G_\infty$ such that $\{p, q\} \subset U_g$. To prove '$\geq$' take two (n, ϵ)-separated orbits x and $y \in Y_0$ and find $g \in G_n$ such that $\{x(e), y(e)\} \subset U_g$ and $d_1(\sigma_g(x), \sigma_g(y)) \geq \epsilon$. Then also $d_0(\sigma_g(x), \sigma_g(y)) \geq \epsilon$. Next, find $l(\epsilon) \in \mathbb{N}$ such that $\sum_{k>l(\epsilon)} 2^{-k} < \epsilon/(2 \operatorname{diam} X)$. Thus,

$$d(g'g(x(e)), g'g(y(e))) = d(x(g', g), y(g', g)) = d(\sigma_g x(g'), \sigma_g y(g')) \geq \frac{\epsilon}{2}$$

for some $g' \in G_{l(\epsilon)}$ and the points $x(e)$, $y(e)$ are $(n + l(\epsilon), \epsilon/2)$-separated on X. Consequently,

$$N(n, \epsilon, Y_0) \leq N(n + l(\epsilon), \epsilon/2, X).$$

Pass to suitable limits to complete the proof. □

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LEMMA 4. *For any $n \in \mathbb{N}$ and any $\epsilon > 0$ there exist $\alpha > 0$ and $l = l(\epsilon)$ such that $N(n+l, \epsilon/6, Y_0) \geq N(n, \epsilon, Y_\alpha)$.*

Proof. Fix $n \in \mathbb{N}$ and $\epsilon > 0$. Let L be the Lipschitz constant for all the generators of G and let $l = l(\epsilon)$ satisfy the inequality $\sum_{m>l} 2^{-m} < (\epsilon/3) \cdot \operatorname{diam} X$. Let $A = \{x_i, i \leq k\}$ be an (n, ϵ)-separated subset of Y_α. For any i, let x_i' be the orbit of $x_i(e)$. An easy induction shows that

$$d(x_i(g), x_i'(g)) \leq \alpha \cdot (1 + L + \cdots + L^{n-1})$$

for any i and any $g \in G_n$. Therefore, for any i and $j \leq k$ there exists $g \in G_{n+l}$ such that

$$\begin{aligned} d(x_i'(g), x_j'(g)) &\geq d(x_i(g), x_j(g)) - d(x_i(g), x_i'(g)) - d(x_j(g), x_j'(g)) \\ &\geq \frac{\epsilon}{3} - 2\alpha \cdot (1 + L + \cdots + L^{n+l-1}) \geq \frac{\epsilon}{6} \end{aligned}$$

whenever α is small enough to satisfy $\alpha \cdot (1 + L + \cdots + L^{n+l-1}) \leq \epsilon/12$. For such α, $A' = \{x_i', i \leq k\} \subset Y_0$ is $(n+l, \epsilon/6)$-separated. □

Now, let us say that α-pseudo-orbits x and y are (n, ϵ)-*strongly separated* just when $d(x(g), y(g)) \geq \epsilon$ for some $g \in G_n \cap D_x \cap D_y$. Let us denote by $N_\alpha(n, \epsilon)$ the maximal cardinality of a set $A \subset Y_\alpha$ such that any two elements $x, y \in A$, $x \neq y$, are (n, ϵ)-strongly separated in this sense. Note that in general $N_\alpha(n, \epsilon) \neq N(n, \epsilon, Y_\alpha)$ since strong separation of α-pseudo-orbits in the above sense is not entirely equivalent to separation of points of Y_α by the action of G. For any sequence $\beta = (\beta_n)$ such that $0 < \beta_n \searrow 0$ we shall put

$$N_\beta(\epsilon) = \limsup_{n\to\infty} \frac{1}{n} \log N_{\beta_n}(n, \epsilon), \tag{8}$$

$$N_{\mathrm{ps}}(\epsilon) = \inf_\beta N_\beta(\epsilon) \tag{9}$$

and

$$h_{\mathrm{ps}}(G, G_1, X) = \lim_{\epsilon\to 0^+} N_{\mathrm{ps}}(\epsilon). \tag{10}$$

Our main result (Theorem 1) in this section says that this 'pseudo-entropy' h_{ps} always coincides with the 'real' entropy h.

LEMMA 5. *$N_\alpha(n, \epsilon) \leq N(n, \epsilon, Y_\alpha)$ for all $n \in \mathbb{N}$, $\epsilon > 0$ and $\alpha \geq 0$.*

Proof. If pseudo-orbits $x, y \in Y_\alpha$ are (n, ϵ)-strongly separated and $d(x(g), y(g)) \geq \epsilon$ for some $g \in G_n \cap D_x \cap D_y$, then

$$d_1(\sigma_g x, \sigma_g y) \geq d(\sigma_g x(e), \sigma_g y(e)) = d(x(g), y(g)) \geq \epsilon,$$

i.e. x and y are separated under the action of G on Y_α. □

THEOREM 1. *For any pseudogroup G on X and any finite symmetric generating set G_1 the equality*

$$h_{\mathrm{ps}}(G, G_1, X) = h(G, G_1, X) \tag{11}$$

holds.

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Proof. The inequality '$\geq$' in (11) is obvious since $Y_0 \subset Y_\alpha$ for all $\alpha \geq 0$ and the notions of separation of points of X under the action of G and strong separation of orbits coincide.

To prove '$\leq$' observe that, by Lemmas 4 and 5,

$$\begin{aligned} N_{\mathrm{ps}}(\epsilon) \leq N_\beta(\epsilon) &\leq \limsup_{n\to\infty} \frac{1}{n} \log N(n, \epsilon, Y_{\beta_n}) \\ &\leq \limsup_{n\to\infty} \frac{1}{n} \log N(n, \epsilon/6, Y_0) = N(\epsilon/6, Y_0) \end{aligned}$$

for any sequence $\beta = (\beta_n)$ such that $\beta_n \leq (\epsilon/12) \cdot (1 + L + \cdots + L^{n+l(\epsilon)-1})^{-1}$, $n \in \mathbb{N}$ and L being—as before—the Lipschitz constant for all the generators of G. Therefore,

$$h_{\mathrm{ps}}(G, G_1, X) \leq h(G, G_1, Y_0) = h(G, G_1, X)$$

by Lemma 3. □

Remark. The proof above shows that

$$h(G, G_1, X) = \lim_{\epsilon\to 0^+} \lim_{n\to\infty} \frac{1}{n} \log N_{\epsilon\alpha_n}(n, \epsilon)$$

with $\alpha_n = (1 + L + \cdots + L^{n+l(\epsilon)})^{-1}$.

3. *Pseudoleaves and holonomy pseudo-orbits*

First, assume that E is a vector space with an inner product $\langle\cdot,\cdot\rangle$. For any $q \geq 1$ and codimension-q subspaces E_1 and E_2 of E put

$$\mathrm{dist}(E_1, E_2) = \max\{\min\{\|v_1 - v_2\|;\, v_2 \in E_2, \|v_2\| = 1\};\, v_1 \in E_1, \|v_1\| = 1\}. \tag{12}$$

Clearly, dist is a distance function on the set of all codimension-q subspaces of E which is nearly proportional to the angle between subspaces.

Let $\mathcal{F}$ be a C^r-foliation, $r \geq 1$, of a compact Riemannian manifold $(M, \langle\cdot,\cdot\rangle)$. Following [**In**] we shall say that a complete submanifold N of M is an α-pseudoleaf when $\dim N = \dim \mathcal{F}$ and

$$\mathrm{dist}(T_pN, T_p\mathcal{F}) \leq \alpha \tag{13}$$

for all $p \in N$.

Next, take a nice cover $\mathcal{U}$ of $(M, \mathcal{F})$ [**HH**] and denote by $H_\mathcal{U}$ the holonomy pseudogroup of $\mathcal{F}$ acting on the space $X = X_\mathcal{U}$ of plaques of the closed charts $\bar{U}$, $U \in \mathcal{U}$. Since $\mathcal{U}$ is finite, X is compact.

The space X can be equipped with some natural distance functions $d = d_\mathcal{U}$ induced from the Riemannian structure of M. For example, one can equip M with the Riemannian distance function d_M and define $d_\mathcal{U}$ as the Hausdorff distance between the closures of the plaques. Or, one can fix a complete transversal $T = T_\mathcal{U}$ intersecting each plaque exactly once, identify X with T and equip T with the Riemannian distance function coming from the induced Riemannian structure. All these choices are equivalent in the sense that any two metrics described above are quasi-isometric and therefore the entropies of $H_\mathcal{U}$ obtained from these metrics are equal [**GLW**, Proposition 2.6]. So, let us fix such a 'good' metric d on X once and use it hereafter.

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Also, we consider the 'standard' generating set $(H_{\mathcal{U}})_1$ for $H_{\mathcal{U}}$. It consists of all the elementary holonomy maps $h_{U,V}$ corresponding to overlapping charts $U, V \in \mathcal{U}$ and is defined by

$$h_{U,V}(P) = Q \text{ iff the plaques } P \subset U \text{ and } Q \subset V \text{ overlap.} \tag{14}$$

Since this is the only generating set to be considered, we shall omit the symbol $(H_{\mathcal{U}})_1$ in all the formulae below.

Now, observe that the space of pseudoleaves embeds naturally into the space of pseudo-orbits of some holonomy pseudogroups $H_{\mathcal{U}}$. To produce such an embedding fix a complete transversal $T_{\mathcal{U}}$ and a sub-bundle E of TM complementary to $T\mathcal{F}$ and such that $E_p = T_p T_{\mathcal{U}}$ for any $p \in T_{\mathcal{U}}$. Also, for any two overlapping charts $U, V \in \mathcal{U}$ fix another distinguished chart $W = W_{U,V}$ containing $\bar{U} \cup \bar{V}$ and assume that $\mathcal{U}$ is small enough for all the plaques of the charts $W_{U,V}$ to be strongly convex subsets of the leaves.

LEMMA 6. *There exist positive constants C, C' and α_0 such that for any $\alpha \leq \alpha_0$, any α-pseudoleaf N and any $p \in N$ there exists a unique $(C\alpha)$-pseudo-orbit x of $H_{\mathcal{U}}$ such that $x(e) = p$, $x(D_x) \subset N$ and $d_N(x(h, g), x(g)) \leq C' d_{\mathcal{F}}(h(x(g)), x(g))$ for any $g \in D_x$ and any generator h of $H_{\mathcal{U}}$ such that $x(g) \in U_h$.*

Proof. Define x on $D_x \cap G_k$ inductively. First, x is already defined on $D_x \cap G_0 = \{e\}$: $x(e) = p$. Then, if $g \in D_x \cap G_k$, $x(g)$ is already defined, $h = h_{U,V}$ is a generator of $H_{\mathcal{U}}$ and $x(g)$ lies in the domain of h, then join $x(g)$ to $h(x(g))$ by the unique leaf geodesic $\gamma : [0, 1] \to W = W_{U,V}$, build from the fibres of the bundle E a tubular neighbourhood E_γ of a leaf L_p around $\gamma([0, 1])$ and lift γ to the unique curve $\tilde{\gamma} : [0, 1] \to E_\gamma$ such that $\pi_\gamma \circ \tilde{\gamma} = \gamma$, $\tilde{\gamma}(0) = p$ and $\tilde{\gamma}(t) \in N$ for all $t \in [0, 1]$, π_γ being the canonical projection in the tubular neighbourhood. Finally, put $x(h, g) = \tilde{\gamma}(1)$ (see Figure 1 with $g = e$).

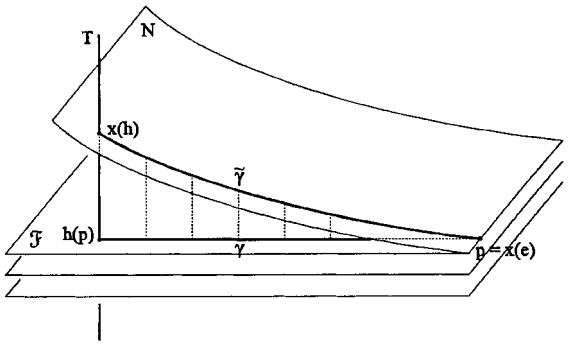


FIGURE 1.

Since the lengths of all such geodesics γ are uniformly bounded, this construction can be performed for α small enough, say $\alpha \leq \alpha_0$. Also, the same reason together with uniform bounds for the geometry (i.e. sectional curvature, injectivity radius etc.) of M implies that $d_M(x(h, g), hx(g)) \leq C\alpha$ and that $l(\tilde{\gamma}) \leq C' l(\gamma)$ for some universal constants C and C'. Therefore, this construction leads to a $C\alpha$-pseudo-orbit of $H_{\mathcal{U}}$ which satisfies all the required conditions.

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The uniqueness of such an x follows from the fact that at each step of the induction the construction is in fact unique if α is so small that the projections $\pi_\gamma|N$ are one-to-one for all minimal geodesics joining q to $h(q)$ for any $q \in M$ and any holonomy map h generating $H_{\mathcal{U}}$. □

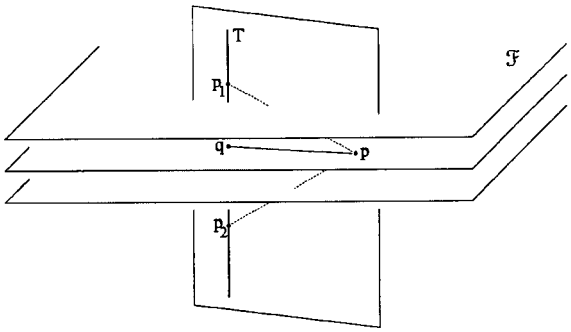


FIGURE 2.

Remark. Clearly, any orbit of the holonomy pseudogroup $\mathcal{H}_{\mathcal{U}}$ is contained in a leaf of $\mathcal{F}$. It seems that embedding a holonomy α-pseudo-orbit x into a pseudoleaf is not so obvious. The difficulty comes from the fact that, even if α is very small, the points $x(h_1)$ and $x(h_2)$, $h_1, h_2 \in (\mathcal{H}_{\mathcal{U}})_1$, may belong to a disc orthogonal to $\mathcal{F}$ (see Figure 2, where $q = h_1(p) = h_2(p)$ and $p_i = x(h_i)$) and therefore any submanifold N containing these points in a reasonable way has to make an angle close to $\pi/2$ with some leaves of $\mathcal{F}$. However, in the case of foliated bundles, one can produce [**In**] immersed pseudoleaves corresponding to pseudo-orbits of the global holonomy group by suitable use of centres of mass [**BK**] of the points of pseudo-orbits.

4. *Geometric entropy*

Again let $(M, \mathcal{F}, \langle\cdot,\cdot\rangle)$ be a compact foliated Riemannian manifold, $\mathcal{U}$ a nice cover of $(M, \mathcal{F})$ and $T = T_{\mathcal{U}}$ a complete transversal. Recall that two points $p, q \in T$ are (R, ϵ)-*separated* w.r.t. $\mathcal{F}$ if either $d_M(p, q) \geq \epsilon$ or there exists a leaf curve $\gamma : [0, 1] \to M$ of length $l(\gamma) \leq R$ such that $\gamma(0) = p$ (respectively, $\gamma(0) = q$), the originated at a $\mathcal{U}$-plaque through q (respectively, through p) orthogonal projection $\tilde{\gamma}$ of γ onto the leaf through q (respectively, through p) exists and satisfies the condition

$$d_M(\gamma(1), \tilde{\gamma}(1)) \geq \epsilon, \tag{15}$$

d_M being, as before, the Riemannian distance function on M. As usual, a subset A of T is (R, ϵ)-separated when any two of its points have this property. Since M is compact, separated sets are always finite and we may define $N(R, \epsilon, \mathcal{F})$ as the maximal cardinality of an (R, ϵ)-separated (w.r.t. $\mathcal{F}$) set. The *geometric entropy* $h(\mathcal{F})$ of $\mathcal{F}$ is defined by (see [**GLW**] and [**LW2**])

$$h(\mathcal{F}) = \lim_{\epsilon \to 0} N(\epsilon, \mathcal{F}), \tag{16}$$

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where

$$N(\epsilon, \mathcal{F}) = \limsup_{R\to\infty} \frac{1}{R} \log N(R, \epsilon, \mathcal{F}). \tag{17}$$

Obviously, $h(\mathcal{F})$ depends on the Riemannian metric $\langle\cdot,\cdot\rangle$ on M or, more precisely, on the Riemannian metric $\langle\cdot,\cdot\rangle|T\mathcal{F}\otimes T\mathcal{F}$ along the leaves: if two Riemannian metrics coincide on $T\mathcal{F}$, then the corresponding entropies are equal.

It is well known that the geometric entropy $h(\mathcal{F})$ is strongly related to the entropies of the holonomy pseudogroups:

$$h(\mathcal{F}) = \sup_{\mathcal{U}} \frac{1}{\Delta(\mathcal{U})} h(H_{\mathcal{U}}), \tag{18}$$

where $\mathcal{U}$ ranges over the family of all nice covers of $(M, \mathcal{F})$ and $\Delta(\mathcal{U})$ is the smallest upper bound for the diameters of the plaques of $\mathcal{U}$ [**GLW**, Theorem 3.4].

Note that since any point p determines the leaf L_p uniquely, we can consider $N(R, \epsilon, \mathcal{F})$ as the number of separated leaves L with base points $p \in T \cap L$. This approach can be adapted to the case of pseudoleaves even if there are infinitely many pseudoleaves passing through a given point.

Fix $\alpha_0 > 0$. Again, since M is compact and has bounded geometry, there exist positive numbers ϵ_0 and ϵ_1 such that for any points $p_1, p_2 \in M$ for which $d_M(p_1, p_2) < \epsilon_0$ and any α-pseudoleaves N_1 and N_2 with $\alpha \le \alpha_0$ and $p_i \in N_i$ there exists a unique vector $v \in T^{\perp}_{p_1} N_1$ such that $p_1' = \exp_M(v) \in N_2$ and $d_{N_2}(p_1', p_2) < \epsilon_1$. The point p_1' can be considered as the orthogonal projection of p_1 onto N_2.

So, let us say that α-pseudoleaves (N_i, p_i) $(\alpha \le \alpha_0,\ i = 1, 2)$ with base points $p_i \in N_i \cap T$ are (R, ϵ)-separated if either $d_M(p_1, p_2) \ge \epsilon_0$ or there exists a curve $\gamma : [0, 1] \to N_1$ (respectively, N_2) originating at p_1 (respectively, at p_2) of length $l(\gamma) \le R$ and such that its orthogonal projection $\tilde{\gamma}$ onto N_2 (respectively, onto N_1) originating in the ball $B_{N_2}(p_2, \epsilon_1)$ (respectively, in $B_{N_1}(p_1, \epsilon_1)$) exists and satisfies inequality (15). Analogous to (16) and (17), let us put

$$h_{\mathrm{ps}}(\mathcal{F}) = \lim_{\epsilon\to 0^+} \inf_{\alpha} N_\alpha(\epsilon, \mathcal{F}), \tag{19}$$

where

$$N_\alpha(\epsilon, \mathcal{F}) = \limsup_{R\to\infty} \frac{1}{R} \log N_{\alpha(R)}(R, \epsilon, \mathcal{F}), \tag{20}$$

$\alpha : \mathbb{R}_+ \to \mathbb{R}_+$ is any function satisfying $\alpha(R) \to 0$ while $R \to \infty$, and $N_{\alpha(R)}(R, \epsilon, \mathcal{F})$ is the maximal number of pairwise (R, ϵ)-separated $\alpha(R)$-pseudoleaves equipped with base points.

Since any leaf becomes an α-pseudoleaf for any $\alpha \ge 0$, it is obvious that our 'geometric pseudo-entropy' $h_{\mathrm{ps}}(\mathcal{F})$ is not less than the 'real' geometric entropy $h(\mathcal{F})$. Below, we shall prove that, similar to the case of a pseudogroup, these two entropies are equal.

THEOREM 2. *For any compact foliated Riemannian manifold* $(M, \mathcal{F}, \langle\cdot,\cdot\rangle)$ *the equality*

$$h_{\mathrm{ps}}(\mathcal{F}) = h(\mathcal{F}) \tag{21}$$

holds.

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Proof. As was observed, we have only to prove the inequality '$\leq$' in (21).

To this end cover M by 'coins' as in the proof of Theorem 3.4 of [**GLW**]. 'Coins' are distinguished charts U_p, $p \in M$, built along the transversals $T_p = \exp_M B^{\perp}(0_p, \rho)$, $B^{\perp}(0_p, \rho)$ being the ball of radius ρ centred at the origin in $T_p^{\perp}\mathcal{F}$, the orthogonal complement of $T_p\mathcal{F}$ in T_pM. More precisely,

$$U_p = \cup_{q \in T_p} B^{\mathcal{F}}(q, \Delta),$$

where $B^{\mathcal{F}}(q, \Delta)$ is a ball of radius Δ centred at q in L_q, the leaf of $\mathcal{F}$ at q. Here, ρ and Δ are sufficiently small such that the corresponding exponential maps are diffeomorphic on all the balls of radii 2ρ and 2Δ.

Choose a finite subset Q of M, $Q = \{q_1, \ldots, q_N\}$, such that the corresponding charts U_{q_i}, $i = 1, \ldots, N$, form a nice covering $\mathcal{U}_\Delta$. Obviously, $\Delta(\mathcal{U}_\Delta) = 2\Delta$. Also, fix a small number $\eta > 0$ and choose Q sufficiently dense in M so that any point of M is at most $\eta/2$ apart in the leaf distance from a point of $T = T_{\mathcal{U}_\Delta}$ and $\eta/2$ apart in d_M-distance from one of the points $q_j \in Q$. The existence of such a set Q is shown in full detail in [**GLW**].

Take $\alpha > 0$ small enough, $\epsilon \in (0, \epsilon_0)$, (R, ϵ)-separated α-pseudoleaves (N_i, p_i), $i = 1, 2$, and the corresponding $C\alpha$-pseudo-orbits x_i of $H_{\mathcal{U}_\Delta}$ (Lemma 6).

If $d_M(p_1, p_2) \geq \epsilon_0$, then $d(x_1(e), x_2(e)) \geq \epsilon_0 \geq \epsilon$ and the pseudo-orbits x_1 and x_2 are (n, ϵ)-separated for all $n \geq 1$. Otherwise, there exists a curve $\gamma : [0, 1] \to N_1$, $\gamma(0) = p_1$, $l(\gamma) \leq R$, separating N_1 from N_2, i.e. such that $d(\gamma(1), \tilde{\gamma}(1)) \geq \epsilon$, where $\tilde{\gamma}$ is the suitable projection of γ onto N_2. Shortening γ if necessary, we may assume that $d(\gamma(1), \tilde{\gamma}(1)) = \epsilon$ and $d(\gamma(t), \tilde{\gamma}(t)) \leq \epsilon$ for all t.

Split γ into $n = [1 + R/(2\Delta - 2\eta)]$ pieces γ_j of length $l(\gamma_j) \leq 2\Delta - 2\eta$. Set $p'_0 = p_1$ and $p'_{n+1} = \gamma(1)$, for each $j = 1, \ldots, n$ denote by p'_j the midpoint of γ_j and find indices $i(j) \in \{1, \ldots, N\}$ such that $d_M(p'_j, q_{i(j)}) \leq \eta/2$ for $j = 0, 1, \ldots, n+1$. Thus, if α is small enough, each piece γ_j is contained in a chart $U_j = U_{q_{i(j)}} \in \mathcal{U}_\Delta$. Let $g = (h_n, \ldots, h_0)$, where $h_j = h_{U_j U_{j+1}}$ for $j = 0, 1, \ldots, n$.

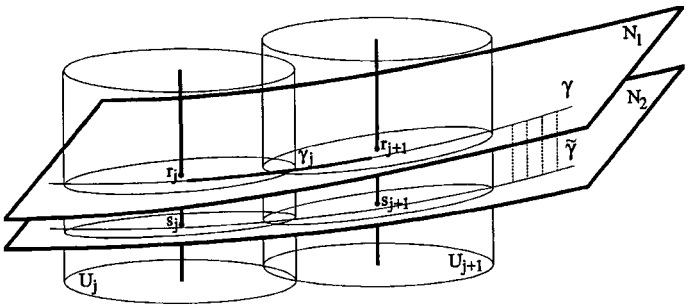


FIGURE 3.

From the construction of the pseudo-orbits x_i it follows that $d_{N_1}(p'_j, x_1(g_j)) \leq \eta$ when $g_j = (h_j, \ldots, h_0)$, $j = 0, 1, \ldots, n$ and α is, again, small enough (see Figure 3, where $r_j = x_1(g_j)$, $r_{j+1} = x_1(g_{j+1})$, $s_j = x_2(g_j)$ and $s_{j+1} = x_2(g_{j+1})$). It follows that

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$x_1(g_j) \in U_j$ for all j if only η is small with respect to Δ and ρ. In particular, $g \in D_{x_1}$. Also, $g \in D_{x_2}$ if only ϵ is small enough. Moreover, since $d_{N_1}(x_1(g), \gamma(1)) \leq \eta$ and the similar inequality holds for $\tilde{\gamma}(1)$ and $x_2(g)$ on N_2, then $d_M(x_1(g), x_2(g)) \geq C''\epsilon$, where $C'' > 0$ is a universal constant depending only on ρ, Δ, η and the geometry of M. Hence, the pseudo-orbits x_1 and x_2 of $H_{\mathcal{U}_\Delta}$ are $(n+1, C''\epsilon)$-strongly separated and consequently

$$N_\alpha(R, \epsilon, \mathcal{F}) \leq N_{C\alpha}([2 + R/(2\Delta - 2\eta)], C''\epsilon, H_{\mathcal{U}_\Delta})$$

for α and ϵ small enough.

Now, pass to suitable limits to arrive at

$$h_{\mathrm{ps}}(\mathcal{F}) \leq \frac{1}{2\Delta - 2\eta} h_{\mathrm{ps}}(H_{\mathcal{U}_\Delta}).$$

Finally, apply Theorem 1 and relation (18) to obtain the inequality

$$h_{\mathrm{ps}}(\mathcal{F}) \leq \frac{\Delta}{\Delta - \eta} h(\mathcal{F})$$

which yields the required result since $\eta > 0$ may be arbitrarily small. □

5. *Groups and foliated bundles*

If G is a group of global transformations of X, then one need not care about domains of maps, $D_x = G_\infty$ for any pseudo-orbit x, $d_0 = d_1$ is defined simply by

$$d_0(x, y) = d_1(x, y) = \sum_{k=0}^{\infty} \frac{1}{2^k} \max\{d(x(g), y(g)); g \in G_k\}$$

and the maps $\sigma_g : Y_\alpha \to Y_\alpha$, $g \in G_\infty$ are also defined globally. In other words, if the group G acts on X, then it acts on the spaces of pseudo-orbits as well.

In particular, if $\mathcal{F}$ is a foliation transverse to the fibres of a fibre bundle $\pi : M \to B$, then its global holonomy group H acts on a fibre F (over a fixed point $b_0 \in B$). H is generated by the maps $h_{[\gamma]}$ with $[\gamma] \in \pi_1(B, b_0)$ generating the fundamental group of B, defined by

$$h_{[\gamma]}(p) = \tilde{\gamma}(1),$$

where $\tilde{\gamma} : [0, 1] \to L_p$ is the lift of γ to the leaf L_p through $p \in F$ originating at p.

Similarly, if N is an α-pseudoleaf and α is small enough, then N is transverse to the fibres of our bundle and the corresponding pseudo-orbit of H can be obtained by lifting to N the loops generating $\pi_1(B, b_0)$.

For such a foliated bundle, a Riemannian metric on M which, when restricted to $T\mathcal{F}$, coincides with the lift of a Riemannian metric on B, and a generating set $H_1 \subset H$ one has the inequality

$$\frac{1}{a} h(H, H_1) \leq h(\mathcal{F}) \leq \frac{1}{b} h(H, H_1), \tag{22}$$

where a (respectively, b) is the maximum (respectively, minimum) of the lengths of the homotopy classes $[\gamma] \in \pi_1(B, b_0)$ with $h_{[\gamma]} \in H_1$ [**GLW**, p. 122]. In the same way,

$$\frac{1}{a} h_{\mathrm{ps}}(H, H_1) \leq h_{\mathrm{ps}}(\mathcal{F}) \leq \frac{1}{b} h_{\mathrm{ps}}(H, H_1). \tag{23}$$

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Since $h_{\mathrm{ps}}(H, H_1) = h(H, H_1)$ by Theorem 1, inequalities (22) and (23) yield

$$h_{\mathrm{ps}}(\mathcal{F}) \leq \frac{a}{b} h(\mathcal{F}). \tag{24}$$

Fix $m \in \mathbb{N}$ and replace $H_1 = \{z_1, \ldots, z_s\}$ by a new generating set

$$H_1' = \{z_1^{m_1}, z_1^{m_1+1}, \ldots, z_s^{m_s}, z_s^{m_s+1}\}$$

with the exponents $m_k = [m/l_k]$, l_k being the length of the homotopy class z_k. For this generating set, the corresponding factor a'/b' in (24) satisfies the inequalities

$$\frac{m}{m+1+b} \leq \frac{a'}{b'} \leq \frac{m+1+b}{m}$$

and is arbitrarily close to 1 when m is sufficiently large.

This implies Theorem 2 for foliated bundles immediately.

6. *An example*

In [**BS**] and [**Mi**], the equality

$$h(G, G_1) = \lim_{\epsilon \to 0^+} \lim_{\alpha \to 0^+} \limsup_{n \to \infty} \frac{1}{n} \log N_\alpha(n, \epsilon) \tag{25}$$

was proved for the group $G = \mathbb{Z}$ generated by $G_1 = \{f\}$, a single transformation of X. Reading the proof of (25) one can observe that some of the argument is very specific: it depends strongly on the fact that the growth of $\#G_n$, $n \in \mathbb{N}$, is linear. Below, we show an example, where the equality (25) does not hold: the limit on the right-hand side is strongly bigger than the entropy.

So, let $X = S^1$, $G = F_2$ be the free group with two generators, f_0 and f_1, f_i being Morse–Smale diffeomorphisms of S^1, each of them with exactly two fixed points, a source p_i and a sink q_i, such that $\{p_0, q_0\} \cap \{p_1, q_1\} = \emptyset$ and the sets $U_i = \{p \in S_1, f_i' > 1 + \delta \text{ on } [p - \alpha_0, p + \alpha_0]\}$, $i = 0, 1$, cover S^1 for some α_0 and $\delta > 0$. Also, let $G_1 = \{\mathrm{id}, f_0^{\pm 1}, f_1^{\pm 1}\}$.

Now, take any $n \in \mathbb{N}$, $\epsilon > 0$, $\alpha \in (0, \alpha_0)$ and an (n, ϵ)-separated set $A \subset S^1$. For any $x \in A$ and any $g = (g_n, \ldots, g_1) \in G_n$ with $g_i \in \{f_0, f_1\}$ define an α-pseudo-orbit $\tilde{x}_g$ of G as follows:

(1) $\tilde{x}_g(h) = h(x)$ for all $h \in G_n$ and any $h \in G_m \setminus G_{m-1}$, $m > n$, of the form $h = (g', g'')$ with $g' \in G_{m-n}$ and $g'' \in G_n$, $g'' \neq g$;

(2) if $g(x) \in U_0$, then $h_1 = f_0$, otherwise $h_1 = f_1$, and, in both cases, $\tilde{x}_g(h_1, g) = h_1(g(x)) + \alpha$;

(3) if $h_1, \ldots, h_j \in G_1$ and $\tilde{x}_g(h_j, \ldots, h_1, g)$ are already defined, and $\tilde{x}_g(h_j, \ldots, h_1, g) \in U_0$, then $h_{j+1} = f_0$, otherwise $h_{j+1} = f_1$, and, in both cases,

$$\tilde{x}_g(h_{j+1}, h_j, \ldots, h_1, g) = h_{j+1}(\tilde{x}_g(h_j, \ldots, h_1, g)) + \alpha;$$

(4) if $j \in \mathbb{N}$, $m \in \mathbb{N}$, $h = (h_m', \ldots, h_1', h_j, \ldots, h_1, g)$ with $h_i' \in G_1$, $h_1' \neq h_{j+1}$ and $h_1' \neq h_j^{-1}$, then $\tilde{x}_g(h) = h_m' \circ \cdots \circ h_1'(\tilde{x}_g(h_j, \ldots, h_1, g))$.

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Roughly speaking, $\tilde{x}_g$ is obtained by a suitable modification of the orbit $G(x)$ along a single branch of the Cayley graph of G originating at g.

A standard induction involving the mean value theorem shows that

$$d(\tilde{x}_g(h_j, \dots, h_1, g), h_j \circ \cdots \circ h_1 \circ g(x)) \geq \alpha \cdot (1+\delta)^j \geq \epsilon \tag{26}$$

whenever $j > n(\epsilon, \alpha)$, $n(\epsilon, \alpha)$ being the largest integer which does not exceed the quotient

$$\frac{\log(\epsilon/\alpha)}{\log(1+\delta)}.$$

It follows that the set A_α of all such pseudo-orbits $\tilde{x}_g$ is $(n+n(\epsilon,\alpha)+1, \epsilon)$-separated. In fact, if $x \neq x' \in A$, then $d(\tilde{x}_g(h), \tilde{x}'_{g'}(h)) = d(h(x), h(x')) \geq \epsilon$ for some $h \in G_n$; if $x = x'$ but $g \neq g'$, then $\tilde{x}_{g'}(h_j, \dots, h_1, g) = h_j \circ \cdots \circ h_1 \circ g(x)$ and the inequality $d(\tilde{x}_{g'}(h), \tilde{x}_g(h)) \geq \epsilon$ for some $h \in G_{n+n(\epsilon,\alpha)+1}$ follows from (26). Clearly,

$$\#A_\alpha = 2^n \cdot \#A.$$

This yields the inequality

$$N_\alpha(n + n(\epsilon,\alpha) + 1, \epsilon) \geq 2^n \cdot N(n, \epsilon, S^1)$$

for all n, $\epsilon > 0$ and $\alpha < \alpha_0$. Passing to suitable limits we end up with

$$\lim_{\epsilon \to 0^+} \lim_{\alpha \to 0^+} \limsup_{n \to \infty} \frac{1}{n} \log N_\alpha(n, \epsilon) \geq h(G, G_1) + \log 2. \tag{27}$$

Inequality (27) shows that, what was already observed in [**GLW**], the entropy of a group (a pseudogroup, a foliation) depends strongly on the growth of the group (the orbits, the leaves). It would be very interesting to obtain a good upper estimate of the right-hand side in (25) in terms of $h(G, G_1)$ and the growth of G. Some indication in this direction is contained in the proof of Proposition 6.13 in [**GLW**].

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REFERENCES

[BS] M. Barge and R. Swanson. Pseudo-orbits and topological entropy. *Proc. Amer. Math. Soc.* **109** (1990), 559–566.

[BK] P. Buser and H. Karcher. Gromov's almost flat manifolds. *Astérisque* **81** (1981), 1–148.

[Eg] S. Egashira. Expansion growth of foliations. *Ann. Fac. Sci. Univ. Toulouse* **2** (1993), 15–52.

[GLW] E. Ghys, R. Langevin and P. Walczak. Entropie géométrique des feuilletages. *Acta Math.* **160** (1988), 105–142.

[HH] G. Hector and u. Hirsch. *Introduction to the Geometry of Foliations, Part A and B.* Vieweg and Sohn, Braunschweig, 1986 and 1987.

[Hu1] S. Hurder. Ergodic theory of foliations and a theorem of Sacksteder. *Dynamical Systems: Proceedings, University of Maryland 1986–87 (Lecture Notes in Mathematics, 1342)* Springer, Berlin, 1988, pp. 291–328.

https://doi.org/10.1017/S0143385798120874 Published online by Cambridge University Press

[Hu2] S. Hurder. Exceptional minimal sets of $C^{1+\alpha}$ group actions on the circle. *Ergod. Th. & Dynam. Sys.* **11** (1991), 455–467.

[Hu3] S. Hurder. Coarse geometry of foliations. *Geometric Study of Foliations (Proc. Tokyo 1993)*. World Scientific, Singapore, 1993, pp. 35–96.

[Hur] M. Hurley. On topological entropy of maps. *Ergod. Th. & Dynam. Sys.* **15** (1995), 557–568.

[In] T. Inaba. Expansivity, pseudoleaf tracing property and topological stability of foliations. *Preprint*, 1995.

[IT] T. Inaba and N. Tsuchiya. Expansive foliations. *Hokkaido Math. J.* **21** (1992), 39–49.

[LW1] R. Langevin and P. Walczak. Entropie d'une dynamique. *C. R. Acad. Sci. Paris* I **312** (1991), 141–144.

[LW2] R. Langevin and P. Walczak. Entropy, transverse entropy and partitions of unity. *Ergod. Th. & Dynam. Sys.* **14** (1994), 551–563.

[Mi] M. Misiurewicz. Remark on the definition of topological entropy. *Dynamical Systems and Partial Differential Equations (Caracas 1984)*. Ed. Equinoccio, Caracas, 1986, pp. 65–67.

https://doi.org/10.1017/S0143385798120874 Published online by Cambridge University Press