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A NOTE ON THE PLANAR SKOROKHOD EMBEDDING PROBLEM

MAHER BOUDABRA

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Abstract

The planar Skorokhod embedding problem was first proposed and solved by Gross ['A conformal Skorokhod embedding', *Electron. Commun. Probab.* **24** (2019), 11 pages; doi:10.1214/19-ECP272]. Gross worked with probability distributions having finite second moment. Boudabra and Markowsky ['Remarks on Gross' technique for obtaining a conformal Skorokhod embedding of planar Brownian motion', *Electron. Commun. Probab.* **25** (2020), 13 pages; doi:10.1214/20-ECP300] extended the solution to all distributions with a finite pth moment for $p > 1$. The case $p = 1$ has remained uncovered since then. In this note, we show that the planar Skorokhod embedding problem is solvable for $p = 1$ when the Hilbert transform of its quantile function is integrable, effectively closing this line of investigation.

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1. Introduction and main results

In 2019, R. Gross [9] posed the following problem. Given a nondegenerate distribution μ with zero mean and finite second moment, find a simply connected domain U such that if $(Z_t)_{t\geq 0}$ is a planar Brownian motion started at 0 and τ_U is its exit time from U, then $\mathrm{Re}(Z_{\tau_U})$ has the law μ.

This is called the planar (or conformal) Skorokhod embedding problem (PSEP). Gross gave a constructive solution using the principle of conformal invariance of planar Brownian motion [11]. The domain he constructed enjoys the property that its underlying exit time has a finite average. In [3], Boudabra and Markowsky showed that Gross' approach remains valid if the distribution μ has a finite pth moment for some $p > 1$. They reformulated the problem by requiring the finiteness of $\mathbb{E}(\tau_U^{p/2})$. We adopt this formulation in this note. Later on, in a subsequent work [2], the same authors provided a second solution to the PSEP with a different geometry from Gross' domain. Both solutions start by crafting a certain univalent function acting on the open unit disc $\mathbb{D}$. Then, the authors show that the range U of such a function fits the statistical

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requirement $\mathrm{Re}(Z_{\tau_U}) \sim \mu$. Our main result gives a sufficient condition for the existence of a $p = 1$ embedding in terms of the Hilbert transform of the quantile function (see Theorem 1.4). Through this work, the following notation is adopted.

- $\mathbb{D}$: the open unit disc in the plane.
- $\mathbb{T}$: the unit circle in the plane.
- $L^p_{2\pi}$ $(p > 0)$: the space of real valued 2π-periodic functions f such that

$$\|f\|_p = \left(\frac{1}{2\pi}\int_0^{2\pi} |f(t)|^p\, dt\right)^{1/p}$$

  is finite; we write $L^p(\mathbb{T})$ for the space of complex-valued functions on $\mathbb{T}$.
- Standard planar Brownian motion: planar Brownian motion started at 0.

The ingredients in the proofs of the results are Hardy spaces and the Hilbert transform. We recall their definitions for the reader's convenience.

DEFINITION 1.1. Let f be an analytic function in the unit disc. The pth Hardy norm of f is defined by

$$\|f\|_{\boldsymbol{H}^p} := \sup_{0\le r<1}\left\{\frac{1}{2\pi}\int_0^{2\pi} |f(re^{\theta i})|^p\, d\theta\right\}^{1/p}.$$

The set of analytic functions on $\mathbb{D}$ with finite $\boldsymbol{H}^p$ norm is denoted by $\boldsymbol{H}^p(\mathbb{D})$ and is called the Hardy space (of index p).

DEFINITION 1.2. The Hilbert transform of a 2π-periodic function f is defined by

$$\mathcal{H}\{f\}(x) := \frac{1}{2\pi} PV\left\{\int_{-\pi}^{\pi} f(x-t)\cot(t/2)\, dt\right\} = \frac{1}{2\pi}\lim_{\eta\to 0}\int_{\eta\le|t|\le\pi} f(x-t)\cot(t/2)\, dt,$$

where PV denotes the Cauchy principal value. Note that the Hilbert transform is an anti-involution on zero-mean functions, that is, $\mathcal{H}^2 = -\mathrm{id}$. The Hilbert transform exists almost everywhere (a.e.) for any function $f \in L^p(\mathbb{T})$ with $p \ge 1$. In addition, if $p > 1$, then

$$\|H\{f\}\|_p \le \lambda_p \|f\|_p \tag{1.1}$$

for some positive constant λ_p (see [5]).

The main results of our work are the following theorems.

THEOREM 1.3. *Let f be a univalent function on $\mathbb{D}$ with $f(0) = 0$ and set $U = f(\mathbb{D})$. Let $(Z_t)_{t\ge0}$ be standard planar Brownian motion and τ_U its exit time from U. Then, Z_{τ_U} and $f(\xi)$ share the same distribution where ξ is any random variable uniformly distributed on the unit circle and $f(\xi)$ denotes the radial boundary value of f at ξ (defined a.e.).*

THEOREM 1.4. *Assume μ is a nondegenerate probability measure with zero mean and let q be its quantile function. Then, there exists a simply connected domain $U \subset \mathbb{C}$ containing the origin such that, for a standard planar Brownian motion $(Z_t)_{t\ge0}$ and* $\tau_U := \inf\{t \ge 0 : Z_t \notin U\}$,

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$$\mathrm{Re}(Z_{\tau_U}) \sim \mu \quad \textit{and} \quad \mathbb{E}(\tau_U^{1/2}) < +\infty$$

whenever $\mathcal{H}\{q(|\cdot|/\pi)\} \in L^1_{2\pi}$.

2. Proofs

Before proving our main results, we recall two celebrated results that we need in the proof.

THEOREM 2.1 [6, Theorem 3.16]. *Any univalent function acting on the unit disc belongs to the Hardy space* $\boldsymbol{H}_p(\mathbb{D})$ *for all* $p < \frac{1}{2}$.

THEOREM 2.2 [12]. *If* $f \in \boldsymbol{H}_p(\mathbb{D})$ *for some positive p, then:*

(1) *the radial limit of f, say* $f^*(\xi)$*, exists a.e. on* $\mathbb{T}$ *and* $f^* \in L^p(\mathbb{T})$*;*
(2) $\lim_{r\to 1^-} \|f(r\cdot) - f^*\|_p = 0$*;*
(3) $\|f^*\|_p$ *equals the pth Hardy norm of f.*

From now on, we do not distinguish notationally between a Hardy function and its boundary values; $f(\xi)$ will denote the radial limit on $\mathbb{T}$.

PROOF OF THEOREM 1.3. The proof is adapted from [3, Lemma 3.2]. The main difference is that in [3], the authors required that f belongs to the Hardy space $\boldsymbol{H}_p(\mathbb{D})$ for some $p > 1$, while this constraint is dropped here.

Let $(W_t)_t$ be a standard planar Brownian motion killed on exiting $\mathbb{D}$. By the conformal invariance principle, the planar process $(Z_t = f(W_{\sigma^{-1}(t)}))_{t\geq 0}$ is a standard planar Brownian motion, where σ is the time change given by

$$\sigma(t) = \int_0^t |f'(W_s)|^2\, ds, \quad t < \tau_{\mathbb{D}}.$$

In particular,

$$Z_{\sigma(\tau_n)} = f(W_{\tau_n}),$$

where τ_n denotes the exit time of W_t from $\mathbb{D}_n := (1 - 1/n)\mathbb{D}$. In fact, $\sigma(\tau_n)$ is the exit time of $(Z_t)_{t\geq 0}$ from $f(\mathbb{D}_n)$. Since $\mathbb{D}_n \uparrow \mathbb{D}$ as $n \to +\infty$, we have $\tau_n \uparrow \tau_{\mathbb{D}}$ almost surely (a.s.). By continuity of the time change, $\sigma(\tau_n)$ converges to $\sigma(\tau_{\mathbb{D}})$ a.s. The quantity $\sigma(\tau_{\mathbb{D}})$ is exactly the exit time of $(Z_t)_{t\geq 0}$ from U, which we denote by τ_U. As the Brownian path is continuous a.s., $Z_{\sigma(\tau_n)}$ converges to Z_{τ_U} a.s. On an auxiliary probability space $\Omega = (0, 2\pi)$ equipped with the normalised Lebesgue measure, define $\xi_n(\omega) = (1 - 1/n)e^{\omega i}$ and $\xi(\omega) = e^{\omega i}$, where $\omega \in \Omega$. As planar Brownian motion is isotropic, the two random variables W_{τ_n} and $\xi_n(\omega) = (1 - 1/n)e^{\omega i}$ share the same distribution. In particular, $f(\xi_n)$ and $Z_{\sigma(\tau_n)}$ have the same law. Now, fix $p \in (0, \frac{1}{2})$. By Theorem 2.1, f belongs to $\boldsymbol{H}_p(\mathbb{D})$ and, hence, it extends radially to the boundary of $\mathbb{D}$. We have

$$\mathbb{E}(|f(\xi_n) - f(\xi)|^p) = \frac{1}{2\pi}\int_0^{2\pi} |f((1 - 1/n)e^{\omega i}) - f(e^{\omega i})|^p\, d\omega = \|f((1 - 1/n)\cdot) - f\|_p^p.$$

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By the second point of Theorem 2.2, $\mathbb{E}(|f(\xi_n) - f(\xi)|^p)$ converges to zero. Hence, $f(\xi_n)$ converges to $f(\xi)$ in distribution. As $f(\xi_n)$ and $Z_{\sigma(\tau_n)}$ are equal in distribution, $Z_{\sigma(\tau_n)}$ converges to $f(\xi)$ in distribution as well. However,

$$\lim_n Z_{\sigma(\tau_n)} = Z_{\tau_U} \text{ a.s.}$$

Therefore, Z_{τ_U} and $f(\xi)$ have the same distribution by uniqueness of weak limits. This ends the proof. □

PROOF OF THEOREM 1.4. Recall that the quantile function q of μ is simply the pseudo-inverse of the cumulative distribution function (c.d.f) F of μ. More precisely,

$$q(u) = \inf\{x \mid F(x) \geq u\}.$$

A crucial property of q is that if q is fed with a uniformly distributed random variable X in $(0, 1)$, then $q(X)$ samples as μ. Set $\varphi(\theta) = q(|\theta|/\pi)$ with $\theta \in (-\pi, \pi)$.

The assumption $H\{\varphi\} \in L^1_{2\pi}$ implies that the analytic map

$$G(z) = \frac{1}{2\pi} \int_0^{2\pi} \frac{e^{ti} + z}{e^{ti} - z} \varphi(t)\, dt \in \boldsymbol{H}^1(\mathbb{D}).$$

The map G is the standard Cauchy–Poisson integral (see [5, Ch. 3]). In particular,

$$\mathrm{Re}(G(e^{\theta i})) = \varphi(\theta) \quad \text{and} \quad \mathrm{Im}(G(e^{\theta i})) = H\{\varphi\} \text{ a.e.} \tag{2.1}$$

Furthermore, the map $G(z)$ is univalent by [9, Proposition 2.2]. In particular, $U := G(\mathbb{D})$ is simply connected and contains the origin. Run a planar Brownian motion $(Z_t)_t$ inside U and set τ_U as the exit time of Z_t from U. By (2.1), the boundary value of G has real part φ, which encodes the quantile of μ. Therefore, by Theorem 1.3,

$$\mathrm{Re}(Z_{\tau_U}) \sim \mu.$$

Since $G \in \boldsymbol{H}^1(\mathbb{D})$, Burkholder's characterisation of the finiteness of the moments of τ_U [4] yields $\mathbf{E}(\tau_U^{1/2}) < +\infty$. Therefore, the PSEP is solvable. □

3. Comments

3.1. Comment 1. Theorem 1.3 might seem obvious as one may think that

$$\lim_n f(W_{\tau_n}) = f(W_{\tau_{\mathbb{D}}})$$

is immediate. However, this is not obvious for the simple reason that W_{τ_n} does not approach the boundary of the unit disc radially! Theorem 1.4 naturally raises the converse question: if the PSEP is solvable for a given law μ, must the Hilbert transform $\mathcal{H}\{q(|\cdot|/\pi)\} \in L^1_{2\pi}$? We conjecture that the answer is yes.

3.2. Comment 2. Theorem 1.4 implies the existence of solutions for the PSEP when μ has a finite pth moment for some $p > 1$. This follows from (1.1).

A natural question is how to find functions with integrable Hilbert transforms in $L^1_{2\pi}$. A naive way to do so is to compute the Hilbert transform and then check its

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integrability. However, this is hard in practice. The following result, due to Zygmund, is handy.

THEOREM 3.1 (see [5, Ch. 4]). *Let u be a harmonic function in the unit disc such that*

$$\sup_{0\le r<1}\int_0^{2\pi}|u(re^{\theta i})|\log^+(|u(re^{\theta i})|)\,d\theta<+\infty. \tag{3.1}$$

Then,

$$\sup_{0\le r<1}\int_0^{2\pi}|H\{u\}(re^{\theta i})|\,d\theta<+\infty.$$

In particular, the condition (3.1) is, in some sense, the weakest that one can impose on u. Note that the $\log^+(x)$ can be replaced by $\log(1+|x|)$ [8]. A classical result in harmonic analysis says that if g is a convex function, then $g(u)$ is subharmonic. In particular, the integral in (3.1) is nondecreasing in r [12]. So, in terms of the boundary function, the condition (3.1) can be replaced by

$$\int_0^{2\pi}|u(e^{\theta i})|\log^+(|u(e^{\theta i})|)\,d\theta<+\infty,$$

which was originally formulated. Based on this fact, the planar Skorokhod embedding problem becomes solvable once $|\varphi|\log^+(|\varphi|)$ belongs to $L^1_{2\pi}$.

Next, we provide an example where the distribution μ has only a first moment. Consider the distribution of the random variable

$$\xi=\frac{5}{(\ln(10)+1)^2}-\frac{10}{X(1-\ln(X/10))^3}$$

with X uniformly distributed on $(0,1)$. The underlying quantile function is

$$q(u)=\frac{5}{(\ln(10)+1)^2}-\frac{10}{u(1-\ln(u/10))^3},\quad u\in(0,1).$$

Using Bertrand's criterion for integrals, q belongs to L^p for $p=1$, but not for any higher p. Near zero, we have the approximation

$$\frac{\log^+(1/u(1-\ln(u))^3)}{u(1-\ln(u))^3}=-\frac{\log(u(1-\ln(u))^3)}{u(1-\ln(u))^3}\approx-\frac{\log(u)}{u|\ln(u)|^3}-3\frac{\log(|\ln(u)|)}{u|\ln(u)|^3}.$$

Thus, the integral of $|q|\log^+(|q|)$ is finite. Therefore, the Hilbert transform of φ is in $L^1_{2\pi}$ (recall $\varphi(\theta)=q(|\theta|/\pi)$). Consequently, the Skorokhod embedding problem is solvable for this μ.

REMARK 3.2. Gross' construction of the map $G(z)$ is based on the calculation of the Fourier coefficients of φ. Now, one can see that $G(z)$ can be simply expressed as

$$G(z)=\frac{1}{2\pi}\int_0^{2\pi}\frac{e^{ti}+z}{e^{ti}-z}\varphi(t)\,dt.$$

Although this paper is primarily addressing the method introduced in [9], the approach is directly applicable, without modification, to the method described in [2].

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3.3. Comment 3. Mathematicians tend to shy away from L^p spaces when $p < 1$ due to their unconventional properties, such as the failure of the triangle inequality, or, in relation to our context, the inability to construct Poisson integrals. However, we present an example that shows something might be done when $p < 1$. Using the formulae in [1, 3], the c.d.f of the stopped planar Brownian motion upon leaving the Koebe domain $\mathcal{K} = \mathbb{C} - (-\infty, -\frac{1}{4}]$ is given by

$$F(w) = 1 - \frac{2}{\pi}\arctan(\sqrt{-1-4w})$$

for $w \in (-\infty, -\frac{1}{4}]$. That is,

$$\varphi(t) = F^{-1}(|t|/\pi) = -\frac{1}{4\sin(t/2)^2}.$$

Interestingly, the function $\varphi(t)$ coincides with the real part of the radial limit of the Koebe function $k(z) = z/(1-z)^2$. In [7], the author gathered some results about the existence of entire functions and harmonic functions with a predefined radial boundary function. The following natural question seems worth being addressed. How can we construct the univalent function whose real part of its radial limit matches φ when the distribution μ has exclusively a finite pth moment for some $p < 1$? Another related issue will be how to reconcile this with the fact that every univalent function belongs to $\boldsymbol{H}^p(\mathbb{D})$ for all $p < \frac{1}{2}$?

3.4. Comment 4. It is worth noting that Theorem 1.3 is still valid if f is a proper map or, more generally, a B-proper map. The notion of B-properness was introduced in [10]. Both types of functions ensure that if a sequence $(z_n)_n$ in the input domain is approaching its boundary, then the image sequence $(f(z_n))_n$ is also approaching the boundary of the range of f.

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MAHER BOUDABRA, Department of Mathematics,
King Fahd University of Petroleum and Minerals, Saudi Arabia
e-mail: maher.boudabra@gmail.com

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