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# Revised-Path Dependence

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In this article, we define a class of revised path–dependent processes and characterize their basic properties. A process exhibits revised-path dependence if the current outcome can revise the value of a past outcome. A revision could be a change to that outcome or a reinterpretation. We first define a revised path–dependent process called the *accumulation process*: in each period, a randomly chosen past outcome is changed to match the current outcome and show that it converges to identical outcomes. We then construct a general class of models that includes the Bernoulli process, the Polya process, and the accumulation process as special cases. For this general class, we show that, apart from knife-edge cases, all processes converge either to homogeneous equilibria or to an equal probability distribution over types. We also show that if random draws advantage one outcome over the other, then the process has a unique equilibrium.

In the familiar definition of path dependence, the value of an outcome depends on the path or the set of previous outcomes (David 1985; Arthur 1994). That outcome can be either an exogenous variable produced by a generating function or an endogenous choice by a strategic actor. Path-dependent outcomes are ubiquitous in social processes. Many decisions and actions depend on prior events. Path-dependent equilibria, in which the long-run state of a system depends on the events along the way, occur less frequently (Page 2006). However, evidence supports path-dependent equilibria in choices over technologies, standards, institutional features, social behaviors, norms, laws, and city locations.[1]

The canonical model producing path-dependent equilibria is the Polya process. In the Polya process, an urn contains two balls: one of each of two colors. A ball is drawn and a ball is returned to the urn along with a second ball of the same color. In future draws, that color ball becomes more likely to be drawn. As the number of balls in the urn grows large, the distribution of the colors of balls within the urn converges, so that each period of drawing the ball and replacing it has an insignificant effect on the overall distribution of ball colors. It can be shown that any proportion of outcomes can be a long-run equilibrium and, in addition, any proportion occurs with equal probability. In other words, anything can happen and is equally likely to happen (Arthur 1994).

Of course, if the outcomes do not differ in their efficiency or fairness, then the existence of path dependence has less relevance. It matters little whether we drive on the left or the right provided that we all drive on the same side of the road. However, in many cases, outcomes are not equally efficient. Evidence suggests that configurations that gain early leads can become locked-in (Arthur 1994).[2]

[1]See, for example, Hacker (2002), Pierson (2000, 2004), Cowan and Gunby (1996), Crouch and Farrell (2004), and North (1990).
[2]See Liebowitz and Margolis (1990, 2002) for critiques of many canonical examples of inefficient lock-in.

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To see how the Polya process can provide insights into the real world, suppose that the Supreme Court's judgments are guided by the principles of *stare decisis* (Hathaway 2001; Schauer 2012). It follows that with sufficient precedent, the legal process converges. If up to now, the Court has decided in favor of business interests 80% of the time, we might then expect it to favor business interests in a current case with similar probability. However, at the outset of the process, say, the launch of the legal system, we could not have predicted the equilibrium distribution: whether the Court will be biased in favor of business interests or not.

In this article, we analyze the potential for path-dependent equilibria within a broader class of processes that allow for past outcomes to be revised. With path revision, current outcomes can change the perceived or interpreted value of previous outcomes. We call this general class of processes *revised-path dependent* to capture that (1) they are path-dependent processes and (2) past outcomes can be altered, or revised, to align with current outcomes. We begin with what we see as the canonical revised path–dependent process, the *accumulation process*, where past outcomes switch to match the current outcome. We then construct a class of models that allows for both revised-path and traditional path dependence and also allow some of the outcomes are random, independent of the state. The infusion of randomness creates both greater contingency and biases the equilibrium toward equally likely outcomes. This general class includes as special cases the Bernoulli process, the Polya process, and the accumulation process.

Our main results are the following: First, we show that the long-run equilibria of the accumulation process consist of all outcomes being of the same type, where the type depends on the order in which outcomes occur. Equilibria are path dependent as in the Polya process, but it is no longer the case that any distribution of types is possible. Second, we show that if we do not include any random outcomes, then the long-run equilibrium of any process that includes path revisions also consists of all outcomes being the same type. Thus, even a small amount of path revision implies that outcomes lie at the extremes, problematizing claims that anything can happen. Finally, when we include the possibility of random outcomes, we find that it is possible to resurrect the result that anything can happen, but that this result only holds in knife-edge cases. And if one outcome is advantaged over another, by creating a biased Bernoulli process, then the the equilibrium is unique.

To see the empirical relevance of revised-path dependence, return for a moment to our earlier example of a court that bases current decisions on prior opinions. When rendering a judgment, path revision would imply that the Court could reinterpret a past decision and yet still lay claim to adherence to precedent. In doing so, it does not change the past decision, but it is "as if" the past decision has changed. These reinterpretations can leap across political institutions. For example, Brandwein (2011) presents evidence that the courts did not abandon the rights of blacks through their use of state action doctrine during the American reconstruction. Yet, those rulings were interpreted by the president and Congress in such a way as to abandon the rights of blacks.

The class of processes that we consider also allow for past decisions and actions to be reversed or undone. In describing the rise of English labor, Acemoglu and Robinson (2012) describe how the Black Death resulted in severe labor shortages, allowing peasants to renegotiate their labor contracts. The force created by the laws of supply and demand toppled the feudal order. Thus, one event, the Black Death, caused previous actions, the establishment of a feudal order, to be revisited. In this particular case, that change was reluctant. In 1351, the English government enacted the wage-fixing State of Laborers law. A labor revolt 30 years later ceased enforcement of this law, resulting in higher wages and effectively ending the feudal system.

Revision also occurs when an ongoing process responds to the current state. An organization might purchase software to do collective ratings of projects. That organization may have relied on a hierarchical procedure for allocating raises, but enabled by their new software, may switch to a collective appraisal system for raises. Path revision occurs regularly in the context of individual political decisions, and is apparent in something as simple as changing an endorsement, as in May 2008 when George McGovern switched his support from Hilary Clinton to Barack Obama.

Finally, path revision can occur in behaviors. In the next section, we describe outcomes from an experimental study in which subjects revised play in one game when a second game was added to their ensemble of strategic play (Bednar et al. 2011). In these experiments, subjects settled into strategies after repeated play in one game, but in some cases revised this settled behavior when a second game was added to their ensemble.

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**Table 1** Game forms used in path dependence experiments

| | | C | S |
|---|---|---|---|
| Prisoner's dilemma: (PD) | C | 7, 7 | 2,10 |
| | S | 10,2 | 4,4 |

| | | C | S |
|---|---|---|---|
| Strong alternation: (SA) | C | 7, 7 | 4,14 |
| | S | 14,4 | 5,5 |

| | | C | S |
|---|---|---|---|
| Self-interest: (SI) | C | 7, 7 | 2,9 |
| | S | 9,2 | 10,10 |

## 1 The Experimental Data

To provide some context for the theoretical results that follow, we begin by presenting evidence of revised-path dependence from behavioral experiments (Bednar et al. 2011). In these experiments, the path revision arises when people change an ongoing behavior to align with new behavior. These data demonstrate existence and also provide a new context, human behavior, within which to contemplate the implications of path and revised-path dependence.

The data shown describe outcomes from experiments in which one of three games, the prisoner's dilemma (PD), a strong alternation game (SA), or a self-interest game (SI), was first played as a control.[3] In the PD, the efficient equilibrium involves both players choosing to cooperate. In the SA game, the efficient equilibrium involved alternating, and in the SI game, the efficient equilibrium required always playing selfish.

Table 2 shows differences in play between a control and a treatment. In the control, players played a single game for approximately 200 rounds with an anonymous opponent. In the treatment, players first played the control game for 100 rounds, and then a second game was added, which they played with a distinct opponent. Players played both games for an additional 100 rounds. Table 2 reports the prevalence of simple strategies: SS corresponds to both players choosing selfish, CC correspond to both cooperating, and ALT corresponds to the players alternating where one player cooperates and the other acts selfishly. The first set of three columns shows the percentage of the outcomes SS, CC, and ALT in the control game. The next set of three columns shows the percentage of the outcomes for the initial game only, but after the second game has been added. In the final three columns, we show statistical $p$ values for the differences in play between the control and the treatment.

We focus attention on the SA game. When played alone, 86% of the time the players alternated (ALT), the payoff-maximizing behavior. Only 6% of outcomes were SS. When the SI is added as the second game, over 27% of the outcomes in SA are SS and only 35% of outcomes are classified as alternating. The increase in SS and decline in the payoff-maximizing ALT are both significant at the 5% level. Thus,

**Table 2** Cognitive load effect on historical game: Treatment versus control

| | *% Simple strategies* | | | | *% Simple strategies* | | | *Control versus treatment* | | |
|---|---|---|---|---|---|---|---|---|---|---|
| *Control* | *SS* | *CC* | *ALT* | *Ensemble* | *SS* | *CC* | *ALT* | *SS* | *CC* | *ALT* |
| PD | 14.50 | 59.50 | 19.17 | (PD,(PD,SA)) | 26.75 | 57.92 | 6.33 | 0.417 | 0.948 | 0.272 |
| | | | | (PD,(PD,SI)) | 59.00 | 34.75 | 1.00 | 0.019 | 0.289 | 0.009 |
| SA | 6.17 | 4.00 | 86.00 | (SA,(SA,PD)) | 14.83 | 14.25 | 65.17 | 0.312 | 0.141 | 0.252 |
| | | | | (SA,(SA,SI)) | 27.17 | 28.67 | 35.50 | 0.022 | 0.003 | 0.005 |
| SI | 100.00 | 0.00 | 0.00 | (SI,(SI,PD)) | 98.50 | 0.25 | 0.00 | | | |
| | | | | (SI,(SI,SA)) | 99.50 | 0.00 | 0.08 | | | |

[3]In SA, the payoff-maximizing play involves agents trading off getting a high payoff and getting a lower payoff. In SI, payoff-maximizing play is a dominant strategy; players do not need to take one another's actions into account when determining their own best play.

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the presence of the SI game led to a significant increase in selfish behavior in the other games.[4] Not only did the human subjects revise their behavior in response to a new behavior, but even more interestingly, the behavioral revision occurred in a game where the subjects had established a payoff-optimizing routine. In these cases, players' net payoffs are reduced. Revised-path dependence not only alters outcomes, but it has the power to reduce payoffs.

## 2 Revised-Path Dependence

We now present formal definitions. To distinguish revised-path dependence from path dependence, we first present a simple definition of the latter. In period $t$, denote the outcome by $x_t$. We assume that $x_t$ belongs to a finite set of outcomes $X$ that have real values. We next define a family of functions $\{A_t\}_{t=1}^{\infty}$. $A_t$ maps a sequence of outcomes in $X$ of length $t$ into the real numbers. $A_t(x_1, x_2, \ldots, x_t) \in R$. This family of functions produces an aggregate statistic. In the cases that we consider, $A_t$ will be the average value of the outcomes, but it could be any statistic.

An *outcome function* $G$ maps $A_t$ into a probability distribution over outcomes in the next period. Thus, formally, the outcome function maps the reals into the set of probability distributions $G : R \rightarrow \Delta(X)$. In other words, the outcome in period $t+1$ is given by $G(A_t(x_1, x_2, \ldots, x_t))$. Implicitly, $G$ is defined over a sequence of outcomes of length $t$. In a slight abuse of notation that simplifies the presentation, we hereafter suppress the $A_t$ and just write $G$ as a function of the sequence of outcomes. Let $\{x_t\}_{t=1}^{\infty}$ denote an infinite sequence of outcomes. We say that a process is *path dependent*; there exists at least two distinct paths of outcomes such that $G$ produces different probability distributions over outcomes under these two paths. Note that we allow for the possibility that $G$ is degenerate.

*A process is* **path dependent** *if* $x_t = G(x_1, x_2, \ldots, x_{t-1})$ *and if there exist* $\{x_t^*\}_{t=1}^{\infty} \neq \{x_t\}_{t=1}^{\infty}$ *s.t.* $\lim_{t\rightarrow\infty} G(x_1, x_2, \ldots, x_t) \neq G(x_1^*, x_2^*, \ldots, x_t^*)$.

Given this definition, the Polya process will be path dependent, as the long-run probability distribution depends on the previous set of outcomes. This definition does not distinguish between set dependence (what Page 2006, calls path dependence) and path dependence, but it suffices for our purposes.

To define revised-path dependence, we must allow for the possibility that the values of past outcomes can be revised. Formally, we define a family of *revision functions*, $\{H_t\}_{t=1}^{\infty}$. A revision function takes the past sequence of outcomes of length $(t-1)$ along with the current outcome, $x_t$, and maps them into a new sequence of outcomes of length $(t-1)$: $H_t : X^{(t-1)} \times X \rightarrow X^{(t-1)}$. If no revision exists, then $H$ is just a projection onto the the first $(t-1)$ dimensions, for example, $H(x_1, x_2, \ldots, x_{t-1}, x_t) = (x_1, x_2, \ldots, x_{t-1})$. If path revision occurs, then the values of one or some of the $x_j$s for $j < t$ will change. To keep track of these changes, we let the variable $x_{j,t}$ denote the value at time $t$ of the outcome that originally occurred at time $j$. If the value of an outcome is never revised, then all the $x_{j,t}$s will be the same. When a revision occurs at time $t$, we have that $x_{j,t} \neq x_{j,t-1}$. If we take all such variables, we obtain an upper triangular matrix of subsequences $X = \{\{x_{j,t}\}_{j=1}^{t}\}_{t=1}^{N}$.

*A process exhibits* **path revision** *if* $H_t : X^{(t-1)} \times X \rightarrow X^{(t-1)}$ *is not a projection onto* $X^{t-1}$ *for all* $t$.

We can then define the *equilibrium* of a revised path–dependent process; it equals the limit of the mean value of the $x_{it}$ as $t$ goes to infinity.

$$\lim_{t\rightarrow\infty} \frac{\sum_{i=1}^{t} x_{i_t}}{t}.$$

Note that this limit is not the mean value of the outcome in period $t$ but the mean value of all the revised outcomes. Thus, a revised-path equilibrium could converge to all revised outcomes being of one type even if with positive probability the outcomes generated by the process are of some other type with positive probability. All that would need to occur would be for those outcomes to be later revised.

[4] Bednar et al. (2012) investigate the behavioral spillovers between games and find a strong influence of SI, a game with a dominant strategy, on other games played simultaneously.

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### 2.1 *The Accumulation Process*

We now define the canonical revised path–dependent process. We call it the *accumulation process* because revisions accumulate. Like the Polya process, this process considers only two possible values for each outcome. Also, like Polya, it can be generalized. In the process, an urn begins with approximately equal numbers of blue (outcome 1) and yellow (outcome 0) balls. In each period, a ball is drawn from the urn. That ball is then replaced in the urn and a new ball is added that matches the color of the chosen ball. In addition, an existing ball has its color switched to match the color of the ball chosen.

*Definition 1.* In the *accumulation process*, let $x_{t,t} \in \{0, 1\}$ with equal probability for all $t$. Let $S$ equal the length of initial history. If $t > S$, let $D = \{j : x_{j,t-1} \neq x_{t,t}\}$. Choose $k \in D$, and set $x_{k,t} = x_{t,t}$. For all other $i < t$, $i \neq k$, let $x_{i,t} = x_{i,t-1}$.

*Claim* 1. In any long-run equilibrium of the accumulation process, all outcomes will be of the same type.

**Proof:** Let $V_t = \sum_{i=1}^{t} x_{i,t}$. In period $T + 1$, if $x_{t,t} = 1$, then $V_{T+1} = V_T + 2$. The old total increases by 2 because a new ball is added *and* an existing ball of value 0 is transformed to value 1. Similarly, if $x_{t,t} = 0$, then $V_{T+1} = V_T - 1$. Therefore, it is possible to write the expected value of the ratio of $V_{T+1}$ to $T + 1$ as follows:

$$E\left[\frac{V_{T+1}}{T+1}\right] = \left[\frac{V_T}{T}\right]\left[\frac{V_T+2}{T+1}\right] + \left[\frac{T-V_T}{T}\right]\left[\frac{V_T-1}{T+1}\right]. \tag{1}$$

This expression can be shown to reduce to the following:

$$E\left[\frac{V_{T+1}}{T+1}\right] = \frac{V_T}{T} + \frac{2V_T - T}{T(T+1)}. \tag{2}$$

The result follows from this expression. The ration $\frac{V_T}{T}$ is either monotonically increasing or decreasing in $T$ depending upon whether $2V_T$ is greater than or less than $T$. Not only does the accumulation process converge to all outcomes being of the same type, it converges relatively quickly. The rate of convergence depends on the length of the initial process in which balls are drawn randomly.

## 3 A General Model with Binary Outcomes

We now construct a general model that allows for both revised-path and traditional path dependent processes. As with the description of the Polya process in the Introduction, to describe these processes, we use standard urn models where different colors of balls represent different types of outcomes and the proportion of balls of each color correspond to the probability of an outcome of that type occurring. Our general class relies on three parameters, $p$, $q$, and $S$.

The parameter $p$ allows us to introduce randomness. It denotes the probability that a new ball is selected from the urn, and so $(1 - p)$ denotes the probability that the new ball is chosen randomly, that is, from an equally weighted distribution. (We relax this equal probability assumption later.) If the ball was chosen from the urn, it is replaced and another ball of the same color is added. If the ball was randomly chosen, it is simply added to the urn. Thus, in either case, whether chosen randomly or from the urn, the urn now contains one more ball, but its color depends on whether the ball was drawn from the urn or randomly.

The second parameter $q$ denotes the probability that a ball of the opposite color within the urn (provided such a ball exists) is switched to match the color of the ball chosen. We refer to this as the *probability of path revision*. For higher values of $q$, the current outcome is more likely to change the state of a past outcome. Finally, we introduce a parameter $S$ to denote the number of initial periods in which all outcomes are determined by a Bernoulli process. In the proofs that follow, we consider cases where $S = 1$, which implies only a single random draw prior to path dependence coming into play. Note that $S$ must be at least one. The first outcome cannot depend on the past, as there is no past on which it can depend. Larger $S$ would imply that the forces that produce path dependence are not in operation for some initial number of periods. Together, $p$, $q$, and $S$ create what we call a *revised path–dependent processes*.

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*Definition 2.* In a *revised path–dependent process* $F(p, q, S)$, the outcome in period $t$ is characterized as follows:

**Step 1:** Choose $x_{t,t}$ according to the following rules:

(i) if $t \leqslant S$, then $x_{t,t} \in \{0, 1\}$ with equal probability.

(ii) if $t > S$, with probability $1 - p$, $x_{t,t} \in \{0, 1\}$ with equal probability, and with probability $(1 - p)$ $x_{t,t} = 1$ with probability $\frac{\sum_{i=1}^{t-1} x_{i,t-1}}{t-1}$ and 0 otherwise.

**Step 2:** Let $D = \{j : x_{j,t-1} \neq x_{t,t}\}$. Set $x_{i,t}$ for $i < t$ according to the following rules:

(i) Set $x_{i,t} = x_{i,t-1}$ for all $i < t$.

(ii) if $t > S$, then with probability $q$, choose $k \in D$, and set $x_{k,t} = x_{t,t}$.

Given this model setup, if $p = 0$ and $q = 0$, we have a Bernoulli process. Balls are chosen randomly and balls in the urn are not changed. If $p = 1$ and $q = 0$, we have the standard Polya process that we described in the Introduction. If $p = 1$ and $q = 1$, we have the accumulation process that we defined in the previous section.

Given this framework, we can prove several results. We first prove a result for the special case $p = 1$, where the current outcome depends entirely on the proportions of the two colors in the urn.

*Claim* 2. In any equilibrium for a process in $F(1, q, S)$ where $q \in (0, 1]$, all outcomes will be of the same type.

**Proof:** Using the notation from the previous claim:

$$E_{F(1,q,S)}\left[\frac{V_{T+1}}{T+1}\right] = \left[\frac{V_T}{T}\right]\left[\frac{V_T + 2q + (1-q)}{T+1}\right] + \left[\frac{T - V_T}{T}\right]\left[\frac{V_T - q}{T+1}\right]. \tag{3}$$

Rearranging terms gives

$$E_{F(1,q,S)}\left[\frac{V_{T+1}}{T+1}\right] = \frac{V_T}{T} + q\frac{2V_T - T}{T(T+1)}. \tag{4}$$

As in the previous claim, the result follows immediately. Unless $2V_T = T$, the ration $\frac{V_T}{T}$ will be either monotonically increasing or decreasing.

The previous result states that if there exists any path revision, then all outcomes converge to the same type if the process does not have any random outcomes. The intuition for this result is as follows. Note that this subclass of processes combines two dynamics, the accumulation process and the Polya process. The former tugs the distribution to the end points. The latter keeps the distribution at its current proportions but only weakly so. The equilibria in Polya are only weakly stable. If you disturb the status quo proportion, the Polya process produces no force to return to the status quo. Metaphorically, you can think of an equilibrium of the Polya process like a ball resting on a flat surface, say a table. If you move the ball a little to the left or right, it will remain in its new location. The accumulation process, in effect, pulls down on both ends of the flat table. In this way, the drift to the extremes created by the accumulation process becomes the dominant dynamic.

This result has implications for empirical claims of path dependence. If one models a dynamic as a Polya process, one would expect a priori that any distribution of types is possible. The previous claim shows that if even a small amount of path revision exists, then that result goes away; equilibria are significantly constrained. We discuss this implication at greater length in the next section.

We next consider the case where $p$ is less than 1. This allows for randomness and resurrects the conventional Polya possibility that anything can happen. However, as the claim shows, such processes are knife-edge in the class of all processes.

*Claim* 3. The equilibrium for a process in $F(p, q, S)$ depends on the value of $p + 2pq$ as follows:

(i) if $p + 2pq > 1$, in equilibrium with probability 1, all $x_{j,t}$ are of the same type.

(ii) if $p + 2pq < 1$, in equilibrium each outcome exists in equal proportions.

(iii) if $p + 2pq = 1$, any proportion of each outcome can be an equilibrium.

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**Proof:** With probability $p$, the outcome depends on the $V_T$, and with probability $(1-p)$, the outcome is equally likely to be a 0 or a 1. Therefore, it follows that

$$E_{F(p,q,S)}\left[\frac{V_{T+1}}{T+1}\right] = pE_{F(1,q,S)}\left[\frac{V_{T+1}}{T+1}\right] + (1-p)E_{F(0,q,S)}\left[\frac{V_{T+1}}{T+1}\right]. \tag{5}$$

Given the definition of the process,

$$E_{F(0,q,S)}\left[\frac{V_{T+1}}{T+1}\right] = \frac{1}{2}\left[\frac{V_T + 2q + (1-q)}{T+1}\right] + \frac{1}{2}\left[\frac{V_T - q}{T+1}\right]. \tag{6}$$

Canceling terms gives

$$E_{F(0,q,S)}\left[\frac{V_{T+1}}{T+1}\right] = \frac{V_T + \frac{1}{2}}{T+1}. \tag{7}$$

This can be rewritten as follows:

$$E_{F(0,q,S)}\left[\frac{V_{T+1}}{T+1}\right] = \frac{V_T}{T} + \frac{1}{2}\frac{T-2V_T}{T(T+1)}. \tag{8}$$

Plugging in the value for $E_{F(1,q,S)}\left[\frac{V_{T+1}}{T+1}\right]$ from the previous claim gives

$$E_{F(p,q,S)}\left[\frac{V_{T+1}}{T+1}\right] = \frac{V_T}{T} + pq\frac{2V_T - T}{T(T+1)} + (1-p)\frac{1}{2}\frac{T-2V_T}{T(T+1)}. \tag{9}$$

It follows that

$$E_{F(p,q,S)}\left[\frac{V_{T+1}}{T+1}\right] = \frac{V_T}{T} \tag{10}$$

if and only if $2pq = 1-p$, which can be rewritten as $p + 2pq = 1$. To complete the proof, it suffices to recognize that if $2pq + p > 1$, then the process amplifies whichever outcome is in greater proportion and if $2pq + p < 1$, the process converges to each outcome being equally likely.

The result would appear to undermine the conventional logic in the path dependence literature that anything can happen. Note that the Polya process, the Bernoulli process, and the accumulation process are all special cases of this general result. One way to read the claim is that if historical processes include a little bit of randomness, dependence, or both, then we should not expect any equilibrium to be possible. We should expect extreme outcomes or equally likely outcomes. The model implies that to get extreme path dependence, the process, be it the Polya process or any other member of the family of processes in which $p + 2pq = 1$, must achieve a delicate balance between heading to the middle or to the extremes. These cases are knife-edge.

One question that might be asked is whether path revision could preclude the "anything can happen" result. Surprisingly, the answer here is "no." Any amount of path revision $q$ can be offset by choosing a $p$, the extent of path dependence of outcomes, to satisfy the constraint $p + 2pq = 1$. In the most extreme case, where $q = 1$, $p$ must equal $\frac{1}{3}$ in order to recover the possibility of any distribution. To see how this process can stabilize at any proportion of outcomes, suppose that 80% of outcomes were of type 1. Two-thirds of the time the outcome would be random. This results in an expected addition of 0.5 to the value of $V_T$. One-third of the time, the outcome would depend on the proportion $\frac{V_T}{T}$ which equals 0.8. Thus, with probability 0.8, $V_T$ would increase by 2, and with probability 0.2, it would decrease by 1. Combining all these expected contributions yields

$$V_{T+1} = V_T + \frac{2}{3}[0.5] + \frac{1}{3}[0.8(2) + 0.2(-1)] = V_T + 0.8. \tag{11}$$

So, the expected proportion remains unchanged.

One counterintuitive corollary of this claim is that all the outcomes can converge to one type even if most of the outcomes are chosen by an equal probability from the Bernoulli process.

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*Corollary* 1. For any $p > \frac{1}{3}$, there exists a probability of path revision sufficiently high such that all outcomes are eventually of the same type.

This corollary implies that path revision can drive the equilibrium to one in which all outcomes become the same type even though when those outcomes first occur they tend to be random. It is worth working through an example to see the logic. Suppose that $p = \frac{1}{2}$, so that half of all draws are Bernoulli with equal likelihood and that $q = 1$, so that in each period, a previous outcome gets revised. Thus, each period half of the time the draw is a random draw. What we will show is that over time, all the outcomes will get revised to be of the same type.

To see how this can happen, first note that an equal distribution is not an equilibrium because whichever outcome gets drawn will become more likely in the next period due to revision. So, let us suppose that we have an equilibrium in which some proportion $r \in (\frac{1}{2}, 1)$ of the balls are of one type or color. For the purposes of this example, let red be the more likely color. In the next period, with probability one-half, we draw a ball from the urn. This means that we will add a red ball with probability $r$. And, with probability one-half, we draw a random ball, which will be red with probability one-half. Combining these probabilities, we get that the expected probability of drawing a red ball equals $\frac{2r+1}{4}$. It follows that the probability of drawing a ball of the other color equals $\frac{3-2r}{4}$.

Recall that revision occurs with probability 1. So, after we draw a ball of either color, we add a ball of that color and also change the color of an existing ball of the opposite color. Therefore, there exists one more ball in the urn than there was previously. Overall then, with probability $\frac{2r+1}{4}$, we have two more red balls and with probability $\frac{3-2r}{4}$, we have one less red ball. Therefore, the expected increase in the number of red balls equals $\frac{(4r+2)}{4} - \frac{(3-2r)}{4}$ which equals to $\frac{(6r-1)}{4}$. Thus, the ratio of the red balls will increase provided $\frac{(6r-1)}{4}$ exceeds $r$. This will be satisfied provided that $2r > 1$, which holds given our assumption that the urn contained mostly red balls. Therefore, even though half of the outcomes are random, those outcomes that are not red initially, eventually are revised to become red.

A second corollary relates the amount of path revision needed to produce homogeneity as a function of the amount of path dependence.

*Corollary* 2. The probability of path revision $q$ required to produce all outcomes of the same type in the long run decreases in $p$, the amount of path dependence.

This corollary states that the more path dependence in a process, that is, the less randomness, the less path revision that needs to occur in order to produce a single type of outcome in the long run. Thus, if a process appears highly path dependent, then it only takes a little bit of revision to drive the system to homogeneity, but if the process only has a little bit of path dependence, the probability of path revision must be rather large for the process to result in a homogeneous distribution.

So far, we have assumed Bernoulli processes in which the probability of the two outcomes are equally likely. One might also consider cases in which one of the outcomes has an advantage. The next claim states a somewhat surprising result. When one of the outcomes has an advantage, this outcome is only guaranteed to be the equilibrium in the case where $p + 2pq = 1$, the knife-edge case. If path revision and dependence are stronger than that, the other outcome could come to dominate. If the forces are weaker, then the equilibrium is internal with the advantaged outcome more prominent. Note that none of the cases support the conventional path dependence claim that any equilibrium distribution is possible.

*Claim* 4. Let $b > \frac{1}{2}$ equal the probability of getting an outcome of one from the Bernoulli process.

(i) if $p + 2pq = 1$, in equilibrium all $x_{jt}$s are of type 1.
(ii) if $p + 2pq > 1$, in equilibrium with probability 1, all $x_{jt}$s are of the same type.
(iii) if $p + 2pq < 1$, the proportion of outcomes of type 1 equals $b + \frac{q(1-2b)}{(2q+p-1)}$.

Recall from Claim 3 that the expected proportion of type 1 outcomes in time $T + 1$ can be written as follows:

$$E_{F(p,q,S)}\left[\frac{V_{T+1}}{T+1}\right] = pE_{F(1,q,S)}\left[\frac{V_{T+1}}{T+1}\right] + (1-p)E_{F(0,q,S)}\left[\frac{V_{T+1}}{T+1}\right]. \quad (12)$$

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With a biased Bernoulli process, $E_{F(1,q,S)}$ does not change. Thus, we need only consider the second term, $E_{F(0,q,S)}$. Define the Bernoulli process such that the probability of choosing outcome 1 is $b$, $\Pr\{x_{t,t} = 1\} = b$. We can now solve for the second term as follows:

$$E_{F(0,q,S)}\left[\frac{V_{T+1}}{T+1}\right] = bq\left[\frac{V_T+2}{T+1}\right] + b(1-q)\left[\frac{V_T+1}{T+1}\right] + (1-b)q\left[\frac{V_T-1}{T+1}\right] + (1-b)(1-q)\left[\frac{V_T}{T+1}\right]. \tag{13}$$

Simplifying this expression, we obtain the following:

$$E_{F(0,q,S)}\left[\frac{V_{T+1}}{T+1}\right] = b\left[\frac{V_T+2q+(1-q)}{T+1}\right] + (1-b)\left[\frac{V_T-q}{T+1}\right]. \tag{14}$$

Plugging this back into the general expectation expression, $E_{F(p,q,S)}$,

$$E_{F(p,q,S)}\left[\frac{V_{T+1}}{T+1}\right] = \frac{V_T}{T} + pq\frac{2V_T-T}{T(T+1)} + (1-p)\frac{bT+qT(2b-1)-V_T}{T(T+1)}. \tag{15}$$

To simplify notation, let $\alpha = (2pq+p-1)$. Substituting the above equilibrium condition back into the equation for $E_{F(p,q,S)}$ and re-arranging, we have,

$$E_{F(p,q,S)}\left[\frac{V_{T+1}}{T+1}\right] = \frac{V_T}{T} + \left[\frac{1}{T(T+1)}\right]\left[\alpha T\left(\frac{V_T}{T}-b\right) + qT(2b-1)\right]. \tag{16}$$

We now consider the three cases:

*Case* (i): $p+2pq=1$: In the expression above, $\alpha = 0$. Rewriting the previous equation gives

$$E_{F(p,q,S)}\left[\frac{V_{T+1}}{T+1}\right] = \frac{V_T}{T} + \left[\frac{1}{T(T+1)}\right][qT(2b-1)]. \tag{17}$$

Given that $(2b-1) > 0$, it follows that in equilibrium, all $x_{j,t}$s are of type 1.

*Case* (ii): $p+2pq > 1$: In equilibrium, $E_{F(p,q,S)}\left[\frac{V_{T+1}}{T+1}\right] = \frac{V_T}{T} = R^*$. Therefore, $\alpha(R^*-b)+q(2b-1)=0$, which can be rewritten as $R^* = b - \frac{q(2b-1)}{\alpha}$.

Given $\alpha > 0$, a straightforward argument shows that $R^*$ is unstable. To see this, let $\frac{V_T}{T} = R^* + \epsilon$. The expression

$$\alpha\left(\frac{V_T}{T}-b\right) + q(2b-1) \tag{18}$$

can then be written as follows:

$$\alpha(R^*+\epsilon-b)+q(2b-1) = \alpha\epsilon > 0. \tag{19}$$

Therefore, a perturbation above $R^*$ leads to an increase in the proportion of type 1 outcomes. Similarly, a perturbation below $R^*$ leads to a decrease. The result follows.

*Case* (iii): $p+2pq < 1$: From the previous case, we have that the equilibrium proportion of outcomes of type 1 equals $b + \frac{q(1-2b)}{(2q+p-1)}$. Using the same argument as above, it can be shown that this equilibrium is stable, completing the proof.

To summarize, if one outcome has an advantage, then in the knife-edge case, it takes over with probability 1. So here, we do not get the possibility that anything can happen. If the forces of path revision and path dependence are strong, then the advantaged outcome is more likely to be the equilibrium, but the other outcome could become the equilibrium if it got an early advantage and then leveraged the revision process. If the forces are weak, then there is an internal mixed equilibrium in which the advantaged outcome occurs with greater frequency.

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## 4 Discussion: The Empirical Content of Path Dependence

In this article, we have asked what would happen if we emend the standard Polya process to include path revision. We find that except in knife-edge cases, equilibria consist of equal numbers of each type or of homogeneous distributions. Our results stands in sharp contrast with conventional interpretations of path-dependent equilibria, where any distribution is equally likely. The results highlight the importance of distinguishing between the concept of path dependence and the idea that anything can happen. A process can be path dependent, but it might not produce any possible outcome. In fact, for the class of processes that we consider, if the forces of path revision and path dependence are strong (formally, if $p + 2pq$ exceeds 1), the outcomes are indeed path dependent, but they can only be one of two extremes. One could then argue that the process is even more path dependent than the Polya process, only two things can happen, and both are extreme. Paradoxically, the process is also more predictable, at least measured by entropy.

We base our conclusion that anything can happen is not likely on the fact that $p + 2pq = 1$ is a knife-edge case; it is of lower dimension than the space of possible distributions. If $p$ and $q$ are exogenously drawn from continuous, nonatomic distributions, lying on the line would be a zero probability event. One might challenge our inference by an argument that processes adjust so as to lie on that line, thus making $p$ and $q$ endogenous. For processes to converge to the line, when $p + 2pq > 1$, actors would have to introduce some randomness (lowering $p$) or revise paths less often. And, when the opposite inequality holds, they would have to decrease randomness and revise outcomes more often. Other than having a goal of creating the possibility of anything happening, we see no reason why either of these forces, let alone both, would emerge in historical processes. Furthermore, if we relax the assumption of equally likely random outcomes, that is, if one of the outcomes has a natural advantage, then the final result in the article states that the anything can happen result goes away entirely. In other words, the idea that anything can happen requires the random outcomes to be equally likely and that the parameters satisfy the linear constraint. Thus, we conclude that perhaps anything may not be as likely to happen as previously thought.

In thinking about the dynamics of the process, the magnitude of $p+2pq-1$ is of central importance. If this value is near 0, or near its maximum of 3, the resulting convergence times will be very fast. If, on the other hand, it takes on a value close to 1, convergence could take a long time. Also, even though we would know that the equilibrium is homogenous, the equilibrium might not be a good predictor for relevant time periods for political processes of interest. Alternatively, a process that is being driven to equal likelihood and is therefore not path dependent could appear to be for a long time. This finding has implications for empiricists who might then over state path dependence.

Of greater import to empiricists attempting to identify path dependence in qualitative or quantitative data (e.g., Bennett and Elman 2006; Jackson and Kollman 2010; Vergne and Durand 2010), our findings suggest the need to test for path revisions in the data. Suppose, for example, that an empirical model assumed no path revision and estimated $p$ to be 0.95. The prediction of such a model would be equal numbers of each type. But suppose that in the process generating the data that $q = 0.07$, so that 7% of the time, a ball in the urn would switch its color to match the current selection. Note that $p + 2pq$ exceeds 1 in this case, so that the process would have an equilibrium of all balls having the same color. Therefore, the statistical finding of $p$ less than 1 would not be in alignment with the empirical finding.

The model that we construct in this article is highly stylized. Yet, we believe that it raises issues much larger than multiple-colored balls being drawn from urns. Beliefs, behaviors, laws, and institutions all exhibit some degree of path dependence. To a lesser extent, they also may exhibit path revision. In the case of ongoing behaviors such as those shown in the experimental data, revisions may occur frequently. In law, where precedent has a long tradition, revisions may be less likely, however, as our results show, a little revision is all that is needed. For the sort of structural path dependence considered by Pierson (2004), revision entails changing past actions, so this too may be rare. However, as the model suggests, even the smallest amount of revision can lead to extreme outcomes.

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