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Continuum-wise expansivity and entropy for flows

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Abstract. We define the concept of continuum-wise expansivity for flows, and we prove that continuum-wise expansive flows on compact metric spaces with topological dimension greater than one have positive entropy.

1. *Introduction*

The notion of expansivity was introduced in the middle of the twentieth century by Utz [**Ut**]. Expansivity is a property shared by a large class of dynamical systems exhibiting chaotic behavior. Roughly speaking, a system is expansive if two orbits cannot remain close to each other under the action of the system. This notion is responsible for many chaotic properties for homeomorphisms defined on compact spaces (see, for instance, [**Hi**, **Hi2**, **Le1**, **Le2**, **Ft**, **Vi**] and the references therein) and there is an extensive literature concerning expansive systems. A classical result establishes that every hyperbolic f-invariant subset M is expansive. There are many variants of the definition of expansivity, all of them of interest, such as *positive expansivity* in [**Sc**], *pointwise expansivity* in [**Re**] and *entropy-expansivity* in [**B3**]. In [**Ma**], Mañé proved that if a compact metric space admits an expansive homeomorphism, then its topological dimension is finite. In the 1990s, Kato introduced the notion of *continuum-wise expansivity* for homeomorphisms [**Ka**], and extended the result of Mañé for continuum-wise (cw-) expansive homeomorphisms.

For flows, a seminal work is [**BW**], where this concept is analyzed for flows and it is proved that some properties valid for discrete dynamics are also valid for flows. But the definition of expansivity in [**BW**] does not admit flows with singularities or equilibrium points. Using this definition, Keynes and Sears [**KS**] extended the results of Mañé for expansive flows. They proved that if a compact metric space admits an expansive flow,

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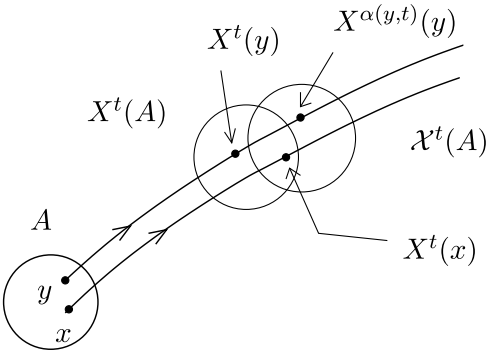


FIGURE 1. The images $X^t(A)$ and $\mathcal{X}^t_\alpha(A)$.

then its topological dimension is finite. They also proved that expansive flows on manifolds with topological dimension greater than one have positive entropy.

The goal of this work is to give a definition of *cw-expansivity* for flows and extend some of the results already established for discrete dynamics for cw-expansive flows. Our main result establishes that cw-expansive continuous flows on compact metric spaces with topological dimension greater than one have positive entropy, which extends the result of Keynes and Sears to continuous dynamics.

Throughout this paper, M denotes a compact metric space. To announce our result precisely, let us recall some concepts and definitions already established.

A *flow* in M is a family of homemorphisms $\{X^t\}_{t\in\mathbb{R}}$ satisfying $X^0(x) = x$, for all $x \in M$ and $X^{t+s}(x) = X^t(X^s(x))$ for all s, $t \in \mathbb{R}$ and $x \in M$. A *continuum* is a compact connected set and it is *non-degenerate* if it contains more than one point. We denote by $\mathcal{C}(M)$ the set of all continuum subsets of M.

Let $\mathcal{H}\text{om}(\mathbb{R}, 0)$ be the set of homeomorphisms on $\mathbb{R}$ fixing the origin and if A is a subset of M, $C^0(A, \mathbb{R})$ denotes the set of real continuous maps defined on A. Define:

- $\mathcal{H}(A) = \{\alpha : A \to \mathcal{H}\text{om}(\mathbb{R}, 0);\ \exists\, x_\alpha \in A \text{ with } \alpha(x_\alpha) = \mathrm{id}_{\mathbb{R}} \text{ and } \alpha(\cdot)(t) \in C^0(A, \mathbb{R}),\ \forall\, t \in \mathbb{R}\}$; and
- if $t \in \mathbb{R}$ and $\alpha \in \mathcal{H}(A)$, $\mathcal{X}^t_\alpha(A) = \{X^{\alpha(x)(t)}(x);\ x \in A\}$ (see Figure 1).

Definition 1.1. A flow X^t is *cw-expansive* if, for any $\varepsilon > 0$, there is a $\delta > 0$ such that if $A \subset M$ is a continuum and $\alpha \in \mathcal{H}(A)$ satisfies

$$\operatorname{diam}(\mathcal{X}^t_\alpha(A)) < \delta \quad \text{for all } t \in \mathbb{R},$$

then $A \subset X^{(-\varepsilon,\varepsilon)}(x_\alpha)$.

Remark 1.2. Note that in the above definition, as $\alpha(x_\alpha) = \mathrm{id}_{\mathbb{R}}$, the orbit $\mathcal{O}(x_\alpha)$ guides the shearing effect caused by displacement of A by the set of maps in $\mathcal{H}(A)$.

Our main result is the following theorem.

THEOREM A. *If X^t is a cw-expansive flow and the topological dimension of M is greater than one, then the topological entropy of X^t is positive.*

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This paper is organized as follows. In §2, we prove basic properties satisfied by cw-expansive flows; in §3 we use Keynes and Sears's techniques to find good properties in cross-sections of cw-expansive flows; in §4 we introduce the notion of entropy for special pairs of cross-sections and we relate the entropy of this cross-sections with the entropy of the flow; and in §5 we prove our main theorem.

2. *Basic properties of cw-expansive flows*

Let M be a compact metric space and let X^t be a cw-expansive flow in M. In this section, we prove basic results satisfied by X^t. Given $x \in M$, let $\mathcal{O}(x) = \{X^t(x),\ t \in \mathbb{R}\}$ be the orbit of x, where x is a *fixed point* of X^t if $\mathcal{O}(x) = \{x\}$. Recall that $\mathcal{O}(x)$ is regular if it is not a unique point.

LEMMA 2.1. *IF X^t is a cw-expansive flow, then each fixed point of X^t is not accumulated by regular orbits of X^t.*

Proof. The proof goes by contradiction. Let $X^t(p) = p$ for all $t \in \mathbb{R}$, and suppose that p is accumulated by regular orbits. Let $\varepsilon = 1$ and let $\delta > 0$ be the corresponding number satisfying the cw-expansive definition. Define

$$h(t) = \begin{cases} t+1 & \text{if } t \notin (-2, 1), \\ 2t & \text{if } t \in [0, 1), \\ \dfrac{t}{2} & \text{if } t \in (-2, 0). \end{cases}$$

By continuity of the flow with respect to initial data, there is $x \in M - \{p\}$ such that if $|t| < 3$, then $d(X^t(x), p) < \delta/2$. If $A = X^{[0,1]}(x)$, then A is a continuum. For each $t \in [0, 1]$ define $h_t : \mathbb{R} \to \mathbb{R}$ by

$$h_t = (1 - t)\,\mathrm{Id} + th.$$

Now define $\alpha : A \to \mathcal{H}\mathrm{om}(\mathbb{R}, 0)$ by $\alpha(X^t(x)) = h_t$. Then $\alpha \in \mathcal{H}(A)$, $X^s_\alpha(A) \subset B_{\delta/2}(x)$ if $s \in (-2, 1)$ and this set reduces to a unique point if $s \notin (-2, 1)$. Therefore, $\mathrm{diam}(\mathcal{X}^s_\alpha(A)) < \delta$ for all $s \in \mathbb{R}$, which contradicts that X^t is cw-expansive. □

Let $\Sigma_{\mathbb{R}}(0) = \{\{x_i\}_{i\in\mathbb{Z}};\ x_i \in \mathbb{R} \text{ and } x_0 = 0\}$. If $A \subset M$, define

$$\mathcal{SQ}(A) = \{\beta : A \to \Sigma_{\mathbb{R}}(0);\ \exists\, x_\beta \text{ such that } \beta(x_\beta)_i \to \infty \text{ when } i \to \infty \text{ and } \beta(x_\beta)_i \to -\infty \text{ when } i \to -\infty\}.$$

Let $\mathcal{SQ}^*(A) \subset \mathcal{SQ}(A)$ so that if $\beta \in \mathcal{SQ}^*(A)$, then, for each $i \in \mathbb{Z}$, the map

$$f_i : A \to \mathbb{R}, \quad f_i(a) = \beta(a)_i \quad \text{is continuous.}$$

If $\beta \in \mathcal{SQ}^*(A)$ and $i \in \mathbb{Z}$, define $\mathcal{X}^i_\beta(A) = \{X^{\beta(x)(i)}(x);\ x \in A\}$.

THEOREM 2.2. *Let X^t be a flow without fixed points. The following properties are equivalent.*

(1) *X^t is cw-expansive.*

(2) *For all $\eta > 0$, there is $\delta > 0$ so that if A is a continuum and there is $\alpha \in \mathcal{H}(A)$ with $\mathrm{diam}(\mathcal{X}^t_\alpha(A)) < \delta$ for all $t \in \mathbb{R}$, then A is an orbit segment inside $B_\eta(x_\alpha)$.*

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(3) *For all* $\varepsilon > 0$, *there is* $\delta > 0$ *such that if* A *is a continuum and there exists* $\beta \in SQ^*(A)$, *with* $\beta(x_\beta)_{i+1} - \beta(x_\beta)_i \le \eta$ *and* $\sup_{a\in A} |\beta(a)_{i+1} - \beta(a)_i| \le \delta$ *for all* $i \in \mathbb{Z}$, *such that* $\operatorname{diam}(\mathcal{X}^i_\beta(A)) < \delta$ *for all* $i \in \mathbb{Z}$, *then* $A \subset X^{(-\varepsilon,\varepsilon)}(x_\beta)$.

Proof. (1) $\Rightarrow$ (3) : Let $\varepsilon > 0$ be given and let $\delta > 0$ by the property of cw-expansivity. Choose $\eta > 0$ such that

$$\eta + \Big(2 \sup_{z\in M, |u|<\eta} d(z, X^u(z))\Big) < \frac{\delta}{2}.$$

Suppose that A is a continuum and that for some $\beta \in \mathcal{SQ}^*(A)$, with $\beta(x_\beta)_{i+1} - \beta(x_\beta)_i \le \eta$ and $\sup_{a\in A} |\beta(a)_{i+1} - \beta(a)_i| \le \delta$ for all $i \in \mathbb{Z}$, we have $\operatorname{diam}(\mathcal{X}^i_\beta(A)) < \eta$, for all $i \in \mathbb{Z}$. For each $a \in A$, we define a homeomorphism $\alpha(a)$ by $\alpha(a)(\beta(x_\beta)_i) = \beta(a)_i$, for all $i \in \mathbb{Z}$ and by linearity on each interval $(\beta(x_\beta)_i, \beta(x_\beta)_{i+1})$. Then $\alpha \in \mathcal{H}(A)$.

Therefore, if $t_i = \beta(x_\beta)_i$, for each $t \in [t_i, t_{i+1})$,

$$\begin{aligned}
\operatorname{diam}(\mathcal{X}^t_\beta(A)) &= \sup_{a,b\in A} d(X^{\beta(a)(t)}(a), X^{\beta(b)(t)}(b)) \\
&\le \sup_{a,b\in A} (d(X^{\beta(a)(t)}(a), X^t(x_\beta)) + d(X^t(x_\beta), X^{\beta(b)(t)}(b))) \\
&\le \sup_{a\in A}(d(X^{\beta(a)(t)}(a), X^t(x_\beta)) + \sup_{b\in A} d(X^t(x_\beta), X^{\beta(b)(t)}(b))) \\
&= 2 \sup_{a\in A} d(X^t(x_\beta), X^{\beta(a)(t)}(a)).
\end{aligned}$$

But, for each $a \in A$,

$$\begin{aligned}
d(X^t(x_\beta), X^{\beta(a)(t)}(a)) &\le d(X^t(x_\beta), X^{t_i}(x_\beta)) + d(X^{t_i}(x_\beta), X^{\beta(a)_i}(a)) \\
&\quad + d(X^{\beta(a)_i}(a), X^{\beta(a)(t)}(a)) \\
&\le \sup_{z\in M, |u|\le\eta} d(z, X^u(z)) + \eta + \sup_{z\in M, |u|\le\eta} (z, X^u(z)) \\
&< \frac{\delta}{2}.
\end{aligned}$$

Therefore

$$\operatorname{diam}(\mathcal{X}^t_\beta(A)) < 2\frac{\delta}{2} = \delta \quad \text{for all } t \in \mathbb{R},$$

and, since X^t is cw-expansive, we obtain $A \subset X^{(-\varepsilon,\varepsilon)}(x_\beta)$.

(3) $\Rightarrow$ (1): Let $\varepsilon > 0$ be given and let $\delta > 0$, as in item (3). Suppose that $A \subset M$ is a continuum such that, for some $\alpha \in \mathcal{H}(A)$, $\operatorname{diam}(\mathcal{X}^t_\alpha(A)) < \delta$ for all $t \in \mathbb{R}$. By induction, define $\beta \in \mathcal{SQ}^*(A)$ by $\beta(x)_0 = 0$ for all $x \in A$. Since A is compact, there is $t_1 > 0$ such that $\alpha(x)(t_1) < \delta$ for all $x \in A$. Define $\beta(x)_1 = \alpha(x)(t_1)$. There is $t_{i+1} > t_i$ such that $|\alpha(x)(t_{i+1}) - \alpha(x)(t_i)| < \delta$ for all $x \in A$. Then define $\beta(x)_{i+1} = \alpha(x)(t_{i+1})$. We can define $\beta(x)_i$ for $i < 0$, similarly. Therefore $\beta \in \mathcal{H}(A)$, with $x_\beta = x_\alpha$. By item (3), $A \subset X^{(-\varepsilon,\varepsilon)}(x_\alpha)$.

(1) $\Rightarrow$ (2): Since M is compact, for all $\eta > 0$ there is $\varepsilon > 0$ such that $X^{(-\varepsilon,\varepsilon)}(x) \subset B_\eta(x)$ for all $x \in M$.

(2) $\Rightarrow$ (1): Since X^t has no fixed points, for all $\varepsilon > 0$ there is $\eta > 0$ so that $\operatorname{diam}(X^{(-\varepsilon,\varepsilon)}(x)) > \eta$ for all $x \in M$. □

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Recall that if M and N are metric spaces, then the flows $X^t : M \to M$ and $Y^t : N \to N$ are *conjugate* if there is a homeomorphism $h : M \to N$ mapping orbits of X^t onto orbits of Y^t.

THEOREM 2.3. *cw-expansivity is a conjugacy invariant.*

Proof. Suppose that X^t and Y^t are conjugate with Y^t cw-expansivity. Let $h : M \to N$ be a homeomorphism conjugating X^t to Y^t. Let $\varepsilon_M > 0$ be given and let $\varepsilon_N > 0$ be such that if $x, y \in N$ satisfies $d(x, y) < \varepsilon_N$, then $d(h^{-1}(x), h^{-1}(y)) < \varepsilon_M$. Let $\delta_N > 0$ be the corresponding number given by cw-expansivity of Y^t to ε_N and let $\delta_M > 0$ be such that if $x, y \in M$ with $d(x, y) < \delta_M$, then $d(h(x), h(y)) < \delta_N$. Therefore, if $A_M \subset M$ is a continuum and $\alpha_M \in \mathcal{H}(A_M)$ is such that $\operatorname{diam}(\mathcal{X}^t_{\alpha_M}(A_M)) < \delta_M$ for all $t \in \mathbb{R}$, then $A_N = h(A_M) \subset N$ is a continuum. For every $x \in A_M$ and $t \in \mathbb{R}$, let $\alpha_N(h(x))(t)$ be the real number such that if

$$h(X^{\alpha_M(x)(t)}(x)) = Y^{\alpha_N(h(x))(t)}(h(x)),$$

then $\alpha_N \in \mathcal{H}(A_N)$. Furthermore, for every $t \in \mathbb{R}$,

$$\begin{aligned}\operatorname{diam}(\mathcal{Y}^t_{\alpha_N}(A_N)) &= \max_{x,y\in A_N} d(Y^{\alpha_N(x)(t)}(x),\, Y^{\alpha_N(y)(t)}(y)) \\ &= \max_{x,y\in A_M} (d(h(X^{\alpha_M(x)(t)}(x)),\, h(X^{\alpha_M(y)(t)})(y))) \\ &= \operatorname{diam}(h(\mathcal{X}^t_{\alpha_M}(A_M))) < \delta_N,\end{aligned}$$

because $\operatorname{diam}(\mathcal{X}^t_{\alpha_M}(A_M)) < \delta_M$. Since Y^t is cw-expansive, A_N is an Y^t-orbit segment inside $B_{\varepsilon_N}(x_{\alpha_N})$. But $A_M = h^{-1}(A_N)$, $x_{\alpha_N} = h(x_{\alpha_M})$ and the choice of ε_N imply that A_M is a X^t-orbit segment inside $B_{\varepsilon_M}(x_{\alpha_M})$, which proves that X^t is cw-expansive. □

3. *Suspensions*

Let (M, d) be a compact metric space and let $f : M \to M$ be a homeomorphism. Let $k : M \to \mathbb{R}^+$ be a continuous function.

Definition 3.1. The *suspension* of f under k is the flow X^t on the space

$$M_k = \bigcup_{0\le t\le k(y)} (y, t)/(y, k(y))(k(y), 0),$$

which is defined for small non-negative time by $X^t(y, s) = (t + s)$, $0 \le t + s < k(y)$.

Each suspension of f is conjugate to the suspension of f under 1, the constant function with value 1. For this reason we shall concentrate on suspensions under the function 1.

Next, following [**BW**], we define a metric on M_1. Suppose that the diameter of M under d is less than 1.

Consider the subset $M \times \{t\}$ of $M \times [0, 1]$ and let d_t denote the metric defined by

$$d_t((y, t), (z, t)) = (1 - t)d(y, t) + td(f(y), f(z)), \quad y, z \in M.$$

Given $x_1, x_2 \in M_1$, consider all finite chains $x_1 = w_0, w_1, \ldots, w_n = x_2$ between x_1 and x_2, where, for each i, either w_i and w_{i+1} belong to $M \times t$ for some t (in which case we

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call $[w_i, w_{i+1}]$ a horizontal segment) or w_i and w_{i+1} are on the same orbit (and then we call $[w_i, w_{i+1}]$ a vertical segment).

Define the length of a chain as the sum of the lengths of its segments, where the length of a horizontal segment $[w_i, w_{i+1}]$ is measured in the metric d_t if w_i and w_{i+1} belongs to $M \times \{t\}$, and the length of a vertical segment $[w_i, w_{i+1}]$ is the shortest distance between w_i and w_{i+1} along the orbit (ignoring the direction of the orbit) using the usual metric on $\mathbb{R}$.

If $w_i \neq w_{i+1}$ and w_i and w_{i+1} are on the same orbit and on the same set $M \times \{t\}$, then the length of the segment $[w_i, w_{i+1}]$ is taken as $d_t(w_i, w_{i+1})$, since this is always less than 1.

Then define $d(x_1, x_2)$ to be the infimum of the lengths of all chains between x_1 and x_2. It is easy to see that d is a metric on M_1. This metric d gives the topology on M_1.

THEOREM 3.2. *Let $\phi : Y \to Y$ be a homeomorphism and let $f : M \to \mathbb{R}^+$ be a continuous map. The suspension of ϕ under f is cw-expansive if and only if ϕ is cw-expansive.*

Proof. We need only show the result when $f \equiv 1$. Suppose that X^t is cw-expansive. Let $\frac{1}{2} > \varepsilon > 0$ be given and let $\delta > 0$ be the corresponding constant determined by cw-expansivity. If $A \subset Y$ is a continuum with $\operatorname{diam}(\phi^n(A)) < \delta$ for all $n \in \mathbb{Z}$, denoting $A_M = A \times \{0\}$ and $x_1 = (x, 0)$, then, for all $t \in \mathbb{R}$,

$$\begin{aligned}\operatorname{diam}(X^t(A_M)) &= \max_{x_1, y_1 \in A_M} d(X^t(x_1), X^t(y_1)) \\ &\leq \max_{x,y \in A} \rho_{t-[t]}((\phi^{[t]}(x), t - [t]), (\phi^{[t]}(y), t - [t])) \\ &= \max_{x,y \in A} ((1 - t + [t])\rho(\phi^{[t]}(x), \phi^{[t]}(y)) \\ &\quad + (t - [t])\rho(\phi^{[t]+1}(x), \phi^{[t]+1}(y))),\end{aligned}$$

where $[t]$ is the greatest integer less than t. But since, for all $x, y \in A$,

$$\phi^{[t]}(x), \phi^{[t]}(y) \in \phi^{[t]}(A) \quad \text{and} \quad \phi^{[t]+1}(x), \phi^{[t]}(y) \in \phi^{[t]+1}(A),$$

we get

$$\operatorname{diam}(X^t(A_M)) \leq (1 - t + [t])\delta + (t - [t])\delta = \delta.$$

Since X^t is cw-expansive, there is $(x, 0)$ such that $A_M \subset X^{(-\varepsilon,\varepsilon)}(x)$. Moreover, since $0 < \varepsilon < \frac{1}{2}$ and $A_M \subset M \times \{0\}$, we obtain $A_M = \{(x, 0)\}$, and so $A = \{x\}$. Therefore ϕ is cw-expansive.

Next, suppose that the suspension ϕ is cw-expansive. Consider in M the metric given by

$$\rho'(x, y) = \min\{\rho(x, y), \rho(\phi(x), \phi(y))\}$$

and let $\delta > 0$ be the cw-expansivity constant to ρ'. Let $\varepsilon > 0$ and $\delta' = \min\{\delta, \varepsilon, \frac{1}{4}\}$. Let $A \subset M_f$ be a continuum and let $\alpha \in \mathcal{H}(A)$ so that $\operatorname{diam} \mathcal{X}_\alpha^t(A) < \delta'$ for all $t \in \mathbb{R}$. We consider two cases: (1) x_α can be represented as $(y_1, \frac{1}{2})$ and (2) x_α can be represented as $(\frac{1}{2}, y_1)$.

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In the first case, define $A_M = \{a \in M;\ (a, s) \in A \text{ com } s \in (0, 1)\}$. Then

$$\begin{aligned}\operatorname{diam}(A_M) &= \max_{y,z\in A_M} \rho'(y, z)\\ &\le \max_{(y,s),(z,r)\in A} d((y, s), (z, r))\\ &= \operatorname{diam}(A) < \delta' < \delta.\end{aligned}$$

By the definition of suspension, $X^1(x_\alpha)$ has representation $(\phi(y_1), \frac{1}{2})$, and since $\operatorname{diam} \mathcal{X}^1_\alpha(A) < \delta < \frac{1}{4}$, $X^{\alpha(y)(1)}(y)$ has representation $(\phi(y), s)$ with $s \in (0, 1)$ for all $y \in A$. So

$$\begin{aligned}\operatorname{diam}(\phi(A_M)) &= \max_{y,z\in A_M} \rho'(\phi(y), \phi(z))\\ &\le \max_{(\phi(y),s),(\phi(z),r)\in\mathcal{X}^1_\alpha(A)} d((\phi(y), s), (\phi(z), r))\\ &= \operatorname{diam}(\mathcal{X}^1_\alpha(A)) < \delta' < \delta.\end{aligned}$$

Similarly, $\operatorname{diam}(\phi^n(A_M)) < \delta$ for all $n \in \mathbb{Z}$. Since ϕ is cw-expansive, $A_M = \{y_1\}$, and so every point in A has the form (y_1, t) with $t \in (0, 1)$. Since $(y_1, \frac{1}{2}) = x_\alpha \in A$ and $\operatorname{diam}(A) < \delta' < \varepsilon$, we conclude that $|t - \frac{1}{2}| < \varepsilon$, i.e., $A \subset X^{-\varepsilon,\varepsilon}(x_\alpha)$, which finishes the proof in the first case.

When x_α does not have a representation as $(y_1, \frac{1}{2})$, then $X^r(x_\alpha)$ has representation as $(y_1, \frac{1}{2})$ for some $|r| < \frac{1}{2}$. Define $\widetilde{A} = \mathcal{X}^r_\alpha(A)$ and, for each $x \in A$ and $t \in \mathbb{R}$, set

$$\widetilde{\alpha}(X^r(x))(t) = \alpha(x)(t + r) - \alpha(x)(r).$$

We know that $\widetilde{A}$ is a continuum and $\widetilde{\alpha} \in \mathcal{H}(\widetilde{A})$. Moreover, for every $t \in \mathbb{R}$,

$$\begin{aligned}\operatorname{diam}(\mathcal{X}^t_{\widetilde{\alpha}}(\widetilde{A})) &= \max_{x,y\in\widetilde{A}} d(X^{\widetilde{\alpha}(x)(t)}(x), X^{\widetilde{\alpha}(y)(t)}(y))\\ &= \max_{x,y\in A} d(X^{\alpha(x)(t+r)-\alpha(x)(r)}(X^{\alpha(x)(r)}(x)), X^{\alpha(y)(t+r)-\alpha(y)(r)}(X^{\alpha(y)(r)}(y)))\\ &= \max_{x,y\in A} d(X^{\alpha(x)(t+r)}(x), X^{\alpha(y)(t+r)}(y))\\ &= \operatorname{diam}(\mathcal{X}^{t+r}_\alpha(A)) < \delta.\end{aligned}$$

By the first case,

$$\widetilde{A} \subset X^{(-\varepsilon,\varepsilon)}(x_{\widetilde{\alpha}}) = X^{(-\varepsilon,\varepsilon)}(X^r(x_\alpha)),$$

and therefore

$$A = X^{-r}(\widetilde{A}) \subset X^{(-\varepsilon,\varepsilon)}(x_\alpha).$$

All together, this completes the proof of Theorem 3.2. □

4. *Pairs of cross-sections families*

In this section, we consider a cw-expansive flow X^t on M and define stable and unstable sets for subsets $A \subset M$. The goal is to extend [**KS**, Theorem 2.7] to this context and establish that, locally, these sets defined for points of M intersect at a unique point. To do so, we use the notation introduced by Keynes and Sears in [**KS**] and start recalling the definition of the δ-adapted family of cross-sections.

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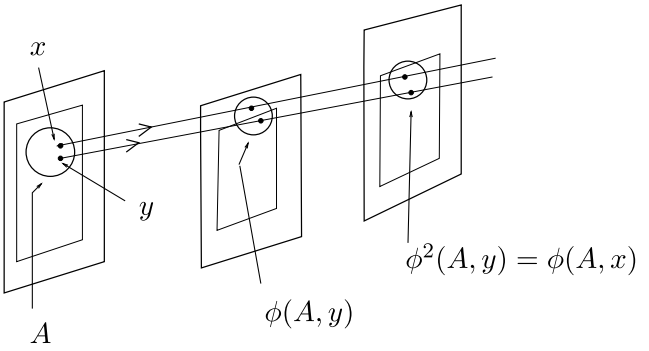


FIGURE 2. The first return map $\phi : \bigcup_{i=1}^{k} T_i \to : \bigcup_{i=1}^{k} T_i$.

A set $S \subset M$ is a *cross-section* of time $\varepsilon > 0$ if it is closed and if, for each $x \in S$, $S \cap X^{(-\varepsilon,\varepsilon)}(x) = \{x\}$. The *interior* of S is the set $S^* = \text{int}(X^{(-\varepsilon,\varepsilon)}(S)) \cap S$.

The proof of the next lemma can be found in [**KS**, Lemma 2.4] as well in [**BW**, Lemma 7].

LEMMA 4.1. *There is $\varepsilon > 0$ such that, for each $\delta > 0$, we can find a pair $(\mathcal{S}, \mathcal{T})$ of finite families $\mathcal{S} = \{S_1, \ldots, S_n\}$ and $\mathcal{T} = \{T_1, \ldots, T_n\}$ of local cross-sections of time $\varepsilon > 0$ and diameter at most δ with $T_i \subset S_i^*$ ($i \in \{1, \ldots, k\}$) such that*

$$M = \bigcup_{i=1}^{k} X^{[0,\varepsilon]}(T_i) = \bigcup_{i=1}^{k} X^{[-\varepsilon,0]}(T_i) = \bigcup_{i=1}^{k} X^{[0,\varepsilon]}(S_i) = \bigcup_{i=1}^{k} X^{[-\varepsilon,0]}(S_i).$$

Definition 4.2. A pair of families of cross-sections $(\mathcal{S}, \mathcal{T})$ as in the previous theorem is called δ-*adequate*.

Given a pair of δ-adequate cross-sections, $(\mathcal{S}, \mathcal{T})$ let

$$\theta = \sup\left\{\delta > 0;\ \forall x \in \bigcup_{i=1}^{k} S_i \text{ it holds } X^{(0,\delta)}(x) \cap \bigcup_{i=1}^{k} S_i = \emptyset\right\}. \tag{1}$$

Let $\rho > 0$ satisfying $5\rho < \varepsilon$ and $2\rho < \theta$. For each S_i, consider $D_\rho^i = X^{(-\rho,\rho)}(S_i)$ and define the projection

$$P_\rho^i : D_\rho^i \to S_i \tag{2}$$

by $P_\rho^i(x) = X^t(x)$, where $X^t(x) \in S_i$ for $|t| < \rho$. Let $\frac{1}{2}\theta > \varepsilon_0 > 0$ be such that if $x, y \in S_i$, $d(x, y) < \varepsilon_0$ and t is a real number with $|t| < 3\delta$ and $X^t(x) \in T_j$, then $X^t(y) \in D_\rho^j$.

Let $\phi : \bigcup_{i=1}^{k} T_i \to \bigcup_{i=1}^{k} T_i$ be the *first-return map* defined as $\phi(x) = X^t(x)$, where $t > 0$ is the smallest positive number such that $X^t(x) \in \bigcup_{i=1}^{k} T_i$. Note that $t \in [\theta, \varepsilon]$.

If $x \in T_i$ and $y \in S_i$ with $d(x, y) < \varepsilon_0$, let $\{y_0^x, \ldots, y_n^x\} \subset \mathcal{O}_X(y)$ such that $y_0^x = y$ and $y_j^x = P_\rho^l(X^t(y_{j-1}^x))$, where $t > 0$ is the smallest positive time such that $\phi^j(x) = X^t(\phi^{j-1}(x))$ and l is such that $\phi^j(x) \in T_l$. We can continue this construction while $d(\phi^j(x), y_j) < \varepsilon_0$ and, similarly, for $j < 0$ (see Figure 2).

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The stable and unstable sets of points are defined in the following way. If $x \in T_i$ and $\eta < \varepsilon_0$, the η-*stable set* of x is defined as

$$W^s_\eta(x) = \{y \in S_i;\, d(\phi^i(x),\, y_i) < \eta\ \forall i \geq 0\}$$

and the η-*unstable set* of x is defined as

$$W^u_\eta(x) = \{y \in S_i;\, d(\phi^i(x),\, y_i) < \eta\ \forall i \leq 0\}.$$

Theorem 2.7 in [**KS**] establishes that a flow X^t is expansive if and only if, given a pair $(\mathcal{S}, \mathcal{T})$ δ-adequate, there is an $\eta > 0$ such that $W^s_\eta(x) \cap W^u_\eta(x) = \{x\}$ for every $x \in \bigcup_{i=1}^k T_i$.

To extend this result in the context of cw-expansive flows, we first introduce the notion of stable and unstable sets of a pair of δ-adapted cross-sections. For this, let $A \subset S_j$ and $x \in A \cap T_j$. If $\operatorname{diam} A < \frac{1}{2}\varepsilon_0$, then $X^t(A) \subset D^i_\rho$, where t and i are such that $X^t(x) = \phi(x) \in T_i$. Define

$$\phi(A, x) = \{P^i_\rho(X^t(y));\ y \in A\} \subset S_i.$$

Fix $n \in \mathbb{N}$. If $\phi^n(A, x)$ is defined with $\operatorname{diam}(\phi^n(A)) < \varepsilon_0$ and $\phi^n(x) \in T^i$, then $X^t(\phi^n(A)) \in D^l_\rho$, where t and l are such that $X^t(\phi^n(x)) = \phi^{n+1}(x) \in T^l$. Define

$$\phi^{n+1}(A, x) = P^l_\rho(X^t(\phi^n(A))).$$

Definition 4.3. For $\varepsilon_0 > \eta > 0$, define the local *stable and unstable sets* with respect to $(\mathcal{T}, \mathcal{S})$ by

$$W^s_\eta(\mathcal{S}, \mathcal{T}) = \{A \in C(M);\ \text{there is } x \in A \text{ such that } \operatorname{diam}(\phi^n(A, x)) < \eta\ \forall n \geq 0\},$$
$$W^u_\eta(\mathcal{S}, \mathcal{T}) = \{A \in C(M);\ \text{there is } x \in A \text{ such that } \operatorname{diam}(\phi^n(A, x)) < \eta\ \forall n \leq 0\}.$$

Here, recall that $C(M)$ is the set of compact and connected sets of M.

THEOREM 4.4. *A flow X^t is cw-expansive if and only if, given a pair $(\mathcal{S}, \mathcal{T})$ δ-adequate, there is an $\eta > 0$ such that*

$$W^s_\eta(\mathcal{S}, \mathcal{T}) \cap W^u_\eta(\mathcal{S}, \mathcal{T}) = \{A \subset M;\ \#A = 1\}.$$

Proof. Suppose that X^t is cw-expansive. Let $a \in (0, \varepsilon)$ be given and let $a_1 > 0$ be the cw-expansive constant. Let $\eta \in (0, \varepsilon_0)$ be such that if $p \in T_i$ and $q \in S_i$, $i \in \{1, \ldots, k\}$, with $d(p, q) < \eta$, then $d(X^t(p), X^s(q)) < a_1/2$, where, if $X^{t_1}(p) = \phi(p)$ and $X^{s_1}(q) = q_1$, then $t \in [0, t_1]$ and $s \in [0, s_1]$ and $|s - t| \leq |s_1 - t_1|$. Suppose that $A \subset W^s_\eta(\mathcal{S}, \mathcal{T}) \cap W^u_\eta(\mathcal{S}, \mathcal{T})$, let $x_s \in A$ be given by the definition of $W^s_\eta(\mathcal{S}, \mathcal{T})$ and x_u be given by the definition of $W^u_\eta(\mathcal{S}, \mathcal{T})$ and fix $x \in A$. Define $t_0 = 0$ and $t_1 > 0$ as the times when $X^{t_1}(x) \in \phi(A, x_s)$ and $t_1 \in (r_1 - \rho, r_1 + \rho)$ with r_1 being the smallest positive time such that $\phi(x_s) = X^{r_1}(x_s)$. If t_n and r_n are well defined, we define t_{n+1} as the real positive number such that $X^{t_{n+1}}(x) \in \phi_{n+1}(A, x_s)$ and $t_{n+1} \in (\sum_{i=1}^n r_i - \rho, \sum_{i=1}^n r_i + \rho)$, with r_{n+1} being the smallest positive real number such that $\phi^{n+1}(x^s) = X^{r_{n+1}}(\phi^n(x^s))$. Similarly, we can define t_n for $n < 0$. For each $y \in A$, we can, with the same process, find a real number sequence s^y_n. Then we define $\alpha(y)$ by $\alpha(y)(t_n) = s_n$ and linearly on each interval (t_n, t_{n+1}).

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Then, for each $t \in \mathbb{R}$ and $y \in A$, $t = t_n + \sigma = s_n^y + \sigma_y$, where $\sigma \in [0, t_{n+1} - t_n]$, $\sigma_y \in [0, s_{n+1}^y - s_n^y]$ and

$$|\sigma - \sigma_y| \leq |(t_{n+1} - t_n) - (s_{n+1}^y - s_n^y)|,$$

for any $n \in \mathbb{Z}$. Therefore

$$\begin{aligned}\operatorname{diam}(\mathcal{X}_\alpha^t(A)) &= \max_{y,z\in A} d(X^{\alpha(y)(t)}(y),\, X^{\alpha(z)(t)}(z)) \\ &\leq \max_{y,z\in A} (d(X^{\alpha(y)(t)}(y),\, X^t(x)) + d(X^t(x),\, X^{\alpha(y)(t)}(y))) \\ &\leq 2 \max_{y\in A} d(X^{\alpha(y)(t)}(y),\, X^t(x)) \\ &< 2\frac{a_1}{2} = a_1.\end{aligned}$$

By the choice of a_1, $A \subset X^{(-a,a)}(x)$, and since $a < \rho$ and A is inside a cross-section, we conclude that $A = \{x\}$.

To prove the reverse implication, suppose that, given a pair $(\mathcal{T}, \mathcal{S})$ and $\rho > 0$, there is $\eta > 0$ such that

$$W_\eta^s(\mathcal{S}, \mathcal{T}) \cap W_\eta^u(\mathcal{S}, \mathcal{T}) = \{A \subset M;\, \#A = 1\}.$$

Given $\varepsilon > 0$, let $0 < a_1 < \min\{\eta/2, \varepsilon\}$. We shall prove that there is $\sigma > 0$ such that if $A \subset M$ is a continuum and $\alpha \in \mathcal{H}(A)$ with $A \not\subset X^{(-a_1,a_1)}(x_\alpha)$, then there is $t \in \mathbb{R}$ such that $\operatorname{diam}(\mathcal{X}_\alpha^t(A)) > \sigma$. For this, we split the proof into two cases.

Case 1. $A \subset S \in \mathcal{S}$, with the guide point $x_\alpha \in T$. For each $y \in A$, let $t_0 = s_0^y = 0$ and, if $i > 0$, define $t_{i+1} = t_i + t$ and $s_{i+1}^y = s_i^y + s$, where t and s are the smallest positive times such that $X^{t_{i+1}}(\phi^i(x_\alpha)) = \phi^{i+1}(x_\alpha)$ and $X^{s_{i+1}}(y_i^{x_\alpha}) = y_{i+1}^{x_\alpha}$. This is similar for $i < 0$.

Choose $\delta_0 \in (0, \varepsilon - \delta - \rho)$ and positive numbers, $a_2 < a_1$ and $a_3, a_4 > 0$ such that, if $u \in T_i$ and $v \in S_i$, then:

(1) $d(u, v) < a_1$ implies $d(u, X^t(v)) > a_1$ for all $|t| \in [\delta_0, \varepsilon]$;
(2) $d(u, v) < a_2$ implies $d(\phi(u), v_1) < a_1$;
(3) $d(u, v) \geq a_2$ implies $d(u, X^t(v)) > a_3$ for all $|t| < \delta$; and
(4) if $x, y \in M$ and $d(x, y) < a_4$, then $d(X^t(x), X^t(y)) < a_1$ for all $|t| < \delta$.

Let $a' = \min\{a_2, a_3, a_4\}$. In this case, we will prove that a' is a cw-expansive constant to X^t. For this we proceed as follows.

(a) Suppose that, for each $i \in \mathbb{Z}$,

$$\sup_{y\in A} |\alpha(y)(t_i) - s_i| < \delta.$$

By hypothesis, there is $n \in \mathbb{Z}$ such that $\operatorname{diam} \phi^n(A) > \eta$. Then take $y \in A$ such that $d(\phi^n(x_\alpha), y_n) > \eta/2$. Therefore

$$d(X^{t_n}(x_\alpha), X^{s_n}(y)) > \frac{\eta}{2} > a_1 > a_2.$$

By (3),

$$\operatorname{diam} \mathcal{X}_\alpha^{t_n}(A) > d(X^{t_n}(x_\alpha), X^{\alpha(y)(t_n)}(y) > a_3 \geq a'.$$

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(b) Suppose that $j \in \mathbb{Z}$ is the integer with smallest modulus such that

$$\sup_{y \in A} |\alpha(y)(t_j) - s_j| \geq \delta.$$

We can assume that $j > 0$. Let $y \in A$ be such that $|\alpha(y)(t_j) - s_j| \geq \delta$.

(b.1) Suppose there is $i \in [0,\ j)$ such that

$$d(\phi^i(x),\ y_i) > a_2.$$

Since $X^{t_n}(x_\alpha)$ and $X^{\alpha(y)(t_n)}(y)$ belong to $\mathcal{X}_\alpha^{t_n}(A)$, reasoning as in (a), we get

$$\operatorname{diam} \mathcal{X}_\alpha^{t_n}(A) \geq d(X^{t_n}(x_\alpha),\ X^{\alpha(y)(t_n)}(y) > a_3 \geq a'.$$

(b.2) Suppose that, for all $i \in [0,\ j)$,

$$d(\phi^i(x),\ y_i) \leq a_2 < a_1.$$

(b.2.1) Suppose that $t = s_j^y - \alpha(y)(t_j) \geq \delta$. If $\alpha(t_j) \geq s_{j-1} - \delta_0$, then

$$\delta_0 \leq t \leq s_j - s_{j-1} + \delta_0 < \delta + \rho + \delta_0 < \varepsilon$$

and, by (1),

$$\begin{aligned} \operatorname{diam} \mathcal{X}_\alpha^{t_j}(A) &\geq d(X^{t_j}(x_\alpha),\ X^{\alpha(y)(t_j)}(y)) \\ &= d(X^{t_j}(x_\alpha),\ X^{(s_j - t)}(y)) > a_1. \end{aligned}$$

If $\alpha(t_j) < s_{j-1}^y - \delta$, we can find $t' \in (t_{j-1}, t_j)$ such that $\alpha(y)(t') = s_{j-1} - \delta_0$. Then, for $\zeta = t' - t_{j-1}$, (1) implies that

$$d(X^{t_{j-1}}(x_\alpha),\ X^{s_{j-1}^y - \delta_0 - \zeta}(y) > a_1.$$

Therefore, by (4),

$$\begin{aligned} \operatorname{diam}(X^{t'}(A)) &\geq d(X^{t'}(x_\alpha),\ X^{\alpha(y)(t')}(y)) \\ &= d(X^{t_{j-1}+\zeta}(x_\alpha),\ X^{s_{j-1}^y - \delta_0}(y))) > a_4. \end{aligned}$$

(b.2.2) Suppose that $t = \alpha(y)(t_j) - s_j^y \geq \delta_0$. Since

$$s_{j-1}^y + \delta \in [\alpha(y)(t_{j-1}),\ s_j^y + \delta_0),$$

there is $t' \in (t_{j-1}, t_j]$ with $\alpha(y)(t') = s_j^y + \delta_0$. Let $\zeta = t_j - t'$. As $d(X^{t_j}(x_\alpha),\ X^{s_j}(y)) < a_1$, (4) implies that

$$d(X^{t_j}(x_\alpha),\ X^{\zeta + \delta_0}(X^{s_j^y}(y))) > a_1.$$

Therefore

$$\begin{aligned} \operatorname{diam} \mathcal{X}_\alpha^{t'}(A) &\geq d(X^{t'}(x_\alpha),\ X^{\alpha(y)(t')}(y)) \\ &= d(X^{t_j+\zeta}(x_\alpha),\ X^{s_j+\delta_0}(y)) > a_4 > a'. \end{aligned}$$

Case 2. Now suppose that A is not inside a cross-section of $\mathcal{S}$. Since X^t is continuous and $\bigcup_{i=1}^k T_i$ is compact, we can take $\delta_1 > 0$ and $a_5 > 0$ such that if $d(x,\ y) < a_5$, then (1) and (2) below hold.

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(1) $d(X^t(x), X^s(y)) < a'/2$, where $t > 0$ is the smallest positive integer such that $X^t(x) \in \bigcup_{i=1}^{k} T_i$, $X^s(y) = D_\rho^i(X^t(y))$ and $|t - s| < \delta_1/16$.
(2) $d(X^w(x), X^v(y)) < a'/2$, for all $|w|, |v| \le \delta_1 + \delta$ and $|v - w| \le \delta_1$.

Let $a_6 > 0$ be such that if $d(x, y) \le a_6$, then $d(X^{t'}(x), X^{t'+t}(y)) > a_6$ for $|t| \in (\delta_1/16, \varepsilon)$ and $|t'| < \delta$.

Take $0 < a_7 < a_5$ such that the connected component of $\mathcal{O}(x) \cap B_{a_7}$ that contains $X^t(x)$ is contained in $X^{(-a_5,a_5)}(x)$.

Now suppose that A is not contained in $X^{(-a_5,a_5)}(x_\alpha)$, and let t be the smallest positive time such that $X^t(x_\alpha) \in \bigcup_{i=1}^{k} T_i$.

(a) Suppose that, for all $y \in A$, $|\alpha(y)(t) - s| < \delta_1/8$. Since A is compact, there is $0 < \delta' < \delta_1/4$ such that

$$\sup_{y \in A} |\alpha(y)(t + t') - s| < \frac{\delta_1}{4}$$

for all $|t'| \le \delta'$. Define, for each $y \in A$, a homeomorphism $\beta(y)(t) = \alpha(y)(t' + t) - s$ for $|t'| \ge \delta'$, $\beta(y)(0) = 0$ and linearly on $(0, \delta')$. Thus, for all $y \in A$ and all $|t'| < \delta'$,

$$\beta(y)(t') - t'| \le \frac{\delta_1}{4} + \delta' < \frac{\delta_1}{2}.$$

By (1) and (2), for every $|t'| < \delta'$,

$$d(X^{t+t'}(x_\alpha), X^{s+\beta(y)(t')}(y)) < \frac{a'}{2}.$$

(b) Suppose that there exists $y_0 \in A$ such that $|\alpha(y_0)(t) - s| \ge \delta_1/8$. Then $|\alpha(y_0)(t) - t| \ge \delta_1/16$ and there is $t' \le t$ such that $|\alpha(y_0)(t') - t'| = \delta_1/16$. Then

$$\begin{aligned} \operatorname{diam}(\mathcal{X}_\alpha^{t'}(A)) &\ge d(X^{t'}(x_\alpha), X^{\alpha(y_0)(t')}(y_0)) \\ &= d(X^{t'}(x_\alpha), X^{t' \pm \delta_1/16}(y_0)). \end{aligned}$$

If $d(X^{t'}(x_\alpha), X^{t' \pm \delta_1/16}(y_0)) > a_6$, then the proof is complete. Otherwise, by the choice of a_6, we get $d(x_\alpha, y_0) > a_6$, which implies that $\operatorname{diam}(A) > a_6$.

Therefore $\sigma = \min(a', a_6, a_7)$ is a cw-expansive constant to X^t relative to ε. □

Roughly speaking, the next result shows that the local stable and unstable sets of pair $(\mathcal{S}, \mathcal{T})$ of δ-adequate families of cross-sections for cw-expansive flows behave as local stable and unstable sets of points when the flow is expansive, for all points in a continuum $A \in W_\eta^s(\mathcal{S}, \mathcal{T})$.

THEOREM 4.5. *Let X^t be a cw-expansive flow. Then:*
(a) *if $A \in W_\eta^s(\mathcal{S}, \mathcal{T})$, then, for all $x \in A$, $\operatorname{diam}(\phi^n(A, x)) \to 0$ when $n \to +\infty$; and*
(b) *if $A \in W_\eta^u(\mathcal{S}, \mathcal{T})$, then, for all $x \in A$, $\operatorname{diam}(\phi^n(A, x)) \to 0$ when $n \to -\infty$.*

Proof. If (a) was not true, then there are $A \subset W_\eta^s(\mathcal{S}, \mathcal{T})$, $x \in A$ and $\delta_0 > 0$ such that $\operatorname{diam}(\phi^{n_i}(A, x)) \ge \delta_0$ for an increasing sequence of integers $\{n_i\}$. Thus, for each i, there is $y(i) \in A$ such that $d(\phi^{n_i}(x), y(i)_{n_i}) \ge \delta_0/2$. Recall that $y(i)_{n_i} \in \phi^{n_i}(A, x)$. There is no loss of generality by assuming that $\phi^{n_i}(x) \to a \in T_j$ and $y(i)_{n_i} \to b \in S_j$ with $a \ne b$. Since $C(\bigcup_{i=1}^{k} S_i)$ is compact, [**N**], we can suppose that $\lim_{n \to \infty} \phi^{n_i}(A, x) \to B \subset S_j$.

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Since $a, b \in B$, and $a \neq b$, B does not reduce to a unique point and hence, for a fixed $k \in \mathbb{Z}$,

$$\operatorname{diam}(\phi^k(B, a)) = \max_{c,d \in B} d(c_k^a, d_k^a)).$$

Fix $c_k^a, d_k^a \in \phi^k(B, a)$. Thus, there are sequences of points $\{c(i)\}$ and $\{d(i)\}$ in A such that $c(i)_{n_i} \to c$ and $d(i)_{n_i} \to d$. Since $\phi^{n_i+k}(x) \to \phi^k(a)$, [**KS**, Lemma 2.9] implies that $c(i)^x_{n_i+k} \to c_k^a$ and $d(i)^x_{n_i+k} \to d_k^a$. Furthermore, since, for all i,

$$d(c(i)^x_{n_i+k}, d(i)^x_{n_i+k}) \leq \operatorname{diam}(\phi^{n_i+k}(A, x)) \leq \eta,$$

we get that $d(c_k^a, d_k^a) \leq \eta$ and, as c, d are arbitrary,

$$\operatorname{diam}(\phi^k(B, a)) < \eta.$$

This is true for every $k \in \mathbb{Z}$ and hence $B \in W^s_\eta(\mathcal{S}, \mathcal{T}) \cap W^u_\eta(\mathcal{S}, \mathcal{T})$. This leads to a contradiction because X^t is cw-expansive and B does not reduce to a point.

Similarly, we prove (b). □

Theorem 4.5 motivates the following definition.

Definition 4.6. The stable $W^s(\mathcal{S}, \mathcal{T})$ and unstable $W^u(\mathcal{S}, \mathcal{T})$ sets of a δ-adequate pair of cross-sections $(\mathcal{S}, \mathcal{T})$ are defined as

$$W^s(\mathcal{S}, \mathcal{T}) = \left\{ A \in \mathcal{C}(M); \exists\, x \in A \cap \bigcup_{i=1}^{k} T_i \text{ and } \operatorname{diam}(\phi^n(A, y)) \overset{n \to +\infty}{\longrightarrow} 0, \forall y \in A \right\},$$

$$W^u(\mathcal{S}, \mathcal{T}) = \left\{ A \in \mathcal{C}(M); \exists\, x \in A \cap \bigcup_{i=1}^{k} T_i \text{ and } \operatorname{diam}(\phi^n(A, y)) \overset{n \to -\infty}{\longrightarrow} 0, \forall y \in A \right\}.$$

Here $\mathcal{C}(M)$ is the set of all continuum subsets of M.

5. *Entropy for flows*

In this section, we define the notion of topological entropy for δ-adequate pairs of cross-sections and relate this notion to the topological entropy of the flow.

Next, we review the definition of topological entropy for flows given in [**B1**, **B2**, **B3**]. Given subsets $E, F \subset M$, we say that E (t, δ)-*spans* F if, for all $x \in F$, there is $e \in E$ such that

$$d(X^s(e)X^s(x)) \leq \delta \quad \text{for all } s \in [0, t].$$

Let $r_t(F, \delta)$ be the smallest cardinality of a (t, δ)-spanning set for F. If F is compact, then $r_t(F, \delta)$ is finite. Define

$$h(X^t|F, \delta) = \limsup_{s \to \infty} \frac{1}{t} \log r_s(F, \delta).$$

The *topological entropy* of X^t on F is defined as

$$h(X^t|F) = \lim_{\delta \to 0} h(X^t|F, \delta).$$

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Given $E, F \subset M$, we say that E (t, γ)-*weakly spans* F if, for every $x \in F$, there are $e \in E$ and $h \in \mathcal{H}\text{om}(\mathbb{R}, 0)$ such that, for every $s \in [0, t]$,

$$d(X^{h(s)}(x), X^s(e)) \leq \delta.$$

Let $R_t(F, \gamma)$ be the smallest cardinality of a (t, γ)-weakly spanning set for F, and define

$$H(X^t|F, \gamma) = \limsup_{t\to\infty} \frac{1}{t} \log R_t(F, \gamma).$$

Define $H(X^t|F) = \lim_{\delta\to 0} H(X^t|F, \gamma)$. If $F = M$, we use the notation $H(X^t|M) = H(X^t)$ and $h(X^t|M) = h(X^t)$.

In [**T**, Theorem 10], it is proved that if X^t is a flow without fixed points, then $H(X^t) = h(X^t)$.

Definition 5.1. We say $h(X^t)$ is the *topological entropy* of the flow X^t.

Next, we shall define entropy for pairs $(\mathcal{S}, \mathcal{T})$ of δ-adequate cross-sections. For this, we start by defining spanning and separated sets for $(\mathcal{S}, \mathcal{T})$. Fix a pair $(\mathcal{S}, \mathcal{T})$ of δ-adequate cross-sections. Given $\gamma > 0$, we say that a set $E \subset \bigcup_{i=1}^k T_i$ is an (n, γ)-*spanning* set for $(\mathcal{S}, \mathcal{T})$, for all $x \in \bigcup_{i=1}^k T_i$, if there exists $e \in E$ such that

$$d(\phi^i(x), e_i^x) < \gamma \quad \text{for all } i \in \{0, \ldots, n\}.$$

Let $r'(n, \gamma)$ be the smallest cardinality of (n, γ)-spanning sets and define

$$H'((\mathcal{S}, \mathcal{T}), \gamma) = \limsup_{n\to\infty} \frac{1}{n} \log r'(n, \gamma), \quad H'((\mathcal{S}, \mathcal{T})) = \lim_{\gamma\to 0} H'((\mathcal{S}, \mathcal{T}), \gamma). \tag{3}$$

We say that $E \subset \bigcup_{i=1}^k T_i$ is an (n, γ)-*separated* set if, for all $x, y \in E$, either y_i^x is not defined for some $i \in \{0, \ldots, n\}$ or there exists $i \in \{0, \ldots, n\}$ such that

$$d(\phi^i(x), y_i^x) \geq \gamma.$$

Let $s'(n, \gamma)$ be the largest cardinality of (n, γ)-separated sets and define

$$s'((\mathcal{S}, \mathcal{T}), \gamma) = \limsup_{n\to\infty} \frac{1}{n} \log s'(n, \gamma), \quad s'((\mathcal{S}, \mathcal{T})) = \lim_{\gamma\to 0} s'((\mathcal{S}, \mathcal{T}), \gamma). \tag{4}$$

LEMMA 5.2. *Let X^t be a flow without fixed points and let $(\mathcal{S}, \mathcal{T})$ be a pair of δ-adequate cross-sections for X^t. Then, for all $\gamma > 0$, $R'(n, \gamma) \leq s'(n, \gamma) \leq R'(n, \gamma/2)$.*

Proof. Let E be the largest (n, γ)-separated set and suppose that it is not an (n, γ)-spanning. Thus there exists $x \in \bigcup_{i=1}^k T_i$ such that if $e \in E$, then, for $i \in \{0, \ldots, n\}$,

$$d(\phi^i(x), e_i^x) \geq \gamma.$$

Then $E \cup \{x\}$ would be (n, γ)-separated, which leads to a contradiction. Then

$$R'(n, \gamma) \leq s'(n, \gamma).$$

To prove the other inequality, let E be an (n, γ)-separated set and let F be an $(n, \gamma/2)$-spanning set. Then, for all $x \in \bigcup_{i=1}^k T_i$, we can take $g(x) \in F$ such that

$$d(\phi^i(x), g(x)_i^x) < \frac{\gamma}{2}$$

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for all $i \in \{0, \dots, n\}$. If $g(x) = g(y)$, $x, y, g(x) \in T_j$ for some $j \in \{1, \dots, k\}$ and $g(x)_i^x = g(y)_i^y$ for all $i \in \{0, \dots, n\}$. The last statement is easy to proof by induction. If, for $i_0 < n$, it is true that, for all $i \in \{0, \dots, i_0\}$, $g(x)_i^x = g(y)_i^y$ then, if $g(x)_{i_0+1}^x \neq g(y)_{i_0+1}^y$ or $x_{i_0+1}^y$ is not defined or $y_{i_0+1}^x$ is not defined, this is absurd because

$$d(\phi^{i_0}(x), \phi^{i_0}(y)) \leq d(\phi^{i_0}(x), g(x)_x^{i_0}) + d(\phi^{i_0}(y), g(y)_y^{i_0}) < \gamma.$$

Thus, for all $i \in \{0, \dots, n\}$,

$$d(\phi^i(x), \phi^i(y)) \leq d(\phi^i(x), g(x)_i^x) + d(\phi^i(y), g(x)_i^y) < \gamma.$$

Since E is $(n, \gamma/2)$-separated and g restricted to E is injective, we obtain that $\#E \leq \#F$ and $s'(n, \gamma) \leq R'(n, \gamma/2)$, which finishes the proof. □

COROLLARY 5.3. $H'((\mathcal{S}, \mathcal{T})) = s'((\mathcal{S}, \mathcal{T}))$.

Definition 5.4. $H'((\mathcal{S}, \mathcal{T}))$ is, by definition, the *topological entropy* of the δ-adequate pair of cross-sections $(\mathcal{S}, \mathcal{T})$.

THEOREM 5.5. *Let X^t be a flow without fixed points and let $(\mathcal{S}, \mathcal{T})$ be a δ-adequate pair of cross-sections. If $H'(\mathcal{S}, \mathcal{T}) > 0$, then $h(X^t) > 0$.*

Proof. Fix $\delta > 0$ such that $s'((\mathcal{S}, \mathcal{T}), \delta) > H'(\mathcal{S}, \mathcal{T})/2 > 0$. Fix $n \in \mathbb{N}$ and consider $E \subset \bigcup_{i=1}^k T_i$ maximal (n, γ)-separated set for $(\mathcal{S}, \mathcal{T})$. Since $X^{[0,n\varepsilon]}(x)$ cross at least n cross-sections $T_i's$ for each x, there is $\gamma > \gamma' > 0$ such that E is an $(n\varepsilon, \gamma')$-separated set for the flow X^t. Thus $s'(n\varepsilon, \gamma', (\mathcal{S}, \mathcal{T}) \leq s(n\varepsilon, \gamma, X^t)$. Here $s'(n\varepsilon, \gamma', (\mathcal{S}, \mathcal{T})$ is the maximal cardinality of $(n\varepsilon, \gamma)$-separated sets of $(\mathcal{S}, \mathcal{T})$ and $s(n\varepsilon, \gamma, X^t)$ is the maximal cardinality of $(n\varepsilon, \gamma)$-separated sets of X^t. Then

$$\begin{aligned} s(\gamma, X^t) &= \limsup_{u \to \infty} s(u, \gamma, X^t) \\ &\geq \limsup_{n \to \infty} s(n\varepsilon, \gamma, X^t) \\ &\geq \limsup_{n \to \infty} s'(n\varepsilon, \gamma, (\mathcal{S}, \mathcal{T})) \\ &= s'(\gamma, (\mathcal{S}, \mathcal{T})) \\ &> \frac{H'((\mathcal{S}, \mathcal{T}))}{2}. \end{aligned}$$

Therefore $h(X^t) = \lim_{\gamma \to 0} s(\gamma, X^t) > 0$. □

6. *Entropy for cw-expansive flows*

In this section, we prove our main theorem. First, we need to prove some technical results.

LEMMA 6.1. *Let X^t be a cw-expansive flow with η as expansive constant. For all $\varepsilon_0 \in (0, \eta/2]$, there is $\delta_0 > 0$ such that if $A \in \mathcal{C}(\bigcup_{i=1}^k S_i)$, $x \in A \cap \bigcup_{i=1}^k T_i$, $\operatorname{diam}(A) \leq \delta_0$, and if there is $n > 0$ such that*

$$\sup\{\operatorname{diam} \phi^i(A, x); i = 1, \dots, n\} \in [\varepsilon_0, 2\varepsilon_0],$$

then $\operatorname{diam}(\phi^n(A, x)) \geq \delta_0$.

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Proof. The proof goes by contradiction. If the hypothesis fails, there is a sequence $\{A_i\}$ of connected and compact sets in $\bigcup_{i=1}^{k} S_i$, a sequence of points $\{x_i\}$ in $A_j \cup \bigcup_{i=1}^{k} S_i$ and an increasing sequence $\{n_i\}$ of natural numbers such that:

(1) $\operatorname{diam} A_i < 1/i$, for all $i \in \mathbb{N}$;

(2 $\sup\{\operatorname{diam} \phi^i(A_i, x_i);\ i = 1, \ldots, n_i\} \in [\varepsilon_0, 2\varepsilon_0]$; and

(3) $\operatorname{diam}(\phi^{n_i}(A_i, x_i)) < 1/i$.

By (1), for each i we can choose $0 < m_i < n_i$ so that $\operatorname{diam}(\phi^{m_i}(A, x_i)) \in [\varepsilon_0, 2\varepsilon_0]$. We will prove that

$$\lim_{i\to\infty} m_i = \lim_{i\to\infty} (n_i - m_i) = \infty. \tag{5}$$

Fix $N > 0$. By continuity, there is $\delta_1 > 0$ such that if $x \in \bigcup_{i=1}^{k} T_i$ and $y \in \bigcup_{i=1}^{k} S_i$ satisfy $d(x, y) < \delta_1$, then $d(\phi^n(x), y_n^x) < \varepsilon_0$ for all $n \in \{-N, \ldots, -1, 0, 1, \ldots, N\}$. Take a natural number i_0 that is sufficiently large so that $1/i_0 < \delta_1$. Then, for all $i > i_0$, $\operatorname{diam}(A_i) < \delta_1$ and $\operatorname{diam}(\phi^{n_i}(A_i, x_i) < \delta_1$. Therefore

$$\operatorname{diam}(\phi^j(A_i, x_i)) < \varepsilon_0 \quad \text{and} \quad \operatorname{diam}(\phi^{n_i - j}(A_i, x_i)) < \varepsilon_0 \quad \text{for all } j \in \{1, \ldots, N\}.$$

Thus $m_i > N$ and $n_i - m_i > N$, and since N was arbitrary, we have proved 5.

We can assume that $\lim \phi^{m_i}(A_i, x_i) \to B$ and $\phi^{m_i}(x_i) \to x \in B$. Since $\operatorname{diam}(B) \geq \varepsilon$, B is not degenerate. Fix an arbitrary integer n. Then, for i sufficiently big, $m_i + n \in \{0, \ldots, n_i\}$. Therefore, for i sufficiently big, $\operatorname{diam}(\phi^{m_i+n}(A_i, x_i)) < 2\varepsilon_0$. So, by [**KS**, Lemma 2.9],

$$\phi^{m_i+n}(A_i, x_i) \to \phi^n(B, x).$$

Since n is arbitrary, $\phi^n(B, x) < 2\varepsilon_0 < \eta$ for all $n \in \mathbb{Z}$ which contradicts the fact that X^t is cw-expansive. □

LEMMA 6.2. *Let X^t be a cw-expansive flow and let ε_0 and δ_0 be as in the previous lemma. If $A \in \mathcal{C}(\bigcup_{i=1}^{k} S_i)$, $A \cap \bigcup_{i=1}^{k} T_i \neq \emptyset$ such that $\operatorname{diam} A < \delta_0$ and, for some integer m, $x \in A$ $\operatorname{diam} \phi^m(A) \geq \varepsilon_0$.*

(1) *If $m \geq 0$, then $\operatorname{diam} \phi^n(A, x) \geq \delta_0$ for all $n \geq m$. More precisely, there is a continuum $B \subset A$ with $x \in B$ such that*

$$\sup\{\operatorname{diam} \varphi^j(B, x);\ j = 1, \ldots, n\} \leq \varepsilon_0$$

e $\operatorname{diam} \phi^n(B, x) = \delta_0$.

(2) *If $m < 0$, then $\operatorname{diam} \phi^{-n}(A, x) \geq \delta_0$ for all $n \geq -m$. More precisely, there is a continuum $B \subset A$ with $x \in B$ such that*

$$\sup\{\operatorname{diam} \phi^{-j}(B, x);\ j = 1, \ldots, n\} \leq \varepsilon_0 \quad \textit{and} \quad \operatorname{diam} \phi^{-n}(B, x) = \delta_0.$$

Proof. Let $m \geq 0$. By [**N**], there is an arc $c : [0, 1] \to \mathcal{C}(\bigcup_{i=1}^{k} S_i)$ from $\{x\}$ to A such that if $r \leq s$, then $c(r) \subset c(s)$. Fix $n \geq m$. Define a map $F : [0, 1] \to [0, \infty]$ by

$$F(r) = \sup\{\operatorname{diam} \phi^j(c(r), x);\ j = 0, 1, \ldots, n\}.$$

Take $r_0 \in [0, 1]$ such that $r_0 \in F^{-1}(\varepsilon_0)$. By Lemma 6.1,

$$\operatorname{diam} \phi^n(A, x) \geq \operatorname{diam} \phi^n(c(r_0)) \geq \delta_0.$$

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Define $D : C(c(r_0)) \to [0, \infty]$ by $D(C) = \operatorname{diam} \phi^n(C)$. Since $C(r(a_0))$ is connected, $D^{-1}(\delta_0)$ is non-empty and therefore all $B \in D^{-1}(\delta_0)$ satisfy $\operatorname{diam} \phi^n(B, x) = \delta_0$ and

$$\sup\{\operatorname{diam} \phi^j(B, x);\ j = 1, \ldots, n\} \leq \varepsilon_0.$$

Similarly, we prove this for $m \leq 0$. □

COROLLARY 6.3. *Let X^t be a cw-expansive flow, and let δ_0 and ε_0 be as in Lemma 6.1. Then, for each $\gamma > 0$, there exists $N > 0$ such that if $A \subset \bigcup_{i=1}^k S_i$ is a continuum with $A \cap \bigcup_{i=1}^k T_i \neq \emptyset$ and $\operatorname{diam} A \geq \gamma$, then, for all $x \in A \cap \bigcup_{i=1}^k T_i$, either $\operatorname{diam} \phi^n(A, x) \geq \delta_0$ for all $n \geq N$ or $\operatorname{diam} \phi^{-n}(A, x) \geq \delta_0$ for all $n \geq N$.*

LEMMA 6.4. *Let X^t be a cw-expansive flow and let $(\mathcal{S}, \mathcal{T})$ be a δ-adequate pair of cross-sections. If there is a cross-section T_i with topological dimension greater than zero, then there exists a non-degenerate continuum $A \subset S_i$ such that either $A \in W^s(\mathcal{S}, \mathcal{T})$ or $A \in W^u(\mathcal{S}, \mathcal{T})$.*

Proof. Let $C \subset T_{i_0}$ be a non-degenerate continuum, with $\operatorname{diam} C \leq \delta_0$. Suppose that any $C' \subset C$ continuum in C does not belong to $W^s_\varepsilon(\mathcal{S}, \mathcal{T})$. Choose a sequence of connected and compact sets

$$C_1 \subset C_2 \subset \cdots \subset C_n \subset \cdots$$

and a sequence of natural numbers $n(1) < n(2) < \cdots$ such that $\operatorname{diam} C_i \to 0$, $C_i \to x$, $x \in C_i$ for all i,

$$\sup\{\operatorname{diam} \phi^j(C_i);\ j = 0, 1, \ldots, n(i)\} \leq \varepsilon_0$$

and $\operatorname{diam} \phi^{n(i)}(C_i) \leq \delta_0$ for all i. We can suppose that $\phi^{n(i)}(x) \to x_0 \in T_{j_0}$ and $\phi^{n(i)}(C_i) \to A \in S_{j_0}$ for some $j_0 \in \{1, \ldots, k\}$. So $\operatorname{diam} A \geq \delta_0$. Given $n \in \mathbb{N}$, [**KS**, Lemma 2.9] implies that $\phi^{n(i)-n}(C_i, x) \to \phi^{-n}(A_i, x_0)$, and since $\operatorname{diam} \phi^{n(i)-n}(C_i, x) < \varepsilon_0$ for all i, $\operatorname{diam} \phi^{-n}(A, x_0) < \varepsilon_0$. Since n is arbitrary, we obtain that $A \in W^u_\varepsilon(\mathcal{S}, \mathcal{T})$. □

Next we give a sketch of the proof of Theorem B. Fix $\delta_1 > 0$ small. Given a δ-adequate pair $(\mathcal{S}, \mathcal{T})$ of cross-sections, we want to exhibit $l \in \mathbb{N}$ (independent of δ_1) such that for each $m \in \mathbb{N}$ there is an $(ml, \delta_1/3)$-separated set E for $(\mathcal{S}, \mathcal{T})$ with 2^m points. Since the topological dimension of M is greater than one, there is a cross-section T_i with topological dimension greater than zero. Lemma 6.4 implies that there is a compact and connected set $A \in W^u_\eta(\mathcal{S}, \mathcal{T})$, $A \subset T_i$ and, by Lemma 6.2, we can assume that $\operatorname{diam} A = \delta_1/3$. Fix a natural number m. By Corollary 6.3 and Lemma 6.2, we can find two connected and compact sets A_1 and A_0 in $\phi^N(A, x)$ such that $d(A_1, A_0) \geq \delta_1/3$ and $\operatorname{diam}(A_{i_1}) = \delta_1/3$, for $i_1 = 0, 1$.

We would like to repeat this process with A_1 and A_2. If so, we would find a finite collection $\{(A_{i_1,i_2,\ldots,i_j}, a_{i_1,i_2,\ldots,i_j})\}$ with $i_k \in \{0, 1\}$, and $j \leq m$, such that:

- $a_{i_1,\ldots,i_k} \in A_{i_1,\ldots,i_k}$ with $A_{i_1,\ldots,i_k}$ compact, connected subsets of $\phi^N(A_{i_1,\ldots,i_{k-1}}, a_{i_1,\ldots,i_{k-1}})$ satisfying

$$\operatorname{diam}(A_{i_1,\ldots,i_k}) = \frac{\delta_1}{3} \quad d(A_{i_1,\ldots,i_{k-1},0}, A_{i_1,\ldots,i_{k-1},1}) \geq \frac{\delta_1}{3}; \text{ and}$$

- $A_{i_1,\ldots,i_j} \in W^u_\eta(\mathcal{S}, \mathcal{T})$.

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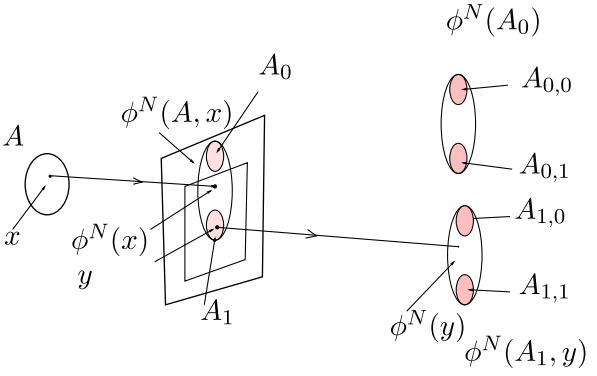


FIGURE 3. Taking a first return map considering only two pairs of δ-adequate family does not work.

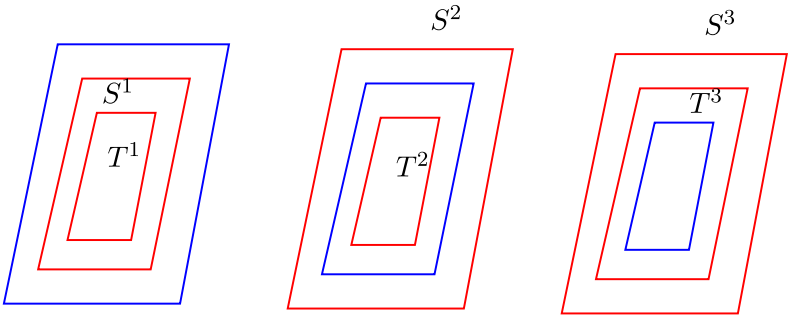


FIGURE 4. The three pairs of δ-adequate cross-sections.

Thus, for each $A_{i_1,\dots,i_m}$ we could choose a point $b(i_1,\dots,i_m) \in A_{i_1}$ such that:

- $c_i(1) = b(i_1,\dots,i_m)_N^{a_{i_1}} \in A_{i_1,i_2}$; and
- $c_i(j) = c(j-1)_N^{a_{i_1,\dots,i_j}} \in A_{i_1,\dots,i_{j+1}}$ for $j \le m-1$.

Then the $E = \{b_{i_1,\dots,i_m}, i_k \in \{0,1\}, 1 \le k \le m\}$ is $(mN, \delta_1/3)$-separated for the δ-adequate pair of cross-sections $(\{T\}, \{S\})$. In this case, $l = N$.

But it is not always possible to repeat the above argument. In fact, for $A_{i_1} \subset (\bigcup S_j) \setminus (\bigcup T_j)$, there are no points in $A_{i_1} \cap \bigcup T_j$, and so $A_{i_1} \cap \mathrm{Dom}(\phi) = \emptyset$. See Figure 3.

To bypass this difficulty, we reason with three pairs of δ-adequate families of cross-sections,

$$(\{T_i^1\}, \{S_i^1\}), (\{T_i^2\}, \{S_i^2\}), (\{T_i^3\}, \{S_i^3\}),$$

such that, for all $i \in \{0,\dots,k\}$, $T_i^1 = T_i^2$, $T_i^3 = S_i^1$, $S_i^2 = S_i^3$ (observe that $T_i^1 \subset T_i^2 \subset S_i^3$). See Figure 4.

This strategy leads us to consider, besides the first-return map ϕ above, three other first-return maps ϕ_1, ϕ_2 and ϕ_3 relative to the distinct pairs of δ-adequate cross-sections $(\mathcal{S}^1, \mathcal{T}^1)$, $(\mathcal{S}^2, \mathcal{T}^2)$ and $(\mathcal{S}^3, \mathcal{T}^3)$. Further, we shall define an auxiliary map $\varphi : \bigcup T_j^3 \to \bigcup T_j^1$ such that, even if $A_{i_1} \cap \mathrm{Dom}(\phi_1) = \emptyset$, we get that $A_{i_1} \subset \phi_1(A,x) \subset \mathrm{Dom}(\varphi)$, and this allows us to continue with the argument. See Figure 5.

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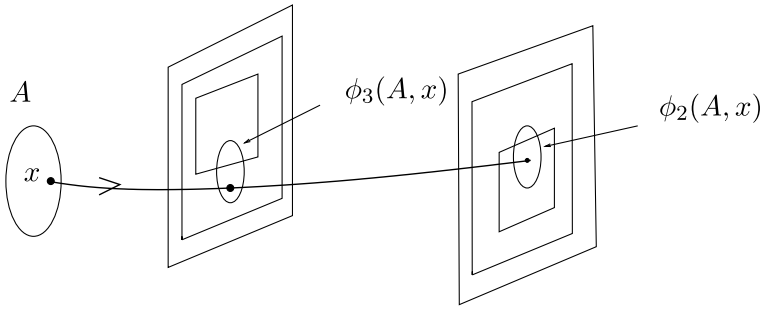


FIGURE 5. The first returns maps acting on A.

Since $(\mathcal{S}^i, \mathcal{T}^i)$, $i = 1, 2, 3$ are finite families of compact sets, there exists $k \in \mathbb{N}$ such that, for each $z \in (\bigcup S_j^2)$, there is $r \in \{1, \dots, k\}$ with $\varphi(A, z) = \phi_3^r(A, z)$. Therefore, reasoning as before, we find an $(lm, \delta_1/3)$-separated set for $(\mathcal{S}^3, \mathcal{T}^3)$ and, in this case, we put $l = kN$. Next, we formalize this in the proof of our main result.

THEOREM B. *If X^t is a cw-expansive flow in compact metric space M with topological dimension greater than one, then the entropy $h(X^t)$ of the flow is positive.*

Proof. Take three pairs of δ-adequate families of cross-sections $(\mathcal{T}^j, \mathcal{S}^j)$, $1 \le j \le 3$,

$$(\mathcal{T}^j, \mathcal{S}^j) = \{(\{T_i^j\}, \{S_i^j\});\ T_i^j \in \mathcal{T}^j,\ S_i^j \in \mathcal{S}^j\}$$

such that, for all $i \in \{0, \dots, k\}$, $T_i^1 = T_i^2$, $T_i^3 = S_i^2$, $S_i^1 = S_i^3$ (observe that $T_i^1 \subset S_i^2 \subset S_i^1$).

For $x \in \bigcup_{i=1}^k S_i^2$, define $\varphi(x) = X^t(x)$, where $t > 0$ is the smallest positive time such that $X^t(x) \in \bigcup_{i=1}^k T_i^1$. Observe that $t \in [\theta_3, \varepsilon]$ (where θ_3 is defined as in (1) relative to the third pair of δ-adequate cross-sections families). Let θ' be the smallest positive time t such that any point $a \in \bigcup_{i=1}^k S_i^2$ satisfies $X^{(0,t)}(a) \cap \bigcup_{i=1}^k T_i^1 \neq \emptyset$. Note that the segment of orbit $X^{[0,\theta_3]}(x)$ intersects cross-sections of type T_i^3 at most $[\varepsilon/\theta']$ times. Hence $\varphi(x) = \phi_1^i(x)$ for some $i \in \{1, \dots, [\varepsilon/\delta']\}$ (where ϕ_1 is the first-return map associated to the pair of families $(\mathcal{T}^1, \mathcal{S}^1)$ of cross-sections).

Let $\varepsilon_1 \in (0, \varepsilon_0)$ be such that if $x, y \in S_i^2$ and $d(x, y) < \varepsilon_1$ and t is a real number with $|t| \le [\varepsilon/\delta']$ and $X^t(x) \in T_j^1$, then $X^t(y) \in D_\rho^j$. If $x \in T_i^1$ and $y \in S_i^2$ with $d(x, y) < \varepsilon_1$, define a set of points $\{y_{\varphi,0}^x, \dots, y_{\varphi,n}^x\} \subset \mathcal{O}(y)$ with $y_{\varphi,0}^x = y$ and $y_{\varphi,l}^x = P_\rho^l(X^t(y_{\varphi,l-1}^x))$, where $t > 0$ is the smallest positive time such that $\varphi^l(x) = X^t(\varphi^{l-1}(x))$, and l is such that $\varphi^j(x) \in T_l$. Recall that P_ρ^l is the projection defined in (2). We can argue as above whenever $d(\varphi^j(x), y_j) < \varepsilon_1$. This is similar for $j < 0$.

Take $\delta_1 \in (0, \delta_0)$ such that if $x \in \bigcup_{i=1}^k T_i^3$ and $y \in \bigcup_{i=1}^k S_i^3$, then $d(\phi_3^i(x), y_i^x) < \varepsilon_1$ for all $i \in \{1, \dots, [\varepsilon/\delta']\}$.

By Lemma 6.4, we can suppose that there exists a non-degenerate compact and connected set $A \in W_\eta^u(\mathcal{S}^2, \mathcal{T}^2)$ in some cross-section T_i^1 and, by Lemma 6.2, we can assume that $\operatorname{diam} A = \delta_1/3$. Let N be as given in Corollary 6.3. Hence, if $D \in \bigcup_{i=1}^k S_i^1$ is a compact and connected set with $D \cap \bigcup_{i=1}^k T_i^1 \neq \emptyset$, $\operatorname{diam} D \ge \delta_1/3$, then, for all $x \in D$, $\max\{\operatorname{diam} \phi_1^j(D, x);\ |j| \le N\} > \eta$. Hence, by Lemma 6.2 and the

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definition of φ, if $D \in W^u_\eta(\mathcal{S}^2, \mathcal{T}^2)$ and $\operatorname{diam} D = \delta_1/3$, we get that, for all $x \in D$, $\operatorname{diam} \varphi^N(D, x) \geq \delta_1$.

Fix $m \in \mathbb{N}$ and $x \in A$. As $\operatorname{diam} \varphi^N(D, x) \geq \delta_1$, Lemma 6.2 implies that there are two connected and compact sets A_1 and A_0 in $\varphi^N(A, x)$ such that $d(A_1, A_0) \geq \delta_1/3$ and $\operatorname{diam}(A_{i_1}) = \delta_1/3$, for $i_1 = 0, 1$. Choosing $a_{i_1} \in A_{i_1}$, we get that $\operatorname{diam} \varphi^N(A_{i_1}, a_{i_1}) \geq \delta_1$ and so Lemma 6.2 implies that there are sets $A_{i_1,0}$ and $A_{i_1,1}$ contained in $\varphi^N(A_{i_1}, a_{i_1})$ such that $\operatorname{diam}(A_{i_1,i_2}) = \delta_1/3$ and $d(A_{i_1,0}, A_{i_1,1}) \geq \delta_1/3$, with $i_1, i_2 \in \{0, 1\}$.

Continuing this construction, we find a finite collection of sets $\{(A_{i_1,i_2,\ldots,i_j}, a_{i_1,i_2,\ldots,i_j})\}$ with $i_k \in \{0, 1\}$, $1 \leq j \leq m$, such that:

- $A_{i_1,\ldots,i_k}$, are compact and connected subsets of $\varphi^N(A_{i_1,\ldots,i_{k-1}}, a_{i_1,\ldots,i_{k-1}})$ with
$$\operatorname{diam}(A_{i_1,\ldots,i_k}) = \frac{\delta_1}{3} \quad \text{and} \quad d(A_{i_1,\ldots,i_{k-1},0}, A_{i_1,\ldots,i_{k-1},1}) \geq \frac{\delta_1}{3}; \text{ and}$$
- $A_{i_1,\ldots,i_j} \in W^u_\eta(\mathcal{S}^2, \mathcal{T}^2)$.

For each $A_{i_1,\ldots,i_m}$, pick a point $b(i_1, \ldots, i_m) \in A_{i_1}$ such that:

- $c_i(1) = b(i_1, \ldots, i_m)^{a_{i_1}}_{\varphi,N} \in A_{i_1,i_2}$; and
- $c_i(j) = c(j-1)^{a_{i_1,\ldots,i_j}}_{\varphi,N} \in A_{i_1,\ldots,i_{j+1}}$ for $j \leq m-1$.

CLAIM 1. *The set E consisting of all points $b_{i_1,\ldots,i_m}$ as above is $(m[\varepsilon/\theta']N, \delta_1/3)$-separated for the δ-adequate pair of families of cross-sections $(\{T^3_i\}, \{S^3_i\})$.*

Proof. Let $b_{i_1,\ldots,i_m} \neq b_{l_1,\ldots,l_m}$ and let k be the smallest natural number such that $i_k \neq l_k$. Then $c_l(k-1) \in A_{i_1,\ldots,i_{k-1},l_k}$ and $c_i(k-1) \in A_{i_1,\ldots,i_{k-1},i_k}$ which implies that
$$d(c_i(k-1), c_l(k-1)) \geq \frac{\delta_1}{3}.$$
Since $c_l(k-1) = \phi_3^M(b_{l_1,\ldots,l_m})$ and $c_i(k-1) = \phi_3^M(b_{i_1,\ldots,i_m})$, for some $M \in \{1, \ldots, [\varepsilon/\theta']Nk\} \subset \{1, \ldots, [\varepsilon/\theta']Nm\}$, the claim follows. □

Now, Claim 1 implies that $s'(mN(\varepsilon/\theta')) \geq 2^m$ and so
$$\begin{aligned} s'\left(\{\mathcal{T}_3, \mathcal{S}_3\}, \frac{\delta_1}{3}\right) &= \limsup_{n\to\infty} \frac{1}{n} \log\left(s'\left(n, \frac{\delta_1}{3}\right)\right) \\ &\geq \limsup_{m\to\infty} \frac{1}{mN[\varepsilon/\theta'] + 1} \log\left(s'\left(mN\left[\frac{\varepsilon}{\theta'}\right] + 1, \frac{\delta_1}{3}\right)\right) \\ &\geq \limsup_{m\to\infty} \frac{m}{mN[\varepsilon/\theta'] + 1} \log(2) = \frac{1}{N[\varepsilon/\theta']} \log(2) > 0. \end{aligned}$$
Thus $H'(\{\mathcal{T}_3, \mathcal{S}_3\}) = s'(\{\mathcal{T}_3, \mathcal{S}_3\}) > 0$ and Theorem 5.5 implies that $h(X^t) > 0$, which finishes the proof. □

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