99.37 Anthropologic

It seems that your work as an anthropologist is getting ever harder. Once you visited ocean islands with just two tribes, Truth Tellers (T) and Liars (L), but now might also be faced with Random Responders (R). On your first visit to such an island, you are certain to meet a group of three with one person of each propensity. Can you sort them out in just three yes/no questions?

Each question can be addressed to any one of the three welcomers, who, by the way, speak perfect English, are experts in logic and know each others' tribes. It seems at first that this is an impossible challenge, because if the first question happens to be addressed to R, you will gain no information and the remaining 2 questions can only have $2^2 = 4$ answers, not enough to distinguish $3! = 6$ permutations. Indeed an eminent mathematician recently produced this argument and was so convinced by it that he refused to listen to our answer.

We can obviously replace yes/no questions by arbitrary *binary* ones, meaning questions that only permit two answers.

Here is our solution: Ask any one of the three welcomers what we shall call the Key Question (KQ): 'Please point to whichever of your colleagues is least likely to tell me the truth?' Then we ask the indicated colleague exactly the same key question. Finally we ask the person now indicated a Last Question (LQ) 'Is 1 + 1 equal to 3?'.

The point of KQ is that the person pointed to cannot be R, since T points to L, L points to T and R points to either L or T. So after KQ has been asked twice, two distinct people have been indicated neither of whom can be R. Also we are now addressing either T or L with LQ (which can obviously be replaced by any question to which we know the answer, for example 'Are you R?').

We thank the referee for pointing out that this can also be established by a simple truth table. This table has eight rows according as the first question (KQ_1) is answered by R truthfully, R falsely, or T or L, and the second (KQ_2) is answered by T or L — exactly enough for 3 binary questions.

We now show that there is no hope for anthropologists who encounter four or more tribes with various veracity propensities, no matter how many questions are asked. We need only consider the case when there are four tribes, say T, T′ (who nearly always tell the truth), L and L′ (who nearly always lie). The anthropologist must be able to handle all cases, in particular ones in which T′ and L′ happen to use their flexibility so as to deliberately confuse the situation.

If asked whether T is more truthful than T′ then T′ *must* lie, for if T and T′ ever both answer truthfully, the anthropologist will know the correct answer and so will know who T is, and can interrogate him to discover exactly who's who. Such arguments prove that when asked to compare T with T′ or L with L′, T′ must lie and L′ must tell the truth, if they are to fool the anthropologist.

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These are essentially the only questions for which T′ differs from T or L′ differs from L. However, the welcomers must also be able to answer arbitrary Boolean combinations of other questions together with these ones. A precise way to do this is for T′ and L′ both to imagine that T has been interchanged with T′ and L with L′. If they do so, then the anthropologist cannot distinguish between this imagined situation and the actual one, and so cannot possibly sort them out in any number of questions.

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99.38 Proof without words: Cotangent expressed as a series of cosecants of multiple angles in powers of two

$$\cot x - \cot\left(2^n x\right) = \operatorname{cosec} 2x + \operatorname{cosec} 4x + \operatorname{cosec} 8x + \ldots + \operatorname{cosec}\left(2^n x\right).$$

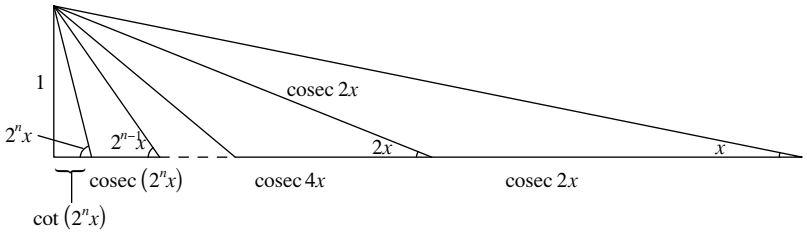


FIGURE 1

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99.39 Proof without words: A delightful water pouring problem

A version of the following little-known problem can be found in [1], a Soviet psychology research project used to gauge the mathematical abilities of middle school students. Although the problem is elementary, it gives rise to an enjoyable and elegant visual solution.

Problem: There are two identical glasses: glass 1 is completely filled with water, and glass 2 is empty. Water is poured back and forth between the glasses as follows. On the first pour, half of the water in glass 1 is poured into glass 2. On the second pour, a third of the water in glass 2 is poured back into glass 1. On the third pour, a fourth of the water in glass 1 is poured into glass 2, and so on. How much water is in glass 2 after 2015 pours?

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