Combinatorics, Probability and Computing (2026), **35**, pp. 40–58
doi:10.1017/S0963548325100229

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ARTICLE

Saturation in random hypergraphs

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(Received 15 May 2024; revised 23 July 2025; accepted 23 July 2025; first published online 7 October 2025)

Abstract

Let K_n^r be the complete r-uniform hypergraph on n vertices, that is, the hypergraph whose vertex set is $[n] := \{1, 2, \dots, n\}$ and whose edge set is $\binom{[n]}{r}$. We form $G^r(n,p)$ by retaining each edge of K_n^r independently with probability p. An r-uniform hypergraph $H \subseteq G$ is *F-saturated* if H does not contain any copy of F, but any missing edge of H in G creates a copy of F. Furthermore, we say that H is *weakly F-saturated* in G if H does not contain any copy of F, but the missing edges of H in G can be added back one-by-one, in some order, such that every edge creates a new copy of F. The smallest number of edges in an F-saturated hypergraph in G is denoted by $sat(G,F)$, and in a weakly F-saturated hypergraph in G by $w\text{-}sat(G,F)$. In 2017, Korándi and Sudakov initiated the study of saturation in random graphs, showing that for constant p, with high probability $sat(G(n,p), K_s) = (1+o(1))n \log_{\frac{1}{1-p}} n$, and $w\text{-}sat(G(n,p), K_s) = w\text{-}sat(K_n, K_s)$. Generalising their results, in this paper, we solve the saturation problem for random hypergraphs $G^r(n,p)$ for cliques K_s^r, for every $2 \le r < s$ and constant p.

Keywords: Saturation; weak saturation; random hypegraphs

2020 MSC Codes: Primary: 05C65, 05C80

1. Introduction and main results

Denote by K_n^r the complete r-uniform hypergraph on n vertices, that is, the hypergraph whose vertex set is $[n] := \{1, 2, \dots, n\}$ and whose edge set is $\binom{[n]}{r}$. For fixed r-uniform hypergraphs F and G, we say that a hypergraph $H \subseteq G$ is (strongly) *F-saturated* in G, if H does not contain any copy of F, but adding any edge $e \in E(G) \setminus E(H)$ to H creates a copy of F. We let $sat(G,F)$ denote the minimum number of edges in an F-saturated hypergraph in G. The problem of determining $sat(K_n^2, K_s^2)$ was raised by Zykov [19] in 1949, and independently by Erdős, Hajnal, and Moon [9] in 1964. They showed that $sat(K_n^2, K_s^2) = \binom{n}{2} - \binom{n-s+2}{2}$. Their result was later generalised by Bollobás [3] who showed that $sat(K_n^r, K_s^r) = \binom{n}{r} - \binom{n-s+r}{r}$.

We say that an r-uniform hypergraph $H \subseteq G$ is *weakly* F-saturated in G if H does not contain any copy of F, but the edges of $E(G) \setminus E(H)$ admit an ordering $e_1, \dots, e_k$ such that for each $i \in [k]$, the hypergraph $H_i = H \cup \{e_1, \dots, e_i\}$ contains a copy of F containing the edge e_i. We define $w\text{-}sat(G,F)$ to be the minimum number of edges in a weakly F-saturated subhypergraph in G. Note that the saturation is in fact a restriction of the weak saturation, where one is allowed only to add all the edges at once (simultaneously). Hence, any H which is F-saturated in G is also weakly F-saturated in G, and thus $w\text{-}sat(G,F) \le sat(G,F)$. As such, the weak saturation can be viewed as a natural extension of the saturation. The problem of determining $w\text{-}sat(K_n^r, K_s^r)$ was first raised by Bollobás [4], who conjectured that $w\text{-}sat(K_n^r, K_s^r) = sat(K_n^r, K_s^r)$. Using ingenious

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algebraic methods, this conjecture was verified by Alon [1], Frankl [10], Kalai [11, 12], and Lovász [17]:

Theorem 1.1 [1, 10–12, 17].

$$w\text{-}sat(K_n^r, K_s^r) = sat(K_n^r, K_s^r) = \binom{n}{r} - \binom{n-s+r}{r}.$$

The binomial random graph $G(n,p)$ is obtained by retaining every edge of K_n independently with probability p. Similarly, the binomial random r-uniform hypergraph $G^r(n,p)$ is obtained by retaining every edge of K_n^r independently with probability p. In 2017, Korándi and Sudakov [15] initiated the study of the saturation and the weak saturation in random graphs.

Theorem 1.2 [15]. *Let $p \in (0,1)$ and let s be a constant. Then,* ***whp****,*

(a) $sat(G(n,p), K_s^2) = (1+o(1))n \log_{\frac{1}{1-p}} n.$

(b) $w\text{-}sat(G(n,p), K_s^2) = w\text{-}sat(K_n^2, K_s^2).$

Note that there is quite a stark difference between the (strong) K_s-saturation number and the weak K_s-saturation number in $G(n,p)$. For the weak K_s-saturation, **whp** the answer is the same as in K_n, whereas for the K_s-saturation **whp** there is an additional $\log n$ factor compared with K_s-saturation in K_n.

Since the work of Korándi and Sudakov, there have been several papers devoted to the study of $sat(G(n,p), F)$ and $w\text{-}sat(G(n,p), F)$ for graphs F which are not cliques, in particular when p is constant. Mohammadian and Tayfeh-Rezaie [18] and Demyanov and Zhukovskii [6] proved tight asymptotics for stars, $F = K_{1,s}$. Considering cycles, Demidovich, Skorkin, and Zhukovskii [5] showed that **whp** $sat(G(n,p), C_m) = n + \Theta\left(\frac{n}{\log n}\right)$ for $m \geq 5$, and **whp** $sat(G(n,p), C_4) = \Theta(n)$. Considering a more general setting, Diskin, Hoshen, and Zhukovskii [8] showed that for every graph F, **whp** $sat(G(n,p), F) = O(n \log n)$, and gave sufficient conditions for graphs F for which **whp** $sat(G(n,p), F) = \Theta(n)$. As for the weak saturation, Kalinichenko and Zhukovskii [14] gave sufficient conditions on F for which **whp** $w\text{-}sat(G(n,p), F) = w\text{-}sat(K_n, F)$, and Kalinichenko, Miralaei, Mohammadian, and Tayfeh-Rezaie [13] showed that, for any graph F, **whp** $w\text{-}sat(G(n,p), F) = (1+o(1))\, w\text{-}sat(K_n, F)$.

Another natural and challenging direction is to extend the results of Korándi and Sudakov to hypergraphs. Indeed, in their paper from 2017, they asked whether their results could be extended to r-uniform hypergraphs. In this paper, we answer that question in the affirmative:

Theorem 1. *Let $p \in (0,1)$, and let $2 \leq r < s$ be constants. Then,* ***whp****,*

(a) $w\text{-}sat(G^r(n,p), K_s^r) = w\text{-}sat(K_n^r, K_s^r).$

(b) $sat(G^r(n,p), K_s^r) = (1+o(1))\binom{n}{r-1} \log_{\frac{1}{1-p^{r-1}}} n.$

Similarly to the case of $r = 2$, for the weak saturation **whp** the answer is the same as in the complete r-uniform hypergraph, whereas for the saturation, there is **whp** an additional $\log n$ factor compared with the complete r-uniform hypergraph. Furthermore, the proof of Theorem 1(a) can be extended to $p \geq n^{-\alpha}$, for some appropriately chosen constant $\alpha > 0$. In fact, we believe there exists a threshold for the property of the weak saturation *stability*, that is, for when $w\text{-}sat(G^r(n,p), K_s^r) = w\text{-}sat(K_n^r, K_s^r)$ – for graphs, the respective result was obtained by Bidgoli, Mohammadian, Tayfeh-Rezaie, and Zhukovskii [2] – though the problem of finding the threshold itself looks very challenging.

The proof of Theorem 1(b) follows the ideas appearing in [15], with some adaptations and more careful treatment. Since the key ideas generalise from the graph setting to the hypergraph

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setting, we present only an outline of the proof in Section 4. The full proof is presented in the Appendix of the ArXiv version of the paper [7].

The weak saturation problem for random hypergraphs turns out to be much harder and requires several new ideas. Indeed, for the upper bound of Theorem 1(a), a novel and delicate construction is needed, see Section 2 for an outline of the proof.

The paper is structured as follows. In Section 2 we demonstrate the key ideas of the proof of Theorem 1(a) by giving a detailed sketch for the case where $r = 3$. In Section 3, we prove Theorem 1(a). In Section 4, we provide a detailed sketch of the proof for Theorem 1(b), with the complete proof appearing in the Appendix of the ArXiv version of the paper [7]. Throughout the paper, we will use the shorthand $G := G^r(n, p)$ and abbreviate r-uniform hypergraph to r-graph.

2. A detailed sketch of the proof of Theorem 1(a) for $r = 3$

Let us begin with the lower bound. To that end, note that **whp** all the edges of $\binom{[n]}{r} \setminus E(G)$ can be activated from the edges of G. Thus, a weakly K_s^r-saturated subgraph of G is **whp** also a weakly K_s^r-saturated of K_n^r, and therefore **whp** $w\text{-}sat(G, K_s^r) \geq w\text{-}sat(K_n^r, K_s^r)$.

The proof of the upper bound follows from a delicate construction. For the sake of clarity of presentation, and since it already illustrates all the main issues one needs to overcome, we consider the case when $r = 3$ in this sketch.

The natural first step for finding a weakly K_s^r-saturated subhypergraph in G, similar in spirit to the approach taken in the case when the host graph is a complete hypergraph, is to choose a *core* $C_0 \subseteq [n]$ of size $s - 3$, such that $G[C_0] \cong K_{s-3}^3$. Observe that, in the case of the complete hypergraph, for every edge $e \subseteq [n] \setminus C_0$ and for every $S \subsetneq e$, we have that $K_n^3[C_0 \cup S] \cong K_{|C_0 \cup S|}^3$. In this case, we can then set H to be the graph whose edges are of the form $f \subseteq [n]$ with $f \cap C_0 \neq \varnothing$. Then, we may activate all the remaining edges $e \subseteq [n] \setminus C_0$, since in K_n^3 we have that $K_n^3[C_0 \cup e] \cong K_s^3$, and all the edges induced by $C_0 \cup e$, except e, are in H. This immediately gives the desired upper bound of $w\text{-}sat(K_n^3, K_s^3) \leq \binom{n}{3} - \binom{n-s+3}{3}$.

However, when the host graph is the random graph G, we may have edges $e \subseteq [n] \setminus C_0$ and sets $S \subsetneq e$ such that $G[C_0 \cup S]$ is not a clique. Thus, in what follows, for every such problematic set S, we will choose an additional core (that is, a special $(s-3)$-subset of $[n]$).

The proof is divided into four main stages. In the first stage, we define appropriate cores for all relevant subsets of vertices $S \subset [n] \setminus C_0$ of size at most two. The second stage describes the construction of a weakly K_s^3-saturated subhypergraph H of G, making use of the properties of the cores established in the first stage. In the third stage, we count the number of edges of H. The fourth and final stage proves that H is indeed a weakly K_s^3-saturated subhypergraph G.

Defining cores

Our cores will be disjoint subsets of size $s - 3$. Let us try to give a rough intuition for the purpose of our cores. For every edge e, we aim to find a set C of size $s - 3$ such that $G[e \cup C] \cong K_s^3$. Further, we want that either $H[e \cup C]$ will form a clique without e, and then e can be immediately activated, or that after several steps of activation of other edges, $H[e \cup C]$ together with the previously activated edges will form a clique without e. To allow for this activation process, we will not choose cores for the edges themselves (as this will be inefficient and require too many edges in H), but rather assign cores to sets of vertices S of size at most two. Each such core allows us to either activate immediately edges containing S, or to do so through some 'chain' of activation. As each core will contribute additional edges to H, we aim to minimise the number of cores. We now turn to choosing the cores.

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We begin by choosing C_0, where the only requirement on the set C_0 is that $G[C_0] \cong K^3_{s-3}$ (**whp** such a set exists in G). We remark that we will use C_0 to activate edges $e \subseteq [n] \setminus C_0$ satisfying that $G[e \cup C_0] = K^3_s$.

We now turn to define a core for every singleton $v \in [n] \setminus C_0$ (noting that cores for different vertices may coincide). For every $v \in [n] \setminus C_0$, if $G[C_0 \cup \{v\}] \cong K^3_{s-2}$, we choose the core $C_v := C_0$. Otherwise, if $G[C_0 \cup \{v\}] \ncong K^3_{s-2}$, we choose a core C_v, disjoint from v, with the following property.

$$\text{Every edge } f \subseteq C_v \cup \{v\} \cup C_0 \text{ with } f \cap C_v \neq \varnothing \text{ is in } G. \tag{1}$$

Indeed, **whp** this is possible for every such $v \in [n] \setminus C_0$.

Let us mention two observations.

(O1) We have that $G[C_0 \cup C_v] \cong K^3_{|C_0 \cup C_v|}$, and $|C_0 \cup C_v| = s - 3$ when $C_v = C_0$, and $|C_0 \cup C_v| = 2(s-3)$ otherwise. Indeed, every $e \subseteq C_0$ is in G since $G[C_0] \cong K^3_{s-3}$, and every remaining edge $e \subseteq C_0 \cup C_v$ with $e \cap C_v \neq \varnothing$ is in G by Property (1).

(O2) If $u \in C_v$ for some v, we have that $G[C_0 \cup \{u\}] \cong K^3_{s-2}$, and therefore $C_u = C_0$.

We remark that in the final activation step of the argument, for every $v \in [n]$, using the edges activated through C_0, we will be able to activate edges of the form $e = \{v, x, y\} \subseteq [n] \setminus C_v$ where $G[e \cup C_v] \cong K^3_s$. Crucially, note that for such edges e (when $C_v \neq C_0$), we have $G[e \cup C_0] \ncong K^3_s$, and thus choosing an additional core was indeed necessary.

Still, there will be edges $e = \{v_1, v_2, v_3\}$, where for every $i \in [3]$, $G[e \cup C_{v_i}] \ncong K^3_s$. We thus turn to defining cores for pairs of vertices. We will define cores for all pairs of vertices $S = \{v_1, v_2\}$ which satisfy that $v_1 \notin C_{v_2}$ and $v_2 \notin C_{v_1}$ (the reason for defining cores for such vertices will become clearer in the last two steps of the argument: when counting the number of edges in H and when considering the activation of edges). We will ensure that C_S satisfies the following property.

$$\text{Every edge } f \subseteq S \cup C_0 \cup C_{v_i} \cup C_S \text{ with } f \cap C_S \neq \varnothing \text{ is in } G, \text{ for every } i \in \{1, 2\}. \tag{2}$$

In all the cases when one of the sets C_0, C_{v_1}, C_{v_2} can be chosen for the role of C_S so that Property (2) is satisfied, we will indeed set C_S to be equal to such a core. However, in other cases, we will have to assign a new core for S. Let us now describe in detail how we define the core C_S.

If $G[C_0 \cup \{v_1, v_2\}] \cong K^3_{|C_0 \cup \{v_1, v_2\}|}$, then we set $C_S = C_0$. Assume now that it is not the case. If for some $i \in \{1, 2\}$, every edge $f \subseteq C_{v_i} \cup S \cup C_0$ with $f \cap C_{v_i} \neq \varnothing$ is in G, then we set $C_S = C_{v_i}$ (since verifying that (2) holds for such a choice of C_S is somewhat technical, we omit the explanation here, and refer to the complete proof in Section 3). Otherwise, a new choice for the core C_S should be made to later activate edges containing S. **Whp** for any such S we may find a core C_S, disjoint from S, which satisfies Property (2).

Let us mention a few observations. Below, we fix $S = \{v_1, v_2\}$.

(O3) For every $i \in \{1, 2\}$ and $c \in C_S$, we have that $C_{\{v_i, c\}} = C_{v_i}$. Indeed, we need to show that every edge $f \subseteq C_{v_i} \cup \{v_i, c\} \cup C_0$ with $f \cap C_{v_i} \neq \varnothing$ is in G. Now, if $c \in f$, then $f \cap C_S \neq \varnothing$ and thus, by Property (2), we have that f is in G. Otherwise, $f \subseteq C_{v_i} \cup \{v_i\} \cup C_0$ (and intersects with C_{v_i}), and thus, by Property (1), we have that f is in G.

(O4) For every $i \in \{1, 2\}$, we have that $G[C_0 \cup C_{v_i} \cup C_S] \cong K^3_{|C_0 \cup C_{v_i} \cup C_S|}$. Indeed, by Observation (O1) we have that $G[C_0 \cup C_{v_i}] \cong K^3_{|C_0 \cup C_{v_i}|}$, and by Property (2), every edge $f \subseteq C_0 \cup C_{v_i} \cup C_S$ with $f \cap C_S \neq \varnothing$ is in G.

(O5) If $X \subseteq [n] \setminus C_0$ satisfies that $G[C_0 \cup X] \cong K^3_{|C_0 \cup X|}$, then $C_S = C_0$ for every $S \subseteq X$ of size at most two. In particular, if $Y \subseteq C_{v_i} \cup C_S$ with $|Y| \leq 2$, by (O4) we have that $G[C_0 \cup Y] \cong K^3_{|C_0 \cup Y|}$ and therefore $C_Y = C_0$.

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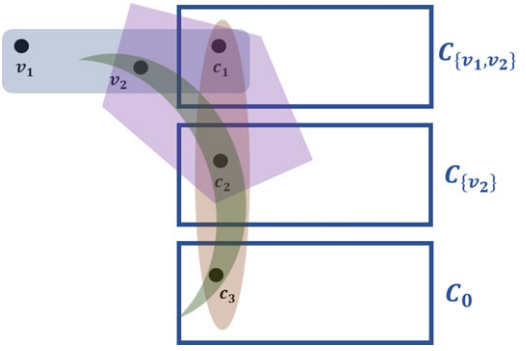


Figure 1. Illustration of a *chain* of cores, C_0, C_{v_2}, and $C_{\{v_1,v_2\}}$, together with edges that were added to H. The edge $\{v_1, v_2, c_1\}$ is added to H by (4). The edge $\{v_2, c_1, c_2\}$ is added to H by (4), noting that $C_{\{v_2,c_1\}} = C_{v_2}$ by (O3). The edge $\{c_1, c_2, c_3\}$ is added to H by (4) since $C_{\{c_1,c_2\}} = C_0$ by (O5). Finally, the edge $\{v_2, c_2, c_3\}$ is added to H by (3).

Constructing H

Let $H \subseteq G$ be a subhypergraph on $V(G) = [n]$, consisting of the following edges:

(1) Every edge $e \subseteq C_0$ is in H.

(2) For every $v \in [n] \setminus C_0$, we add to H all edges of the form $\{v\} \cup C'$ for every $C' \subseteq C_v$ of size two. These edges are in G by Property (1).

(3) For every $v \in [n] \setminus C_0$, we add to H all edges of the form $\{v, c_0, c\}$ for every $c_0 \in C_0$ and $c \in C_v$. These edges are in G by Property (1).

(4) For every $S = \{v_1, v_2\} \subseteq [n]$ for which C_S is defined, we add to H all edges of the form $S \cup \{c\}$ for every $c \in C_S$. These edges are in G by Property (2).

While Steps (1) and (2) are rather transparent, we refer to Figure 1 for an illustration of edges added to H at Steps (3) and (4).

Number of edges in H

The number of edges we have added to H is at most

$$\binom{s-3}{3} + \binom{n-s+3}{1}\binom{s-3}{2} + \binom{n-s+3}{2}\binom{s-3}{1} = \binom{n}{3} - \binom{n-s+3}{3}.$$

Indeed, the first term comes from the edges induced by C_0, which we have added to H at Step (1).

The second term comes from the edges which were added at Step (2). For every $v \in [n] \setminus C_0$, we have chosen C_v (possibly $C_v = C_0$), $|C_v| = s - 3$, and added all edges of the form $\{v\} \cup C'$ where $C' \subseteq C_v$ with $|C'| = 2$. Thus, we have $\binom{n-s+3}{1}$ choices for $v \in [n] \setminus C_0$ and $\binom{s-3}{2}$ choices for $C' \subseteq C_v$ of size two.

Finally, the third term comes from the edges which were added at Step (3) and at Step (4). To see this, consider a pair of vertices $S = \{u, v\} \subseteq [n] \setminus C_0$. Recall that we define C_S only if $v \notin C_u$ and $u \notin C_v$. Thus, if C_S is not defined, then WLOG $v \in C_u$. In this case, since $v \notin C_0$, we must have $C_u \cap C_0 = \varnothing$ (recall that the cores are either identical or vertex-disjoint). In Step (3), we add all the edges $\{v, u, c_0\}$ for each of the $\binom{s-3}{1}$ possibilities for $c_0 \in C_0$ (note here, that when considering all pairs $\{u, v\}$ in this manner, we in fact cover all edges added at Step (3)). Otherwise, C_S is defined, and at Step (4) we add all edges $\{u, v, c\}$ for each of the $\binom{s-3}{1}$ possibilities for $c \in C_S$.

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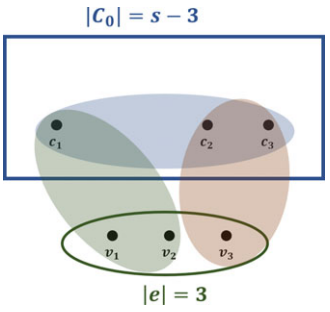


Figure 2. The edge e closes a copy with C_0 and can thus be activated. The three types of edges that are in H appear in shaded colours. The edges induced by C_0 appear in blue, and were added to H at Step (1). The edges that contain one vertex from e and two vertices from C_0 appear in red, and were added to H at Step (2). The edges that contain two vertices from e and one vertex from C_0 appear in green, and were added to H at Step (4).

Activating the edges in $E(G) \setminus E(H)$

First, note that we can activate all edges $e \subseteq [n] \setminus C_0$, such that $G[C_0 \cup e] \cong K_s^3$. Indeed, by Observation (O5), for every non-empty $S \subsetneq e$ we have $C_S = C_0$. Thus, by Steps (1), (2), and (4), all edges induced by $C_0 \cup \{e\}$, except e, are in H. We can thus activate e, which closes a copy of K_s^3 with C_0. See Figure 2 for illustration.

We can now activate the remaining edges $e = \{v_1, v_2, v_3\} \subseteq [n] \setminus C_0$. To that end, **whp** we can choose an *auxiliary clique* $\tilde{C} := \tilde{C}(e)$ disjoint with $e \cup \bigcup_{i,j\in[3]} (C_{v_i} \cup C_{\{v_i,v_j\}})$ (in fact, the union is over $i, j \in [3]$ for which such cores are defined), such that $G[\tilde{C} \cup e] \cong K_s^3$ and for every $i \neq j \in \{1, 2, 3\}$, every edge $f \subseteq \tilde{C} \cup \{v_i, v_j\} \cup C_{\{v_i,v_j\}} \cup C_{v_i} \cup C_0$ with $f \cap \tilde{C} \neq \varnothing$ is in G. We will utilise the following observation.

(O6) For every $c \in \tilde{C}$ and every $i \in [3]$, we have that $c \notin C_{v_i}$ and thus $C_{\{v_i,c\}}$ is defined. Further, $C_{\{v_i,c\}} = C_{v_i}$. To show that, we need to argue that every edge $f \subseteq C_{v_i} \cup \{v_i, c\} \cup C_0$ which intersects with C_{v_i} is in G. Indeed, if $c \in f$, then this follows from our choice of $\tilde{C}$, and otherwise, this follows from (1).

Our goal is to show that all edges induced by $e \cup \tilde{C}$, except for e, are either in H or can be activated. We will then have that e closes a copy of K_s^3 with $\tilde{C}$, and can thus be activated as well. We do so in several activation steps, where some activation steps may depend on previous activation steps (see also Figure 3 for illustration). In what follows, we always assume $i, j \in [3]$, $i \neq j$.

(A1) **Edges f induced by $\tilde{C} \cup C_{\{v_i,v_j\}} \cup C_{v_i}$.**
We claim that such f closes a copy of K_s^3 with C_0 and can thus be activated. By the choice of $\tilde{C}$ and Observation (O4),

$$G\left[C_0 \cup \left(\tilde{C} \cup C_{\{v_i,v_j\}} \cup C_{v_i}\right)\right] \cong K^3_{\left|C_0 \cup \left(\tilde{C} \cup C_{\{v_i,v_j\}} \cup C_{v_i}\right)\right|}.$$

Now, consider an edge $f' = \{c_0, x, y\} \subseteq C_0 \cup \left(\tilde{C} \cup C_{\{v_i,v_j\}} \cup C_{v_i}\right)$ with $c_0 \in C_0$ and $f' \nsubseteq C_0$. We have that, in particular, $G[\{x, y\} \cup C_0] \cong K^3_{|\{x,y\}\cup C_0|}$, and thus $C_{\{x,y\}} = C_0$ by (O5). Hence, we added f' to H at Step (4) (where $\{x, y\}$ play the role of S). Therefore, we can now activate any edge $f \subseteq \tilde{C} \cup C_{\{v_i,v_j\}} \cup C_{v_i}$ since it closes a copy of K_s^3 with C_0 (together with the edges in H).

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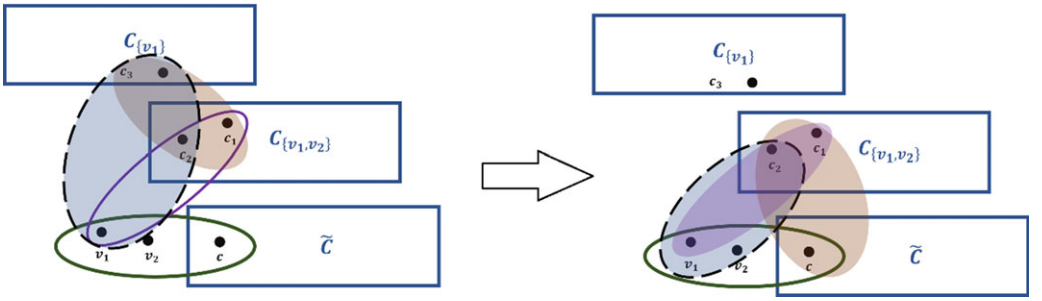


Figure 3. Illustration of a *chain* of activation, with the complexity of the construction evident already when $r = 3$. Towards activating an edge $\{v_1, v_2, v_3\}$ with $\tilde{C}$, we need to activate edges of the form $\{v_1, v_2, c\}$ where $c \in \tilde{C}$. To that end, we first activate all edges that form a clique with C_0, and in particular, all edges induced by $C_{v_1} \cup C_{\{v_1,v_2\}} \cup \tilde{C}$. Then, as the left side illustrates, we can activate the edges of the form $\{v_1, c_1, c_2\}$ as they form a clique with C_{v_1}. Indeed, the edge $\{c_1, c_2, c_3\}$ forms a clique with C_0, and the edge $\{v_1, c_2, c_3\}$ is in H since $C_{\{v_1,c_2\}} = C_{v_1}$ (and was added at Step (4)). We can then, as the right side illustrates, turn our attention to edges of the form $\{v_1, v_2, c\}$, which will close a clique with $C_{\{v_1,v_2\}}$. Here, the edge $\{c, c_1, c_2\}$ closes a clique with C_0 and thus has already been activated, and the edge $\{v_1, v_2, c_2\}$ is in H by Step (4).

(A2) **Edges of the form $f = \{v, c_1, c_2\}$ where $v \in \{v_i, v_j\}$, $c_1, c_2 \in C_S \cup \tilde{C}$.**
We claim that f closes a copy of K_s^3 with C_v and can thus be activated. Set $S = \{v_i, v_j\}$. We have already activated edges induced by $C_S \cup C_v \cup \tilde{C}$ during (A1). In Step (2), we added to H all edges of the form $\{v, x_1, x_2\}$ where $x_1, x_2 \in C_v$. As for edges $\{v, c, x\}$ where $c \in \{c_1, c_2\}$ and $x \in C_v$, note that by (O3) and (O6), $C_{\{v,c\}} = C_v$. Thus, $\{v, c, x\}$ is in H by Step (4). Thus, f closes a copy of K_s^3 with C_v (together with edges of H and edges activated during (A1)) and may thus be activated. See the left-hand side of Figure 3 for illustration.

(A3) **Edges of the form $f = \{v_i, v_j, c\}$ where $c \in \tilde{C}$.** Set $S = \{v_i, v_j\}$. We claim that f closes a copy of K_s^3 with C_S and can thus be activated. Edges of the form $\{v_i, v_j, x\}$ where $x \in C_S$ have been added to H at Step (4). Edges induced by $\{c\} \cup C_S$ were activated in (A1). Edges of the form $\{v, x_1, x_2\}$ where $v \in S$ and $x_1, x_2 \in C_S$ were activated in (A2). Furthermore, edges of the form $\{v, c, x\}$ where $v \in S$ and $x \in C_S$ were activated in (A2). Thus, f closes a copy of K_s^3 with C_S (together with edges of H and edges activated during (A1) and (A2)), and may thus be activated. See the right-hand side of Figure 3 for illustration.

Thus, e closes a copy of K_s^3 together with $\tilde{C}$ (with the edges of H and edges activated during (A1)–(A3)).

Finally, we are left with edges $e = \{v_1, v_2, v_3\}$ which intersect with C_0. The argument for these edges is similar to before, and we omit the details here, referring to the complete proof present in Section 3.

3. Weak saturation

Let $r \geq 3$ and $s \geq r + 1$ be integers. Let $p \in (0, 1]$ be a constant. Recall that $G := G^r(n, p)$. Set $\ell := s - r$.

We start we the lower bound, which is the easiest part of the paper, and follows from the fact that **whp** G is weakly saturated in K_n

Claim 3.1. ***Whp*** $w\text{-}sat\left(G, K_s^r\right) \geq w\text{-}sat(K_n^r, K_s^r)$.

Proof. Let us first show that **whp** G is weakly saturated in K_n^r. To that end, fix some edge $e \in K_n^r \setminus G$. The probability that for a fixed set S of size ℓ we have that $e \cup G[S \cup e] \not\supseteq K_s^r$ is $1 - p^{\binom{s}{r}-1}$. This event is independent for every two disjoint sets of size ℓ, and thus the probability that e does

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not form a copy of K_s^r in $G \cup e$ is at most $\left(1-p^{\binom{s}{r}-1}\right)^{\lfloor \frac{n}{\ell} \rfloor} = \exp(-\Theta(n))$. The union bound over $\binom{n}{r} \le n^r$ edges $e \in K_n^r$ shows that **whp** every edge $e \in K_n^r$ belongs to a copy of K_s^r in $G \cup e$, and therefore **whp** G is weakly saturated in K_n^r.

Now, if H is weakly saturated in G, then we can add to H edges one-by-one until we obtain G, and then keep adding edges until we reach K_n^r. Thus, **whp** any such H is weakly saturated in K_n^r as well. But then, $w\text{-}sat\left(G, K_s^r\right) \ge w\text{-}sat(K_n^r, K_s^r)$. □

The rest of this section is devoted to the proof of the upper bound. In Section 3.1, we lay the groundwork for our proof: define the cores, construct the weakly saturated subhypergraph H, and count its edges. In Section 3.2, we show that there exists an ordering of the edges under which we can activate all edges of G that are not in H.

3.1 Laying the groundwork

Before we delve into the fine details, we note that what follows will be a natural extension of the construction in Section 2 to general r (together with their formalisation). We will once again define a core C_0, and inductively define cores for sets $S \subseteq V(G) \setminus C_0$ with size $1 \le |S| \le r-1$. The construction of the cores will naturally have a *chain* property, that is, the core of S may (and will) depend on the cores of $S' \subsetneq S$. Once again, there will be several sets S for which we will not define a core, and instead draw relevant edges with vertices from C_0.

We begin by partitioning $V(G)$ into $\lfloor \frac{n}{\ell} \rfloor$ sets of size ℓ, denoted by $Q_1, \dots, Q_{\lfloor \frac{n}{\ell} \rfloor}$. Let us further write $\mathcal{Q} := \{Q_1, \dots, Q_{\lfloor \frac{n}{\ell} \rfloor}\}$. Throughout the proof, we will use the following probabilistic lemma.

Lemma 3.2. *Let $k \ge 0$ be a constant. Then, **whp**, for every $S \subseteq V(G)$ with $|S| = k$, there exists Q_i, with $1 \le i \le \lfloor \frac{n}{\ell} \rfloor$, such that $S \cap Q_i = \varnothing$ and every edge $e \subseteq Q_i \cup S$ with $e \cap Q_i \ne \varnothing$ is in G.*

Proof. Fix S and fix i such that $Q_i \cap S = \varnothing$. The probability that every edge $e \subseteq Q_i \cup S$ with $e \cap Q_i \ne \varnothing$ is in G is at least $p^{\binom{\ell+k}{r}}$. There are at least $\lfloor \frac{n}{\ell} \rfloor - k$ different i such that $Q_i \cap S = \varnothing$. Therefore, the probability there doesn't exist such a Q_i is at most

$$\left(1-p^{\binom{\ell+k}{r}}\right)^{\lfloor \frac{n}{\ell} \rfloor - k} = \exp(-\Theta(n)),$$

where we used our assumptions that k, r, ℓ, and p are constants. There are $\binom{n}{k}$ ways to choose S, and thus by the union bound, the probability of violating the statement of the lemma is at most

$$\binom{n}{k} \exp(-\Theta(n)) = o(1),$$

as required. □

Defining the cores

Let i_0 be the first index in $\left[\lfloor \frac{n}{\ell} \rfloor\right]$ such that $G[Q_{i_0}] \cong K_\ell^r$, and let us set $C_0 := Q_{i_0}$ (note that by Lemma 3.2 **whp** such an i_0 exists). For every $j > 0$, let i_j be the j-th index in $\left[\lfloor \frac{n}{\ell} \rfloor\right]$ for which we have that $G[C_0 \cup Q_{i_j}] \cong K_{2\ell}^r$ (that is, the j-th occurrence of some $Q \in \mathcal{Q}$ for which it holds). We then set $C_j := Q_{i_j}$ (once again, note that since $G[C_0] \cong K_\ell^r$, by Lemma 3.2 **whp** such a C_j exists). We call these sets *cores*, and we enumerate them $C_0, C_1, \dots, C_m$. Note that for every $i \ne j \in [m]$, $C_i \ne C_j$, as they are two different $Q \ne Q' \in \mathcal{Q}$. We continue assuming these cores have been defined deterministically.

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Assigning cores to sets

Set $C_\varnothing = C_0$. For every vertex $v \in V \setminus C_0$, let $i(v)$ be the first index such that the following holds.

1. Every edge $e \subseteq \{v\} \cup C_0 \cup C_{i(v)}$ with $e \cap C_{i(v)} \neq \varnothing$ is in G.
2. $\{v\} \cap C_{i(v)} = \varnothing$.

We then set $C_{\{v\}} = C_{i(v)}$. Note that by the properties of the cores and by Lemma 3.2, **whp** such a $C_{\{v\}}$ exists for every $v \in V(G) \setminus C_0$. Furthermore, observe that the properties of $C_{\{v\}}$ imply that $G[\{v\} \cup C_{\{v\}}] \cong K^r_{\ell+1}$.

Now, we define cores for suitable subsets outside of C_0 by induction on their size. For every $j \in [2, r-1]$ and for every $S \subseteq V(G) \setminus C_0$ of size j, if

$$\text{there is no } S' \subsetneq S, \text{ such that } C_{S'} \text{ is defined and } S \setminus S' \subseteq C_{S'}, \qquad (S \text{ is core-definable})$$

we define C_S in the following way. Let $i(S)$ be the first index in $[m]$ such that the following holds for every integer $t \in [j-1]$ and a sequence $\varnothing = S_0 \subsetneq \ldots \subsetneq S_t \subsetneq S$ for which $C_{S_0}, \ldots, C_{S_t}$ are defined.

(P1) Every edge $e \subseteq S \cup C_{i(S)} \cup C_{S_0} \cup \ldots \cup C_{S_t}$ with $e \cap C_{i(S)} \neq \varnothing$ is in G.

(P2) $G\left[C_{i(S)} \cup C_{S_0} \cup \ldots \cup C_{S_t}\right] \cong K^r_{|C_{i(S)} \cup C_{S_0} \cup \ldots \cup C_{S_t}|}$.

(P3) $S \cap C_{i(S)} = \varnothing$.

We then set $C_S = C_{i(S)}$. Note that we allow $S_0 = \varnothing$ in order to include edges intersecting with $C_\varnothing = C_0$.

Note that the condition in Property (P1) applies only to edges intersecting with $C_{i(S)}$. Furthermore, by induction, we have that $G\left[S_0 \cup C_{S_0} \cup \ldots \cup C_{S_t}\right] \cong K^r_{|S_0 \cup C_{S_0} \cup \ldots \cup C_{S_t}|}$, and thus the condition in Property (P2) also concerns only edges intersecting with $C_{i(S)}$. Thus, by these properties and by Lemma 3.2, **whp** for every $S \subseteq V(G) \setminus C_0$ of size j such that there is no $S' \subsetneq S$ with $C_{S'}$ defined and $S \setminus S' \subseteq C_{S'}$, we can find such a C_S. In what follows, we assume this holds deterministically.

Note that we have defined a core C_S for every set $S \subseteq V(G) \setminus C_0$ of size at most $r-1$ which is core-definable. Furthermore, if C_S is not defined, then there exists $S' \subseteq S$ for which $C_{S'}$ is defined and $S \setminus S' \subseteq C_{S'}$. Finally, note that

$$i(S') \leq i(S) \quad \text{for every } S' \subseteq S \text{ such that } C_{S'} \text{ and } C_S \text{ are defined.} \qquad (3)$$

Indeed, the Properties (P1) through (P3) are closed under inclusion, and thus any index satisfying the properties for S must already satisfy the properties for S'.

Constructing H

Let $H \subseteq G$ be a subhypergraph consisting of the following edges:

(E1) Every edge $e \subseteq C_0$ is in H.

(E2) For every $\varnothing \neq S \subseteq V(G) \setminus C_0$ for which C_S is defined, we add to H all edges of the form $S \cup C'$ for every $C' \subseteq C_S$ of size $r - |S|$.

(E3) For every $\varnothing \neq S \subseteq V(G) \setminus C_0$ for which C_S is defined, we add to H all edges of the form $S \cup C' \cup C'_0$ for every $\varnothing \neq C' \subseteq C_S$ and $\varnothing \neq C'_0 \subseteq C_0$ satisfying $|C' \cup C'_0| = r - |S|$.

Note that edges of type (E1) are in G by the definition of the cores, and edges of type (E2) and type (E3) are in G by Property (P1) (and the first property when defining cores for singletons).

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Number of edges in H

Let us bound from above the number of edges added in each step of the construction.

First, at Step (E1), we add $\binom{\ell}{r}$ edges to H induced by C_0. We now turn to Steps (E2) and (E3). Let us set

$$A = \left\{S\colon \varnothing \neq S \subseteq V(G) \setminus C_0 \wedge C_S \textit{ is defined}\right\} \quad \text{and}$$
$$B = \left\{S \cup C'\colon \varnothing \neq S \subseteq V(G) \setminus C_0 \wedge C_S \textit{ is defined} \wedge C' \subseteq C_S\right\}.$$

We have the following.

- In Step (E2), we add the edges $S \cup C'$ for every $S \in A$ and $C' \subseteq C_S$ of size $r - |S|$.
- In Step (E3), we add the edges $X \cup C'_0$ for every $X \in B$ and $C'_0 \subseteq C_0$ of size $r - |X|$.

Thus, the number of edges considered at Step (E3) is at most

$$\sum_{S \in A} \binom{\ell}{r - |S|} = \sum_{\substack{S \subseteq V(G)\setminus C_0 \\ C_S \text{ is defined}}} \binom{\ell}{r - |S|}.$$

Note that, for every $S \subseteq V(G) \setminus C_0$ such that C_S is defined and for every $\varnothing \neq C' \subseteq C_S$, we have that $C_{S \cup C'}$ is not defined. Indeed, in that case, we have that $S \subseteq S \cup C'$, C_S is defined, and $(S \cup C') \setminus S \subseteq C_S$. Thus, by definition, $S \cup C'$ is not core-definable. Hence, we have that

$$B \subseteq \{X \subseteq V(G) \setminus C_0\colon C_X \text{ is not defined}\}.$$

Moreover, the number of added edges at Step (E3) is at most

$$\sum_{\substack{X \subseteq V(G)\setminus C_0 \\ C_X \text{ is not defined}}} \binom{\ell}{r - |X|}.$$

Therefore, the number of edges in H is at most

$$\binom{\ell}{r} + \sum_{i=1}^{r-1} \left(\sum_{\substack{S \subseteq V(G)\setminus C_0, \quad |S|=i \\ C_S \text{ is defined}}} \binom{\ell}{r-i} + \sum_{\substack{X \subseteq V(G)\setminus C_0, \quad |X|=i \\ C_X \text{ is not defined}}} \binom{\ell}{r-i} \right)$$
$$= \binom{\ell}{r} + \sum_{i=1}^{r-1} \binom{n-\ell}{i}\binom{\ell}{r-i} = \binom{n}{r} - \binom{n-\ell}{r}.$$

3.2 Activating the remaining edges

The argument for activating the remaining edges will be a natural extension of the argument given for the case $r = 3$ in Section 2, together with its formalisation.

Assigning an auxiliary core to every edge

Let us fix an edge $e \in E(G)$. Note that we only assign cores to sets of size at most $r - 1$, that is, there is no C_e defined. By Lemma 3.2 applied with

$$S = e \cup \bigcup_{\substack{A \subsetneq e \\ A \text{ is core-definable}}} C_A,$$

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we can choose an auxiliary core $\tilde{C} := \tilde{C}(e)$ from $Q_1, \ldots, Q_{\lfloor \frac{n}{\ell} \rfloor}$, such that the following holds. For every $S \subsetneq e$, every $t \in [0, |S| - 1]$, and every $S_0 \subsetneq \cdots \subsetneq S_t \subsetneq S$ such that C_S is defined and C_{S_i} is defined for every $i \in [0, t]$:

($\overline{P}1$) Every edge $f \subseteq S \cup C_S \cup C_{S_0} \cup \cdots \cup C_{S_t} \cup \tilde{C}$ with $f \cap \tilde{C} \neq \varnothing$ is in G.
($\overline{P}2$) $G\left[S_0 \cup C_S \cup \tilde{C} \cup C_{S_0} \cup \cdots \cup C_{S_t}\right] \cong K^r_{|S_0 \cup C_S \cup \tilde{C} \cup C_{S_0} \cup \cdots \cup C_{S_t}|}$.
($\overline{P}3$) $(S \cup C_S \cup C_{S_0} \cup \cdots \cup C_{S_t}) \cap \tilde{C} = \varnothing$.

Lemma 3.2 guarantees the (typical) existence of edges intersecting with $\tilde{C}$. Indeed, the above properties concern only edges intersecting with $\tilde{C}$. To see that in Property ($\overline{P}2$), note that $G\left[S_0 \cup C_S \cup C_{S_0} \cup \cdots \cup C_{S_t}\right] \cong K^r_{|S_0 \cup C_S \cup \tilde{C} \cup C_{S_0} \cup \cdots \cup C_{S_t}|}$ by Property (P2) and thus Property ($\overline{P}2$) indeed concern only edges intersecting with $\tilde{C}$. Lemma 3.2 further guarantees the non-intersection of $\tilde{C}$ with S, which yields Property ($\overline{P}3$).

Core extension property

Recall that, by (3), given $S \subseteq S \cup U \subseteq V(G)$ such that C_S and $C_{S \cup U}$ are defined, then $i(S) \leq i(S \cup U)$. If U satisfies $i(S \cup U) = i(S)$, then $C_{S \cup U} = C_S$. In the following claim, we show that for a given S for which C_S is defined, we have that U is satisfies $i(S \cup U) = i(S)$ whenever $U \subseteq C_{S_0} \cup \cdots \cup C_{S_t}$, for some $S \subsetneq S_0 \subsetneq \ldots \subsetneq S_t$. This 'extension' property of the cores will be important for us throughout the activation process, and in fact, we required Properties (P1) and (P2) when choosing our cores so that this extension property will hold.

Claim 3.3. *Let e_0 be an edge in G. Let $S \subseteq V(G) \setminus C_0$ be such that C_S is defined. For every set U and $t \in [r-1]$ such that $|S \cup U| \leq r - 1$ and $U \subseteq C_{S_0} \cup \cdots \cup C_{S_t} \cup \tilde{C}(e_0)$ for some $S \subsetneq S_0 \subsetneq \ldots \subsetneq S_t \subsetneq e$ for which $C_{S_0}, \ldots, C_{S_t}$ are defined and different than C_S, we have that*

$$C_{S \cup U} = C_S.$$

Proof. We prove by induction on $|S \cup U|$, where the case when $|S \cup U| = 0$ follows trivially.

Let $S' \cup U' \subsetneq S \cup U$ with $S' \subseteq S$ and $U' \subseteq U$ such that $C_{S' \cup U'}$ is defined. Suppose towards contradiction that S' is not core-definable. Then, there exists $S'' \subsetneq S'$ such that $C_{S''}$ is defined and $S' \setminus S'' \subseteq C_{S''}$. But then, by the induction hypothesis, $C_{S'' \cup U'} = C_{S''}$, and by our assumption $(S' \cup U') \setminus (S'' \cup U') \subseteq C_{S''} = C_{S'' \cup U'}$. Therefore, $C_{S' \cup U'}$ is not defined – contradiction. Therefore, $C_{S'}$ is defined, and by the induction hypothesis, we have that $C_{S' \cup U'} = C_{S'}$.

Let us verify that $C_{S \cup U}$ is defined. Suppose towards contradiction that $S \cup U$ is not core-definable. Then, there exists $S' \cup U' \subsetneq S \cup U$ such that $S' \subseteq S$, $U' \subseteq U$, $C_{S' \cup U'}$ is defined, and $(S \cup U) \setminus (S' \cup U') \subseteq C_{S' \cup U'}$. We then have by the above that $C_{S' \cup U'} = C_{S'}$. Suppose first that $U' = U$. Then $(S \cup U) \setminus (S' \cup U') = S \setminus S' \subseteq C_{S'}$, contradicting the fact that C_S is defined. Otherwise, $U' \subsetneq U$. Then, by the induction hypothesis, $C_{S \cup U'} = C_S$. By our assumption, $(S \cup U') \setminus (S' \cup U') \subseteq C_{S' \cup U'}$, contradicting the fact that $C_{S \cup U'}$ is defined. Therefore, we conclude that $C_{S \cup U}$ is defined.

Therefore, it suffices to verify that $S \cup U$ satisfies Properties (P1) through (P3) with respect to C_S. Let $k \geq 1$ be an integer and let $A_1 \subsetneq A_2 \subsetneq \cdots \subsetneq A_k \subsetneq S \cup U$ be such that C_{A_i} is defined for every $i \in [k]$. We can then write $A_i = (A_i \cap S) \cup (A_i \cap U)$. Noting that $|A_i| < |S \cup U|$, by the above and by the hypothesis, we then have that $C_{A_i} = C_{S \cap A_i}$, and therefore when we consider C_{A_i}, we may assume that $A_i \subseteq S$.

1. First, let us show that every edge $e \subseteq (S \cup U) \cup C_S \cup C_{A_1} \cup \cdots \cup C_{A_k}$ with $e \cap C_S \neq \varnothing$ is in G, that is, let us verify Property (P1). Note that, by the above, there is some sequence of $S'_1 \subsetneq \cdots \subsetneq S'_k \subsetneq S$ such that $C_{A_j} = C_{S'_j}$ for every $j \in [k]$. Since $S \subsetneq S_0$, we have that $B_1 \subsetneq \cdots \subsetneq B_m$ where $m = k + t + 2$ and $B_1 = S'_1, \ldots, B_k = S'_k, B_{k+1} = S, B_{k+1} = S_0, \ldots, B_m = S_t$. Thus, if

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$e \cap \tilde{C} \neq \varnothing$, then by Property ($\overline{P}1$) with B_m we have that e is in G. Otherwise, let $\tau \in [t]$ be the maximal index such that $e \cap C_{S_\tau} \neq \varnothing$. We may assume that $\tau \geq 1$, otherwise $e \subseteq S \cup C_S \cup C_{A_1} \cup \ldots \cup C_{A_k}$ and then the above follows from Property (P1) with respect to S and C_S. Then, choosing $B_1 = A_1, \ldots, B_k = A_k, B_{k+2} = S, B_{k+3} = S_1, \ldots, B_m = S_\tau$, since $e \cap C_{S_\tau} \neq \varnothing$, we have that e is in G by Property (P1). Therefore, $S \cup U$ satisfies property (P1) with respect to C_S.

2. We now turn to show that $G[A_1 \cup C_S \cup C_{A_1} \cup \cdots \cup C_{A_k}] \cong K^r_{|A_1 \cup C_S \cup C_{A_1} \cup \cdots \cup C_{A_k}|}$. It suffices to show that $G[(A_1 \cup U) \cup C_S \cup C_{A_1} \cup \cdots \cup C_{A_k}] \cong K^r_{|(A_1 \cup U) \cup C_S \cup C_{A_1} \cup \cdots \cup C_{A_k}|}$, where we may assume, as discussed above, that $A_i \subseteq S$ for all $i \in [k]$.
   To that end, note that by Property (P2) (or by Property ($\overline{P}2$) if U intersects with $\tilde{C}$) with respect to S_t and C_{S_t}, we have that for every $B_1 \subsetneq B_2 \subsetneq \cdots \subsetneq B_m \subseteq S_t$ such that C_{B_i} is defined for every $i \in [m]$, $G[B_1 \cup C_{S_t} \cup C_{B_1} \cup \cdots \cup C_{B_m} \cup \tilde{C}] \cong K^r_{|B_1 \cup C_{S_t} \cup C_{B_1} \cup \cdots \cup C_{B_m} \cup \tilde{C}|}$.
   We may thus choose $B_1 = A_1, \ldots, B_k = A_k, B_{k+1} = S, B_{k+2} = S_0, \ldots, B_m = S_t$, and since $U \subseteq C_{S_0} \cup \cdots \cup C_{S_t} \cup \tilde{C}$, we conclude that $S \cup U$ satisfies Property (P2).
3. Finally, let us show that $(S \cup U) \cap C_S = \varnothing$. Indeed, since C_S is defined, we have that $S \cap C_S = \varnothing$, and by our assumption, $C_{S_0}, \ldots, C_{S_t}$ are different from C_S (and in particular disjoint, due to the definition of the sequence Q_i, $i \in \left[\lfloor \frac{n}{\ell} \rfloor\right]$), and by property ($\overline{P}3$), $\tilde{C}$ is disjoint from C_S. Therefore, $U \cap C_S = \varnothing$ and $(S \cup U) \cap C_S = \varnothing$. □

Activating the edges

With this at hand, we are ready to prove that we can activate all the edges of G outside of H. The argument here will be a natural extension, and formalisation, of the argument for $r = 3$ detailed in Section 2. We consider separately edges that intersect with C_0 and edges that do not.

Claim 3.4. *One can activate all edges $e \in V(G) \setminus C_0$.*

Proof. Let $e \subseteq V(G) \setminus C_0$ be of size r. We will show that e closes a copy of K^r_s with $\tilde{C} = \tilde{C}(e)$.

Let us prove by induction on $|S|$ that for every set $S \subsetneq e$, one can activate all edges (that are not in H) of the form $S \cup U$, where $U \subseteq \tilde{C} \cup C_{S_0} \cup \ldots \cup C_{S_t}$ for $S \subsetneq S_0 \subsetneq S_1 \subsetneq \ldots \subsetneq S_t \subsetneq e$ for which C_{S_i} is defined for all $i \in [t]$. Note that we need the above only for $U \subseteq \tilde{C}$, however, for the induction argument it will be easier to prove the aforementioned stronger statement.

For the base case of the induction, we consider $S = \varnothing$. Let $f \subseteq \tilde{C} \cup C_{S_0} \cup \ldots \cup C_{S_t}$ be an edge for some $\varnothing \subsetneq S_0 \subsetneq S_1 \subsetneq \ldots \subsetneq S_t \subsetneq e$. Note that by Property (P2), $G[f \cup C_0] \cong K^r_{|f \cup C_0|}$, and thus if $f \cap C_0 \neq \varnothing$, we have that f is in H. Let us thus suppose that $f \cap C_0 = \varnothing$. By (E1), all the edges induced by C_0 are in H. Moreover, for every $S' \subsetneq f$, since $G[f \cup C_0] \cong K^r_{|f \cup C_0|}$, we have that $C_{S'} = C_0$. Thus, by (E2), every edge of the form $S' \cup C'_0$, for every $C'_0 \subseteq C_0$ of size $r - |S'|$, is in H. Hence, $f \cup C_0$ is a clique in G, and all its edges but f are in H. Thus, we can activate f.

For the induction step, let $i \in [r-1]$. Suppose that for every $S' \subsetneq e$, $|S'| < i$, and $S' \subsetneq A_1 \subsetneq A_2 \subsetneq \ldots \subsetneq A_k \subsetneq e$ such that C_{A_i} exists for all $i \in [k]$, we have activated (or added to H) all edges of the form $S' \cup U'$ where $U' \subseteq \tilde{C} \cup C_{A_1} \cup \ldots \cup C_{A_k}$. Let $S \subsetneq e$ of size i. Let f be an edge of the form $S \cup U$ where $U \subseteq \tilde{C} \cup C_{S_0} \cup \ldots \cup C_{S_t}$ for some $S \subsetneq S_0 \subsetneq S_1 \subsetneq \ldots \subsetneq S_t \subsetneq e$ for which $C_{S_0}, \ldots, C_{S_t}$ are defined. We will consider two separate cases, determined by whether C_S is defined or not.

Assume first that C_S is defined. We will show that we can activate f by closing a copy of K^r_s induced by $f \cup C_S$. Indeed, note if $f \cap C_S \neq \varnothing$ then f is in H (indeed, by Claim 3.3, $C_{f \setminus C_S} = C_S$). We may thus suppose $f \cap C_S = \varnothing$. By Claim 3.3, we have that $C_{S \cup U'} = C_S$ for every $U' \subseteq U$. Thus, we added to H all edges of the form $S \cup U' \cup C'$ for every $U' \subsetneq U$ and $C' \subseteq C_S$. By the induction

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hypothesis, we have already activated (or have in H) all edges of the form $S' \cup U'$ for every $S' \subsetneq S$, $U' \subseteq U \cup C_S$. Hence, we can activate f by closing a copy of K_s^r induced by $f \cup C_S$.

Assume now that C_S is not defined. Then, there exists $S' \subsetneq S$ such that $C_{S'}$ is defined and $S \setminus S' \subseteq C_{S'}$. By Claim 3.3, we have that $C_{S' \cup U'} = C_{S'}$ for every $U' \subseteq U$. Hence, we have already added to H all edges of the form $S' \cup U' \cup C'$ for every $U' \subseteq U$ and $\varnothing \neq C' \subseteq C_{S'}$. In particular, letting $C' = S \setminus S'$, we get that the edge $f = S \cup U$ is already in H.

Altogether, all edges induced by $e \cup \tilde{C}$, other than e, are either in H or have been activated, and thus e can be activated by closing a copy of K_s^r with $\tilde{C}$. □

Claim 3.5. *One can activate all the edges intersecting with* C_0.

Proof. Let e be an edge such that $e \cap C_0 \neq \varnothing$. Once again, we will show that e closes a copy of K_s^r with $\tilde{C} = \tilde{C}(e)$ in H.

Let us prove by induction on $|S|$ that for every set $S \subsetneq e \setminus C_0$, one can activate (if not already in H) all edges of the form $S \cup U$ where $U \subseteq C_0 \cup \tilde{C} \cup C_{S_0} \cup \ldots \cup C_{S_t}$ for some $S \subsetneq S_0 \subsetneq S_1 \subsetneq \cdots \subsetneq S_t \subsetneq e \setminus C_0$ for which $C_{S_0}, \ldots, C_{S_t}$ were defined. We can then activate e as it closes a copy of K_S^r with $\tilde{C}$.

For the base case of the induction, we consider $S = \varnothing$. Let $f \subseteq C_0 \cup \tilde{C} \cup C_{S_0} \cup \ldots \cup C_{S_t}$ be an edge for some $S \subsetneq S_0 \subsetneq S_1 \subsetneq \cdots \subsetneq S_t \subsetneq e \setminus C_0$ for which $C_{S_0}, \ldots, C_{S_t}$ were defined. By Claim 3.4, we may assume that $f \cap C_0 \neq \varnothing$. By Claim 3.3, we have that $C_{f \setminus C_0} = C_S = C_0$ since $S = \varnothing$. Therefore, by (E2), f is in H.

For the induction step, let $i \in [r-1]$ and take $S \subsetneq e \setminus C_0$ of size i. Let f be an edge of the form $S \cup U$, where $U \subseteq C_0 \cup \tilde{C} \cup C_{S_0} \cup \ldots \cup C_{S_t}$ for some $S \subsetneq S_0 \subsetneq S_1 \subsetneq \cdots \subsetneq S_t \subsetneq e \setminus C_0$ for which $C_{S_0}, \ldots, C_{S_t}$ were defined. We will consider two separate cases, determined by whether C_S is defined or not.

Assume first that C_S is defined. If $U \cap C_0 = \varnothing$, then we can activate $S \cup U$ by Claim 3.4. If $U \cap C_0 \neq \varnothing$, note that by Claim 3.3, $C_{S \cup (U \setminus C_0)} = C_S$. We thus have that f closes a copy of K_s^r with C_S. Indeed, let $h \subseteq f \cup C_S$ be an edge with $h \neq f$. If $S \cap h \neq S$, then $|S \cap h| \leq i - 1$. Writing $S' = S \cap h$ and $U' = h \setminus S$, we want to apply the induction hypothesis with $S' \cup U'$. Indeed, note that $U' \subseteq C_0 \cup \tilde{C} \cup C_S \cup C_{S_0} \cup \ldots \cup C_{S_t}$, and in particular where $S' \subsetneq S \subsetneq S_0 \subsetneq \cdots \subsetneq S_t \subsetneq e$. Thus, by the induction hypothesis, we have already activated (or added to H) the edge h. Otherwise, we have $S \subseteq h$. Noting that by Claim 3.3, $C_{h \setminus C_0} = C_S$, by (E3) we have already added the edge h to H.

Assume now that C_S is not defined. Then, there exists $S' \subseteq S$ such that $C_{S'}$ is defined and $S \setminus S' \subseteq C_{S'}$. By Claim 3.3, $C_{S' \cup (U \setminus C_0)} = C_{S'}$. Thus, by (E3), we added to H all edges of the form $S' \cup (U \setminus C_0) \cup C' \cup C_0'$ for every $\varnothing \neq C' \subseteq C_{S'}$ and $C_0' \subseteq C_0$. In particular, letting $C' = S \setminus S' \subseteq C_{S'}$ and $C_0' = U \cap C_0$, we get that the edge $f = S' \cup (U \setminus C_0) \cup (S \setminus S') \cup (U \cap C_0) = S \cup U$ was added to H. □

4. Strong saturation

We begin with the proof of the lower bound of Theorem 1(b), which sheds some light as to why typically $sat(G^r(n,p), K_s^r) = (1 + o(1))\binom{n}{r-1} \log_{\frac{1}{1-p^{r-1}}}(n)$.

Proof of the lower bound of Theorem 1(b). Let us show that if H is K_s^r-saturated in G, then $e(H) \geq (1 + o(1))\binom{n}{r-1} \log_{\frac{1}{1-p^{r-1}}} n$. Note that if H is K_s^r-saturated in G, then adding any $e \in E(G) \setminus E(H)$ creates a new copy of K_s^r, and in particular, a new copy of K_{r+1}^r. Let $\alpha = \frac{1}{1-p^{r-1}}$.

Given an $(r-1)$-subset $S \subseteq V(G)$, let $N_H(S) = \{v \in V(G) \colon \{v\} \cup S \in E(H)\}$. Let

$$A := \{S \subseteq V(G) \colon |S| = r - 1 \text{ and } |N_H(S)| \geq \log_\alpha^2 n\},$$

and set $B = \binom{V(G)}{r-1} \setminus A$. Note that if $|A| = \Omega\left(n^{r-1}\right)$, then $|E(H)| = \Omega\left(n^{r-1} \log_\alpha^2 n\right)$, and we are done. We may thus assume that $|A| = o\left(n^{r-1}\right)$, and thus $|B| = (1 + o(1))\binom{n}{r-1}$.

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Let $S \in B$. We claim that there are at least $(1+o(1))\log_\alpha n$ edges $e \in E(H)$, such that S is the only $(r-1)$-subset of e which is in B. This will imply that the number of edges in H is at least $|B|\log_\alpha n = (1+o(1))\binom{n}{r-1}\log_\alpha n$, as required. Let $u \in V(G)$ be such that $e = \{u\} \cup S \in E(G) \setminus E(H)$. Then $\{u\} \cup S$ closes a copy of K^r_{r+1} together with H, in particular, there is some $w \in V(G)$ such that $e \cup H[S \cup \{u\} \cup \{w\}] \cong K^r_{r+1}$. Note that for all but at most $\log^4_\alpha n$ choices of u, we have that the only $(r-1)$-subset of $S \cup \{w\}$ that is in B is S. Indeed, $S \in B$, so there are at most $\log^2_\alpha n$ choices of w such that $\{w\} \cup S \in E(H)$. Then, if at least one other $(r-1)$-subset $S \neq S' \subseteq S \cup \{w\}$ is such that $S' \in B$, then we have at most $\log^2_\alpha n$ choices for this u as well, since $S' \cup \{u\} \in E(H)$.

Thus, we have a set U of at least $|N_G(S)| - \log^4_\alpha - O(1)$ vertices u such that there exists $w = w(u)$ forming an edge with S and with every $\{u\} \cup S'$, where $S' \subset S$, $|S'| = r-2$, and such that S is the only $(r-1)$-subset of $S \cup \{w\}$ that is in B. Let us prove that $|\{w(u)\colon u \in U\}| \geq (1+o(1))\log_\alpha n$. This will immediately imply the desired upper bound.

To that end, it suffices to show that **whp** for any $(r-1)$-set S, and W of size at most $\log_\alpha n - 5\log_\alpha \log_\alpha n$, there are at least $2\log^4_\alpha n$ vertices $u \in V(G) \setminus (S \cup W)$ such that $S \cup \{u\} \in E(G)$, and for every $w \in W$ there is some $X \subset S \cup \{u\}$ with $|X| = r-1, X \neq S$, such that $X \cup \{w\} \notin E(G)$. Fix S and W. The probability that u satisfies the above is $p' = p(1-p^{r-1})^{|W|} \geq p \cdot \frac{\ln^5 n}{n}$. These events are independent for different u, so the number of vertices satisfying this property is distributed as $Bin(n-(r-1)-|W|, p')$. Thus, by a standard Chernoff's bound, the probability that there are less than $2\log^4_\alpha n$ such vertices is at most $\exp\left(-\Omega(\ln^5 n)\right)$. There are $\binom{n}{r-1}$ ways to choose S and $\sum_{i=1}^{\log_\alpha n}\binom{n}{i} \leq n^{\log_\alpha n}$ ways to choose W, and thus by the union bound there are **whp** at least $2\log^4_\alpha n$ such vertices. □

We now turn to the upper bound of Theorem 1(b). As we follow here the proof strategy from [15] with minor adjustments, in the next section we give an outline of the proof. For the full proof, see the Appendix of the ArXiv version [7].

4.1 Outline of the upper bound of Theorem 1(b)

We will utilise two lemmas. The first one generalises a result of Krivelevich [16] to the case of hypergraphs:

Lemma 4.1. *Let $r \geq 3$ and $t \geq r$ be integers. There exist positive constants c_0 and c_1 such that if*

$$\rho \geq c_1 n^{-\frac{t+1-r}{\binom{t+1}{r}-1}} \quad \text{and} \quad k \geq c_0 n^{\frac{\binom{t}{r}(t+1-r)}{\left(\binom{t+1}{r}-1\right)(t-1)}} \log^{\frac{1}{t-1}} n,$$

*then **whp** $G^r(n,\rho)$ contains a subhypergraph H on $[n]$ such that*

- *H is K^r_{t+1}-free,*
- *every induced k-subhypergraph of H contains a copy of K^r_t.*

The proof of Lemma 4.1 follows, in general, ideas present in [16], utilising Janson's inequality.

We also require the following lemma, which shows that for ρ chosen as in that in Lemma 4.1, typically every two vertices u and v are not both contained in 'many' copies of K^r_t in $G^r(n,\rho)$. This lemma is a new ingredient in the proof – this was not needed in the graph setting of [15], but is necessary in the hypergraph setting.

Lemma 4.2. *Let $r \geq 3$ and $t \geq 4$ satisfying $t \geq r$ be integers. Let $c > 0$ be a constant and let $p = cn^{-(t+1-r)/\left(\binom{t+1}{r}-1\right)}$. Then, **whp**, for every two vertices u and v, the number of copies of K^r_t in $G^r(n,\rho)$, which contains u and v, is at most $2\binom{n}{t-2}\rho^{\binom{t}{r}}$.*

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Proof. Fix two vertices u and v. Denote by X the number of copies of K_t^r in $G \sim G^r(n,\rho)$ which contain u and v. By the union bound over all the choices of u and v, it suffices to prove that

$$\mathbb{P}\left(X > 2\binom{n}{t-2}p^{\binom{t}{r}}\right) \ll n^{-2}.$$

Set $L = K\log n$ where K is a sufficiently large constant. For every subset $T \subset V(G)$ of size t, denote by Y_T the indicator random variable of the event $G[T] \cong K_t^r$. Below, we consider only those t-subsets of $V(G)$ that contain u and v (in particular, sum up over such sets in the computation of the L-th moment of X below). We have

$$\begin{aligned}
\mathbb{E}[X^L] &= \sum_{\substack{T_1,\ldots,T_L\subset V(G)\\ |T_1|=\cdots=|T_L|=t}} \mathbb{E}[Y_{T_1}\cdot\cdots\cdot Y_{T_L}]\\
&= \sum_{\substack{T_1,\ldots,T_{L-1}\subset V(G)\\ |T_1|=\cdots=|T_{L-1}|=t}} \mathbb{E}[Y_{T_1}\cdot\cdots\cdot Y_{T_{L-1}}] \sum_{T_L\subset V(G),|T_L|=t} \mathbb{E}[Y_{T_L}\mid Y_{T_1}=\cdots=Y_{T_{L-1}}=1]\\
&= \sum_{\substack{T_1,\ldots,T_{L-1}\subset V(G)\\ |T_1|=\ldots=|T_{L-1}|=t}} \mathbb{E}[Y_{T_1}\cdot\ldots\cdot Y_{T_{L-1}}]\cdot\mathbb{E}[X\mid Y_{T_1}=\ldots=Y_{T_{L-1}}=1]. \qquad (4)
\end{aligned}$$

Let H be a hypergraph with $e(H) \le L\cdot t$. By FKG inequality, $\mathbb{E}[X \mid H\subset G] \ge \mathbb{E}X$. On the other hand,

$$\mathbb{E}[X\mid H\subset G] \le \binom{n}{t-2}\rho^{\binom{t}{r}} + \sum_{j=r}^{t} e(H)^{\binom{j}{r}} n^{t-j}\rho^{\binom{t}{r}-\binom{j}{r}}.$$

The first summand in the right-hand side above is an upper bound to the expectation of the number of copies of K_t^r which does not share any edges with H while the second summand is an upper bound to the expectation of the number of copies of K_t^r which does share at least one edge with H. The index j indicates how many vertices in the copy of K_t^r belongs to H. So we have at most $\binom{j}{r}$ edges which are already in H. The term $e(H)^{\binom{j}{r}}$ is an upper bound to the number of choices of the common j vertices, n^{t-j} is an upper bound to the number of choices of the remaining $t-j$ vertices and $p^{\binom{t}{r}-\binom{j}{r}}$ is an upper bound to the probability that this copy appears in G conditioned on $H\subseteq G$. We will show that the second summand on the right-hand side of the inequality is asymptotically smaller than

$$\mathbb{E}[X] = (1+o(1))\binom{n}{t-2}\rho^{\binom{t}{r}}, \qquad (5)$$

implying $\mathbb{E}[X\mid H\subset G] \le (1+o(1))\mathbb{E}[X]$. Note that, for every $j = r, r+1,\ldots,t$, we have

$$\frac{n^{t-j}\rho^{\binom{t}{r}-\binom{j}{r}}}{n^{t-2}\rho^{\binom{t}{r}}} = n^{2-j}\rho^{-\binom{j}{r}} = \Theta\left(n^{2-j+\frac{\binom{j}{r}(t+1-r)}{\binom{t+1}{r}-1}}\right).$$

Thus, it is sufficient to show that the last expression tends to 0 as $n\to\infty$, which is true if

$$\frac{\binom{j}{r}(t+1-r)}{\binom{t+1}{r}-1} < j-2.$$

To show the latter, we will use the following claim.

Claim 4.3. $\frac{\binom{t}{r}(t+1-r)}{\left(\binom{t+1}{r}-1\right)(t-1)} < 1-\frac{r-1}{t}$.

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Proof. We have

$$\begin{aligned}\frac{\binom{t}{r}(t+1-r)}{\left(\binom{t+1}{r}-1\right)(t-1)} &< \frac{\binom{t}{r}}{\binom{t+1}{r}-1} = \frac{\binom{t}{r}}{\binom{t}{r}+\binom{t}{r-1}-1} \\ &= \frac{\binom{t}{r}}{\binom{t}{r}+\binom{t}{r}\cdot\frac{r}{t-r+1}-1} = \frac{1}{1+\frac{r}{t-r+1}-\frac{1}{\binom{t}{r}}} \\ &\le 1-\frac{r}{t-r+1}+\frac{1}{\binom{t}{r}} \le 1-\frac{r}{t-r+1}+\frac{1}{t-r+1} \\ &= 1-\frac{r-1}{t-r+1} < 1-\frac{r-1}{t}.\end{aligned}$$

□

By Claim 4.3,

$$\begin{aligned}\frac{\binom{j}{r}(t+1-r)}{\binom{t+1}{r}-1} &< \frac{\binom{j}{r}}{\binom{t}{r}}\left(1-\frac{r-1}{t}\right)(t-1) \le \frac{\binom{j}{r}}{\binom{t}{r}}(t-2) = \frac{j(j-1)\dots(j-r+1)}{t(t-1)\dots(t-r+1)}(t-2) \\ &\overset{r\ge 3}{\le} j-2,\end{aligned}$$

as needed.

Thus, we conclude that $\mathbb{E}[X \mid H \subset G] = (1+o(1))\mathbb{E}[X]$ uniformly over all H with $e(H) \le L \cdot t$. By induction, from (4), we have

$$\mathbb{E}[X^L] = \Big((1+o(1))\mathbb{E}[X]\Big)^L.$$

From (5), by Markov's inequality, we get

$$\mathbb{P}\left(X > 2\binom{n}{t-2}\rho^{\binom{t}{r}}\right) \le \frac{\mathbb{E}[X^L]}{((2+o(1))\mathbb{E}[X])^L} = \left(\frac{(1+o(1))\mathbb{E}[X]}{2\mathbb{E}[X]}\right)^L = (1/2+o(1))^L \ll n^{-2},$$

where the last asymptotical inequality is true if K is sufficiently large. □

With these two lemmas at hand, we can now describe our construction. We say that an edge $e \in E(G) \setminus E(H)$ can be *completed* if it closes a copy of K_s^r in $H \cup \{e\}$. Recall that we aim to construct a K_s^r-free subhypergraph $H \subseteq G$ with $(1+o(1))\binom{n}{r-1}\log_{\frac{1}{1-p^{r-1}}} n$ edges, such that every edge $e \in E(G) \setminus E(H)$ can be completed. Throughout the proof, we make use of the following observation, already appearing in [15].

Observation 4.4. *It suffices to construct a K_s^r-free graph that completes all but $o(n^{r-1}\log n)$ edges. Indeed, by adding each of the uncompleted edges to H, if necessary, we obtain a K_s^r-saturated graph with an asymptotically equal number of edges.*

We set $\alpha = (1-p^{r-1})^{-1}$ and $\beta = (1-p^{\binom{s}{r}-\binom{s-r}{r}-1})^{-1}$. Throughout the rest of this section, unless explicitly stated otherwise, the base of the logarithms is α. We then set aside three disjoint subsets of $[n]$, A_1, A_2, and A_3, of sizes $a_1 := \frac{1}{p}\log n\left(1+\frac{3}{\log\log n}\right)$, $a_2 := sr\log_\beta n$, and $a_3 = \frac{a_2}{\log^{1/r} a_2}$, respectively. We let $B = [n] \setminus (A_1 \cup A_2 \cup A_3)$, shorthand $t = s-r$, and recall that $G \sim G^r(n,p)$.

We now define H to be a subhypergraph of G with the following edges:

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- all edges of G intersecting both A_1 and B with at most t vertices from A_1; and,
- if $t \geq r$, we take $H[A_1] \subsetneq G[A_1]$ to be K^r_{t+1}-free, such that there exists a copy of K^r_t in every induced subhypergraph of $H[A_1]$ of size at least $c_0 a_1^{\frac{\binom{t}{r}(t+1-r)}{\left(\binom{t+1}{r}-1\right)(t-1)}} \log^{1/(t-1)} a_1$. Moreover, every two vertices in A_1 are contained in at most $2\binom{a_1}{t-2}(c_1 a_1)^{-\frac{t+1-r}{\binom{t+1}{r}-1}}$ copies of K^r_t in $H[A_1]$. Otherwise, if $t < r$, we take $H[A_1]$ to be the empty hypergraph.

Note that, **whp**, by Lemmas 4.1 and 4.2 the subhypergraph described in the second bullet exists. Furthermore, the number of edges we have in H now is indeed $(1+o(1))\binom{n}{r-1}\log_{\frac{1}{1-p^{r-1}}} n$. Moreover, since $H[A_1]$ is K^r_{t+1}-free, if $t+1 \geq r$ we see that H is K^r_s-free. If $t+1 \leq r-1$, we note that we do not have edges intersecting both A_1 and B with at least $t+1$ vertices in A_1, and thus H is K^r_s-free.

Given an $(r-1)$-set of vertices $S \subseteq [n]$, the neighbourhood of S in $X \subseteq [n]$ is given by $N_X(S) := \{v \in X\colon \{v\} \cup S \in E(G)\}$. Given an $(r-1)$-set of vertices $S \subseteq B$, we say that S is *good* if $|N_{A_1}(S)| \geq pa_1 - \frac{\log n}{\log\log n}$, and otherwise we say that S is *bad*. Finally, we say that an edge $e \in G[B]$ is good if there exists at least one good $(r-1)$-subset $S \subseteq e$, and otherwise say that the edge e is *bad*.

We first utilise the subhypergraph in $H[A_1]$ in order to activate almost all the good edges $e \in E(G[B])$. Observe that good edges e have some $S \subseteq e$, with $|S| = r-1$ and such that S has a large neighbourhood in A_1. Fix an r-subset $e \subseteq B$. Note that given $S \subseteq e$ with $|S| = r-1$ and its neighbourhood in A_1, we have that

$$N_{S,e} := \left|\left\{v \in N_{A_1}(S)\colon \{v\} \cup S' \in E(G) \text{ for all } S' \subseteq e, |S'| = r-1\right\}\right| \sim Bin(|N_{A_1}(S)|, p^{r-1}).$$

By the choice of $H[A_1]$, we can then expose the edges of the form $S' \cup \{v\}$ in G, where $S' \subseteq e$ and $v \in N_{A_1}(S)$, and then using Chernoff's bound find in $N_{S,e}$ many copies of K^r_t. Now, note that given two copies of K^r_t in $H[N_{S,e}]$ the events that each of them closes a copy of K^r_s with e are dependent only if these copies intersect in two vertices. To bound these dependencies, we utilise the fact that in $H[A_1]$ every two vertices do not have many copies of K^r_t sharing them. This, together with Janson's inequality, allows us to show that **whp**, for all but $o(n^{r-1}\log n)$ good edges in $e \in E(G[B])$, there is at least one copy of K^r_t that closes a copy of K^r_s with e. Note that by Observation 4.4, this in fact completes our treatment for all the good edges in $G[B]$.

Before turning to bad edges in $G[B]$, let us discuss the difference in the treatment of good edges above from the setting of graphs in [15]. Indeed, in the case of graphs, a key part of the argument for the upper bound is to consider vertices v with a large neighbourhood in A_1. Then, fixing v and considering edges $uv \in E(G[B]))$, it suffices to find a copy of K^2_{s-2} in $N_v \cap N_u$ – whose size is distributed according to $Bin(|N_v|, p)$. On the other hand, when considering hypergraphs, we consider instead an $(r-1)$-set $S \subseteq e$ with a substantial neighbourhood in A_1, $N_{S,e}$. Then, when considering edges $\{u\} \cup S \in E(G[B])$, it no longer suffices to find a copy of K^r_{s-r} in $\tilde{N}_{S,u} = \bigcap_{S' \subset S \cup \{u\}, |S'| = r-1} N_{S',e}$. Indeed, one needs to further consider edges with $1 \leq \ell < r-1$ vertices from $S \cup \{u\}$, and $r-\ell$ vertices from $\tilde{N}_{S,u}$. As noted above, this further creates possible dependencies which do not appear in the case of graphs, and, in particular, this is why Lemma 4.2 is important to us in the hypergraph setting.

We now turn to bad edges in $G[B]$. As there is a non-negligible portion of them, we require the set A_2 to complete them. By Lemma 4.1 there exists a subhypergraph $H_2 \subseteq G[A_2]$ which is K^r_{t+1}-free and every large enough subset in it induces a copy of K^r_t, when $t \geq r$. Moreover, there are at least $r\log_\beta n$ vertex-disjoint copies of K^r_t in $H_2[A_2]$. We can thus add to H the edges of $H_2[A_2]$, edges intersecting both A_2 and B in G with at least two and at most t vertices from A_2, and edges of the form $S \cup \{v\}$ where $S \subseteq B$ is bad and $v \in A_2$. The first two are of order $o(n^{r-1}\log n)$, and

https://doi.org/10.1017/S0963548325100229 Published online by Cambridge University Press

using Chernoff's bound, one can show that there are at most $\frac{n^{r-1}}{\log n}$ bad $(r-1)$-subsets $S \subseteq B$, and thus the last set of edges is also of order $o(n^{r-1}\log n)$. Similarly to before, H is still K_s^r-free. Using Markov's inequality, **whp** we can now close all bad edges in $G[B]$ (recall that a bad edge has that all its $(r-1)$-subsets are bad).

Note that we have not added any edge to H intersecting both B and A_2, whose intersection with B is a good $(r-1)$-subset. As there are $\Omega(n^{r-1})$ good $(r-1)$-subsets in B, and $\Omega(\log n)$ vertices in A_2, we cannot ignore these edges, and this is the reason for setting aside the set A_3. Once again, we find in $G[A_3]$ a subhypergraph H_3 which is K_{t+1}^r-free, and every large enough subset in it induces a copy of K_t^r, when $t \geq r$.

The choice of edges which we add to H now is slightly more delicate. We add to H the edges of H_3, all edges intersecting both B and A_3 with at least two and at most t vertices from A_3, edges of the form $S \cup \{v\}$ where $S \subseteq B$ is good and $v \in A_3$, and edges intersecting both $B \cup A_2$ and A_3 with one vertex from A_2 and at least one and at most t vertices from A_3. This careful choice of edges allows us to show that the hypergraph H is still K_s^r-free. As the argument here is slightly more delicate, let us state it explicitly. Since there are no edges intersecting both A_1 and $A_2 \cup A_3$, it suffices to consider $X \subseteq B \cup A_2 \cup A_3$ of size s, and suppose towards contradiction that X induces a clique in H. Since $H[B]$ is an empty hypergraph, we must have that $|X \cap B| \leq r-1$. Since we only added edges intersecting A_3, we may assume that X contains vertices from A_3. If $|X \cap A_3| \geq t+1$, we are done by similar arguments to before. Furthermore, if there are at least two vertices in $A_2 \cap X$, note that there are no edges in H containing two vertices from A_2 and at least one vertex from A_3, leading to a contradiction. Thus, we are left with the case where $|X \cap A_2| = 1$, $|X \cap A_3| = t$, $|X \cap B| = r-1$. Now, if $X \cap B$ is bad, there are no edges of the form $(X \cap B) \cup \{v\}$, $\{v\} \in A_3$, in H, and if it is good, then there are no edges of the form $(X \cap B) \cup \{v\}$, $\{v\} \in A_2$, in H – either way, we get that X cannot induce a clique. Let us note that this last part of the argument is also quite different from the argument in the setting of graphs. Indeed, in [15], to ensure that H remains K_s^2-free, one needs to consider a partition of A_2 into independent sets, and partition A_3 accordingly. Here, adding edges intersecting both $B \cup A_2$ and A_3 which intersect A_2 in at most one vertex suffices to show that H remains K_s^r-free.

Furthermore, due to the size of A_3, the number of edges we added at this stage is of order $o(n^{r-1}\log n)$. Using similar arguments to before, we can show that we can activate all but $o(n^{r-1}\log n)$ of the edges of the form $S \cup \{v\}$ where $S \subseteq B$ is good and $v \in A_2$. Again, by Observation 4.4, this completes our treatment for all edges of the form $S \cup \{v\}$ where $S \subseteq B$ is good and $v \in A_2$.

Finally, note that we have not completed all the edges: we did not complete edges of the form $S \cup \{v\}$ where $S \subseteq B$ is bad and $v \in A_3$ as well as some of the edges induced by $A_1 \cup A_2 \cup A_3$. Furthermore, if $t < r$, we have not completed edges instersecting both B and $A_1 \cup A_2 \cup A_3$ with more than t vertices in either A_1, A_2, or A_3. However, there are at most $o(n^{r-1}\log n)$ edges of these types, and we are thus done by Observation 4.4.

Funding statement

B. Sudakov is research supported in part by SNSF grant 200021_228014.

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Cite this article: Diskin S, Hoshen I, Korándi D, Sudakov B, and Zhukovskii M (2026). Saturation in random hypergraphs. *Combinatorics, Probability and Computing* **35**, 40–58. https://doi.org/10.1017/S0963548325100229

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