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TWO-INPUT CONTROL SYSTEMS ON THE EUCLIDEAN GROUP $\mathsf{SE}(2)$

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Abstract. Any two-input left-invariant control affine system of full rank, evolving on the Euclidean group $\mathsf{SE}(2)$, is (detached) feedback equivalent to one of three typical cases. In each case, we consider an optimal control problem which is then lifted, *via* the Pontryagin Maximum Principle, to a Hamiltonian system on the dual space $\mathfrak{se}(2)^*$. These reduced Hamilton−Poisson systems are the main topic of this paper. A qualitative analysis of each reduced system is performed. This analysis includes a study of the stability nature of all equilibrium states, as well as qualitative descriptions of all integral curves. Finally, the reduced Hamilton equations are explicitly integrated by Jacobi elliptic functions. Parametrisations for all integral curves are exhibited.

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1. Introduction

A general left-invariant control affine system on the Euclidean group $\mathsf{SE}(2)$ has the form $\dot{g} = g(A + u_1B_1 + \cdots + u_\ell B_\ell)$, where $A, B_1, \ldots, B_\ell \in \mathfrak{se}(2)$, $1 \leq \ell \leq 3$. (The elements $B_1, \ldots, B_\ell$ are assumed to be linearly independent). Specific left-invariant optimal control problems on the Euclidean group $\mathsf{SE}(2)$, associated with the above mentioned control systems, have been studied by several authors (see, *e.g.*, [2,10–12,17,22,23,25–28]).

In this paper, we consider only *two-input* control systems, *i.e.*, systems of the form $\dot{g} = g\,(A + u_1B_1 + u_2B_2)$. Any such *homogeneous* full-rank control system is (detached feedback) equivalent to the control system $\Sigma_0 : \dot{g} = g(u_1E_2 + u_2E_3)$. Then again, any such *inhomogeneous* control system is (detached feedback) equivalent to exactly one of the control systems $\Sigma_1 : \dot{g} = g(E_1 + u_1E_2 + u_2E_3)$ and $\Sigma_{2,\alpha} : \dot{g} = g(\alpha\, E_3 + u_1E_1 + u_2E_2)$, $\alpha > 0$. Here E_1, E_2 and E_3 denote elements of the standard basis for $\mathfrak{se}(2)$. In each typical case, we consider an optimal control problem (with quadratic cost) of the form

$$\dot{g} = g\,(A + u_1B_1 + u_2B_2)\,, \qquad g \in \mathsf{SE}(2),\ u = (u_1, u_2) \in \mathbb{R}^2$$
$$g(0) = g_0, \quad g(T) = g_T$$
$$\mathcal{J} = \tfrac{1}{2}\int_0^T \big(c_1u_1^2(t) + c_2u_2^2(t)\big)\ \mathrm{d}t \to \min\,.$$

Keywords and phrases. Left-invariant control system, (detached) feedback equivalence, Lie−Poisson structure, energy-Casimir method, Jacobi elliptic function.

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Each problem is lifted, *via* the Pontryagin Maximum Principle, to a Hamiltonian system on the dual space $\mathfrak{se}(2)^*$. Then the (minus) Lie−Poisson structure on $\mathfrak{se}(2)^*$ is used to derive the equations for extrema (*cf.* [3,11,14]). The stability nature of all equilibrium states for the reduced system is then investigated by the energy-Casimir method. Also, a qualitative description of all integral curves of the reduced system is given. Finally, these equations are explicitly integrated by Jacobi elliptic functions. A brief description of this process is given now.

First, we partition the set of initial conditions in terms of simple inequalities. Specifically, we distinguish between various ways that the level sets, defined by the constants of motion, intersect. When required, this set is further partitioned in order to facilitate integration. This enables one to distinguish between solution curves with different explicit expressions. In each case, the extremal equations are reduced to a (separable) differential equation and then transformed into standard form (see, *e.g.*, [4] or [15]). Thereafter, an integral formula is applied. Consequently, by use of the constants of motion (and allowing for possible changes in sign), an explicit expression for the solution is obtained.

The paper is organized as follows. In Section 2 we review some basic facts regarding left-invariant control systems, optimal control, the energy-Casimir method and Jacobi elliptic functions. In Section 3 we classify all two-input left-invariant control affine systems on $\mathsf{SE}(2)$ and then introduce a general optimal control problem (with quadratic cost) to be considered for each equivalence class. In Section 4 a qualitative analysis of the reduced Hamiltonian systems is given and in Section 5 the reduced systems are explicitly integrated. A tabulation of integral curves is included as an appendix. We conclude the paper with a summary and a few remarks.

2. Preliminaries

2.1. Invariant control systems and optimal control

Invariant control systems on Lie groups were first considered in 1972 by Brockett [8] and by Jurdjevic and Sussmann [13]. A *left-invariant control system* Σ is a (smooth) control system evolving on a (real, finite-dimensional) Lie group G, whose dynamics $\Xi : \mathsf{G} \times U \to T\mathsf{G}$ are invariant under left translations. (The tangent bundle $T\mathsf{G}$ is identified with $\mathsf{G} \times \mathfrak{g}$, where $\mathfrak{g}$ is the Lie algebra of G). For the sake of convenience, we shall assume that G is a *matrix* Lie group. Also, for the purposes of this paper, we may assume that $U = \mathbb{R}^\ell$. Such a control system is described as follows (*cf.* [3,11,24])

$$\dot{g} = \Xi(g,u), \qquad g \in \mathsf{G},\ u \in \mathbb{R}^\ell \tag{2.1}$$

where $\Xi(g,u) = g\,\Xi(\mathbf{1},u) \in T_g\mathsf{G}$.

Admissible controls are bounded and measurable maps $u(\cdot) : [0,T] \to \mathbb{R}^\ell$, whereas the parametrisation map $\Xi(\mathbf{1},\cdot) : \mathbb{R}^\ell \to \mathfrak{g}$ is an embedding. The trace $\Gamma = \operatorname{im}\Xi(\mathbf{1},\cdot)$ is a submanifold of $\mathfrak{g}$ so that $\Gamma = \{\Xi_u = \Xi(\mathbf{1},u) : u \in \mathbb{R}^\ell\}$ (*cf.* [5,6]). A left-invariant control *affine* system is one whose parametrisation map is affine. For such a system, the trace Γ is an affine subspace of $\mathfrak{g}$. We say that the system has *full rank* if the Lie algebra generated by its trace, $\mathsf{Lie}(\Gamma)$, coincides with $\mathfrak{g}$. A *trajectory* for an admissible control $u(\cdot) : [0,T] \to \mathbb{R}^\ell$ is an absolutely continuous curve $g(\cdot) : [0,T] \to \mathsf{G}$ such that $\dot{g}(t) = g(t)\,\Xi(\mathbf{1},u(t))$ for almost every $t \in [0,T]$.

We shall denote a (left-invariant control) system Σ by (G,Ξ) (see, *e.g.*, [5,6]). We say that a system $\Sigma = (\mathsf{G},\Xi)$ is *connected* if its state space G is connected. Let $\Sigma = (\mathsf{G},\Xi)$ and $\Sigma' = (\mathsf{G}',\Xi')$ be two connected full-rank systems with traces $\Gamma \subseteq \mathfrak{g}$ and $\Gamma' \subseteq \mathfrak{g}'$, respectively. We say that Σ and Σ' are *locally detached feedback equivalent* if there exist open neighbourhoods N and N' of (the unit elements) $\mathbf{1}$ and $\mathbf{1}'$, respectively, and a diffeomorphism $\Phi = \phi \times \varphi : N \times \mathbb{R}^\ell \to N' \times \mathbb{R}^\ell$ such that $\phi(\mathbf{1}) = \mathbf{1}'$ and $T_g\phi \cdot \Xi(g,u) = \Xi'(\phi(g),\varphi(u))$ for $g \in N$ and $u \in \mathbb{R}^\ell$. Two detached feedback equivalent systems have the same trajectories (up to a diffeomorphism in the state space), which are parametrised differently by admissible controls. We recall the following result.

Proposition 2.1 ([6]). *$\Sigma = (\mathsf{G},\Xi)$ and $\Sigma' = (\mathsf{G}',\Xi')$ are locally detached feedback equivalent if and only if there exists a Lie algebra isomorphism $\psi : \mathfrak{g} \to \mathfrak{g}'$ such that $\psi \cdot \Gamma = \Gamma'$.*

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Now, consider an *optimal control problem* given by the specification of (i) a left-invariant control system $\Sigma = (\mathsf{G}, \Xi)$, (ii) a cost function $L : \mathbb{R}^\ell \to \mathbb{R}$, and (iii) boundary data, consisting of an initial state $g_0 \in \mathsf{G}$, a target state $g_1 \in \mathsf{G}$ and a terminal time $T > 0$. Explicitly, we want to *minimize* the functional $\mathcal{J} = \int_0^T L(u(t))\,\mathrm{d}t$ over the trajectory-control pairs of Σ subject to the boundary conditions

$$g(0) = g_0, \quad g(T) = g_1. \tag{2.2}$$

The *Pontryagin Maximum Principle* is a necessary condition for optimality which is most naturally expressed in the language of the geometry of the cotangent bundle $T^*\mathsf{G}$ of G (*cf.* [3, 11]). The cotangent bundle $T^*\mathsf{G}$ can be trivialized (from the left) such that $T^*\mathsf{G} = \mathsf{G} \times \mathfrak{g}^*$, where $\mathfrak{g}^*$ is the dual space of the Lie algebra $\mathfrak{g}$. The dual space $\mathfrak{g}^*$ has a natural *Poisson structure*, called the "minus Lie–Poisson structure", given by

$$\{F, G\}(p) = -p\left([\mathrm{d}F(p), \mathrm{d}G(p)]\right)$$

for $p \in \mathfrak{g}^*$ and $F, G \in C^\infty(\mathfrak{g}^*)$. (Note that $\mathrm{d}F(p)$ is a linear function on $\mathfrak{g}^*$ and so is an element of $\mathfrak{g}$). The Poisson space $(\mathfrak{g}^*, \{\cdot,\cdot\})$ is denoted by $\mathfrak{g}^*_-$. Each left-invariant Hamiltonian on the cotangent bundle $T^*\mathsf{G}$ is identified with its reduction on the dual space $\mathfrak{g}^*_-$. To an optimal control problem (with fixed terminal time)

$$\int_0^T L(u(t))\,\mathrm{d}t \to \min \tag{2.3}$$

subject to (2.1) and (2.2), we associate, for each real number λ and each control parameter $u \in \mathbb{R}^\ell$, a Hamiltonian function on $T^*\mathsf{G} = \mathsf{G} \times \mathfrak{g}^*$:

$$\begin{aligned} H^\lambda_u(\xi) &= \lambda\, L(u) + \xi\ (g\, \Xi\,(\mathbf{1}, u)) \\ &= \lambda\, L(u) + p\ (\Xi\,(\mathbf{1}, u)), \quad \xi = (g, p) \in T^*\mathsf{G}. \end{aligned}$$

The Maximum Principle can be stated, in terms of the above Hamiltonians, as follows.

Maximum Principle. *Suppose the trajectory-control pair $(\bar g(\cdot), \bar u(\cdot))$ defined over the interval $[0, T]$ is a solution for the optimal control problem (2.1)–(2.3). Then, there exists a curve $\xi(\cdot) : [0, T] \to T^*\mathsf{G}$ with $\xi(t) \in T^*_{\bar g(t)}\mathsf{G}$, $t \in [0, T]$, and a real number $\lambda \le 0$, such that the following conditions hold for almost every $t \in [0, T]$:*

$$(\lambda, \xi(t)) \not\equiv (0, 0) \tag{2.4}$$

$$\dot\xi(t) = \vec H^\lambda_{\bar u(t)}(\xi(t)) \tag{2.5}$$

$$H^\lambda_{\bar u(t)}\left(\xi(t)\right) = \max_u H^\lambda_u\left(\xi(t)\right) = \text{const}. \tag{2.6}$$

An optimal trajectory $\bar g(\cdot) : [0, T] \to \mathsf{G}$ is the projection of an integral curve $\xi(\cdot)$ of the (time-varying) Hamiltonian vector field $\vec H^\lambda_{\bar u(t)}$ defined for all $t \in [0, T]$. A trajectory-control pair $(\xi(\cdot), u(\cdot))$ defined on $[0, T]$ is said to be an *extremal pair* if $\xi(\cdot)$ satisfies the conditions (2.4), (2.5) and (2.6). The projection $\xi(\cdot)$ of an extremal pair is called an extremal. An extremal curve is called normal if $\lambda = -1$ and abnormal if $\lambda = 0$. In this paper, we shall be concerned only with normal extremals. Suppose the maximum condition (2.6) eliminates the parameter u from the family of Hamiltonians (H_u), and as a result of this elimination, we obtain a smooth function H (without parameters) on $T^*\mathsf{G}$ (in fact, on $\mathfrak{g}^*_-$). Then the whole (left-invariant) optimal control problem reduces to the study of integral curves of a fixed Hamiltonian vector field $\vec H$.

2.2. The energy-Casimir method

The *energy-Casimir method* [9] gives sufficient conditions for Lyapunov stability of equilibrium states for certain types of Hamilton–Poisson dynamical systems (*cf.* [16, 21]). The method is restricted to certain types of systems, since its implementation relies on an abundant supply of Casimir functions.

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The standard energy-Casimir method states that if z_e is an equilibrium point of a Hamiltonian vector field $\vec{H}$ (associated with an energy function H) and if there exists a Casimir function C such that z_e is a critical point of $H+C$ and $d^2(H+C)(z_e)$ is (positive or negative) definite, then z_e is Lyapunov stable.

Ortega and Ratiu have obtained a generalisation of the standard energy-Casimir method (*cf.* [18, 19]). This extended version states that if $C = \lambda_1 C_1 + \cdots + \lambda_k C_k$, where $\lambda_1, \ldots, \lambda_k \in \mathbb{R}$ and $C_1, \ldots, C_k$ are conserved quantities (*i.e.*, they Poisson commute with the energy function H), then definiteness of $d^2(\lambda_0 H + C)(z_e)$, $\lambda_0 \in \mathbb{R}$ is only required on the intersection (subspace) $W = \ker \mathrm{d}H(z_e) \cap \ker \mathrm{d}C_1(z_e) \cap \cdots \cap \ker \mathrm{d}C_k(z_e)$.

2.3. Jacobi elliptic functions

Given the modulus $k \in [0,1]$, the basic *Jacobi elliptic functions* $\mathrm{sn}(\cdot,k)$, $\mathrm{cn}(\cdot,k)$ and $\mathrm{dn}(\cdot,k)$ can be defined as

$$\begin{aligned} \mathrm{sn}(x,k) &= \sin \mathrm{am}(x,k) \\ \mathrm{cn}(x,k) &= \cos \mathrm{am}(x,k) \\ \mathrm{dn}(x,k) &= \sqrt{1-k^2 \sin^2 \mathrm{am}(x,k)} \end{aligned}$$

where $\mathrm{am}(\cdot,k) = F(\cdot,k)^{-1}$ is the amplitude and $F(\varphi,k) = \int_0^{\varphi} \frac{\mathrm{d}t}{\sqrt{1-k^2\sin^2 t}}$. (For the degenerate cases $k=0$ and $k=1$, we recover the circular functions and the hyperbolic functions, respectively). The complementary modulus k' and the number K are then defined as $k' = \sqrt{1-k^2}$ and $K = F(\frac{\pi}{2}, k)$. (The functions $\mathrm{sn}(\cdot,k)$ and $\mathrm{cn}(\cdot,k)$ are $4K$ periodic, whereas $\mathrm{dn}(\cdot,k)$ is $2K$ periodic). Nine other elliptic functions are defined by taking reciprocals and quotients; in particular, we get $\mathrm{nd}(\cdot,k) = \frac{1}{\mathrm{dn}(\cdot,k)}$, $\mathrm{sd}(\cdot,k) = \frac{\mathrm{sn}(\cdot,k)}{\mathrm{dn}(\cdot,k)}$ and $\mathrm{cd}(\cdot,k) = \frac{\mathrm{cn}(\cdot,k)}{\mathrm{dn}(\cdot,k)}$. Simple elliptic integrals can be expressed in terms of appropriate inverse (elliptic) functions. The following formulas hold true (see [4] or [15]):

$$\int_0^x \frac{\mathrm{d}t}{\sqrt{(a^2-t^2)(b^2-t^2)}} = \tfrac{1}{a}\,\mathrm{sn}^{-1}\left(\tfrac{1}{b}\,x,\ \tfrac{b}{a}\right), \qquad 0 \le x \le b < a \tag{2.7}$$

$$\int_0^x \frac{\mathrm{d}t}{\sqrt{(a^2+t^2)(b^2-t^2)}} = \tfrac{1}{\sqrt{a^2+b^2}}\,\mathrm{sd}^{-1}\left(\tfrac{\sqrt{a^2+b^2}}{ab}\,x,\ \tfrac{b}{\sqrt{a^2+b^2}}\right), \qquad 0 \le x \le b \tag{2.8}$$

$$\int_x^a \frac{\mathrm{d}t}{\sqrt{(a^2-t^2)(t^2-b^2)}} = \tfrac{1}{a}\,\mathrm{dn}^{-1}\left(\tfrac{1}{a}\,x,\ \tfrac{\sqrt{a^2-b^2}}{a}\right), \qquad b \le x \le a. \tag{2.9}$$

3. Control systems on $\mathsf{SE}(2)$

We consider two-input left-invariant control affine systems on $\mathsf{SE}(2)$. Such a system is fully specified by its parametrisation map $\Xi(\mathbf{1}, u) = A + u_1 B_1 + u_2 B_2$. A system is said to be *homogeneous* if $A \in \langle B_1, B_2 \rangle$, *i.e.*, the trace Γ is a linear subspace of $\mathfrak{se}(2)$. (In this paper, the notation $\langle \cdot, \cdot \rangle$ is used for the linear span of two vectors). Otherwise, the system is said to be *inhomogeneous*. A classification of all full-rank two-input systems, under detached feedback equivalence, is provided. We then introduce a general optimal control problem (with diagonal cost) to be considered for each equivalence class.

3.1. The Euclidean group $\mathsf{SE}(2)$

The Euclidean group

$$\mathsf{SE}(2) = \left\{ \begin{bmatrix} 1 & 0 \\ \mathbf{v} & R \end{bmatrix} \,:\, \mathbf{v} \in \mathbb{R}^{2\times 1},\ R \in \mathsf{SO}(2) \right\}$$

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is a (real) three-dimensional connected matrix Lie group. The associated Lie algebra is given by

$$\mathfrak{se}(2) = \left\{ \begin{bmatrix} 0 & 0 & 0 \\ x_1 & 0 & -x_3 \\ x_2 & x_3 & 0 \end{bmatrix} \,:\, x_1, x_2, x_3 \in \mathbb{R} \right\}.$$

Let

$$E_1 = \begin{bmatrix} 0\,0\,0 \\ 1\,0\,0 \\ 0\,0\,0 \end{bmatrix}, \qquad E_2 = \begin{bmatrix} 0\,0\,0 \\ 0\,0\,0 \\ 1\,0\,0 \end{bmatrix}, \qquad E_3 = \begin{bmatrix} 0\,0 & 0 \\ 0\,0 & -1 \\ 0\,1 & 0 \end{bmatrix}$$

be the standard basis of $\mathfrak{se}(2)$. (The bracket operation is given by $[E_2, E_3] = E_1$, $[E_3, E_1] = E_2$ and $[E_1, E_2] = 0$). With respect to this basis, the group of Lie algebra automorphisms of $\mathfrak{se}(2)$ is given by

$$\mathsf{Aut}(\mathfrak{se}(2)) = \left\{ \begin{bmatrix} x & y & v \\ -\varsigma y & \varsigma x & w \\ 0 & 0 & \varsigma \end{bmatrix} \,:\, x, y, v, w \in \mathbb{R},\, x^2 + y^2 \neq 0,\, \varsigma = \pm 1 \right\}.$$

We use the non-degenerate bilinear form

$$\left\langle\!\!\left\langle \begin{bmatrix} 0 & 0 & 0 \\ x_1 & 0 & -x_3 \\ x_2 & x_3 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ y_1 & 0 & -y_3 \\ y_2 & y_3 & 0 \end{bmatrix} \right\rangle\!\!\right\rangle = x_1y_1 + x_2y_2 + x_3y_3$$

to identify $\mathfrak{se}(2)$ with $\mathfrak{se}(2)^*$ (*cf.* [11]). Then each extremal curve $p(\cdot)$ in $\mathfrak{se}(2)^*$ is identified with a curve $P(\cdot)$ in $\mathfrak{se}(2)$ *via* the formula $\langle\!\langle P(t), X \rangle\!\rangle = p(t)(X)$ for all $X \in \mathfrak{se}(2)$. Thus

$$P(t) = \begin{bmatrix} 0 & 0 & 0 \\ P_1(t) & 0 & -P_3(t) \\ P_2(t) & P_3(t) & 0 \end{bmatrix}$$

where $P_i(t) = \langle\!\langle P(t), E_i \rangle\!\rangle = p(t)(E_i) = p_i(t), \quad i = 1, 2, 3.$

Now consider a Hamiltonian H on $\mathfrak{se}(2)^*_-$. The equations of motion take the following form

$$\dot{p}_i = -p([E_i, \mathrm{d}H(p)]), \quad i = 1, 2, 3$$

or, explicitly,

$$\begin{cases} \dot{p}_1 = \dfrac{\partial H}{\partial p_3}\, p_2 \\ \dot{p}_2 = -\dfrac{\partial H}{\partial p_3}\, p_1 \\ \dot{p}_3 = \dfrac{\partial H}{\partial p_2}\, p_1 - \dfrac{\partial H}{\partial p_1}\, p_2. \end{cases}$$

We note that $C : \mathfrak{se}(2)^*_- \to \mathbb{R}$, $C(p) = p_1^2 + p_2^2$ is a Casimir function.

3.2. Classification of systems

It turns out that there is only one homogeneous two-input system on $\mathsf{SE}(2)$, up to equivalence. Furthermore, in the inhomogeneous case there are only two types. The characterisation of detached feedback equivalence in Proposition 2.1 is used to prove both these results.

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Theorem 3.1. *Any full-rank homogeneous two-input system Σ is locally detached feedback equivalent to the system Σ_0 with parametrisation*

$$\Xi_0(\mathbf{1}, u) = u_1 E_2 + u_2 E_3.$$

Proof. Let the trace of Σ be given by $\Gamma = \left\langle \sum_{i=1}^3 b_i E_i, \sum_{i=1}^3 c_i E_i \right\rangle$. First, as either $b_3 \neq 0$ or $c_3 \neq 0$, we may assume $b_3 \neq 0$. Then

$$\Gamma = \left\langle \tfrac{b_1}{b_3} E_1 + \tfrac{b_2}{b_3} E_2 + E_3, (c_1 - \tfrac{b_1 c_3}{b_3}) E_1 + (c_2 - \tfrac{b_2 c_3}{b_3}) E_2 \right\rangle.$$

Now let $x = c_1 - \frac{b_1 c_3}{b_3}$ and $y = c_2 - \frac{b_2 c_3}{b_3}$. Then

$$\psi = \begin{bmatrix} y & x & \frac{b_1}{b_3} \\ -x & y & \frac{b_2}{b_3} \\ 0 & 0 & 1 \end{bmatrix}$$

is a Lie algebra automorphism mapping $\Gamma_0 = \langle E_2, E_3 \rangle$ to Γ. □

Theorem 3.2. *Any inhomogeneous two-input system Σ is locally detached feedback equivalent to exactly one of the following systems: Σ_1 or $\Sigma_{2,\alpha}$ $(\alpha > 0)$ with respective parametrisations*

$$\Xi_1(\mathbf{1}, u) = E_1 + u_1 E_2 + u_2 E_3, \qquad \Xi_{2,\alpha}(\mathbf{1}, u) = \alpha E_3 + u_1 E_1 + u_2 E_2.$$

Proof. Let the trace of Σ be given by

$$\Gamma = \sum_{i=1}^3 a_i E_i + \left\langle \sum_{i=1}^3 b_i E_i, \sum_{i=1}^3 c_i E_i \right\rangle.$$

First, consider the case $b_3 \neq 0$ or $c_3 \neq 0$. We may assume $b_3 \neq 0$ and so

$$\Gamma = a_1' E_1 + a_2' E_2 + \langle b_1' E_1 + b_2' E_2 + E_3, c_1' E_1 + c_2' E_2 \rangle$$

for some constants a_i', b_i', $c_i' \in \mathbb{R}$, $i = 1, 2$. Now either $c_1' \neq 0$ or $c_2' \neq 0$ and so

$$\begin{bmatrix} c_1' & -c_2' \\ c_2' & c_1' \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} -a_1' \\ -a_2' \end{bmatrix}$$

has a unique solution. (Note that $v_2 = 0$ leads to a contradiction). Hence

$$\psi = \begin{bmatrix} v_2 c_2' & v_2 c_1' & b_1' \\ -v_2 c_1' & v_2 c_2' & b_2' \\ 0 & 0 & 1 \end{bmatrix}$$

is a Lie algebra automorphism mapping $\Gamma_1 = E_1 + \langle E_2, E_3 \rangle$ to Γ.

Next, consider the case $b_3 = 0$ and $c_3 = 0$. Then

$$\Gamma = a_1 E_1 + a_2 E_2 + a_3 E_3 + \langle b_1 E_1 + b_2 E_2, c_1 E_1 + c_2 E_2 \rangle.$$

Since $a_3 \neq 0$ and either $b_1 \neq 0$ or $b_2 \neq 0$, we get that

$$\psi = \begin{bmatrix} b_1 & -\operatorname{sgn}(a_3) b_2 & \frac{a_1}{\alpha} \\ b_2 & \operatorname{sgn}(a_3) b_1 & \frac{a_2}{\alpha} \\ 0 & 0 & \operatorname{sgn}(a_3) \end{bmatrix}$$

is a Lie algebra automorphism. If we set $\alpha = |a_3|$, then ψ maps $\Gamma_{2,\alpha} = \alpha E_3 + \langle E_1, E_2 \rangle$ to Γ.

Finally, a simple argument shows that Σ_1 is not equivalent to any system $\Sigma_{2,\alpha}$, and that $\Sigma_{2,\alpha}$ is not equivalent to $\Sigma_{2,\beta}$, for any $\alpha \neq \beta$, $\alpha, \beta > 0$. □

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3.3. Left-invariant control problems

Henceforth, we consider only the systems Σ_0, Σ_1 and $\Sigma_{2,\alpha}$. In each of these typical cases, we shall investigate the optimal control problem corresponding to an arbitrary diagonal cost $L(u) = c_1u_1^2 + c_2u_2^2$, where c_1, $c_2 > 0$. Specifically, we shall consider the left-invariant control problems:

$$\left.\begin{aligned} \dot g &= g\,(u_1E_2 + u_2E_3)\\ g(0) &= g_0, \quad g(T) = g_T\\ \mathcal{J} &= \tfrac12\int_0^T \big(c_1u_1^2(t) + c_2u_2^2(t)\big)\,\mathrm{d}t \to \min \end{aligned}\right\} \qquad \mathsf{LiCP}(1)$$

$$\left.\begin{aligned} \dot g &= g\,(E_1 + u_1E_2 + u_2E_3)\\ g(0) &= g_0, \quad g(T) = g_T\\ \mathcal{J} &= \tfrac12\int_0^T \big(c_1u_1^2(t) + c_2u_2^2(t)\big)\,\mathrm{d}t \to \min \end{aligned}\right\} \qquad \mathsf{LiCP}(2)$$

and

$$\left.\begin{aligned} \dot g &= g\,(\alpha\, E_3 + u_1E_1 + u_2E_2)\\ g(0) &= g_0, \quad g(T) = g_T\\ \mathcal{J} &= \tfrac12\int_0^T \big(c_1u_1^2(t) + c_2u_2^2(t)\big)\,\mathrm{d}t \to \min\,. \end{aligned}\right\} \qquad \mathsf{LiCP}(3)$$

Remark 3.3. Each member of a significant subclass of left-invariant control problems on $\mathsf{SE}\,(2)$ is equivalent to one of the above three problems, up to cost-equivalence [7]. (If two cost-extended systems are cost-equivalent, then they have the same extremal trajectories, up to a Lie group isomorphism between their state spaces. The corresponding controls are mapped by an affine isomorphism). More specifically, any full-rank cost-extended system

$$\Xi\,(\mathbf{1}, u) = u_1B_1 + u_2B_2, \qquad L(u) = u^\top\, Q\, u$$

is cost-equivalent to

$$\Xi_0(\mathbf{1}, u) = u_1E_2 + u_2E_3, \qquad L_0(u) = u_1^2 + u_2^2. \tag{3.1}$$

(Here $Q \in \mathbb{R}^{2\times2}$, Q is positive definite; a proof can be found in [7]). $\mathsf{LiCP}(1)$ corresponds to (3.1). On the other hand, any cost-extended system

$$\Xi\,(\mathbf{1}, u) = A + u_1B_1 + u_2B_2, \qquad L(u) = (u-\mu)^\top\, Q\,(u-\mu)$$

where $A \notin \langle B_1, B_2\rangle$ is cost-equivalent to one of the cost-extended systems

$$\Xi_1(\mathbf{1}, u) = E_1 + u_1E_2 + u_2E_3, \qquad L_{1,\beta_1}(u) = (u_1-\mu_1)^2 + \beta_1(u_2-\mu_2)^2 \tag{3.2}$$

$$\Xi_{2,\alpha}(\mathbf{1}, u) = \alpha E_3 + u_1E_1 + u_2E_2, \qquad L_{2,\beta_2}(u) = u_1^2 + \beta_2u_2^2 \tag{3.3}$$

where $\alpha, \beta_1 > 0$ and $\beta_2 \geq 1$. (Here $\mu \in \mathbb{R}^2$, $Q \in \mathbb{R}^{2\times2}$, and Q is positive definite). $\mathsf{LiCP}(2)$ corresponds to (3.2) with $\mu_1 = \mu_2 = 0$, whereas $\mathsf{LiCP}(3)$ corresponds to (3.3).

The following three results easily follow.

Proposition 3.4. *For the* $\mathsf{LiCP}(1)$*, the (normal) extremal control is given by* $u_1 = \frac{1}{c_1}p_2$, $u_2 = \frac{1}{c_2}p_3$, *where* $H_1(p) = \frac12\left(\frac{1}{c_1}p_2^2 + \frac{1}{c_2}p_3^2\right)$ *and*

$$\begin{cases} \dot p_1 = \frac{1}{c_2}p_2p_3\\ \dot p_2 = -\frac{1}{c_2}p_1p_3\\ \dot p_3 = \frac{1}{c_1}p_1p_2. \end{cases} \tag{3.4}$$

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Proposition 3.5. *For the* LiCP(2)*, the (normal) extremal control is given by* $u_1 = \frac{1}{c_1}p_2$, $u_2 = \frac{1}{c_2}p_3$, *where* $H_2(p) = p_1 + \frac{1}{2}\left(\frac{1}{c_1}p_2^2 + \frac{1}{c_2}p_3^2\right)$ *and*

$$\begin{cases} \dot{p}_1 = \frac{1}{c_2}p_2p_3 \\ \dot{p}_2 = -\frac{1}{c_2}p_1p_3 \\ \dot{p}_3 = \left(\frac{1}{c_1}p_1 - 1\right)p_2. \end{cases} \tag{3.5}$$

Proposition 3.6. *For the* LiCP(3)*, the (normal) extremal control is given by* $u_1 = \frac{1}{c_1}p_1$, $u_2 = \frac{1}{c_2}p_2$, *where* $H_3(p) = \alpha\, p_3 + \frac{1}{2}\left(\frac{1}{c_1}p_1^2 + \frac{1}{c_2}p_2^2\right)$ *and*

$$\begin{cases} \dot{p}_1 = \alpha\, p_2 \\ \dot{p}_2 = -\alpha\, p_1 \\ \dot{p}_3 = \left(\frac{1}{c_2} - \frac{1}{c_1}\right)p_1p_2. \end{cases} \tag{3.6}$$

4. Qualitative analysis

In this section a qualitative analysis of the reduced Hamilton–Poisson systems (3.4)–(3.6) is performed. The stability nature of every equilibrium state is determined. The vector fields $\vec{H}_1$, $\vec{H}_2$, and $\vec{H}_3$ are shown to be complete. Subsequently, each maximal integral curve is described as a constant, periodic or bounded curve.

4.1. Equilibrium states

The equilibrium states for (3.4) are

$$e_1^\mu = (\mu, 0, 0), \quad e_2^\nu = (0, \nu, 0) \quad \text{and} \quad e_3^\mu = (0, 0, \mu)$$

where $\mu, \nu \in \mathbb{R}$, $\nu \neq 0$.

Theorem 4.1. *The equilibrium states have the following behaviour:*

(*i*) *Each equilibrium state* e_1^μ *is stable.*
(*ii*) *Each equilibrium state* e_2^ν *is unstable.*
(*iii*) *Each equilibrium state* e_3^μ *is stable.*

Proof. The linearization of the system is given by

$$\begin{bmatrix} 0 & \frac{1}{c_2}p_3 & \frac{1}{c_2}p_2 \\ -\frac{1}{c_2}p_3 & 0 & -\frac{1}{c_2}p_1 \\ \frac{1}{c_1}p_2 & \frac{1}{c_1}p_1 & 0 \end{bmatrix}.$$

(i) Assume $\mu \neq 0$. (The state $e_1^0 = e_3^0$ is dealt with in (iii)). Let $H_\chi = H + \chi(C)$ be an energy-Casimir function, *i.e.*,

$$H_\chi(p_1, p_2, p_3) = \tfrac{1}{2c_1}p_2^2 + \tfrac{1}{2c_2}p_3^2 + \chi(p_1^2 + p_2^2)$$

where $\chi \in C^\infty(\mathbb{R})$. The derivative

$$\mathrm{d}H_\chi = \left[2p_1\dot{\chi}(p_1^2 + p_2^2)\ \tfrac{1}{c_1}p_2 + 2p_2\dot{\chi}(p_1^2 + p_2^2)\ \tfrac{1}{c_2}p_3\right]$$

vanishes at e_1^μ if $\dot{\chi}(\mu^2) = 0$. Then the Hessian (at e_1^μ)

$$\mathrm{d}^2H_\chi(\mu, 0, 0) = \mathrm{diag}\left(2\dot{\chi}(\mu^2) + 4\mu^2\ddot{\chi}(\mu^2),\ \tfrac{1}{c_1} + 2\dot{\chi}(\mu^2),\ \tfrac{1}{c_2}\right)$$

is positive definite if $\ddot{\chi}(\mu^2) > 0$ (and $\dot{\chi}(\mu^2) = 0$). The function $\chi(x) = \frac{1}{2}x^2 - \mu^2 x$ satisfies these requirements. Hence, by the standard energy-Casimir method, e_1^μ is stable.

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(ii) The linearization of the system at e_2^ν has eigenvalues $\lambda_1 = 0$, $\lambda_{2,3} = \pm \frac{\nu}{\sqrt{c_1 c_2}}$. Thus e_2^ν is unstable.

(iii) Let $H_\lambda = \lambda_0 H + \lambda_1 C$. Then $\mathrm{d}H_\lambda(0,0,\mu) = \begin{bmatrix} 0 & 0 & \frac{\lambda_0 \mu}{c_2} \end{bmatrix}$ and $\mathrm{d}^2 H_\lambda(0,0,\mu) = \mathrm{diag}(2\lambda_1, \frac{\lambda_0}{c_1} + 2\lambda_1, \frac{\lambda_0}{c_2})$. Suppose $\mu = 0$ and let $\lambda_0 = \lambda_1 = 1$. Then $\mathrm{d}H_\lambda(0,0,0) = 0$ and $\mathrm{d}^2 H_\lambda(0,0,0)$ is positive definite. On the other hand, suppose $\mu \neq 0$ and let $\lambda_0 = 0$, $\lambda_1 = 1$. Then $\mathrm{d}H_\lambda(0,0,\mu) = 0$ and $\mathrm{d}^2 H_\lambda(0,0,\mu) = \mathrm{diag}\,(2,2,0)$. Also,

$$\ker \mathrm{d}H(e_3^\mu) \cap \ker \mathrm{d}C(e_3^\mu) = \mathrm{span}\,\{(1,0,0),(0,1,0)\}$$

and so $\mathrm{d}^2 H_\lambda(0,0,\mu)\big|_{W\times W} = \mathrm{diag}\,(2,\,2)$ is positive definite. Hence, by the extended energy-Casimir method, e_3^μ is stable. □

The equilibrium states for (3.5) are

$$e_1^\mu = (\mu,0,0), \quad e_2^\mu = (c_1,\mu,0) \quad \text{and} \quad e_3^\nu = (0,0,\nu)$$

where $\mu, \nu \in \mathbb{R}$, $\nu \neq 0$.

Theorem 4.2. *The equilibrium states have the following behaviour:*

(*i*) *Each equilibrium state* e_1^μ *is unstable if* $\mu \in [0, c_1]$ *and stable if* $\mu \in (-\infty, 0) \cup (c_1, \infty)$.
(*ii*) *Each equilibrium state* e_2^μ *is unstable.*
(*iii*) *Each equilibrium state* e_3^ν *is stable.*

Proof. The linearization of the system is given by

$$\begin{bmatrix} 0 & \frac{1}{c_2}p_3 & \frac{1}{c_2}p_2 \\ -\frac{1}{c_2}p_3 & 0 & -\frac{1}{c_2}p_1 \\ \frac{1}{c_1}p_2 & \frac{1}{c_1}p_1 - 1 & 0 \end{bmatrix}.$$

(i) Assume $\mu \in (0, c_1)$. The linearization of the system (at e_1^μ) has eigenvalues $\lambda_1 = 0$, $\lambda_{2,3} = \pm\sqrt{\frac{(c_1-\mu)\mu}{c_1 c_2}}$. Thus e_1^μ is unstable. Now, assume $\mu = c_1$ or $\mu = 0$. Then the linearization of the system (at e_1^μ) has eigenvalues $\lambda_{1,2,3} = 0$. Thus, as the geometric multiplicity is strictly less than the algebraic multiplicity, e_1^μ is unstable.
Assume $\mu \in (-\infty, 0) \cup (c_1, \infty)$. Let $H_\chi = H + \chi(C)$ be an energy-Casimir function, *i.e.*,

$$H_\chi(p_1,p_2,p_3) = p_1 + \tfrac{1}{2c_1}p_2^2 + \tfrac{1}{2c_2}p_3^2 + \chi(p_1^2 + p_2^2)$$

where $\chi \in C^\infty(\mathbb{R})$. The derivative

$$\mathrm{d}H_\chi = \begin{bmatrix} 1 + 2p_1\dot{\chi}(p_1^2+p_2^2) & \frac{1}{c_1}p_2 + 2p_2\dot{\chi}(p_1^2+p_2^2) & \frac{1}{c_2}p_3 \end{bmatrix}$$

vanishes at e_1^μ if $\dot{\chi}(\mu^2) = -\frac{1}{2\mu}$. Then the Hessian (at e_1^μ)

$$\begin{aligned} \mathrm{d}^2 H_\chi(\mu,0,0) &= \mathrm{diag}\,(4\mu^2\ddot{\chi}(\mu^2) + 2\dot{\chi}(\mu^2),\ \tfrac{1}{c_1} + 2\dot{\chi}(\mu^2),\ \tfrac{1}{c_2}) \\ &= \mathrm{diag}(4\mu^2\ddot{\chi}(\mu^2) - \tfrac{1}{\mu},\ \tfrac{1}{c_1} - \tfrac{1}{\mu},\ \tfrac{1}{c_2}) \end{aligned}$$

is positive definite if $\ddot{\chi}(\mu^2) > \frac{1}{4\mu^3}$. The function $\chi(x) = (\frac{1}{8\mu^3} + 1)x^2 - \frac{3+8\mu^3}{4\mu}x$ satisfies these requirements. Hence, by the standard energy-Casimir method, e_1^μ is stable.

(ii) Assume $\mu \neq 0$. (The case $\mu = 0$ has already been dealt with). The linearization of the system (at e_2^μ) has eigenvalues $\lambda_1 = 0$, $\lambda_{2,3} = \pm \frac{\mu}{\sqrt{c_1 c_2}}$. Thus e_2^μ is unstable.

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(iii) Let $H_\lambda = \lambda_0 H + \lambda_1 C$, where $\lambda_0 = 0$, $\lambda_1 = 1$. Now $\mathrm{d}H = \left[1 \ \frac{1}{c_1}p_2 \ \frac{1}{c_2}p_3\right]$ and $\mathrm{d}C = \left[2p_1 \ 2p_2 \ 0\right]$. Hence $\mathrm{d}H_\lambda(0,0,\nu) = 0$ and $\mathrm{d}^2 H_\lambda(0,0,\nu) = \mathrm{diag}(2,\,2,\,0)$. Also,

$$\ker \mathrm{d}H(e_3^\nu) \cap \ker \mathrm{d}C(e_3^\nu) = \mathrm{span}\,\{(-\nu, 0, c_2), (0,1,0)\}$$

and so $\mathrm{d}^2 H_\lambda(0,0,\nu)\big|_{W\times W} = \mathrm{diag}\left(2\nu^2,\,2\right)$ is positive definite. Hence, by the extended energy-Casimir method, e_3^ν is stable. □

The equilibrium states for (3.6) are

$$e^\mu = (0,0,\mu), \quad \mu \in \mathbb{R}.$$

Theorem 4.3. *Each equilibrium state e^μ is stable.*

Proof. Let $H_\lambda = \lambda_0 H + \lambda_1 C$, where $\lambda_0 = 0$, $\lambda_1 = 1$. Now $\mathrm{d}H = \left[\frac{1}{c_1}p_1 \ \frac{1}{c_2}p_2 \ \alpha\right]$ and $\mathrm{d}C = \left[2p_1 \ 2p_2 \ 0\right]$. Hence $\mathrm{d}H_\lambda(0,0,\mu) = 0$ and $\mathrm{d}^2 H_\lambda(0,0,\mu) = \mathrm{diag}(2,\,2,\,0)$. Also,

$$\ker \mathrm{d}H(e^\mu) \cap \ker \mathrm{d}C(e^\mu) = \mathrm{span}\,\{(1,0,0), (0,1,0)\}$$

and so $\mathrm{d}^2 H_\lambda(0,0,\mu)\big|_{W\times W} = \mathrm{diag}\,(2,\,2)$ is positive definite. Hence, by the extended energy-Casimir method, e^μ is stable. □

4.2. Integral curves

We give qualitative descriptions of the integral curves of $\vec{H}_1$, $\vec{H}_2$, and $\vec{H}_3$. Let $\mathcal{E}_1$, $\mathcal{E}_2$, and $\mathcal{E}_3$ denote the set of equilibrium points for $\vec{H}_1$, $\vec{H}_2$, and $\vec{H}_3$, respectively.

Proposition 4.4. *The level sets*

$$\mathcal{C}_i = \left(C^{-1}(c_0) \cap H_i^{-1}(h_i)\right) \setminus \mathcal{E}_i, \qquad i = 1,2,3$$

are bounded embedded 1-submanifolds of $\mathfrak{se}(2)^*$ for $c_0 > 0$, $h_1 > 0$, $h_2 > -\sqrt{c_0}$, and $h_3 \in \mathbb{R}$.

(Some typical cases for these sets are graphed in Figs. 1–3).

Proof. Let $F_1 : \mathfrak{se}(2)^* \setminus \mathcal{E}_1 \to \mathbb{R}^2$, $p \mapsto (C(p), H_1(p))$. Note that, as $\mathcal{E}_1$ is closed, $\mathfrak{se}(2)^* \setminus \mathcal{E}_1$ is open and thus an embedded 3-submanifold of $\mathfrak{se}(2)^*$. We have

$$DF_1(p) = \begin{bmatrix} 2p_1 & 2p_2 & 0 \\ 0 & \frac{1}{c_1}p_2 & \frac{1}{c_2}p_3 \end{bmatrix}$$

which has full rank unless $p_1 = 0$ and $p_3 = 0$. However, $(0, p_2, 0) \in \mathcal{E}_1$. Thus $\mathcal{C}_1 = F^{-1}(c_0, h_1)$ is an embedded 1-submanifold of $\mathfrak{se}(2)^*$. For $p = (p_1, p_2, p_3) \in \mathcal{C}_1$, we have $p_1^2 + p_2^2 = c_0$ and $\frac{1}{2}\left(\frac{1}{c_1}p_2^2 + \frac{1}{c_2}p_3^2\right) = h_1$. Hence $p_1^2 \le c_0$, $p_2^2 \le c_0$, and $p_3^2 \le 2h_1c_2$. Thus $\mathcal{C}_1$ is bounded.

A similar argument shows that $\mathcal{C}_2$ and $\mathcal{C}_3$ are bounded embedded 1-submanifolds of $\mathfrak{se}(2)^*$. The conditions on c_0, h_1, and h_2 are required such that the sets $\mathcal{C}_1$, $\mathcal{C}_2$, and $\mathcal{C}_3$ are nonempty. □

As H_i and C are constants of the motion, any non-constant integral curve $p(\cdot)$ of $\vec{H}_i$ evolves on $\mathcal{C}_i$ (where $c_0 = C(p(0))$ and $h_i = H_i(p(0))$). Moreover, as each $\mathcal{C}_i$ is bounded, any integral curve lies in a compact subset of $\mathfrak{se}(2)^*$. Hence (see, *e.g.*, [1])

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Corollary 4.5. *The vector fields* $\vec{H}_1$, $\vec{H}_2$ *and* $\vec{H}_3$ *are complete.*

We now describe the integral curves of $\vec{H}_1$, $\vec{H}_2$, and $\vec{H}_3$. Let $p(\cdot)$ be a maximal integral curve of a complete vector field on $\mathfrak{se}(2)^*$. A point $p \in \mathfrak{se}(2)^*$ is an *ω-limit* point of $p(\cdot)$ if there exists a sequence (t_n) such that $t_n \to \infty$ and $p(t_n) \to p$. Similarly, if there exists a sequence (t_n) such that $t_n \to -\infty$ and $p(t_n) \to q$, then q is an *α-limit* point. The set of all α-limit points of $p(\cdot)$ is denoted $\lim_\alpha p(\cdot)$; the set of all ω-limit points of $p(\cdot)$ is denoted $\lim_\omega p(\cdot)$.

Theorem 4.6. *Let* $p(\cdot) : \mathbb{R} \to \mathfrak{se}(2)^*$ *be a non-constant maximal integral curve of* $\vec{H}_i$. *Let* $\mathcal{C}_i^0$ *be the corresponding connected component of* $\mathcal{C}_i$ *containing* $p(0)$.

1. *Suppose* $p(\cdot)$ *is an integral curve of* $\vec{H}_1$.
   (a) *If* $\mathcal{C}_1^0$ *is closed (or equivalently* $\operatorname{cl}\mathcal{C}_1^0 \cap \mathcal{E}_1 = \emptyset$*), then* $p(\cdot)$ *is periodic.*
   (b) *If* $\mathcal{C}_1^0$ *is not closed (or equivalently* $\operatorname{cl}\mathcal{C}_1^0 \cap \mathcal{E}_1 \neq \emptyset$*), then* $p(\cdot)$ *is bounded and the limit sets* $\lim_\omega p(\cdot)$ *and* $\lim_\alpha p(\cdot)$ *are singletons in* $\mathcal{E}_1$.
   In either case, $p(\cdot)$ *has image* $\mathcal{C}_1^0$.
2. *Suppose* $p(\cdot)$ *is an integral curve of* $\vec{H}_2$.
   (a) *If* $\mathcal{C}_2^0$ *is closed (or equivalently* $\operatorname{cl}\mathcal{C}_2^0 \cap \mathcal{E}_2 = \emptyset$*), then* $p(\cdot)$ *is periodic.*
   (b) *If* $\mathcal{C}_2^0$ *is not closed (or equivalently* $\operatorname{cl}\mathcal{C}_2^0 \cap \mathcal{E}_2 \neq \emptyset$*), then* $p(\cdot)$ *is bounded and the limit sets* $\lim_\omega p(\cdot)$ *and* $\lim_\alpha p(\cdot)$ *are singletons in* $\mathcal{E}_2$.
   In either case, $p(\cdot)$ *has image* $\mathcal{C}_2^0$.
3. *If* $p(\cdot)$ *is an integral curve of* $\vec{H}_3$, *then* $p(\cdot)$ *is periodic and has image* $\mathcal{C}_3^0$.

Proof. First we show that $\mathcal{C}_i^0$ is closed if and only if $\operatorname{cl}\mathcal{C}_i^0 \cap \mathcal{E}_i = \emptyset$ (for $i = 1, 2, 3$). If $\mathcal{C}_i^0$ is closed, then $\operatorname{cl}\mathcal{C}_i^0 = \mathcal{C}_i^0 \subseteq \big(C^{-1}(c_0) \cap H_i^{-1}(h_i)\big) \setminus \mathcal{E}_i$ and so $\operatorname{cl}\mathcal{C}_i^0 \cap \mathcal{E}_i = \emptyset$. Conversely, suppose $\operatorname{cl}\mathcal{C}_i^0 \cap \mathcal{E}_i = \emptyset$. Then $\operatorname{cl}\mathcal{C}_i^0 \subset \mathfrak{se}(2)^* \setminus \mathcal{E}_i$ and $\operatorname{cl}\mathcal{C}_i^0 \subseteq C^{-1}(c_0) \cap H_i^{-1}(h_i)$, *i.e.*, $\operatorname{cl}\mathcal{C}_i^0 \subseteq \mathcal{C}_i$. However, as $\mathcal{C}_i^0$ is connected, $\operatorname{cl}\mathcal{C}_i^0$ is connected. Thus $\operatorname{cl}\mathcal{C}_i^0 = \mathcal{C}_i^0$.

(1) We have that $p(\cdot)$ evolves on $\mathcal{C}_1^0$. As $\mathcal{C}_1^0$ is a connected 1-manifold, it is diffeomorphic to either the circle $\mathbb{S}$ or the real line $\mathbb{R}$. (1a) Suppose $\mathcal{C}_1^0$ is closed. Then $\mathcal{C}_1^0$ is diffeomorphic to $\mathbb{S}$. Let X be the push-forward (to $\mathbb{S}$) of the restriction of $\vec{H}_1$ to $\mathcal{C}_1^0$. X is complete and nonzero everywhere. As $\mathbb{S}$ is compact, $\|X\|$ attains a positive minimum. Hence there is a finite interval $[t_0, t_1]$ in which any integral curve of X covers $\mathbb{S}$. Hence any maximal integral curve of X is periodic. Consequently, $p(\cdot)$ is periodic (as it is diffeomorphic to a periodic integral curve) and has image $\mathcal{C}_1^0$.

(1b) Suppose $\mathcal{C}_1^0$ is not closed. As $\mathcal{C}_1$ is bounded, the limit sets $\lim_\alpha p(\cdot)$ and $\lim_\omega p(\cdot)$ are non-empty, connected, compact subsets of $\mathfrak{se}(2)^*$ (*cf.* [20]). Also $p(\cdot)$ is bounded. Now $\mathcal{C}_1^0$ is diffeomorphic to $\mathbb{R}$. Let X be the push-forward (to $\mathbb{R}$) of the restriction of $\vec{H}_1$ to $\mathcal{C}_1^0$. X is complete and nonzero everywhere. Hence, we may assume $X(q) > 0$ for $q \in \mathbb{R}$. Let $q(\cdot) : \mathbb{R} \to \mathbb{R}$ be a maximal integral curve of X. For every compact interval $[q(t_0), q_1]$ or $[q_0, q(t_1)]$ in $\mathbb{R}$, $\|X\|$ attains a positive minimum. Hence (for every such interval) there exists a $T \geq 0$ such that $q(t_0 + T) > q_1$ or $q(t_1 - T) < q_0$, respectively. From this we draw two conclusions. First, any maximal integral curve of X covers $\mathbb{R}$. Second, if (t_n) is a sequence in $\mathbb{R}$ such that $t_n \to \pm\infty$ and $q(\cdot) : \mathbb{R} \to \mathbb{R}$ is a maximal integral curve, then $(q(t_n))$ does not converge in $\mathbb{R}$. Consequently, the image of $p(\cdot)$ is $\mathcal{C}_1^0$. Also, there are no α- or ω-limit points of $p(\cdot)$ in $\mathcal{C}_1^0$. However, any α- or ω-limit point must be in $\operatorname{cl}\mathcal{C}_1^0 \subseteq C^{-1}(c_0) \cap H_1^{-1}(h_1)$.

We claim that $\lim_\alpha p(\cdot) \cap \mathcal{C}_1 = \emptyset$ and $\lim_\omega p(\cdot) \cap \mathcal{C}_1 = \emptyset$. Suppose there exists $p \in \lim_\alpha p(\cdot) \cap \mathcal{C}_1$ or $p \in \lim_\omega p(\cdot) \cap \mathcal{C}_1$. Then $p \in \operatorname{cl}\mathcal{C}_1^0$, $p \in \mathcal{C}_1$, and $p \notin \mathcal{C}_1^0$. Hence $\mathcal{C}_1^0 \cup \{p\}$ is connected and $\mathcal{C}_1^0 \cup \{p\} \subseteq \mathcal{C}_1$. Thus $\mathcal{C}_1^0 \cup \{p\}$ is contained in the connected component $\mathcal{C}_1^0$ of $\mathcal{C}_1$ containing $p(0)$, a contradiction. So we have

$$\lim_\alpha p(\cdot) \subseteq C^{-1}(c_0) \cap H_1^{-1}(h_1) \qquad \lim_\alpha p(\cdot) \cap \big(\big(C^{-1}(c_0) \cap H_1^{-1}(h_1)\big) \setminus \mathcal{E}_1\big) = \emptyset$$

and similarly for $\lim_\omega p(\cdot)$. Hence

$$\lim_\alpha p(\cdot),\ \lim_\omega p(\cdot) \subseteq C^{-1}(c_0) \cap H_1^{-1}(h_1) \cap \mathcal{E}_1.$$

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A simple calculation shows that $C^{-1}(c_0)\cap H_1^{-1}(h_1)\cap\mathcal{E}_1$ is finite and hence completely disconnected. Therefore, as $\lim_\alpha p(\cdot)$ and $\lim_\omega p(\cdot)$ are non-empty and connected, $\lim_\alpha p(\cdot)$ and $\lim_\omega p(\cdot)$ are singletons.

For (2), a similar argument yields the result. (3) We show that $\mathcal{C}_3^0$ is closed by showing that $\operatorname{cl}\mathcal{C}_3^0\cap\mathcal{E}_3=\emptyset$. The result then follows in the same way as for (1a). Suppose there exists $p\in\operatorname{cl}\mathcal{C}_3^0\cap\mathcal{E}_3$. Then $p\in C^{-1}(c_0)\cap H_3^{-1}(h_3)\cap\mathcal{E}_3$. Hence, as $p\in\mathcal{E}_3$, $p=(0,0,\mu)$ for some $\mu\in\mathbb{R}$. Hence, as $p\in C^{-1}(c_0)$, we get $c_0=0$, a contradiction. □

5. Explicit integration

The reduced Hamilton equations (3.4)–(3.6) can be integrated by Jacobi elliptic functions. (In fact, (3.6) can easily be integrated by trigonometric functions). In each of these cases, we obtain explicit expressions for the integral curves of $\vec{H}$. Before producing these results, we outline the basic approach employed in obtaining them. (In each case, MATHEMATICA is utilised to facilitate calculations). A similar approach was used in [2] for single-input systems on $\mathsf{SE}(2)$.

Note 5.1. In this section we consider only non-constant integral curves.

First, we fix a Hamiltonian vector field $\vec{H}$ (specifically (3.4) or (3.5)). We then partition the set of all initial conditions; this enables one to produce a single explicit expression for each (sub)case. The first separation is made by considering when the level surfaces, defined by the constants of motion C and H, are tangent to one another. This level of separation is sufficient for solving (3.4), but further partitioning (made retrospectively) is needed in solving (3.5).

Next, we suppose that $\bar{p}(\cdot):(-\varepsilon,\varepsilon)\to\mathfrak{se}(2)^*$ is an integral curve of $\vec{H}$ (satisfying some appropriate conditions). We let $h_0=H(\bar{p}(0))$ and $c_0=C(\bar{p}(0))>0$. Then, as $\bar{p}(\cdot)$ solves (3.4) or (3.5), we get that

$$\frac{\mathrm{d}}{\mathrm{d}t}\bar{p}_2=\pm\sqrt{\tfrac{1}{c_1c_2}\,(c_0-\bar{p}_2^2)\,(2h_0c_1-\bar{p}_2^2)} \tag{5.1}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\bar{p}_1=\pm\sqrt{\tfrac{1}{c_1c_2}\,(c_0-\bar{p}_1^2)\,(2c_1h_0-c_0-2c_1\bar{p}_1+\bar{p}_1^2)} \tag{5.2}$$

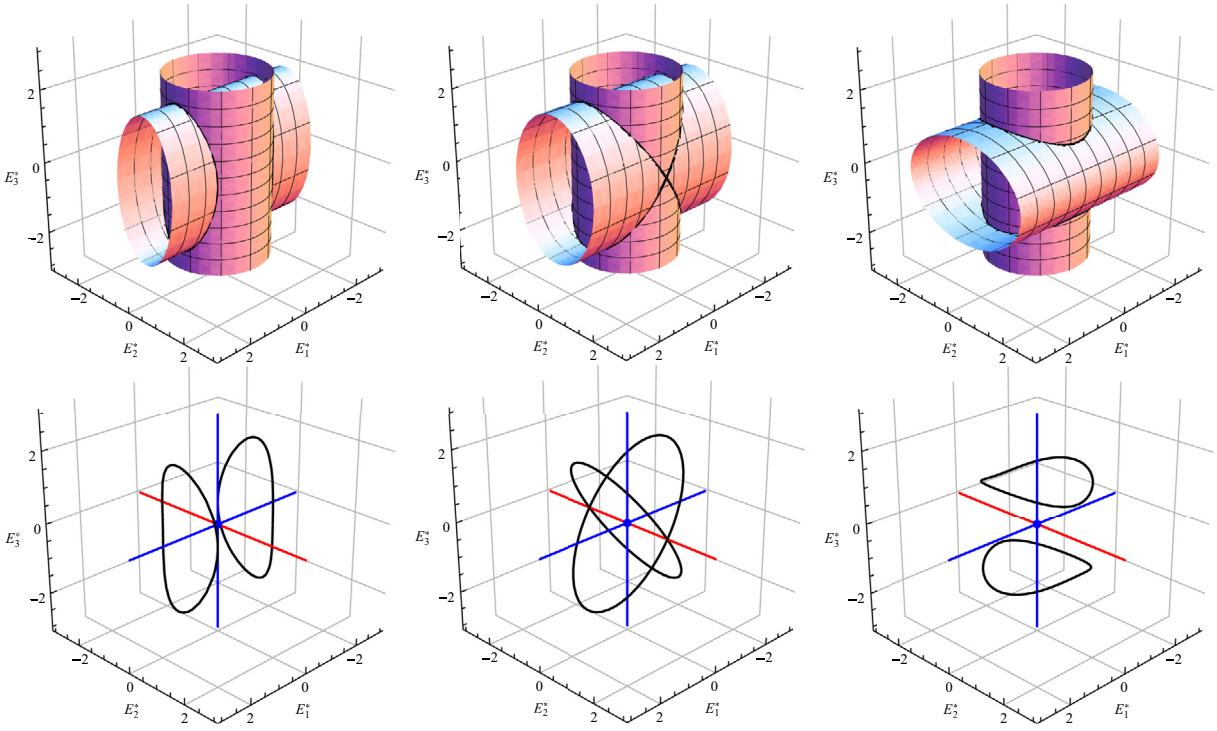
respectively. In most cases, the respective (separable) differential equation is transformed into standard form (see [4] or [15]). A formula for an elliptic integral is then applied to obtain an expression for $\bar{p}_2(t)$ or $\bar{p}_1(t)$, respectively. (Observe however that (5.1) is already in standard form). Often, a good deal of further simplification is then performed. Next, by use of the constants of motion C and H, expressions for $\bar{p}_1(t)$, $\bar{p}_2(t)$ and $\bar{p}_3(t)$ are determined up to a choice of sign and organised so as to be smooth (again involving further simplification).

Accordingly, we get a prospective (smooth) integral curve $\bar{p}(\cdot)$ whose domain is extended to $\mathbb{R}$. In some special cases, the prospective integral curve $\bar{p}(\cdot)$ may be produced by a limiting process from other results already obtained, or by directly solving the differential equation, as is the case with (3.6). Then, by explicitly differentiating $\bar{p}(\cdot)$, we verify for which choices of sign $\bar{p}(\cdot)$ is an integral curve of $\vec{H}$. (This is then further verified by solving the respective differential equation numerically for some suitable initial condition). Finally, we show that any other integral curve $p(\cdot):(-\varepsilon,\varepsilon)\to\mathfrak{se}(2)^*$ of $\vec{H}$, evolving on $C^{-1}(c_0)\cap H^{-1}(h_0)$, is identical to $\bar{p}(\cdot)$ up to a translation in the independent variable and an allowable choice of sign.

Various properties of the Jacobi elliptic functions are involved in making the above mentioned calculations (*cf.* [4, 15]). In particular, we use the periodicity properties (*e.g.*, $\operatorname{sn}(x+K,k)=\operatorname{cd}(x,k)$), relations of squares (*e.g.*, $1-k^2\operatorname{sn}^2(x,k)=\operatorname{dn}^2(x,k)$) and half-angle formulas (*e.g.*, $\operatorname{cn}^2\left(\frac{1}{2}\,x,\,k\right)=\frac{\operatorname{cn}(x,k)+\operatorname{dn}(x,k)}{1+\operatorname{dn}(x,k)}$).

We now produce the results for each typical case. (A summary of these results may be found in the appendix). Only for Theorem 5.10 will a proof detailing the method used to obtain the result be provided. For the remaining results we omit details pertaining to finding a maximal integral curve (they follow the same approach or are easy to obtain by straightforward integration) and only verify that every other integral curve is identical up to a translation in the independent variable.

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(A) $c_0 > 2c_1h_0$ (B) $c_0 = 2c_1h_0$ (C) $c_0 < 2c_1h_0$

FIGURE 1. Typical cases of LiCP(1).

5.1. Homogeneous systems

There are three typical cases for the reduced extremal equations (3.4), corresponding to (a) $c_0 > 2c_1h_0$, (b) $c_0 = 2c_1h_0$ and (c) $c_0 < 2c_1h_0$. In Figure 1, we graph the level sets of H and C and their intersection for some suitable values of h_0, c_0, c_1 and c_2. The stable equilibrium points (illustrated in blue) and unstable equilibrium points (illustrated in red), as presented in Theorem 4.1, are also plotted in each case.

We begin our presentation of the integral curves of (3.4) with case (a). (In the following theorem, formula (2.7) is used to obtain a prospective integral curve).

Theorem 5.2 (case a). *Suppose* $p(\cdot) : (-\varepsilon, \varepsilon) \to \mathfrak{se}(2)^*$ *is an integral curve of* $\vec{H}$ *such that* $H(p(0)) = h_0 > 0$, $C(p(0)) = c_0 > 0$ *and* $c_0 > 2c_1h_0$. *Then there exists* $t_0 \in \mathbb{R}$ *and* $\sigma \in \{-1, 1\}$ *such that* $p(t) = \bar{p}(t + t_0)$ *for* $t \in (-\varepsilon, \varepsilon)$, *where*

$$\begin{cases} \bar{p}_1(t) = \sigma\sqrt{c_0}\,\mathrm{dn}\,(\Omega\, t,\, k) \\ \bar{p}_2(t) = \sqrt{2c_1h_0}\,\mathrm{sn}\,(\Omega\, t,\, k) \\ \bar{p}_3(t) = -\sigma\sqrt{2c_2h_0}\,\mathrm{cn}\,(\Omega\, t,\, k)\,. \end{cases}$$

Here $k = \sqrt{\frac{2c_1h_0}{c_0}}$ *and* $\Omega = \sqrt{\frac{c_0}{c_1c_2}}$.

Proof. Verification that $\bar{p}(\cdot)$ is an integral curve satisfying $H(\bar{p}(0)) = h_0$, $C(p(0)) = c_0$ whenever $c_0 > 2c_1h_0 > 0$ is straightforward. We have $p_1(0)^2 + p_2(0)^2 = c_0$ and $\frac{1}{c_1}p_2(0)^2 + \frac{1}{c_2}p_3(0)^2 = 2h_0$. Thus $p_2(0)^2 \le 2h_0c_1$ and so

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$p_1(0)^2 = c_0 - p_2(0)^2 \geq c_0 - 2h_0c_1 > 0$. Let $\sigma = \operatorname{sgn}(p_1(0))$ (we have $\sigma \neq 0$). Furthermore, $-\sqrt{2h_0c_2} \leq p_3(0) \leq \sqrt{2h_0c_2}$, $\bar{p}_3(0) = -\sigma\sqrt{2c_2h_0}$ and $\bar{p}_3(\frac{2K}{\Omega}) = \sigma\sqrt{2c_2h_0}$. Hence there exists $t_1 \in \mathbb{R}$ such that $\bar{p}_3(t_1) = p_3(0)$. Now

$$p_2(0)^2 = 2h_0c_1 - \tfrac{c_1}{c_2}p_3(0)^2 = 2h_0c_1 - \tfrac{c_1}{c_2}\bar{p}_3(t_1)^2 = \bar{p}_2(t_1)^2.$$

Thus $p_2(0) = \pm\bar{p}_2(t_1)$. However, $\bar{p}_2(-t) = -\bar{p}_2(t)$ and $\bar{p}_3(-t) = \bar{p}_3(t)$. Therefore there exists $t_0 \in \mathbb{R}$ (either $t_0 = t_1$ or $t_0 = -t_1$) such that $p_3(0) = \bar{p}_3(t_0)$ and $p_2(0) = \bar{p}_2(t_0)$. Also

$$p_1(0)^2 = c_0 - p_2(0)^2 = c_0 - \bar{p}_2(t_0)^2 = \bar{p}_1(t_0)^2.$$

Hence, as $\operatorname{sgn}(p_1(0)) = \sigma = \operatorname{sgn}(\bar{p}_1(t_0))$, we get $p_1(0) = \bar{p}_1(t_0)$. Thus $p(0) = \bar{p}(t_0)$. Consequently the integral curves $t \mapsto p(t)$ and $t \mapsto \bar{p}(t + t_0)$ solve the same Cauchy problem, and therefore are identical. □

Next, by limiting $c_0 \to 2c_1h_0$ in Theorem 5.2 (and allowing for possible changes in sign), we get a prospective integral curve for case (b).

Proposition 5.3 (case b)**.** *Suppose $p(\cdot) : (-\varepsilon, \varepsilon) \to \mathfrak{se}(2)^*$ is an integral curve of $\vec{H}$ such that $H(p(0)) = h_0 > 0$, $C(p(0)) = c_0 > 0$ and $c_0 = 2c_1h_0$. Then there exists $t_0 \in \mathbb{R}$ and $\sigma_1, \sigma_2 \in \{-1, 1\}$ such that $p(t) = \bar{p}(t + t_0)$ for $t \in (-\varepsilon, \varepsilon)$, where*

$$\begin{cases} \bar{p}_1(t) = \sigma_1\sigma_2\sqrt{c_0}\,\operatorname{sech}(\Omega t) \\ \bar{p}_2(t) = \sigma_1\sqrt{c_0}\,\tanh(\Omega t) \\ \bar{p}_3(t) = -\sigma_2\sqrt{\frac{c_0c_2}{c_1}}\,\operatorname{sech}(\Omega t)\,. \end{cases}$$

Here $\Omega = \sqrt{\frac{c_0}{c_1c_2}}$.

Proof. Let $\sigma_2 = -\operatorname{sgn}(p_3(0))$ and let $\sigma_1 = \sigma_2 \operatorname{sgn}(p_1(0))$. We have $p_1(0)^2 + p_2(0)^2 = c_0$. Thus $-\sqrt{c_0} \leq p_2(0) \leq \sqrt{c_0}$ and so there exists $t_0 \in \mathbb{R}$ such that $p_2(0) = \bar{p}_2(t_0)$. A simple computation then yields $p(0) = \bar{p}(t_0)$. (It is also simple to verify that $\sigma_1 \neq 0$ and $\sigma_2 \neq 0$ provided that $p(\cdot)$ is not constant). Consequently the integral curves $t \mapsto p(t)$ and $t \mapsto \bar{p}(t + t_0)$ solve the same Cauchy problem, and therefore are identical. □

Lastly, for case (c), we obtain a prospective integral curve by use of formula (2.7).

Theorem 5.4 (case c)**.** *Suppose $p(\cdot) : (-\varepsilon, \varepsilon) \to \mathfrak{se}(2)^*$ is an integral curve of $\vec{H}$ such that $H(p(0)) = h_0 > 0$, $C(p(0)) = c_0 > 0$ and $c_0 < 2c_1h_0$. Then there exists $t_0 \in \mathbb{R}$ and $\sigma \in \{-1, 1\}$ such that $p(t) = \bar{p}(t + t_0)$ for $t \in (-\varepsilon, \varepsilon)$, where*

$$\begin{cases} \bar{p}_1(t) = \sigma\sqrt{c_0}\,\operatorname{cn}(\Omega t,\, k) \\ \bar{p}_2(t) = \sqrt{c_0}\,\operatorname{sn}(\Omega t,\, k) \\ \bar{p}_3(t) = -\sigma\sqrt{2c_2h_0}\,\operatorname{dn}(\Omega t,\, k)\,. \end{cases}$$

Here $k = \sqrt{\frac{c_0}{2c_1h_0}}$ and $\Omega = \sqrt{\frac{2h_0}{c_2}}$.

Proof. Let $\sigma = -\operatorname{sgn}(p_3(0))$. We have $p_1(0)^2 + p_2(0)^2 = c_0$ and $\frac{1}{c_1}p_2(0)^2 + \frac{1}{c_2}p_3(0)^2 = 2h_0$. Hence $p_2(0)^2 \leq c_0$ and so $p_3(0)^2 \geq \frac{c_2}{c_1}(2c_1h_0 - c_0) > 0$. Thus $\sigma \neq 0$. Also, $-\sqrt{c_0} \leq p_2(0) \leq \sqrt{c_0}$, $\bar{p}_2(\frac{K}{\Omega}) = \sqrt{c_0}$ and $\bar{p}_2(\frac{3K}{\Omega}) = -\sqrt{c_0}$. Therefore there exists $t_1 \in [\frac{K}{\Omega}, \frac{3K}{\Omega}]$ such that $p_2(0) = \bar{p}_2(t_1)$. We have

$$p_1(0)^2 = c_0 - p_2(0)^2 = c_0 - \bar{p}_2(t_1)^2 = \bar{p}_1(t_1)^2.$$

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Thus $p_1(0) = \pm\bar{p}_1(t_1)$. Now $\bar{p}_1(-t+\frac{2K}{\Omega}) = -\bar{p}_1(t)$ and $\bar{p}_2(-t+\frac{2K}{\Omega}) = \bar{p}_2(t)$. Hence there exists $t_0 \in \mathbb{R}$ ($t_0 = t_1$ or $t_1 = -t_1 + \frac{2K}{\Omega}$) such that $p_1(0) = \bar{p}_1(t_0)$ and $p_2(0) = \bar{p}_2(t_0)$. Next

$$p_3(0)^2 = 2h_0c_2 - \tfrac{c_2}{c_1}p_2(0)^2 = 2h_0c_2 - \tfrac{c_2}{c_1}\bar{p}_2(t_0)^2 = \bar{p}_3(t_0)^2.$$

Hence, as $\operatorname{sgn}(p_3(0)) = \sigma = \operatorname{sgn}(\bar{p}_3(t_0))$, $p_3(0) = \bar{p}_3(t_0)$. Thus $p(0) = \bar{p}(t_0)$. Consequently the integral curves $t \mapsto p(t)$ and $t \mapsto \bar{p}(t + t_0)$ solve the same Cauchy problem, and therefore are identical. □

Remark 5.5. These three results are similar to those found by Sachkov [26]. However, our approach and formulation are different. The control problem LiCP(1), or rather, the associated sub-Riemannian problem, is further studied in [17, 27, 28].

5.2. Inhomogeneous systems

In order to separate qualitatively different cases of (3.5), we determine at which points (and for what values of h_0, c_0, c_1 and c_2) the cylinder $C^{-1}(c_0)$ and the paraboloid $H^{-1}(h_0)$ are tangent to one another. If they are tangent at a point $p = (p_1, p_2, p_3)$, then the gradients of the functions defining these level surfaces at p must be parallel, *i.e.*,

$$\nabla C(p) = \begin{bmatrix} 2p_1 & 2p_2 & 0 \end{bmatrix} = r \begin{bmatrix} 1 & \frac{1}{c_1}p_2 & \frac{1}{c_2}p_3 \end{bmatrix} = r\nabla H(p)$$

for some $r \in \mathbb{R}$. There are three distinct possibilities:

$$\begin{cases} p_1 = 0 \\ p_2 = 0 \\ p_3 = \pm\sqrt{2h_0c_2} \end{cases} \qquad \begin{cases} p_1 = c_1 \\ p_2 = \pm\sqrt{c_0 - c_1^2} \neq 0 \\ p_3 = 0 \end{cases} \qquad \begin{cases} p_1 \neq 0 \\ p_2 = 0 \\ p_3 = 0. \end{cases}$$

The first case corresponds to a constant solution and is therefore ignored. By back substitution (into C and H), the third case yields $h_0^2 = c_0$; this motivates us to distinguish between the three signs for $h_0^2 - c_0$. (Observe however that the cases $h_0 = -\sqrt{c_0}$ and $c_0 = 0$ correspond to constant solutions, whereas the situation $h_0 < -\sqrt{c_0}$ is impossible). The second case only occurs when $c_0 - c_1^2 > 0$ (and is hence distinguished from the case $c_0 - c_1^2 \leq 0$). Back substitution yields $h_0 = \frac{1}{2c_1}(c_0 + c_1^2)$, motivating another separation of three cases. However, not all combinations of these cases are possible.

Lemma 5.6. *If* $c_0 - c_1^2 > 0$ *and* $h_0 \geq \frac{1}{2c_1}(c_0 + c_1^2)$, *then* $h_0 > \sqrt{c_0}$.

Proof. It suffices to show that $\frac{1}{2c_1}(c_0 + c_1^2) > \sqrt{c_0}$. Now

$$\begin{aligned} & \tfrac{1}{2c_1}(c_0 + c_1^2) > \sqrt{c_0} \\ \Leftrightarrow \quad & (c_0 + c_1^2)^2 > 4c_0c_1^2 \\ \Leftrightarrow \quad & c_0^2 - 2c_0c_1^2 + c_1^4 > 0 \\ \Leftrightarrow \quad & (c_0 - c_1^2)^2 > 0 \end{aligned}$$

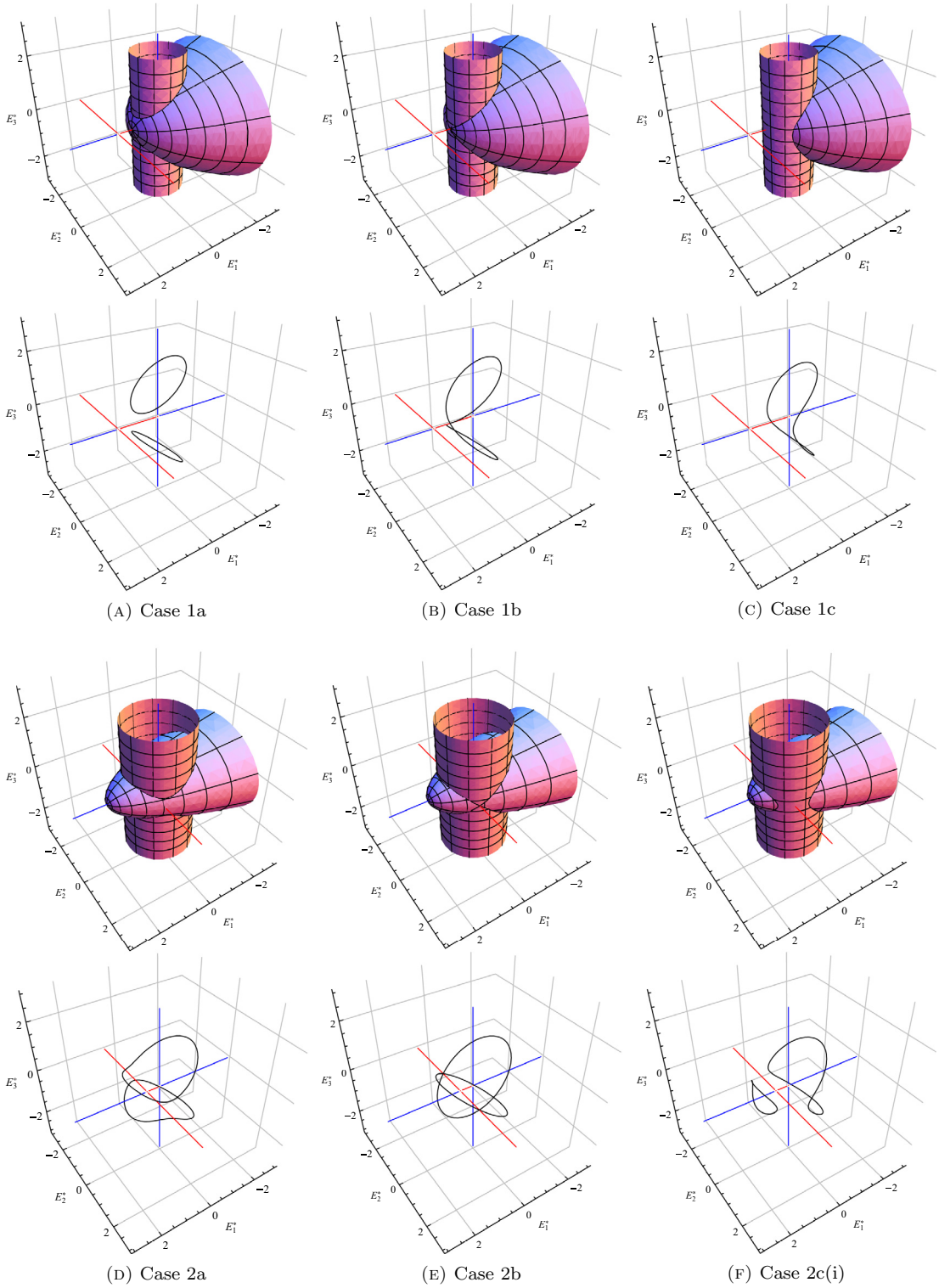
thus yielding the result. □

Some of the above mentioned cases are (retrospectively) further subdivided to facilitate integration. An index of the final subdivision of cases is provided in Table 1.

In Figure 2, we graph the level sets of H and C and their intersection for each major case (*i.e.*, by choosing some suitable values for h_0, c_0, c_1 and c_2). The stable equilibrium points (illustrated in blue) and unstable equilibrium points (illustrated in red), as presented in Theorem 4.2, are also plotted in each case.

We start our presentation of the integral curves of (3.5) by considering case 1a(i). (Formula (2.7) is utilised in the following theorem).

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(A) Case 1a (B) Case 1b (C) Case 1c

(D) Case 2a (E) Case 2b (F) Case 2c(i)

FIGURE 2. Typical cases of LiCP(2).

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TABLE 1. Index of typical cases for reduced extremals of LiCP(2).

Conditions			Index
$c_0 - c_1^2 \leq 0$	$h_0 > \sqrt{c_0}$	$c_1 - h_0 > \sqrt{h_0^2 - c_0}$	1a(i)
		$c_1 - h_0 = \sqrt{h_0^2 - c_0}$	1a(ii)
		$c_1 - h_0 < \sqrt{h_0^2 - c_0}$	1a(iii)
	$h_0 = \sqrt{c_0}$	$h_0 < c_1$	1b(i)
		$h_0 = c_1$	1b(ii)
	$-\sqrt{c_0} < h_0 < \sqrt{c_0}$		1c
$c_0 - c_1^2 > 0$	$h_0 > \frac{1}{2c_1}(c_0 + c_1^2)$		2a
	$h_0 = \frac{1}{2c_1}(c_0 + c_1^2)$		2b
	$h_0 < \frac{c_0 + c_1^2}{2c_1}$	$h_0 > \sqrt{c_0}$	2c(i)
		$h_0 = \sqrt{c_0}$	2c(ii)
		$-\sqrt{c_0} < h_0 < \sqrt{c_0}$	2c(iii)

Theorem 5.7 (case 1a(i))**.** *Suppose* $p(\cdot) : (-\varepsilon, \varepsilon) \to \mathfrak{se}(2)^*$ *is an integral curve of* $\vec{H}$ *such that* $H(p(0)) = h_0$, $C(p(0)) = c_0 > 0$ *and the conditions of case 1a(i) are satisfied. Then there exists* $t_0 \in \mathbb{R}$ *and* $\sigma \in \{-1, 1\}$ *such that* $p(t) = \bar{p}(t + t_0)$ *for* $t \in (-\varepsilon, \varepsilon)$, *where*

$$\begin{cases} \bar{p}_1(t) = \sqrt{c_0}\, \dfrac{\sqrt{h_0 - \delta} - \sqrt{h_0 + \delta}\, \mathrm{sn}\,(\Omega\, t,\, k)}{\sqrt{h_0 + \delta} - \sqrt{h_0 - \delta}\, \mathrm{sn}\,(\Omega\, t,\, k)} \\ \bar{p}_2(t) = -\sigma \sqrt{2c_0\delta}\, \dfrac{\mathrm{cn}\,(\Omega\, t,\, k)}{\sqrt{h_0 + \delta} - \sqrt{h_0 - \delta}\, \mathrm{sn}\,(\Omega\, t,\, k)} \\ \bar{p}_3(t) = \dfrac{2\sigma\delta\sqrt{c_2}}{k'}\, \dfrac{\mathrm{dn}\,(\Omega\, t,\, k)}{\sqrt{h_0 + \delta} - \sqrt{h_0 - \delta}\, \mathrm{sn}\,(\Omega\, t,\, k)}\,. \end{cases}$$

Here $\delta = \sqrt{h_0^2 - c_0}$, $\Omega = \sqrt{\frac{(h_0+\delta)(c_1-h_0+\delta)}{c_1 c_2}}$, $k = \sqrt{\frac{(h_0-\delta)(c_1-h_0-\delta)}{(h_0+\delta)(c_1-h_0+\delta)}}$ *and* $k' = \sqrt{\frac{2\delta c_1}{(h_0+\delta)(c_1-h_0+\delta)}}$.

Proof. Again, verification that $\bar{p}(\cdot)$ is a maximal integral curve satisfying $H(\bar{p}(0)) = h_0$ and $C(p(0)) = c_0$ whenever $c_0 - c_1^2 \leq 0$, $h_0 > \sqrt{c_0}$ and $c_1 - h_0 > \sqrt{h_0^2 - c_0}$ is straightforward. (Also, note that $\sqrt{h_0 + \delta} > \sqrt{h_0 - \delta}$ and so $\bar{p}(t)$ is defined for all $t \in \mathbb{R}$). We have $p_1(0)^2 + p_2(0)^2 = c_0$, $p_1(0) + \frac{1}{2}\left(\frac{1}{c_1}p_2(0)^2 + \frac{1}{c_2}p_3(0)^2\right) = h_0$ and $h_0 > \sqrt{c_0}$. Thus $-\sqrt{c_0} \leq p_1(0) \leq \sqrt{c_0}$. Also $p_3(0)^2 = h_0 - p_1(0) - \frac{1}{2c_1}(c_0 - p_1(0)^2)$. Now

$$\min_{-\sqrt{c_0} \leq p_1 \leq \sqrt{c_0}} \left(-p_1 - \tfrac{1}{2c_1}(c_0 - p_1^2)\right) = -\sqrt{c_0}.$$

Hence $p_3(0)^2 \geq h_0 - \sqrt{c_0} > 0$. Hence $p_3(0) \neq 0$. Let $\sigma = \mathrm{sgn}\,(p_3(0))$. Now $\bar{p}_1(-\frac{K}{\Omega}) = \sqrt{c_0}$ and $\bar{p}_1(\frac{K}{\Omega}) = -\sqrt{c_0}$. Thus there exists $t_1 \in [-\frac{K}{\Omega}, \frac{K}{\Omega}]$ such that $\bar{p}_1(t_1) = p_1(0)$. Next

$$p_2(0)^2 = c_0 - p_1(0)^2 = c_0 - \bar{p}_1(t_1)^2 = \bar{p}_2(t_1)^2.$$

Hence $p_2(0) = \pm\bar{p}_2(t_1)$. Now $\bar{p}_1(-t + \frac{2K}{\Omega}) = \bar{p}_1(t)$ and $\bar{p}_2(-t + \frac{2K}{\Omega}) = -\bar{p}_2(t)$. Thus there exists $t_0 \in \mathbb{R}$ ($t_0 = t_1$ or $t_0 = -t_1 + \frac{2K}{\Omega}$) such that $p_1(0) = \bar{p}_1(t_0)$ and $p_2(0) = \bar{p}_2(t_0)$. We have

$$p_3(0)^2 = 2c_2h_0 - 2c_2p_1(0) - \tfrac{c_2}{c_1}p_2(0)^2 = 2c_2h_0 - 2c_2\bar{p}_1(t_0) - \tfrac{c_2}{c_1}\bar{p}_2(t_0)^2 = \bar{p}_3(t_0)^2.$$

Thus, as $\mathrm{sgn}\,(p_3(0)) = \sigma = \mathrm{sgn}\,(\bar{p}_3(t_0))$, we get $p_3(0) = \bar{p}_3(t_0)$. Hence $p(0) = \bar{p}(t_0)$. Consequently the integral curves $t \mapsto p(t)$ and $t \mapsto \bar{p}(t + t_0)$ solve the same Cauchy problem, and therefore are identical. □

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Next, by limiting $\delta \to c_1 - h_0$ (from the right) in the above result, we get a prospective integral curve for case 1a(ii). The proof is omitted as it is essentially the same as for Theorem 5.7 (K is replaced by $\frac{\pi}{2}$).

Proposition 5.8 (case 1a(ii))**.** *Suppose $p(\cdot) : (-\varepsilon, \varepsilon) \to \mathfrak{se}(2)^*$ is an integral curve of $\vec{H}$ such that $H(p(0)) = h_0$, $C(p(0)) = c_0 > 0$ and the conditions of case 1a(ii) are satisfied. Then there exists $t_0 \in \mathbb{R}$ and $\sigma \in \{-1, 1\}$ such that $p(t) = \bar{p}(t + t_0)$ for $t \in (-\varepsilon, \varepsilon)$, where*

$$\begin{cases} \bar{p}_1(t) = \sqrt{c_0}\, \dfrac{\sqrt{h_0 - \delta} - \sqrt{h_0 + \delta}\sin(\Omega t)}{\sqrt{h_0 + \delta} - \sqrt{h_0 - \delta}\sin(\Omega t)} \\ \bar{p}_2(t) = -\sigma\sqrt{2c_0\delta}\, \dfrac{\cos(\Omega t)}{\sqrt{h_0 + \delta} - \sqrt{h_0 - \delta}\sin(\Omega t)} \\ \bar{p}_3(t) = \dfrac{2\sigma\delta\sqrt{c_2}}{\sqrt{h_0 + \delta} - \sqrt{h_0 - \delta}\sin(\Omega t)}\,. \end{cases}$$

Here $\delta = \sqrt{h_0^2 - c_0}$ and $\Omega = \sqrt{\frac{2(c_1 - h_0)}{c_2}}$.

It turns out that integral curves satisfying the conditions of case 1a(iii) or 2a take the same explicit expression. We shall provide a detailed proof for the result.

Lemma 5.9. *For cases 1a*(iii) *and 2a we have*

$$h_0 - c_1 + \sqrt{h_0^2 - c_0} > 0 \qquad \text{and} \qquad c_1 - h_0 + \sqrt{h_0^2 - c_0} > 0.$$

Theorem 5.10 (cases 1a(iii) and 2a)**.** *Suppose $p(\cdot) : (-\varepsilon, \varepsilon) \to \mathfrak{se}(2)^*$ is an integral curve of $\vec{H}$ such that $H(p(0)) = h_0$, $C(p(0)) = c_0 > 0$ and the conditions of case 1a*(iii) *or 2a are satisfied. Then there exists $t_0 \in \mathbb{R}$ and $\sigma \in \{-1, 1\}$ such that $p(t) = \bar{p}(t + t_0)$ for $t \in (-\varepsilon, \varepsilon)$, where*

$$\begin{cases} \bar{p}_1(t) = \sqrt{c_0}\, \dfrac{\sqrt{h_0 - \delta} - \sqrt{h_0 + \delta}\,\mathrm{cn}\,(\Omega t,\, k)}{\sqrt{h_0 + \delta} - \sqrt{h_0 - \delta}\,\mathrm{cn}\,(\Omega t,\, k)} \\ \bar{p}_2(t) = \sigma\sqrt{2c_0\delta}\, \dfrac{\mathrm{sn}\,(\Omega t,\, k)}{\sqrt{h_0 + \delta} - \sqrt{h_0 - \delta}\,\mathrm{cn}\,(\Omega t,\, k)} \\ \bar{p}_3(t) = 2\sigma\delta\sqrt{c_2}\, \dfrac{\mathrm{dn}\,(\Omega t,\, k)}{\sqrt{h_0 + \delta} - \sqrt{h_0 - \delta}\,\mathrm{cn}\,(\Omega t,\, k)}\,. \end{cases}$$

Here $\delta = \sqrt{h_0^2 - c_0}$, $\Omega = \sqrt{\frac{2\delta}{c_2}}$ and $k = \sqrt{\frac{(h_0 - \delta)(h_0 - c_1 + \delta)}{2\delta c_1}}$.

Proof. We start by explaining how the expression for $\bar{p}(\cdot)$ was found. A number of convenient assumptions are made implicitly and translations in the independent variable are discarded. We shall verify that $\bar{p}(\cdot)$ is a maximal integral curve (defined for any h_0, c_0, c_1 and c_2 satisfying the conditions of case 1a(iii) or 2a) only at the end of the construction.

Suppose $\bar{p}(\cdot)$ is an integral curve of $\vec{H}$ satisfying the conditions of 1a(iii) or 2a, where $h_0 = H(\bar{p}(0))$ and $c_0 = C(\bar{p}(0))$. We transform (5.2) into standard form (see, *e.g.*, [4] or [15]). First, we can rewrite (5.2) as

$$\frac{\mathrm{d}\bar{p}_1}{\mathrm{d}t} = \sqrt{(A_1(\bar{p}_1 - r_1)^2 + B_1(\bar{p}_1 - r_2)^2)\,(A_2(\bar{p}_1 - r_1)^2 + B_2(\bar{p}_1 - r_2)^2)}$$

where

$$A_1 = \frac{h_0 + \delta}{2c_1c_2\delta} > 0 \qquad A_2 = -\frac{h_0 - c_1 + \delta}{2\delta} < 0$$

$$B_1 = -\frac{h_0 - \delta}{2c_1c_2\delta} < 0 \qquad B_2 = -\frac{c_1 - h_0 + \delta}{2\delta} < 0$$

$$r_1 = h_0 - \delta \qquad r_2 = h_0 + \delta.$$

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Then we have

$$\int \mathrm{d}t = \int \frac{\mathrm{d}\bar{p}_1}{\sqrt{(A_1(\bar{p}_1 - r_1)^2 + B_1(\bar{p}_1 - r_2)^2)\,(A_2(\bar{p}_1 - r_1)^2 + B_2(\bar{p}_1 - r_2)^2)}}\,.$$

We make a change of variable $s = \frac{\bar{p}_1 - r_1}{\bar{p}_1 - r_2}$, which then yields

$$t = \frac{1}{(r_1 - r_2)\sqrt{-A_1 A_2}} \int_0^{\frac{\bar{p}_1(t) - r_1}{\bar{p}_1(t) - r_2}} \frac{ds}{\left(\frac{B_2}{A_2} + s^2\right)\left(-\frac{B_1}{A_1} - s^2\right)}\,.$$

Applying formula (2.8), we get

$$(r_1 - r_2)\sqrt{-A_1 A_2}\,t = \frac{1}{\sqrt{\frac{B_2}{A_2} - \frac{B_1}{A_1}}}\,\mathrm{sd}^{-1}\left(\frac{\sqrt{\frac{B_2}{A_2} - \frac{B_1}{A_1}}}{\sqrt{-\frac{B_1}{A_1}}\sqrt{\frac{B_2}{A_2}}}\,\frac{\bar{p}_1(t) - r_1}{\bar{p}_1(t) - r_2}, \frac{\sqrt{-\frac{B_1}{A_1}}}{\sqrt{\frac{B_2}{A_2} - \frac{B_1}{A_1}}}\right).$$

Solving for $\bar{p}_1(t)$ we find

$$\bar{p}_1(t) = \frac{r_2\sqrt{\frac{B_1 B_2}{B_1 A_2 - B_2 A_1}}\,\mathrm{sd}\left((r_1 - r_2)\sqrt{B_1 A_2 - B_2 A_1}\,t, \sqrt{\frac{B_1 A_2}{B_1 A_2 - B_2 A_1}}\right) - r_1}{\sqrt{\frac{B_1 B_2}{B_1 A_2 - B_2 A_1}}\,\mathrm{sd}\left((r_1 - r_2)\sqrt{B_1 A_2 - B_2 A_1}\,t, \sqrt{\frac{B_1 A_2}{B_1 A_2 - B_2 A_1}}\right) - 1}\,.$$

Substituting the values for $A_1, A_2, B_1, B_2, r_1, r_2$ and simplifying then yields

$$\bar{p}_1(t) = \sqrt{h_0^2 - \delta^2}\,\frac{\sqrt{h_0 - \delta} - \sqrt{h_0 + \delta}\,k'\,\mathrm{sd}\,(\Omega\,t,\,k)}{\sqrt{h_0 + \delta} - \sqrt{h_0 - \delta}\,k'\,\mathrm{sd}\,(\Omega\,t,\,k)}\,.$$

Now $\mathrm{cn}\,(x + 3K, k) = k'\,\mathrm{sd}\,(x, k)$. Thus, by making a suitable translation in t, we get the following (prospective) expression for $\bar{p}_1$

$$\bar{p}_1(t) = \sqrt{h_0^2 - \delta^2}\,\frac{\sqrt{h_0 - \delta} - \sqrt{h_0 + \delta}\,\mathrm{cn}\,(\Omega\,t,\,k)}{\sqrt{h_0 + \delta} - \sqrt{h_0 - \delta}\,\mathrm{cn}\,(\Omega\,t,\,k)}\,.$$

Next, as $C(\bar{p}(t)) = c_0$, we get

$$\bar{p}_2(t)^2 = c_0 - \bar{p}_1(t)^2 = \frac{2\delta\left(h_0^2 - \delta^2\right)\,\mathrm{sn}\,(\Omega\,t,\,k)^2}{\left(\sqrt{h_0 + \delta} - \sqrt{h_0 - \delta}\,\mathrm{cn}\,(\Omega\,t,\,k)\right)^2}\,.$$

Hence

$$\bar{p}_2(t) = \sigma_2\,\frac{\sqrt{2c_0\delta}\,\mathrm{sn}\,(\Omega\,t,\,k)}{\sqrt{h_0 + \delta} - \sqrt{h_0 - \delta}\,\mathrm{cn}\,(\Omega\,t,\,k)}$$

for some $\sigma_2 \in \{-1, 1\}$. By a similar computation, using the constant of motion H, we obtain

$$\bar{p}_3(t) = \sigma_3\,\frac{2\delta\sqrt{c_2}\,\mathrm{dn}\,(\Omega\,t,\,k)}{\sqrt{h_0 + \delta} - \sqrt{h_0 - \delta}\,\mathrm{cn}\,(\Omega\,t,\,k)}$$

for some $\sigma_3 \in \{-1, 1\}$. We now show that $\bar{p}(\cdot)$ is an integral curve for certain values of σ_2 and σ_3. From (3.5), we require that $\frac{\mathrm{d}}{\mathrm{d}t}\bar{p}_1(t) = \frac{1}{c_2}\bar{p}_2(t)\bar{p}_3(t)$. Now

$$\frac{\mathrm{d}}{\mathrm{d}t}\bar{p}_1(t) - \tfrac{1}{c_2}\bar{p}_2(t)\bar{p}_3(t) = \frac{2\delta\sqrt{2\delta}\,\mathrm{dn}\,(\Omega\,t,\,k)\,\mathrm{sn}\,(\Omega\,t,\,k)\sqrt{\frac{h_0^2 - \delta^2}{c_2}}\,(1 - \sigma_2\sigma_3)}{\left(\sqrt{h_0 + \delta} - \sqrt{h_0 - \delta}\,\mathrm{cn}\,(\Omega\,t,\,k)\right)^2}\,.$$

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Therefore, if $\sigma_2 = \sigma_3 = \sigma \in \{-1, 1\}$, then $\frac{\mathrm{d}}{\mathrm{d}t}\bar{p}_1(t) = \frac{1}{c_2}\bar{p}_2(t)\bar{p}_3(t)$. For this choice of σ_2 and σ_3, further calculation shows that $\frac{\mathrm{d}}{\mathrm{d}t}\bar{p}_2(t) = -\frac{1}{c_2}\bar{p}_1(t)\bar{p}_3(t)$ and $\frac{\mathrm{d}}{\mathrm{d}t}\bar{p}_3(t) = \left(\frac{1}{c_1}\bar{p}_1(t) - 1\right)\bar{p}_2(t)$. This motivates $\bar{p}(\cdot)$ as a prospective integral curve.

Elementary calculations show that $\bar{p}(\cdot)$ is defined over $\mathbb{R}$ when either set of conditions (case 1a(iii) or 2a) is satisfied. In particular, note that the denominator in each expression is strictly positive, the constants δ, k and Ω are real, and $0 < k < 1$. Thus $\bar{p}(\cdot)$ is a maximal integral curve of $\vec{H}$ for any h_0, c_0, c_1 and c_2 satisfying the conditions of case 1a(iii) or 2a.

Finally, we claim that any integral curve $p(\cdot)$ (as described in the statement of the theorem) must be of the form $p(t) = \bar{p}(t + t_0)$ for some $\sigma \in \{-1, 1\}$ and $t_0 \in \mathbb{R}$. By Theorem 4.6, $\bar{p}(\cdot)$ covers each connected component of $H^{-1}(h_0) \cap C^{-1}(c_0)$ (for each $\sigma \in \{-1, 1\}$, respectively) as illustrated in Figures 2a and 2d. Thus, as $p(0) \in H^{-1}(h_0) \cap C^{-1}(c_0)$ by assumption, there must exists $t_0 \in \mathbb{R}$ and $\sigma \in \{-1, 1\}$ satisfying the conditions of the theorem. We now give a proof for this claim.

We have $p_1(0)^2 + p_2(0)^2 = c_0$ and $p_1(0) + \frac{1}{2}\left(\frac{1}{c_1}p_2(0)^2 + \frac{1}{c_2}p_3(0)^2\right) = h_0$. Thus $-\sqrt{c_0} \le p_1(0) \le \sqrt{c_0}$. Also $\frac{1}{2c_2}p_3(0)^2 = h_0 - p_1(0) - \frac{1}{2c_1}(c_0 - p_1(0)^2)$. Suppose the conditions of case 1a(iii) hold. Then

$$\min_{-\sqrt{c_0} \le p_1 \le \sqrt{c_0}} \left(-p_1 - \tfrac{1}{2c_1}(c_0 - p_1^2)\right) = -\sqrt{c_0}.$$

Hence $\frac{1}{2c_2}p_3(0)^2 \ge h_0 - \sqrt{c_0} > 0$ and so $p_3(0) \ne 0$. On the other hand, suppose the conditions of case 2a hold. Then

$$\min_{-\sqrt{c_0} \le p_1 \le \sqrt{c_0}} \left(-p_1 - \tfrac{1}{2c_1}(c_0 - p_1^2)\right) = -\tfrac{1}{2c_1}(c_0 + c_1^2).$$

Hence $\frac{1}{2c_2}p_3(0)^2 \ge h_0 - \frac{1}{2c_1}(c_0 + c_1^2) > 0$ and so $p_3(0) \ne 0$.

Let $\sigma = \operatorname{sgn}(p_3(0))$. Now $\bar{p}_1(0) = -\sqrt{c_0}$ and $\bar{p}_1(\frac{2K}{\Omega}) = \sqrt{c_0}$. Thus there exists $t_1 \in [0, \frac{2K}{\Omega}]$ such that $\bar{p}_1(t_1) = p_1(0)$. Next we have

$$p_2(0)^2 = c_0 - p_1(0)^2 = c_0 - \bar{p}_1(t_1)^2 = \bar{p}_2(t_1)^2.$$

Therefore $p_2(0) = \pm\bar{p}_2(t_1)$. Now $\bar{p}_1(-t) = \bar{p}_1(t)$ and $\bar{p}_2(-t) = -\bar{p}_2(t)$. Hence there exists $t_0 \in \mathbb{R}$ ($t_0 = t_1$ or $t_0 = -t_1$) such that $p_1(0) = \bar{p}_1(t_0)$ and $p_2(0) = \bar{p}_2(t_0)$. Next

$$p_3(0)^2 = 2c_2h_0 - 2c_2p_1(0) - \tfrac{c_2}{c_1}p_2(0)^2 = 2c_2h_0 - 2c_2\bar{p}_1(t_0) - \tfrac{c_2}{c_1}\bar{p}_2(t_0)^2 = \bar{p}_3(t_0)^2.$$

Thus, as $\operatorname{sgn}(p_3(0)) = \sigma = \operatorname{sgn}(\bar{p}_3(t_0))$, we get $p_3(0) = \bar{p}_3(t_0)$. Hence $p(0) = \bar{p}(t_0)$. Consequently the integral curves $t \mapsto p(t)$ and $t \mapsto \bar{p}(t + t_0)$ solve the same Cauchy problem, and therefore are identical. □

Limiting processes were unsuccessful in producing a prospective integral curve for case 1b. Therefore, we resorted to integrating case 1b(i) independently by use of formula (2.9).

Proposition 5.11 (case 1b(i))**.** *Suppose $p(\cdot) : (-\varepsilon, \varepsilon) \to \mathfrak{se}(2)^*$ is an integral curve of $\vec{H}$ such that $H(p(0)) = h_0$, $C(p(0)) = c_0 > 0$ and the conditions of case 1b(i) are satisfied. Then there exists $t_0 \in \mathbb{R}$ and $\sigma \in \{-1, 1\}$ such that $p(t) = \bar{p}(t + t_0)$ for $t \in (-\varepsilon, \varepsilon)$, where*

$$\begin{cases} \bar{p}_1(t) = \dfrac{h_0}{2}\,\dfrac{c_1(\cosh(2\Omega t) - 3) + 2h_0}{c_1\cosh^2(\Omega t) - h_0} \\ \bar{p}_2(t) = 2\sigma h_0\sqrt{c_1(c_1 - h_0)}\,\dfrac{\sinh(\Omega t)}{c_1\cosh^2(\Omega t) - h_0} \\ \bar{p}_3(t) = 2\sigma(c_1 - h_0)\sqrt{c_2h_0}\,\dfrac{\cosh(\Omega t)}{c_1\cosh^2(\Omega t) - h_0}. \end{cases}$$

Here $\Omega = \sqrt{\frac{h_0(c_1 - h_0)}{c_1c_2}}$.

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Proof. We have $p_1(0)^2 + p_2(0)^2 = c_0$, $p_1(0) + \frac{1}{2}\left(\frac{1}{c_1}p_2(0)^2 + \frac{1}{c_2}p_3(0)^2\right) = h_0$ and $h_0 = \sqrt{c_0}$. If $p_3(0) = 0$, then a simple calculation shows that $p_2(0) = 0$ and so $p(\cdot)$ is constant. Hence $p_3(0) \neq 0$ (if $p(\cdot)$ is not constant). Let $\sigma = \operatorname{sgn}(p_3(0))$.

Now $-\sqrt{c_0} \leq p_1(0) < \sqrt{c_0}$, $\bar{p}_1(0) = -\sqrt{c_0}$ and $\lim_{t\to\infty} \bar{p}_1(t) = \sqrt{c_0}$. Thus there exists $t_1 \in \mathbb{R}$ such that $\bar{p}_1(t_1) = p_1(0)$. Next

$$p_2(0)^2 = c_0 - p_1(0)^2 = c_0 - \bar{p}_1(t_1)^2 = \bar{p}_2(t_1)^2.$$

Hence $p_2(0) = \pm\bar{p}_2(t_1)$. Now $\bar{p}_1(-t) = \bar{p}_1(t)$ and $\bar{p}_2(-t) = -\bar{p}_2(t)$. Thus there exists $t_0 \in \mathbb{R}$ ($t_0 = t_1$ or $t_0 = -t_1$) such that $p_1(0) = \bar{p}_1(t_0)$ and $p_2(0) = \bar{p}_2(t_0)$. We have

$$p_3(0)^2 = 2c_2h_0 - 2c_2p_1(0) - \tfrac{c_2}{c_1}p_2(0)^2 = 2c_2h_0 - 2c_2\bar{p}_1(t_0) - \tfrac{c_2}{c_1}\bar{p}_2(t_0)^2 = \bar{p}_3(t_0)^2.$$

Thus, as $\operatorname{sgn}(p_3(0)) = \sigma = \operatorname{sgn}(\bar{p}_3(t_0))$, we get $p_3(0) = \bar{p}_3(t_0)$. Hence $p(0) = \bar{p}(t_0)$. Consequently the integral curves $t \mapsto p(t)$ and $t \mapsto \bar{p}(t + t_0)$ solve the same Cauchy problem, and therefore are identical. □

A simple calculation (either limiting $h_0 \to c_1$ from the left in the above case, or integrating (5.2) directly) then yields a prospective integral curve for case 1b(ii). The proof is omitted as it is essentially the same as for Proposition 5.11.

Proposition 5.12 (case 1b(ii))**.** *Suppose $p(\cdot) : (-\varepsilon, \varepsilon) \to \mathfrak{se}(2)^*$ is an integral curve of $\vec{H}$ such that $H(p(0)) = h_0$, $C(p(0)) = c_0 > 0$ and the conditions of case 1b*(ii) *are satisfied. Then there exists $t_0 \in \mathbb{R}$ and $\sigma \in \{-1, 1\}$ such that $p(t) = \bar{p}(t + t_0)$ for $t \in (-\varepsilon, \varepsilon)$, where*

$$\begin{cases} \bar{p}_1(t) = c_1\,\dfrac{c_1t^2 - c_2}{c_1t^2 + c_2} \\ \bar{p}_2(t) = 2\sigma c_1\sqrt{c_1c_2}\,\dfrac{t}{c_1t^2 + c_2} \\ \bar{p}_3(t) = 2\sigma c_2\sqrt{c_1c_2}\,\dfrac{1}{c_1t^2 + c_2}\,. \end{cases}$$

For case 1c, the roots of the two quadratics in (5.2) need to be deinterlaced before this differential equation can be transformed into standard form. Consequently, the expression for this integral curve is quite involved. It turns out that integral curves satisfying the conditions of case 2c(iii), take the same explicit expression as those of case 1c. (The following result utilises formula (2.7)).

Theorem 5.13 (cases 1c and 2c(iii))**.** *Suppose $p(\cdot) : (-\varepsilon, \varepsilon) \to \mathfrak{se}(2)^*$ is an integral curve of $\vec{H}$ such that $H(p(0)) = h_0$, $C(p(0)) = c_0 > 0$ and the conditions of case 1c or 2c(iii) are satisfied. Then there exists $t_0 \in \mathbb{R}$ such that $p(t) = \bar{p}(t + t_0)$ for $t \in (-\varepsilon, \varepsilon)$, where*

$$\begin{cases} \bar{p}_1(t) = \eta_1\,\dfrac{\frac{2\sqrt{c_0}c_1 - \rho_2}{2\sqrt{c_0}c_1 + \rho_2}\sqrt{\rho_1 + \rho_2} - \sqrt{\rho_1 - \rho_2}\,\mathrm{cd}(\Omega t,\, k)}{\sqrt{\rho_1 + \rho_2} - \sqrt{\rho_1 - \rho_2}\,\mathrm{cd}(\Omega t,\, k)} \\ \bar{p}_2(t) = -\dfrac{\eta_2\,k'}{\sqrt{k}}\,\dfrac{\mathrm{sd}\left(\frac{1}{2}\Omega t,\, k\right)\sqrt{1 + \mathrm{nd}(\Omega t,\, k)}\sqrt{1 + k\,\mathrm{cd}(\Omega t,\, k)}}{\sqrt{\rho_1 + \rho_2} - \sqrt{\rho_1 - \rho_2}\,\mathrm{cd}(\Omega t,\, k)} \\ \bar{p}_3(t) = -\dfrac{\eta_3}{\sqrt{k}}\,\dfrac{\mathrm{cn}\left(\frac{1}{2}\Omega t,\, k\right)\sqrt{1 + \mathrm{nd}(\Omega t,\, k)}\sqrt{1 - k\,\mathrm{cd}(\Omega t,\, k)}}{\sqrt{\rho_1 + \rho_2} - \sqrt{\rho_1 - \rho_2}\,\mathrm{cd}(\Omega t,\, k)}\,. \end{cases}$$

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Here

$$\delta = \sqrt{c_1^2 - 2c_1h_0 + c_0} \qquad k = \sqrt{\frac{\rho_1 - \frac{1}{2}\left(\sqrt{c_0} + c_1 - \delta\right)^2 - \rho_2}{\rho_1 - \frac{1}{2}\left(\sqrt{c_0} + c_1 - \delta\right)^2 + \rho_2}}$$

$$\rho_1 = (c_1 + 2\delta)\sqrt{c_0} + \delta(\delta - c_1) + c_0 \qquad k' = \sqrt{\frac{2\rho_2}{\rho_1 - \frac{1}{2}\left(\sqrt{c_0} + c_1 - \delta\right)^2 + \rho_2}}$$

$$\rho_2 = 2\sqrt[4]{c_0}\sqrt{\delta\left(\left(\sqrt{c_0} + \delta\right)^2 - c_1^2\right)} \qquad \Omega = \sqrt{\frac{\rho_1 - \frac{1}{2}\left(\sqrt{c_0} + c_1 - \delta\right)^2 + \rho_2}{2c_1c_2}}$$

and

$$\eta_1 = \frac{\left(2c_1\sqrt{c_0} + \rho_2\right)}{2\left(\delta + \sqrt{c_0}\right)}$$

$$\eta_2 = \sqrt{\sqrt{c_0}\left(\sqrt{c_0} + c_1 - \delta\right)\left(\rho_2 + \left(\sqrt{c_0} + c_1 - \delta\right)c_1 - \rho_1\right)}$$

$$\eta_3 = \sqrt{\frac{c_2\left(\rho_1 - \rho_2\right)\rho_2\left(4\delta c_1 + \left(\frac{\delta}{\sqrt{c_0}} - 1\right)\rho_2\right)}{4\left(\delta + \sqrt{c_0}\right)^2 c_1}}.$$

Proof. Verification that $\bar{p}(\cdot)$ is an integral curve is quite involved. However, it is sufficient to show that $\dot{\bar{p}}_1(t) = \frac{1}{c_2}\bar{p}_2(t)\bar{p}_3(t)$, $\bar{p}_1(t)^2 + \bar{p}_2(t)^2 = c_0$ and $\bar{p}_1(t) + \frac{1}{2}\left(\frac{1}{c_1}\bar{p}_2(t)^2 + \frac{1}{c_2}\bar{p}_3(t)^2\right) = h_0$.

We have $p_1(0)^2 + p_2(0)^2 = c_0$ and $p_1(0) + \frac{1}{2}\left(\frac{1}{c_1}p_2(0)^2 + \frac{1}{c_2}p_3(0)^2\right) = h_0$. Hence $-\sqrt{c_0} \le p_1(0) \le \sqrt{c_0}$. Also, $p_1(0) + \frac{1}{2c_1}(c_0 - p_1(0)^2) - h_0 \le 0$. Thus $\frac{1}{2c_1}\left(p_1(0) - (c_1 - \delta)\right)\left(p_1(0) - (c_1 + \delta)\right) \ge 0$, and so $p_1(0) \le c_1 - \delta$ or $p_1(0) \ge c_1 + \delta$. It turns out that $-\sqrt{c_0} \le c_1 - \delta \le \sqrt{c_0} \le c_1 + \delta$. Therefore $-\sqrt{c_0} \le p_1(0) \le c_1 - \delta$.

We have that

$$\bar{p}_1(-t) = \bar{p}_1(t) \qquad \bar{p}_1\left(t + \tfrac{4K}{\Omega}\right) = \bar{p}_1(t) \qquad \bar{p}_1\left(-t + \tfrac{4K}{\Omega}\right) = \bar{p}_1(t)$$

$$\bar{p}_2(-t) = -\bar{p}_2(t) \qquad \bar{p}_2\left(t + \tfrac{4K}{\Omega}\right) = -\bar{p}_2(t) \qquad \bar{p}_2\left(-t + \tfrac{4K}{\Omega}\right) = \bar{p}_2(t)$$

$$\bar{p}_3(-t) = \bar{p}_3(t) \qquad \bar{p}_3\left(t + \tfrac{4K}{\Omega}\right) = -\bar{p}_3(t) \qquad \bar{p}_3\left(-t + \tfrac{4K}{\Omega}\right) = -\bar{p}_3(t).$$

Now $\bar{p}_1(0) = -\sqrt{c_0}$ and $\bar{p}_1(\frac{2K}{\Omega}) = c_1 - \delta$. Hence there exists $t_1 \in [0, \frac{2K}{\Omega}]$ such that $\bar{p}_1(t_1) = p_1(0)$. Then (by using the constants of motion C and H) we get $p_2(0) = \pm\bar{p}_2(t_1)$ and $p_3(0) = \pm\bar{p}_3(t_1)$. Therefore there exists $t_0 \in \mathbb{R}$ ($t_0 = t_1$, $t_0 = -t_1$, $t_0 = t_1 + \frac{2K}{\Omega}$, or $t_0 = -t_1 + \frac{2K}{\Omega}$) such that $p(0) = \bar{p}(t_0)$. Consequently the integral curves $t \mapsto p(t)$ and $t \mapsto \bar{p}(t + t_0)$ solve the same Cauchy problem, and therefore are identical. □

We now proceed to case 2b. In this case, a prospective integral curve is found by limiting $h_0 \to \frac{1}{2c_1}\left(c_0 + {c_1}^2\right)$ from the left in case 2a (Thm. 5.10) and allowing for possible changes in sign.

Proposition 5.14 (case 2b)**.** *Suppose $p(\cdot) : (-\varepsilon, \varepsilon) \to \mathfrak{se}(2)^*$ is an integral curve of $\vec{H}$ such that $H(p(0)) = h_0$, $C(p(0)) = c_0 > 0$ and the conditions of case 2b are satisfied. Then there exists $t_0 \in \mathbb{R}$ and $\sigma_1, \sigma_2 \in \{-1, 1\}$ such that $p(t) = \bar{p}(t + t_0)$ for $t \in (-\varepsilon, \varepsilon)$, where*

$$\begin{cases} \bar{p}_1(t) = c_1 + \dfrac{c_0 - c_1^2}{c_1 - \sigma_1\sqrt{c_0}\cosh(\Omega t)} \\ \bar{p}_2(t) = \sqrt{c_0}\,\dfrac{\sigma_1\sigma_2\sqrt{c_0 - c_1^2}\sinh(\Omega t)}{c_1 - \sigma_1\sqrt{c_0}\cosh(\Omega t)} \\ \bar{p}_3(t) = \dfrac{\sigma_2\left(c_0 - c_1^2\right)\sqrt{\frac{c_2}{c_1}}}{c_1 - \sigma_1\sqrt{c_0}\cosh(\Omega t)}. \end{cases}$$

Here $\Omega = \sqrt{\frac{c_0 - {c_1}^2}{c_1c_2}}$.

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Proof. We have $p_1(t)^2 + p_2(t)^2 = c_0$ and $p_1(t) + \frac{1}{2}\left(\frac{1}{c_1}p_2(t)^2 + \frac{1}{c_2}p_3(t)^2\right) = h_0$. Hence we have $2c_1p_1(t) + p_2(t)^2 - c_0 - c_1^2 = -\frac{c_1}{c_2}p_3(t)^2$. Let $s(p_1, p_2) = 2c_1p_1 + p_2^2 - c_0 - c_1^2$. Consider the restriction of s to the circle $\mathbb{S} = \{(p_1, p_2) : p_1^2 + p_2^2 = c_0\}$; $s|_{\mathbb{S}}$ has exactly two roots $(p_1, p_2) = (c_1, \pm\sqrt{c_0 - c_1^2})$.

We claim that $p_1(t) \neq c_1$ and $p_3(t) \neq 0$ for any $t \in (-\varepsilon, \varepsilon)$. Suppose $p_3(t_1) = 0$ for some t_1. Then $p_1(t_1) = c_1$ and so $p(t_1)$ is a equilibrium point. Hence we have $p(\cdot)$ is a constant trajectory, a contradiction. On the other hand suppose $p_1(t_1) = c_1$ for some t_1. Then $p_2(t_1)^2 = c_0 - c_1^2$ and so $p_3(t_1) = 0$. Hence $p(\cdot)$ is a constant trajectory, a contradiction. Now $\operatorname{sgn}(\bar{p}_1(t) - c_1) = -\sigma_1$ and $\operatorname{sgn}(\bar{p}_3(t)) = -\sigma_1\sigma_2$. Let $\sigma_1 = -\operatorname{sgn}(p_1(0) - c_1)$ and $\sigma_2 = -\sigma_1 \operatorname{sgn}(p_3(0))$.

We claim that there exists $t_0 \in \mathbb{R}$ such that $p_2(0) = \bar{p}_2(t_0)$ and $\operatorname{sgn}(p_1(0)) = \operatorname{sgn}(\bar{p}_1(t_0))$. We need to consider two cases for $p_1(0)$. First suppose $p_1(0) > c_1$. Then $\sigma_1 = -1$. We have $-\sqrt{c_0 - c_1^2} < p_2(0) < \sqrt{c_0 - c_1^2}$,

$$\lim_{t\to-\infty} \bar{p}_2(t) = \sigma_2\sqrt{c_0 - c_1^2} \qquad \lim_{t\to\infty} \bar{p}_2(t) = -\sigma_2\sqrt{c_0 - c_1^2}$$

and $\bar{p}_1(t) > 0$ for $t \in \mathbb{R}$. Thus there exists $t_0 \in \mathbb{R}$ satisfying the claim. Now suppose $p_1(0) < c_1$. Then $\sigma_1 = 1$. Let $\alpha = \frac{1}{\Omega}\cosh^{-1}\left(\frac{\sqrt{c_0}}{c_1}\right) > 0$. We have

$$\bar{p}_2(-\alpha) = \sigma_2\sqrt{c_0} \qquad \bar{p}_2(\alpha) = -\sigma_2\sqrt{c_0}$$
$$\lim_{t\to-\infty} \bar{p}_2(t) = \sigma_2\sqrt{c_0 - c_1^2} \qquad \lim_{t\to\infty} \bar{p}_2(t) = -\sigma_2\sqrt{c_0 - c_1^2}$$

and

$$\bar{p}_1(t) < 0 \quad \text{for } t \in (-\infty, -\alpha) \cup (\alpha, \infty)$$
$$\bar{p}_1(t) > 0 \quad \text{for } t \in (-\alpha, \alpha).$$

Note that $-\infty < -\alpha < \alpha < \infty$. We have $-\sqrt{c_0} \leq p_2(0) \leq \sqrt{c_0}$. If $p_1(0) \leq 0$ then there exists $t_0 \in [-\alpha, \alpha]$ satisfying the claim. If $0 < p_1 < c_1$, then $-\sqrt{c_0} < p_2(0) < -\sqrt{c_0 - c_1^2}$ or $\sqrt{c_0 - c_1^2} < p_2(0) < \sqrt{c_0}$. In either case there exists $t_0 \in (-\infty, -\alpha) \cup (\alpha, \infty)$ satisfying the claim.

By using the constant of motion C, we find $p_1(0)^2 = \bar{p}_1(t_0)^2$. However, we have that $\operatorname{sgn}(p_1(0)) = \operatorname{sgn}(\bar{p}_1(t_0))$ and so $p_1(0) = \bar{p}_1(t_0)$. Next, by using the constant of motion H, we find $p_3(0)^2 = \bar{p}_3(t_0)^2$. However $\operatorname{sgn}(p_3(0)) = -\sigma_1\sigma_2 = \operatorname{sgn}(\bar{p}_3(t_0))$ and so $p_3(0) = \bar{p}_3(t_0)$. Consequently the integral curves $t \mapsto p(t)$ and $t \mapsto \bar{p}(t + t_0)$ solve the same Cauchy problem, and therefore are identical. □

Moving on to case 2c(i), we again find a prospective integral curve by use of formula (2.9).

Theorem 5.15 (case 2c(i))**.** *Suppose $p(\cdot) : (-\varepsilon, \varepsilon) \to \mathfrak{se}(2)^*$ is an integral curve of $\vec{H}$ such that $H(p(0)) = h_0$, $C(p(0)) = c_0 > 0$ and the conditions of case 2c(i) are satisfied. Then there exists $t_0 \in \mathbb{R}$ and $\sigma \in \{-1, 1\}$ such that $p(t) = \bar{p}(t + t_0)$ for $t \in (-\varepsilon, \varepsilon)$, where*

$$\begin{cases} \bar{p}_1(t) = (h_0 - \delta)\,\dfrac{\frac{h_0+\delta}{h_0-\delta}k'\sqrt{h_0 - \delta} - \sigma\sqrt{h_0 + \delta}\,\mathrm{dn}(\Omega t,\, k)}{k'\sqrt{h_0 - \delta} - \sigma\sqrt{h_0 + \delta}\,\mathrm{dn}(\Omega t,\, k)} \\ \bar{p}_2(t) = -k\sqrt{2\delta\,(h_0^2 - \delta^2)}\,\dfrac{\mathrm{cn}(\Omega t,\, k)}{k'\sqrt{h_0 - \delta} - \sigma\sqrt{h_0 + \delta}\,\mathrm{dn}(\Omega t,\, k)} \\ \bar{p}_3(t) = 2\sigma\delta k'\sqrt{c_2}\,\dfrac{\mathrm{sn}(\Omega t,\, k)}{k'\sqrt{h_0 - \delta} - \sigma\sqrt{h_0 + \delta}\,\mathrm{dn}(\Omega t,\, k)}. \end{cases}$$

Here $\delta = \sqrt{h_0^2 - c_0}$, $\Omega = \sqrt{\frac{(h_0-\delta)(h_0-c_1+\delta)}{c_1c_2}}$, $k = \sqrt{\frac{2\delta c_1}{(h_0-\delta)(h_0-c_1+\delta)}}$ and $k' = \sqrt{\frac{(h_0+\delta)(h_0-c_1-\delta)}{(h_0-\delta)(h_0-c_1+\delta)}}$.

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Proof. We have $p_1(0)+\frac{1}{2}\left(\frac{1}{c_1}p_2(0)^2+\frac{1}{c_2}p_3(0)^2\right)=h_0$ and $p_1(0)^2+p_2(0)^2=c_0$. Hence $p_1(0)+\frac{1}{2c_1}p_2(0)^2-h_0\le 0$. Let $s(p_1,p_2)=p_1+\frac{1}{2c_1}p_2^2-h_0$. Consider the restriction of s to the circle $\mathbb{S}=\{(p_1,p_2)\,:\,p_1^2+p_2^2=c_0\}$; $s|_{\mathbb{S}}$ has exactly four roots $(p_1,p_2)=(\alpha_1,\pm\alpha_2)$ and $(p_1,p_2)=(\beta_1,\pm\beta_2)$, where

$$\alpha_1=c_1+\sqrt{c_0+c_1^2-2c_1h_0} \qquad \alpha_2=\sqrt{2c_1}\sqrt{h_0-c_1-\sqrt{c_0+c_1^2-2c_1h_0}}$$

$$\beta_1=c_1-\sqrt{c_0+c_1^2-2c_1h_0} \qquad \beta_2=\sqrt{2c_1}\sqrt{h_0-c_1+\sqrt{c_0+c_1^2-2c_1h_0}}.$$

Consequently, if $p_1(0)>c_1$, then $\alpha_1\le p_1(0)\le\sqrt{c_0}$. On the other hand, if $p_1(0)<c_1$, then $-\sqrt{c_0}\le p_1(0)\le\beta_1$.

It is easy to show, under the conditions of 2c(i), that $p_1(0)\neq c_1$. Let $\sigma=-\operatorname{sgn}(p_1(0)-c_1)$. Suppose $p_1(0)>c_1$, *i.e.*, $\sigma=-1$. A somewhat involved computation yields $\bar{p}_1(0)=\alpha_1$ and $\bar{p}_1(\frac{K}{\Omega})=\sqrt{c_0}$. Hence there exists $t_1\in[0,\frac{K}{\Omega}]$ such that $\bar{p}_1(t_1)=p_1(0)$. On the other hand, suppose $p_1(0)<c_1$, *i.e.*, $\sigma=1$. Again, a somewhat involved computation yields $\bar{p}_1(0)=\beta_1$ and $\bar{p}_1(\frac{K}{\Omega})=-\sqrt{c_0}$. Hence there again exists $t_1\in[0,\frac{K}{\Omega}]$ such that $\bar{p}_1(t_1)=p_1(0)$.

Finally, we claim that there exists $t_0\in\mathbb{R}$ such that $\bar{p}(t_0)=p(0)$. By using the constants of motion C and H, we get $p_2(0)=\pm\bar{p}_2(t_1)$ and $p_3(0)=\pm\bar{p}_3(t_1)$. Now

$$\begin{array}{lll}\bar{p}_1(-t)=\bar{p}_1(t) & \bar{p}_1\left(t+\frac{2K}{\Omega}\right)=\bar{p}_1(t) & \bar{p}_1\left(-t+\frac{2K}{\Omega}\right)=\bar{p}_1(t)\\ \bar{p}_2(-t)=\bar{p}_2(t) & \bar{p}_2\left(t+\frac{2K}{\Omega}\right)=-\bar{p}_2(t) & \bar{p}_2\left(-t+\frac{2K}{\Omega}\right)=-\bar{p}_2(t)\\ \bar{p}_3(-t)=-\bar{p}_3(t) & \bar{p}_3\left(t+\frac{2K}{\Omega}\right)=-\bar{p}_3(t) & \bar{p}_3\left(-t+\frac{2K}{\Omega}\right)=\bar{p}_3(t).\end{array}$$

Therefore there exists $t_0\in\mathbb{R}$ ($t_0=t_1$, $t_0=-t_1$, $t_0=t_1+\frac{2K}{\Omega}$, or $t_0=-t_1+\frac{2K}{\Omega}$) such that $p(0)=\bar{p}(t_0)$. Consequently the integral curves $t\mapsto p(t)$ and $t\mapsto\bar{p}(t+t_0)$ solve the same Cauchy problem, and therefore are identical. □

We are left to consider case 2c(ii). By limiting $c_0\to h_0^2$ in case 2c(iii) (Thm. 5.13) and allowing for possible changes in sign, we obtain a prospective integral curve.

Proposition 5.16 (case 2c(ii))**.** *Suppose $p(\cdot):(-\varepsilon,\varepsilon)\to\mathfrak{se}(2)^*$ is an integral curve of $\vec{H}$ such that $H(p(0))=h_0$, $C(p(0))=c_0>0$ and the conditions of case 2c(ii) are satisfied. Then there exists $t_0\in\mathbb{R}$ such that $p(t)=\bar{p}(t+t_0)$ for $t\in(-\varepsilon,\varepsilon)$, where*

$$\begin{cases}\bar{p}_1(t)=-\sqrt{c_0}\,\dfrac{\sqrt{c_0}+\frac{1}{2}c_1(-3+\cos(2\Omega t))}{\sqrt{c_0}-\frac{1}{2}c_1(1+\cos(2\Omega t))}\\[2ex] \bar{p}_2(t)=\dfrac{2\sqrt{c_0\left(\sqrt{c_0}-c_1\right)c_1}\sin(\Omega t)}{\sqrt{c_0}-\frac{1}{2}c_1(1+\cos(2\Omega t))}\\[2ex] \bar{p}_3(t)=\dfrac{2\left(\sqrt{c_0}-c_1\right)\sqrt{\sqrt{c_0}c_2}\cos(\Omega t)}{\sqrt{c_0}-\frac{1}{2}c_1(1+\cos(2\Omega t))}.\end{cases}$$

Here $\Omega=\sqrt{\frac{\sqrt{c_0}(\sqrt{c_0}-c_1)}{c_1c_2}}$.

Proof. We have $p_1(0)+\frac{1}{2}\left(\frac{1}{c_1}p_2(0)^2+\frac{1}{c_2}p_3(0)^2\right)=h_0$ and $p_1(0)^2+p_2(0)^2=c_0$. Hence $p_1(0)+\frac{1}{2c_1}p_2(0)^2-h_0\le 0$. Let $s(p_1,p_2)=p_1+\frac{1}{2c_1}p_2^2-h_0$. Consider the restriction of s to the circle $\mathbb{S}=\{(p_1,p_2)\,:\,p_1^2+p_2^2=c_0\}$; $s|_{\mathbb{S}}$ has exactly three roots

$$(p_1,p_2)=(\sqrt{c_0},0) \qquad (p_1,p_2)=\left(2c_1-\sqrt{c_0},\pm2\sqrt{c_1}\sqrt{\sqrt{c_0}-c_1}\right).$$

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Consequently $p_1(0) = \sqrt{c_0}$ or $-\sqrt{c_0} \leq p_1(0) \leq 2c_1 - \sqrt{c_0}$. However, if $p_1(0) = \sqrt{c_0}$, then $p_2(0) = 0$ and $p_3(0) = 0$, *i.e.*, $p(\cdot)$ is a constant trajectory. Thus this case can be discarded.

Now $\bar{p}_1(0) = -\sqrt{c_0}$ and $\bar{p}_1(\frac{\pi}{2\Omega}) = 2c_1 - \sqrt{c_0}$. Thus there exists $t_1 \in [0, \frac{\pi}{2\Omega}]$ such that $\bar{p}_1(t_1) = p_1(0)$. By using the constants of motion C and H, we get $p_2(0) = \pm\bar{p}_2(t_1)$ and $p_3(0) = \pm\bar{p}_3(t_1)$. Now

$$\begin{array}{lll} \bar{p}_1(-t) = \bar{p}_1(t) & \bar{p}_1\left(t + \frac{\pi}{\Omega}\right) = \bar{p}_1(t) & \bar{p}_1\left(-t + \frac{\pi}{\Omega}\right) = \bar{p}_1(t) \\ \bar{p}_2(-t) = -\bar{p}_2(t) & \bar{p}_2\left(t + \frac{\pi}{\Omega}\right) = -\bar{p}_2(t) & \bar{p}_2\left(-t + \frac{\pi}{\Omega}\right) = \bar{p}_2(t) \\ \bar{p}_3(-t) = \bar{p}_3(t) & \bar{p}_3\left(t + \frac{\pi}{\Omega}\right) = -\bar{p}_3(t) & \bar{p}_3\left(-t + \frac{\pi}{\Omega}\right) = -\bar{p}_3(t). \end{array}$$

Therefore there exists $t_0 \in \mathbb{R}$ ($t_0 = t_1$, $t_0 = -t_1$, $t_0 = t_1 + \frac{\pi}{\Omega}$, or $t_0 = -t_1 + \frac{\pi}{\Omega}$) such that $p(0) = \bar{p}(t_0)$. Consequently the integral curves $t \mapsto p(t)$ and $t \mapsto \bar{p}(t + t_0)$ solve the same Cauchy problem, and therefore are identical. □

Finally, let us consider the invariant control problem LiCP(3). There is only one typical case which is easy to integrate. Again, we graph the level sets of H and C and their intersection.

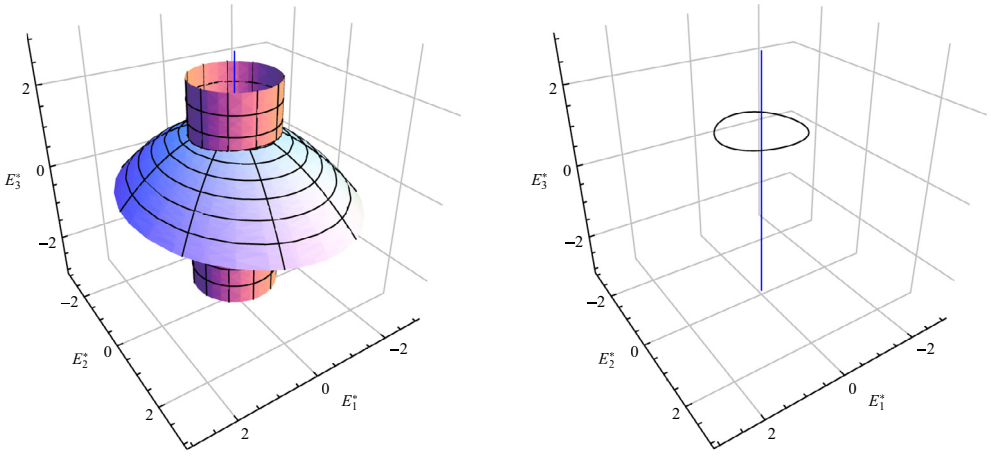


FIGURE 3. Typical reduced extremal of LiCP(3).

Proposition 5.17. *The reduced Hamilton equations* (3.6) *have the solutions*

$$\begin{cases} p_1(t) = \sqrt{c_0}\sin(\alpha t + t_0) \\ p_2(t) = \sqrt{c_0}\cos(\alpha t + t_0) \\ p_3(t) = \frac{h_0}{\alpha} - \frac{c_0}{2\alpha}\left(\frac{1}{c_1}\sin^2(\alpha t + t_0) + \frac{1}{c_2}\cos^2(\alpha t + t_0)\right) \end{cases}$$

where $c_0 = C(p(0))$, $h_0 = H(p(0))$ *and* $t_0 \in \mathbb{R}$.

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6. Conclusion

This paper shows that there are only three types of two-input left-invariant control affine systems on $\mathsf{SE}(2)$ (up to equivalence), and then studies a general optimal control problem (with quadratic cost) for each type. In each case the problem is reduced (*via* the Pontryagin Maximum Principle) to the study of a single Hamiltonian H on the Poisson space $\mathfrak{se}(2)^*_-$. The stability nature of all equilibrium states (of the reduced system) is investigated by use of the energy-Casimir method. $\vec{H}$ is shown to be complete. Also, every maximal integral curve is shown to be a constant, periodic or bounded curve.

For each problem, explicit expressions for all integral curves of $\vec{H}$ are found, up to a choice of sign and a translation in the independent variable. This is achieved (in most instances) by reducing the given system of differential equations to a single (separable) differential equation and then transforming it into standard form. Thereafter, an application of an appropriate integral formula yields a solution involving Jacobi elliptic functions. Using the constants of motion C and H (and allowing for possible changes in sign) an explicit expression for an integral curve of $\vec{H}$ is then obtained. Finally, we verify that any other integral curve, satisfying the same partitioning conditions, is identical to the one produced (up to a translation).

We now have explicit expressions for all (normal) extremal controls. A natural next step would be to solve the equations on the base $\mathsf{SE}(2)$, *i.e.*, to obtain expressions for the (normal) extremal trajectories. For the control problem $\mathsf{LiCP}(1)$, this was essentially accomplished in [26]. The inhomogeneous case has yet to be considered.

Single-input systems evolving on $\mathsf{SE}(2)$ have been considered in [2]. Specifically, it was shown that there are only two typical cases (up to equivalence). Likewise, stability and explicit integration were addressed. The three-input case is a topic for future research.

Appendix A. Tabulation of integral curves

Table A.1. Integral curves of $\vec{H}$ for reduced extremals of $\mathsf{LiCP}(1)$ and $\mathsf{LiCP}(3)$.

Case a	$c_0 > 2c_1h_0$
$\bar{p}_1(t) = \sigma\sqrt{c_0}\,\mathrm{dn}\,(\Omega\,t,\,k)$ $\bar{p}_2(t) = \sqrt{2c_1h_0}\,\mathrm{sn}\,(\Omega\,t,\,k)$ $\bar{p}_3(t) = -\sigma\sqrt{2c_2h_0}\,\mathrm{cn}\,(\Omega\,t,\,k)$	$k = \sqrt{\frac{2c_1h_0}{c_0}}$ $\Omega = \sqrt{\frac{c_0}{c_1c_2}}$ $\sigma \in \{-1, 1\}$
Case b	$c_0 = 2c_1h_0$
$\bar{p}_1(t) = \sigma_1\sigma_2\sqrt{c_0}\,\mathrm{sech}\,(\Omega\,t)$ $\bar{p}_2(t) = \sigma_1\sqrt{c_0}\,\tanh\,(\Omega\,t)$ $\bar{p}_3(t) = -\sigma_2\sqrt{\frac{c_0c_2}{c_1}}\,\mathrm{sech}\,(\Omega\,t,)$	$\Omega = \sqrt{\frac{c_0}{c_1c_2}}$ $\sigma_1 \in \{-1, 1\}$ $\sigma_2 \in \{-1, 1\}$
Case c	$c_0 < 2c_1h_0$
$\bar{p}_1(t) = \sigma\sqrt{c_0}\,\mathrm{cn}\,(\Omega\,t,\,k)$ $\bar{p}_2(t) = \sqrt{c_0}\,\mathrm{sn}\,(\Omega\,t,\,k)$ $\bar{p}_3(t) = -\sigma\sqrt{2c_2h_0}\,\mathrm{dn}\,(\Omega\,t,\,k)$	$k = \sqrt{\frac{c_0}{2c_1h_0}}$ $\Omega = \sqrt{\frac{2h_0}{c_2}}$
$\mathsf{LiCP}(3)$	
$p_1(t) = \sqrt{c_0}\sin(\alpha\,t + t_0)$ $p_2(t) = \sqrt{c_0}\cos(\alpha\,t + t_0)$ $p_3(t) = \frac{h_0}{\alpha} - \frac{c_0}{2\alpha}\left(\frac{1}{c_1}\sin^2(\alpha\,t + t_0) + \frac{1}{c_2}\cos^2(\alpha\,t + t_0)\right)$	

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TABLE A.2. Integral curves of $\vec{H}$ for reduced extremals of LiCP(2).

Case 1a(i) $\quad c_0 - c_1^2 \leq 0, \quad h_0 > \sqrt{c_0}, \quad c_1 - h_0 > \sqrt{h_0^2 - c_0}$

$$\bar{p}_1(t) = \sqrt{c_0}\ \frac{\sqrt{h_0-\delta}-\sqrt{h_0+\delta}\,\mathrm{sn}(\Omega\,t,k)}{\sqrt{h_0+\delta}-\sqrt{h_0-\delta}\,\mathrm{sn}(\Omega\,t,k)}$$
$$\bar{p}_2(t) = -\sigma\sqrt{2c_0\delta}\ \frac{\mathrm{cn}(\Omega\,t,k)}{\sqrt{h_0+\delta}-\sqrt{h_0-\delta}\,\mathrm{sn}(\Omega\,t,k)}$$
$$\bar{p}_3(t) = \frac{2\sigma\delta\sqrt{c_2}}{k'}\ \frac{\mathrm{dn}(\Omega\,t,k)}{\sqrt{h_0+\delta}-\sqrt{h_0-\delta}\,\mathrm{sn}(\Omega\,t,k)}$$

$$\delta = \sqrt{h_0^2 - c_0}$$
$$\Omega = \sqrt{\frac{(h_0+\delta)(c_1-h_0+\delta)}{c_1 c_2}}$$
$$k = \sqrt{\frac{(h_0-\delta)(c_1-h_0-\delta)}{(h_0+\delta)(c_1-h_0+\delta)}}$$
$$\sigma \in \{-1, 1\}$$

Case 1a(ii) $\quad c_0 - c_1^2 \leq 0, \quad h_0 > \sqrt{c_0}, \quad c_1 - h_0 = \sqrt{h_0^2 - c_0}$

$$\bar{p}_1(t) = \sqrt{c_0}\ \frac{\sqrt{h_0-\delta}-\sqrt{h_0+\delta}\sin(\Omega\,t)}{\sqrt{h_0+\delta}-\sqrt{h_0-\delta}\sin(\Omega\,t)}$$
$$\bar{p}_2(t) = -\sigma\sqrt{2c_0\delta}\ \frac{\cos(\Omega\,t)}{\sqrt{h_0+\delta}-\sqrt{h_0-\delta}\sin(\Omega\,t)}$$
$$\bar{p}_3(t) = \frac{2\sigma\delta\sqrt{c_2}}{\sqrt{h_0+\delta}-\sqrt{h_0-\delta}\sin(\Omega\,t)}$$

$$\delta = \sqrt{h_0^2 - c_0}$$
$$\Omega = \sqrt{\frac{2(c_1-h_0)}{c_2}}$$
$$\sigma \in \{-1, 1\}$$

Cases 1a(iii) & 2a $\quad c_0 - c_1^2 \leq 0, \quad h_0 > \sqrt{c_0}, \quad c_1 - h_0 < \sqrt{h_0^2 - c_0}$ $\quad c_0 - c_1^2 > 0, \quad h_0 > \frac{c_0+c_1^2}{2c_1}$

$$\bar{p}_1(t) = \sqrt{c_0}\ \frac{\sqrt{h_0-\delta}-\sqrt{h_0+\delta}\,\mathrm{cn}(\Omega\,t,k)}{\sqrt{h_0+\delta}-\sqrt{h_0-\delta}\,\mathrm{cn}(\Omega\,t,k)}$$
$$\bar{p}_2(t) = \sigma\sqrt{2c_0\delta}\ \frac{\mathrm{sn}(\Omega\,t,k)}{\sqrt{h_0+\delta}-\sqrt{h_0-\delta}\,\mathrm{cn}(\Omega\,t,k)}$$
$$\bar{p}_3(t) = 2\sigma\delta\sqrt{c_2}\ \frac{\mathrm{dn}(\Omega\,t,k)}{\sqrt{h_0+\delta}-\sqrt{h_0-\delta}\,\mathrm{cn}(\Omega\,t,k)}$$

$$\delta = \sqrt{h_0^2 - c_0}$$
$$\Omega = \sqrt{\frac{2\delta}{c_2}}$$
$$k = \sqrt{\frac{(h_0-\delta)(h_0-c_1+\delta)}{2\delta c_1}}$$
$$\sigma \in \{-1, 1\}$$

Case 1b(i) $\quad c_0 - c_1^2 \leq 0, \quad h_0 = \sqrt{c_0}, \quad h_0 < c_1$

$$\bar{p}_1(t) = \frac{h_0}{2}\ \frac{c_1(\cosh(2\Omega\,t)-3)+2h_0}{c_1\cosh^2(\Omega\,t)-h_0}$$
$$\bar{p}_2(t) = 2\sigma h_0\sqrt{c_1\,(c_1-h_0)}\ \frac{\sinh(\Omega\,t)}{c_1\cosh^2(\Omega\,t)-h_0}$$
$$\bar{p}_3(t) = 2\sigma\,(c_1-h_0)\sqrt{c_2 h_0}\ \frac{\cosh(\Omega\,t)}{c_1\cosh^2(\Omega\,t)-h_0}$$

$$\Omega = \sqrt{\frac{h_0(c_1-h_0)}{c_1 c_2}}$$
$$\sigma \in \{-1, 1\}$$

Case 1b(ii) $\quad c_0 - c_1^2 \leq 0, \quad h_0 = \sqrt{c_0}, \quad h_0 = c_1$

$$\bar{p}_1(t) = c_1\ \frac{c_1 t^2 - c_2}{c_1 t^2 + c_2}$$
$$\bar{p}_2(t) = 2\sigma c_1\sqrt{c_1 c_2}\ \frac{t}{c_1 t^2 + c_2}$$
$$\bar{p}_3(t) = 2\sigma c_2\sqrt{c_1 c_2}\ \frac{1}{c_1 t^2 + c_2}$$

$$\sigma \in \{-1, 1\}$$

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TABLE A.3. Integral curves of $\vec{H}$ for reduced extremals of LiCP(2) (continued).

Cases 1c & 2c(iii) $\quad c_0 - c_1^2 \le 0, \quad -\sqrt{c_0} < h_0 < \sqrt{c_0}$

$c_0 - c_1^2 > 0, \quad h_0 < \frac{c_0+c_1^2}{2c_1}, \quad -\sqrt{c_0} < h_0 < \sqrt{c_0}$

$$\bar{p}_1(t) = \eta_1 \, \frac{\frac{2\sqrt{c_0}c_1-\rho_2}{2\sqrt{c_0}c_1+\rho_2}\sqrt{\rho_1+\rho_2}-\sqrt{\rho_1-\rho_2}\,\mathrm{cd}(\Omega\,t,\,k)}{\sqrt{\rho_1+\rho_2}-\sqrt{\rho_1-\rho_2}\,\mathrm{cd}(\Omega\,t,\,k)}$$

$$\bar{p}_2(t) = -\frac{\eta_2\,k'}{\sqrt{k}} \, \frac{\mathrm{sd}\left(\frac{1}{2}\Omega\,t,\,k\right)\sqrt{1+\mathrm{nd}(\Omega\,t,\,k)}\sqrt{1+k\,\mathrm{cd}(\Omega\,t,\,k)}}{\sqrt{\rho_1+\rho_2}-\sqrt{\rho_1-\rho_2}\,\mathrm{cd}(\Omega\,t,\,k)}$$

$$\bar{p}_3(t) = -\frac{\eta_3}{\sqrt{k}} \, \frac{\mathrm{cn}\left(\frac{1}{2}\Omega\,t,\,k\right)\sqrt{1+\mathrm{nd}(\Omega\,t,\,k)}\sqrt{1-k\,\mathrm{cd}(\Omega\,t,\,k)}}{\sqrt{\rho_1+\rho_2}-\sqrt{\rho_1-\rho_2}\,\mathrm{cd}(\Omega\,t,\,k)}$$

$$\delta = \sqrt{c_1^2-2c_1h_0+c_0} \qquad k = \sqrt{\frac{\rho_1-\frac{1}{2}\left(\sqrt{c_0}+c_1-\delta\right)^2-\rho_2}{\rho_1-\frac{1}{2}\left(\sqrt{c_0}+c_1-\delta\right)^2+\rho_2}}$$

$$\rho_1 = (c_1+2\delta)\sqrt{c_0}+\delta(\delta-c_1)+c_0 \qquad k' = \sqrt{\frac{2\rho_2}{\rho_1-\frac{1}{2}\left(\sqrt{c_0}+c_1-\delta\right)^2+\rho_2}}$$

$$\rho_2 = 2\sqrt[4]{c_0}\sqrt{\delta\left(\left(\sqrt{c_0}+\delta\right)^2-c_1^2\right)} \qquad \Omega = \sqrt{\frac{\rho_1-\frac{1}{2}\left(\sqrt{c_0}+c_1-\delta\right)^2+\rho_2}{2c_1c_2}}$$

$$\eta_1 = \frac{2c_1\sqrt{c_0}+\rho_2}{2\left(\delta+\sqrt{c_0}\right)}$$

$$\eta_2 = \sqrt{\sqrt{c_0}\left(\sqrt{c_0}+c_1-\delta\right)\left(\rho_2+\left(\sqrt{c_0}+c_1-\delta\right)c_1-\rho_1\right)}$$

$$\eta_3 = \sqrt{\frac{c_2(\rho_1-\rho_2)\rho_2\left(4\delta c_1+\left(\frac{\delta}{\sqrt{c_0}}-1\right)\rho_2\right)}{4\left(\delta+\sqrt{c_0}\right)^2c_1}}$$

Case 2b $\quad c_0 - c_1^2 > 0, \quad h_0 = \frac{c_0+c_1^2}{2c_1}$

$$\bar{p}_1(t) = c_1 + \frac{c_0-c_1^2}{c_1-\sigma_1\sqrt{c_0}\cosh(\Omega\,t)}$$

$$\bar{p}_2(t) = \sqrt{c_0}\,\frac{\sigma_1\sigma_2\sqrt{c_0-c_1^2}\sinh(\Omega\,t)}{c_1-\sigma_1\sqrt{c_0}\cosh(\Omega\,t)}$$

$$\bar{p}_3(t) = \frac{\sigma_2\left(c_0-c_1^2\right)\sqrt{\frac{c_2}{c_1}}}{c_1-\sigma_1\sqrt{c_0}\cosh(\Omega\,t)}$$

$$\Omega = \sqrt{\frac{{c_0-c_1}^2}{c_1c_2}}$$

$$\sigma_1 \in \{-1,1\}$$

$$\sigma_2 \in \{-1,1\}$$

Case 2c(i) $\quad c_0 - c_1^2 > 0, \quad h_0 < \frac{c_0+c_1^2}{2c_1}, \quad h_0 > \sqrt{c_0}$

$$\bar{p}_1(t) = \frac{k'(h_0+\delta)\sqrt{h_0-\delta}-\sigma(h_0-\delta)\sqrt{h_0+\delta}\,\mathrm{dn}(\Omega\,t,\,k)}{k'\sqrt{h_0-\delta}-\sigma\sqrt{h_0+\delta}\,\mathrm{dn}(\Omega\,t,\,k)}$$

$$\bar{p}_2(t) = -\frac{k\sqrt{2\delta\left(h_0^2-\delta^2\right)}\,\mathrm{cn}(\Omega\,t,\,k)}{k'\sqrt{h_0-\delta}-\sigma\sqrt{h_0+\delta}\,\mathrm{dn}(\Omega\,t,\,k)}$$

$$\bar{p}_3(t) = \frac{2\sigma k'\sqrt{c_2}\,\mathrm{sn}(\Omega\,t,\,k)}{k'\sqrt{h_0-\delta}-\sigma\sqrt{h_0+\delta}\,\mathrm{dn}(\Omega\,t,\,k)}$$

$$\delta = \sqrt{h_0^2-c_0}$$

$$\Omega = \sqrt{\frac{(h_0-\delta)(h_0-c_1+\delta)}{c_1c_2}}$$

$$k = \sqrt{\frac{2\delta c_1}{(h_0-\delta)(h_0-c_1+\delta)}}$$

$$\sigma \in \{-1,1\}$$

Case 2c(ii) $\quad c_0 - c_1^2 > 0, \quad h_0 < \frac{c_0+c_1^2}{2c_1}, \quad h_0 = \sqrt{c_0}$

$$\bar{p}_1(t) = -\sqrt{c_0}\,\frac{\sqrt{c_0}+\frac{1}{2}c_1(-3+\cos(2\Omega\,t))}{\sqrt{c_0}-\frac{1}{2}c_1(1+\cos(2\Omega\,t))}$$

$$\bar{p}_2(t) = \frac{2\sqrt{c_0\left(\sqrt{c_0}-c_1\right)c_1}\sin(\Omega\,t)}{\sqrt{c_0}-\frac{1}{2}c_1(1+\cos(2\Omega\,t))}$$

$$\bar{p}_3(t) = \frac{2\left(\sqrt{c_0}-c_1\right)\sqrt{\sqrt{c_0}c_2}\cos(\Omega\,t)}{\sqrt{c_0}-\frac{1}{2}c_1(1+\cos(2\Omega\,t))}.$$

$$\Omega = \sqrt{\frac{\sqrt{c_0}\left(\sqrt{c_0}-c_1\right)}{c_1c_2}}$$

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