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On a Problem of Danzer†

NABIL H. MUSTAFA[1‡] and SAURABH RAY[2]

[1]Université Paris-Est, Laboratoire d'Informatique Gaspard-Monge, Equipe A3SI, ESIEE Paris, France
(e-mail: mustafan@esiee.fr)
[2]Computer Science, New York University, Abu Dhabi, United Arab Emirates
(e-mail: saurabh.ray@nyu.edu)

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Let C be a bounded convex object in $\mathbb{R}^d$, and let P be a set of n points lying outside C. Further, let c_p, c_q be two integers with $1 \leqslant c_q \leqslant c_p \leqslant n - \lfloor d/2 \rfloor$, such that every $c_p + \lfloor d/2 \rfloor$ points of P contain a subset of size $c_q + \lfloor d/2 \rfloor$ whose convex hull is disjoint from C. Then our main theorem states the existence of a partition of P into a small number of subsets, each of whose convex hulls are disjoint from C. Our proof is constructive and implies that such a partition can be computed in polynomial time.

In particular, our general theorem implies polynomial bounds for Hadwiger–Debrunner (p,q) numbers for balls in $\mathbb{R}^d$. For example, it follows from our theorem that when $p > q = (1+\beta)\cdot d/2$ for $\beta > 0$, then any set of balls satisfying the (p,q)-property can be hit by $O((1+\beta)^2 d^2 p^{1+1/\beta} \log p)$ points. This is the first improvement over a nearly 60 year-old exponential bound of roughly $O(2^d)$.

Our results also complement the results obtained in a recent work of Keller, Smorodinsky and Tardos where, apart from improvements to the bound on $\mathrm{HD}(p,q)$ for convex sets in $\mathbb{R}^d$ for various ranges of p and q, a polynomial bound is obtained for regions with low union complexity in the plane.

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1. Introduction

Given a finite set $\mathcal{C}$ of geometric objects in $\mathbb{R}^d$, we say that $\mathcal{C}$ satisfies the (p,q)-property if for any set $\mathcal{C}' \subseteq \mathcal{C}$ of size p, there exists a point in $\mathbb{R}^d$ common to at least q objects of $\mathcal{C}'$. The goal then is to show that there exists a small set Q of points in $\mathbb{R}^d$ such that each object of $\mathcal{C}$ contains some point of Q; such a Q is called a hitting set for $\mathcal{C}$.

These bounds for a set $\mathcal{C}$ of convex sets in $\mathbb{R}^d$ have been studied since the 1950s (see the surveys [7, 8, 16]), and it was only in 1991 that Alon and Kleitman [1], in a breakthrough result, gave an upper bound that is *independent* of $|\mathcal{C}|$. Unfortunately it depends exponentially on p, q

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and d. For the case where $\mathcal{C}$ consists of arbitrary convex objects, the current best bounds remain exponential in p, q and d.

Theorem 1.1 ([1, 10]). *Let $\mathcal{C}$ be a finite set of convex objects in $\mathbb{R}^d$ satisfying the (p,q)-property, where p,q are two integers with $p \geqslant q \geqslant d+1$. Then there exists a hitting set for $\mathcal{C}$ of size*

$$\begin{cases} O\left(p^{d(q-1)/(q-d)} \cdot \log^{c' d^3 \log d} p\right), & \\ (p-q) + O\left(\left(\frac{p}{q}\right)^d \log^{c' d^3 \log d}\left(\frac{p}{q}\right)\right) & \text{for } q \geqslant \log p, \\ p-q+2 & \text{for } q \geqslant p^{1-1/d+\varepsilon}, p \geqslant p(d,\varepsilon). \end{cases}$$

where c' is an absolute constant independent of $|\mathcal{C}|, p, q$ and d, and $p(d,\varepsilon)$ is a function depending only on d and ε.

Consider the basic case where $\mathcal{C}$ is a set of balls in $\mathbb{R}^d$ satisfying the (p,q)-property. Theorem 1.1 implies – ignoring logarithmic factors and for general values of p and q – the existence of a hitting set of size no better than $O(p^d)$. Furthermore, it requires $q \geqslant d+1$ – a necessary condition for arbitrary convex objects[1] but not for balls.

Almost 60 years ago, Danzer [5, 6] considered the $\mathrm{HD}(p,q)$ problem for balls. The best bound that we are aware of, derived from the survey of Eckhoff [7] by combining inequalities (4.2), (4.4) and (4.5), is stated below. It is better than the one from Theorem 1.1 quantitatively, but also in that it gives a bound requiring only that $q \geqslant 2$. Further, for a very specific case – namely when $p = q$ and $(d-q)$ is $O(\log d)$ – it succeeds in giving polynomial bounds.

Theorem 1.2 ([7]). *Let $\mathcal{B}$ be a finite set of balls in $\mathbb{R}^d$. If $\mathcal{B}$ satisfies the (p,q)-property for some $d \geqslant p \geqslant q \geqslant 2$, then there exists a hitting set for $\mathcal{B}$ of size at most*

$$\sqrt{\frac{3\pi}{2}} \cdot 2^{d-q} \cdot \left((p-q) \cdot 2^q \cdot d^{3/2} \cdot g(d) + 4(d-q+2)^{3/2} \cdot g(d-q+2)\right)$$

where $g(x) = \log x + \log\log x + 1$. Ignoring logarithmic terms in the above bound, it is of the form $\Theta((p-q) \cdot 2^d \cdot d^{3/2} + 2^{d-q} \cdot (d-q)^{3/2})$. If $p \neq q$ the first term dominates, otherwise the second term dominates.

Turning towards the lower bound for the case where $\mathcal{C}$ is a set of unit balls in $\mathbb{R}^d$, Bourgain and Lindenstrauss [2] proved a lower bound of 1.0645^d when $p = q = 2$ in $\mathbb{R}^d$, that is, one needs at least 1.0645^d points to hit all pairwise intersecting unit balls in $\mathbb{R}^d$.

Our result. We consider a set-up for the $\mathrm{HD}(p,q)$ problem that is more general than the one for balls, as follows.

Let C be a convex object in $\mathbb{R}^d$, and let P be a set of n points lying outside C. For each $p \in P$, let H_p be the set of hyperplanes separating p from C. Let C_p be the set of points in $\mathbb{R}^d$ *dual* to the hyperplanes in H_p (see [12, Chapter 5.1]), and let $\mathcal{S} = \{C_p : p \in P\}$.

[1] There are easy examples, *e.g.* when the convex objects are hyperplanes in $\mathbb{R}^d$.

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Our goal is to study the (p,q)-property for $\mathcal{S}$ – namely, that out of every p objects of $\mathcal{S}$, there exists a point in $\mathbb{R}^d$ common to at least q of them. This is equivalent to the property of C and P that out of every p-sized set $P' \subseteq P$, there exists a hyperplane separating C from a q-sized subset $P'' \subset P'$, or equivalently, $\mathrm{conv}(P'')$ is disjoint from C.

Our main theorem is the following. For a simpler expression, let c_q, c_p be two positive integers such that $p = c_p + \lfloor d/2 \rfloor$ and $q = c_q + \lfloor d/2 \rfloor$.

Theorem 1.3. *Let C be a bounded convex object in $\mathbb{R}^d$ and let P be a set of n points lying outside C. Further, let c_p, c_q be two integers, with $1 \leqslant c_q \leqslant c_p \leqslant n - \lfloor d/2 \rfloor$, such that for every $c_p + \lfloor d/2 \rfloor$ points of P, there exists a subset of size $c_q + \lfloor d/2 \rfloor$ whose convex hull is disjoint from C. Then the points of P can be partitioned into*

$$\lambda_d(c_p, c_q) = K_2 \frac{d}{c_q} \cdot (\sqrt{2}K_1)^{d/c_q} \cdot \left(\left\lfloor \frac{d}{2} \right\rfloor + c_q \right)^2 \cdot \left(\left\lfloor \frac{d}{2} \right\rfloor + c_p \right)^{1+(\lfloor d/2 \rfloor - 1)/c_q} \cdot \log\left(\left\lfloor \frac{d}{2} \right\rfloor + c_p \right)$$

sets, each of whose convex hull is disjoint from C. Here K_1, K_2 are absolute constants independent of n, d, c_p and c_q. Furthermore, such a partition can be computed in polynomial time.

The proof, presented in Section 2, is a combination of three ingredients: the Alon–Kleitman technique [1], bounds on independent sets in hypergraphs [10] and bounds on $(\leqslant k)$-sets for half-spaces [4]. It is an extension of the proof in [15], which studied Carathéodory's theorem in this setting.

Remarks. **(1)** The restriction that $q \geqslant \lfloor d/2 \rfloor + 1$ is necessary, as can be seen when P form the vertices of a cyclic polytope in $\mathbb{R}^d$ and C is a slightly shrunken copy of $\mathrm{conv}(P)$.

(2) Note that when $c_q = \beta \cdot d/2$ for any absolute constant $\beta > 0$, the above bound is *polynomial* in the dimension d: it is upper-bounded by $O((1+\beta)^2 d^2 p^{1+1/\beta} \log p)$.

(3) It was shown in [14] that C_p is a convex object in $\mathbb{R}^d$ and thus the bounds of Theorem 1.1 apply. As before, Theorem 1.3 substantially improves upon this, as the bounds following from Theorem 1.1 are exponential in d and furthermore, require $q \geqslant d+1$.

As an immediate corollary of Theorem 1.3, we obtain the first improvements to the old bound on the (p,q) problem for balls in $\mathbb{R}^d$. The bound in Theorem 1.2 is exponential in d, except in special cases where $p = q$ *and* $(d-q)$ is[2] $O(\log d)$. On the other hand, our result gives polynomial bounds as long as $q \geqslant \beta d$ for any constant $\beta > 1/2$.

Corollary 1.4 (Hadwiger–Debrunner (p,q) bound for balls in $\mathbb{R}^d$). *Let $\mathcal{B}$ be collection of balls in $\mathbb{R}^d$ such that for every subset of $c_p + \lfloor (d+1)/2 \rfloor$ balls in $\mathcal{B}$, some $c_q + \lfloor (d+1)/2 \rfloor$ have a common intersection, where c_p and c_q are integers such that $1 \leqslant c_q \leqslant c_p \leqslant n - \lfloor (d+1)/2 \rfloor$. Then there exists a set X of $\lambda_{d+1}(c_p, c_q)$ points that form a hitting set for the balls in $\mathcal{B}$. Here $\lambda_{d+1}(\cdot,\cdot)$ is the function defined in the statement of Theorem 1.3.*

[2] Recall that Theorem 1.2 assumes $q \leqslant p \leqslant d$.

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Proof. Observe that one can stereographically 'lift' balls in $\mathbb{R}^d$ to caps of a sphere S in $\mathbb{R}^{d+1}$, where a cap of a sphere is a portion of the sphere contained in a half-space that does not contain the centre of the sphere. Thus we will prove a slightly more general result where $\mathcal{B}$ consists of caps of a d-dimensional sphere S embedded in $\mathbb{R}^{d+1}$.

For a point $x \in S$, let h_x denote the hyperplane tangent to S at x. For any point y lying outside S, define the *separating set of* y to be

$$S_y = \{z \in S\colon h_z \text{ separates } y \text{ and } S\}.$$

Geometrically, S_y is the set of points of S 'visible' from y, and form a cap of S. Furthermore, for any cap K of S, there is a unique point w such that $K = S_w$. We denote this point w by $\mathrm{apex}(K)$.

Given the set of caps $\mathcal{B}$ on S, consider the point set

$$\mathrm{apex}(\mathcal{B}) = \{\mathrm{apex}(B)\colon B \in \mathcal{B}\}.$$

Observe that for a point $x \in S$ and a cap $B \in \mathcal{B}$, $x \in B$ if and only if $x \in S_{\mathrm{apex}(B)}$. As $\mathcal{B}$ satisfies the (p,q) property – namely that for every p-sized subset $\mathcal{B}'$ of $\mathcal{B}$, there exists a point $x \in S$ lying in some q elements of $\mathcal{B}'$ – we have that for every p-sized subset A' of $\mathrm{apex}(\mathcal{B})$, there exists a point $x \in S$ lying in the separating set of some q points of A'. In other words, h_x separates these q points from S.

Applying Theorem 1.3 with $C = S$ and $P = \mathrm{apex}(\mathcal{B})$ in dimension $d+1$, we conclude that P can be partitioned into a family Ξ of $\lambda_{d+1}(c_p, c_q)$ sets, each of whose convex hulls are disjoint from S. Consider a set $P' \in \Xi$. Since the convex hull of P' is disjoint from S, we can find a hyperplane h_x tangent to S at x such that h_x separates P' from S. This implies that all the caps in $\mathcal{B}$ corresponding to the points in P' contain the point x. Thus for each set of Ξ we obtain a point which is contained in all the caps corresponding to the points in that set. These $|X| = \lambda_{d+1}(c_p, c_q)$ points form the required set X. □

Our results complement the recent results of Keller, Smorodinsky and Tardos [9, 10], who obtain polynomial bounds for regions of low union complexity in the plane.

2. Proof of Theorem 1.3

We first briefly sketch the Alon–Kleitman technique [1], on which our proof is based, as well as the other theorems needed for the proof.

Let $\mathcal{C}$ be a finite collection of objects in $\mathbb{R}^d$ satisfying the (p,q)-property. Pick an arbitrary point from each cell of the arrangement induced by the objects of $\mathcal{C}$, and let U be this set of points. Alon and Kleitman proved that if $\mathcal{C}$ satisfies the (p,q)-property, then there exists a subset of U, of size independent of $|\mathcal{C}|$, that is a hitting set for $\mathcal{C}$. Their proof proceeds in three steps.

(1) For any multi-subset $\mathcal{C}'$ of $\mathcal{C}$ (namely, multiple copies of an object in $\mathcal{C}$ can exist in $\mathcal{C}'$), there exists a point $q \in U$ such that q lies in at least $|\mathcal{C}'|/c_{d,p,q}$ objects of $\mathcal{C}'$, where $c_{d,p,q}$ is a constant depending only on d, p and q.

(2) Next, using fact 1. together with LP-duality, it is possible to assign a rational positive weight to each point in U such that the total weight of the points of U contained in *each* object of $\mathcal{C}$ is least 1. Furthermore, the total weight of all the points in U is at most W, where W depends only on d, p and q.

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(3) Finally, using weak ε-nets with $\varepsilon = 1/W$, one can pick a subset of U, of size depending only on d, p and q, that hits all objects of $\mathcal{C}$.

Our improvement comes from two sources: first, improving the quantitative bound in 1. using bounds on independent sets in hypergraphs together with bounds on $(\leqslant k)$-sets for half-spaces, and second, improving the quantitative bound in 3. using ε-nets for half-spaces. We now define these formally.

Given a set P of points in $\mathbb{R}^d$, and a parameter $\varepsilon > 0$, a set $Q \subseteq P$ is an ε-net for P with respect to half-spaces if any half-space containing at least $\varepsilon \cdot |P|$ points of P contains at least one point of Q. It is well known that there always exists such an ε-net of size independent of $|P|$ (see also [13]).

Theorem 2.1 ([8, 11]). *Let P be a finite set of points in $\mathbb{R}^d$, and $\varepsilon > 0$ a given parameter. Then there exists an ε-net for P, with respect to half-spaces, of size $O(d/\varepsilon \log(1/\varepsilon))$. Furthermore, this bound is optimal.*

By the standard point–hyperplane duality (see *e.g.* [12, Chapter 5.1]), Theorem 2.1 implies the following.

Corollary 2.2. *Let $\mathcal{H}$ be a finite set of half-spaces in $\mathbb{R}^d$, and $\varepsilon > 0$ a given parameter. Then there exists a set $\mathcal{H}' \subseteq \mathcal{H}$ of size $O(d/\varepsilon \log(1/\varepsilon))$ such that any point $q \in \mathbb{R}^d$ contained in at least $\varepsilon \cdot |\mathcal{H}|$ half-spaces of $\mathcal{H}$ is contained in at least one half-space of $\mathcal{H}'$.* □

Given an integer $k \geqslant 1$, a set $P' \subseteq P$ is called a k-set of P if $|P'| = k$ and if there exists a half-space h in $\mathbb{R}^d$ such that $P' = P \cap h$. A set $P' \subseteq P$ is called a $(\leqslant k)$-set if P' is an l-set for some $l \leqslant k$. The next well-known theorem gives an upper bound on the number of $(\leqslant k)$-sets in a point set (see also [18]).

Theorem 2.3 (Clarkson and Shor [4]). *For any integer $k \geqslant \lfloor d/2 \rfloor + 1$, the number of $(\leqslant k)$-sets of any set of n points in $\mathbb{R}^d$ is at most*

$$\kappa_d(n,k) = 2\left(\frac{K_1}{\lceil d/2 \rceil}\right)^{\lceil d/2 \rceil} \binom{n}{\lfloor d/2 \rfloor} (k + \lceil d/2 \rceil)^{\lceil d/2 \rceil} \leqslant \kappa'_d(k) \cdot n^{\lfloor d/2 \rfloor}, \tag{2.1}$$

where

$$\kappa'_d(k) = 2K_1^d \left\lfloor \frac{d}{2} \right\rfloor^{-\lfloor d/2 \rfloor} \left(1 + \frac{k}{\lceil d/2 \rceil}\right)^{\lceil d/2 \rceil}$$

and $K_1 \geqslant e$ is an absolute constant independent of n, d and k.

Definition 2.4 (depth). Given a set P of n points in $\mathbb{R}^d$ and any set $Q \subseteq P$, define the *depth* of Q with respect to P, denoted $\mathrm{depth}_P(Q)$, to be the minimum number of points of P contained in any half-space containing Q.

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For two parameters $l \geqslant k \geqslant 2$, let $\tau_d(n,k,l)$ denote the maximum number of subsets of size k and depth at most l with respect to P in any set P of n points in $\mathbb{R}^d$:

$$\tau_d(n,k,l) = \max_{\substack{P\subseteq\mathbb{R}^d \\ |P|=n}} |\{Q \subseteq P\colon |Q| = k \text{ and } \operatorname{depth}_P(Q) \leqslant l\}|.$$

The following statement is easily implied by an application of the Clarkson–Shor technique [4] (see *e.g.*[17]).

Theorem 2.5. *For parameters $l \geqslant k \geqslant \lfloor d/2\rfloor + 1$,*

$$\tau_d(n,k,l) \leqslant e \cdot \kappa_d(n,k) \cdot l^{k-\lfloor d/2\rfloor},$$

where the function $\kappa(\cdot,\cdot)$ is as defined in equation (2.1).

Proof. Let P be any set of n points in $\mathbb{R}^d$. Let t be the number of sets of P of size k and depth at most l. Pick each element of P independently with probability $\rho = 1/l$ to get a random sample R. The expected number of k-sets in R satisfies

$$\begin{aligned}
\rho^k \cdot (1-\rho)^{l-k} \cdot t &\leqslant \mathbb{E}[\text{number of } k\text{-sets in } R\,] \\
&\leqslant 2\left(\frac{K_1}{\lceil d/2\rceil}\right)^{\lceil d/2\rceil} \mathbb{E}\left[\binom{|R|}{\lfloor d/2\rfloor}\right]\left(k+\left\lceil\frac{d}{2}\right\rceil\right)^{\lceil d/2\rceil} \\
&= 2\left(\frac{K_1}{\lceil d/2\rceil}\right)^{\lceil d/2\rceil} \binom{n}{\lfloor d/2\rfloor}\rho^{\lfloor d/2\rfloor}\left(k+\left\lceil\frac{d}{2}\right\rceil\right)^{\lceil d/2\rceil} \\
&= \kappa_d(n,k)\cdot\rho^{\lfloor d/2\rfloor} \\
\Longrightarrow\ t &\leqslant \frac{\kappa_d(n,k)\cdot\rho^{\lfloor d/2\rfloor}}{\rho^k\cdot(1-\rho)^{l-k}} \leqslant e\cdot\kappa_d(n,k)\cdot l^{k-\lfloor d/2\rfloor},
\end{aligned}$$

as $(1-1/l)^{-(l-k)} \leqslant e$ for any $l \geqslant k \geqslant 2$. □

Lemma 2.6. *Let C be a bounded convex object in $\mathbb{R}^d$, and let P be a set of n points lying outside C. Let $p \geqslant q \geqslant \lfloor d/2\rfloor + 1$ be parameters such that for every subset $Q \subseteq P$ of size p, there exists a set $Q' \subset Q$ of size q such that Q' can be separated from C by a hyperplane. Then there exists a hyperplane separating at least*

$$(2q\,p^{q-1}\cdot e\,\kappa'_d(q))^{1/(\lfloor d/2\rfloor - q)}$$

fraction of the points of P from C.

Proof. From [3, 10], it follows that the number of distinct q-tuples of P that can be separated from C by a hyperplane, assuming that $n \geqslant 2p$, is at least

$$\frac{n-p+1}{n-q+1}\frac{\binom{n}{q}}{\binom{p-1}{q-1}} \geqslant \frac{n^q}{2q\,p^{q-1}}.$$

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Let l be the maximum depth (Definition 2.4) of any of these separable q-tuples. The number of such tuples is therefore at most $\tau_d(n,q,l)$. Thus by Theorem 2.5 we must have

$$\frac{n^q}{2q\,p^{q-1}} \leqslant \tau_d(n,q,l) \leqslant e\,\kappa_d(n,q)\,l^{q-\lfloor d/2\rfloor}.$$

Rearranging the terms and from inequality (2.1), we get

$$\begin{aligned} l \geqslant \left(\frac{n^q}{2q\,p^{q-1}\cdot e\,\kappa_d(n,q)}\right)^{1/(q-\lfloor d/2\rfloor)} &\geqslant \left(\frac{n^q}{2q\,p^{q-1}\cdot e\,\kappa'_d(q)\,n^{\lfloor d/2\rfloor}}\right)^{1/(q-\lfloor d/2\rfloor)} \\ &= n\cdot(2q\,p^{q-1}\cdot e\,\kappa'_d(q))^{1/(\lfloor d/2\rfloor-q)}. \end{aligned}$$

Thus one of the separable q-tuples, say $P'\subseteq P$, must have depth at least l; in other words, the hyperplane separating P' from C must contain at least l points of P. This is the required hyperplane. □

We now prove a weighted version of the above statement.

Corollary 2.7. *Let C be a bounded convex object in $\mathbb{R}^d$, and let P be a weighted set of n points lying outside C, where the weight of each point $p\in P$ is a non-negative rational number. Let $p\geqslant q\geqslant\lfloor d/2\rfloor+1$ be parameters such that for every subset $Q\subseteq P$ of size p, there exists a set $Q'\subset Q$ of size q such that Q' can be separated from C by a hyperplane. Then there exists a hyperplane separating a set of points whose weight is at least*

$$\alpha_d(p,q) = (2e\,\kappa'_d(q)\,q^q\,p^{q-1})^{1/(\lfloor d/2\rfloor-q)}$$

fraction of the total weight of the points in P.

Proof. By appropriately scaling all the rational weights, we may assume that each weight is a non-negative integer and we replace a point with weight m by m unweighted copies of the point. Let P' be the new set of points. Observe that any set S of pq points in P' either contains q copies of some point in P or it contains p distinct points from P. In either case, there is a hyperplane separating q points of S from C. Thus, we can apply Lemma 2.6 to the point set P' with the parameter p in the lemma replaced by pq. The result follows. □

Finally we return to the proof of the main theorem.

Proof of Theorem 1.3.. Let $p = c_p+\lfloor d/2\rfloor$ and $q = c_q+\lfloor d/2\rfloor$. Let $\mathcal{H}$ be the set of all hyperplanes separating a distinct subset of points of P from C. As the number of subsets of P is finite, one can assume that $\mathcal{H}$ is also finite. Consider the following linear program on $|\mathcal{H}|$ variables $\{u_h\geqslant 0\colon h\in\mathcal{H}\}$:

$$\min\sum_{h\in\mathcal{H}} u_h, \quad \text{such that for all } r\in P \sum_{\substack{h\in\mathcal{H}\\ h \text{ separates } r \text{ from } C}} u_h \geqslant 1. \tag{2.2}$$

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The LP-dual to the above program, on $|P|$ variables $\{w_r \geqslant 0 \colon r \in P\}$, is

$$\max \sum_{p \in P} w_p, \quad \text{such that for all } h \in \mathcal{H} \sum_{\substack{r \in P \\ h \text{ separates } r \text{ from } C}} w_r \leqslant 1. \tag{2.3}$$

Consider an optimal solution w^* of the dual linear program and interpret w_r^* as the weight of each $r \in P$. Since the weights are rational, by Corollary 2.7, there exists a hyperplane $h \in \mathcal{H}$ separating a subset of P of combined weight at least $\varepsilon = \alpha_d(p,q)$ fraction of the total weight of all the points. Since the total weight of the points in any half-space is constrained to be at most 1 by the linear program, the total weight of all the points of P must be at most $1/\varepsilon$. In other words, the optimal value of linear program (2.3) is at most $1/\varepsilon$. Since the optimal values of both linear programs are equal due to strong duality, the optimal value of linear program (2.2) is also at most $1/\varepsilon$.

Let u^* be the optimal solution of linear program (2.2). If we interpret u_h as the weight of the hyperplane h, the constraints of the program imply that each point is separated by a set of hyperplanes in $\mathcal{H}$ whose combined weight is at least 1 out of a total weight of at most $1/\varepsilon$ – in other words, at least εth fraction of the total weight of $\mathcal{H}$. By associating with each hyperplane the half-space bounded by it and not containing C, and using the ε-net theorem for half-spaces in $\mathbb{R}^d$ (Corollary 2.2), there exists a set of $O(d/\varepsilon \log(1/\varepsilon))$ hyperplanes which together separate all points of P from C. Recalling that

$$\frac{1}{\varepsilon} = \frac{1}{\alpha_d(p,q)} = (2e\,\kappa'_d(q)\,q^q\,p^{q-1})^{1/(q-\lfloor d/2 \rfloor)} = (2e\,\kappa'_d(q)\,q^q\,p^{q-1})^{1/c_q}.$$

and that

$$\kappa'_d(q) = 2K_1^d \left\lfloor \frac{d}{2} \right\rfloor^{-\lfloor d/2 \rfloor} \left(1 + \frac{q}{\lceil d/2 \rceil}\right)^{\lceil d/2 \rceil},$$

we get

$$\begin{aligned}
\frac{1}{\varepsilon} &= \left(4K_1^d e \left\lfloor \frac{d}{2} \right\rfloor^{-\lfloor d/2 \rfloor} \left(1 + \frac{q}{\lceil d/2 \rceil}\right)^{\lceil d/2 \rceil} q^q\,p^{q-1}\right)^{1/c_q} \\
&\leqslant \left(4K_1^{d+1} \left\lfloor \frac{d}{2} \right\rfloor^{-d} (c_q + d)^{\lceil d/2 \rceil}\, q^q\,p^{q-1}\right)^{1/c_q} \quad \text{(using } e \leqslant K_1 \text{ and } q = c_q + \lfloor d/2 \rfloor\text{)} \\
&\leqslant \left(4K_1^{d+1} \left\lfloor \frac{d}{2} \right\rfloor^{-d} (c_q + d)^{\lceil d/2 \rceil} q^{c_q + \lfloor d/2 \rfloor}\, p^{c_q + \lfloor d/2 \rfloor - 1}\right)^{1/c_q} \\
&= O\left(K_1^{d/c_q} \left\lfloor \frac{d}{2} \right\rfloor^{-d/c_q} (c_q + d)^{\lceil d/2 \rceil / c_q} \left(c_q + \left\lfloor \frac{d}{2} \right\rfloor\right)^{1 + \lfloor d/2 \rfloor / c_q} \left(c_p + \left\lfloor \frac{d}{2} \right\rfloor\right)^{1 + (\lfloor d/2 \rfloor - 1)/c_q}\right) \\
&= O\Bigg(K_1^{d/c_q} d^{2 + (\lfloor d/2 \rfloor - 1)/c_q} \\
&\qquad \times \left(1 + \frac{c_q}{d}\right)^{\lceil d/2 \rceil / c_q} \left(1 + \frac{c_q}{\lfloor d/2 \rfloor}\right)^{1 + \lfloor d/2 \rfloor / c_q} \left(1 + \frac{c_p}{\lfloor d/2 \rfloor}\right)^{1 + (\lfloor d/2 \rfloor - 1)/c_q}\Bigg)
\end{aligned}$$

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$$
\begin{aligned}
&= O\Bigg(K_1^{d/c_q} d^{2+(\lfloor d/2\rfloor-1)/c_q} \ \exp\left(\frac{c_q}{d}\cdot\frac{\lceil d/2\rceil}{c_q}\right)\\
&\qquad\times\left(1+\frac{c_q}{\lfloor d/2\rfloor}\right)\exp\left(\frac{c_q}{\lfloor d/2\rfloor}\cdot\frac{\lfloor d/2\rfloor}{c_q}\right)\left(1+\frac{c_p}{\lfloor d/2\rfloor}\right)^{1+(\lfloor d/2\rfloor-1)/c_q}\Bigg)\\
&= O\left(K_1^{d/c_q} d^{2+(\lfloor d/2\rfloor-1)/c_q}\ \left(1+\frac{c_q}{\lfloor d/2\rfloor}\right)\left(1+\frac{c_p}{\lfloor d/2\rfloor}\right)^{1+(\lfloor d/2\rfloor-1)/c_q}\right)\\
&= O\left(K_1^{d/c_q} 2^{d/2c_q}\ \left(\left\lfloor\frac{d}{2}\right\rfloor+c_q\right)\left(\left\lfloor\frac{d}{2}\right\rfloor+c_p\right)^{1+(\lfloor d/2\rfloor-1)/c_q}\right)\\
&= O\left((\sqrt{2}K_1)^{d/c_q}\ \left(\left\lfloor\frac{d}{2}\right\rfloor+c_q\right)\left(\left\lfloor\frac{d}{2}\right\rfloor+c_p\right)^{1+(\lfloor d/2\rfloor-1)/c_q}\right).
\end{aligned}
$$

The 'big Oh' notation here does not hide dependences on d, namely we do not treat d as a constant. From the above it follows that

$$\log\frac{1}{\varepsilon} = O\left(c_q^{-1}\left(\left\lfloor\frac{d}{2}\right\rfloor+c_q\right)\log\left(\left\lfloor\frac{d}{2}\right\rfloor+c_p\right)\right).$$

Thus, $d/\varepsilon\log(1/\varepsilon)$ is

$$
\begin{aligned}
O\Bigg(d\cdot\Bigg((\sqrt{2}K_1)^{d/c_q}\ &\left(\left\lfloor\frac{d}{2}\right\rfloor+c_q\right)\left(\left\lfloor\frac{d}{2}\right\rfloor+c_p\right)^{1+(\lfloor d/2\rfloor-1)/c_q}\Bigg)\\
&\times\left(c_q^{-1}\left(\left\lfloor\frac{d}{2}\right\rfloor+c_q\right)\log\left(\left\lfloor\frac{d}{2}\right\rfloor+c_p\right)\right)\Bigg),
\end{aligned}
$$

which simplifies to

$$O\left(\frac{d}{c_q}(\sqrt{2}K_1)^{d/c_q}\left(\left\lfloor\frac{d}{2}\right\rfloor+c_q\right)^2\left(\left\lfloor\frac{d}{2}\right\rfloor+c_p\right)^{1+(\lfloor d/2\rfloor-1)/c_q}\log\left(\left\lfloor\frac{d}{2}\right\rfloor+c_p\right)\right).$$

Since linear programs can be solved in polynomial time and epsilon nets can be computed in polynomial time, the partition of P into the above number of sets can be achieved in polynomial time. The theorem follows. □

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