TLP **14** (2): 165–213, 2014. © Cambridge University Press 2012
doi:10.1017/S1471068412000373 First published online 30 October 2012

Rewriting and narrowing for constructor systems with call-time choice semantics[1]

FRANCISCO J. LÓPEZ-FRAGUAS, ENRIQUE MARTIN-MARTIN,
JUAN RODRÍGUEZ-HORTALÁ and JAIME SÁNCHEZ-HERNÁNDEZ

Departamento de Sistemas Informáticos y Computación
Universidad Complutense de Madrid, Madrid, Spain
(*e-mail:* fraguas@sip.ucm.es, emartinm@fdi.ucm.es,
juan.rodriguez.hortala@gmail.com, jaime@sip.ucm.es)

submitted 30 November 2010; revised 14 November 2011; accepted 4 July 2012

Abstract

Non-confluent and non-terminating constructor-based term rewriting systems are useful for the purpose of specification and programming. In particular, existing functional logic languages use such kinds of rewrite systems to define possibly non-strict non-deterministic functions. The semantics adopted for non-determinism is *call-time choice*, whose combination with non-strictness is a non-trivial issue, addressed years ago from a semantic point of view with the Constructor-based Rewriting Logic (CRWL), a well-known semantic framework commonly accepted as suitable semantic basis of modern functional logic languages. A drawback of CRWL is that it does not come with a proper notion of one-step reduction, which would be very useful to understand and reason about how computations proceed. In this paper, we develop thoroughly the theory for the first-order version of let-rewriting, a simple reduction notion close to that of classical term rewriting, but extended with a let-binding construction to adequately express the combination of call-time choice with non-strict semantics. Let-rewriting can be seen as a particular textual presentation of term graph rewriting. We investigate the properties of let-rewriting, most remarkably their equivalence with respect to a conservative extension of the CRWL-semantics coping with let-bindings, and we show by some case studies that having two interchangeable formal views (reduction/semantics) of the same language is a powerful reasoning tool. After that, we provide a notion of let-narrowing, which is adequate for call-time choice as proved by soundness and completeness results of let-narrowing with respect to let-rewriting. Moreover, we relate those let-rewriting and let-narrowing relations (and hence CRWL) with ordinary term rewriting and narrowing, providing in particular soundness and completeness of let-rewriting with respect to term rewriting for a class of programs which are deterministic in a semantic sense.

KEYWORDS: term rewriting systems, constructor-based rewriting logic, narrowing, non-determinism, call-time choice semantics, sharing, local bindings

[1] This work has been partially supported by the Spanish projects FAST-STAMP (TIN2008-06622-C03-01/TIN), PROMETIDOS-CM (S2009TIC-1465) and GPD-UCM (UCM-BSCH-GR58/08-910502).

https://doi.org/10.1017/S1471068412000373 Published online by Cambridge University Press

$coin \to 0$	$repeat(X) \ \to \ X : repeat(X)$
$coin \to 1$	$heads(X : Y : Ys) \ \to \ (X, Y)$

Fig. 1. A non-terminating and non-confluent program.

1 Introduction

Term rewriting systems (TRS; Baader and Nipkow 1998) are a well-known and useful formalism from the point of view of specification and programming. The theory of TRS underlies many of the proposals made in the last decades for so-called *functional logic programming*, attempting to integrate into a single language the main features of both functional and logic programming—see DeGroot and Lindstrom (1986) and Hanus (1994, 2007) for surveys corresponding to different historical stages of the development of functional logic languages. Typically, functional logic programs are modeled by some kind of TRS to define functions, and logic programming capabilities are achieved by using some kind of *narrowing* as operational mechanism. Narrowing, a notion coming from the field of automated theorem proving, generalizes rewriting by using unification instead of matching in reduction steps. Up to 14 different variants of narrowing were identified in Hanus (1994) as being used in different proposals for the integration of functional and logic programming.

Modern functional logic languages like *Curry* (Hanus *et al.* 1995; Hanus 2006) or *Toy* (López-Fraguas and Sánchez-Hernández 1999; Caballero and Sánchez 2006) consider that programs are constructor-based term rewriting systems, possibly non-terminating and non-confluent, thus defining possibly non-strict non-deterministic functions. For instance, in the program of Figure 1, non-confluence comes from the two rules of *coin* and non-termination is due to the rule for *repeat*.

For non-determinism, those systems adopt *call-time choice* semantics (Hussmann 1993; González-Moreno *et al.* 1999), also called sometimes *singular* semantics (Søndergaard and Sestoft 1992). Loosely speaking, call-time choice means to pick a value for each argument of a function application before applying it. Call-time choice is easier to understand and implement in combination with strict semantics and eager evaluation in terminating systems as in Hussmann (1993), but can be made also compatible—via partial values and sharing—with non-strictness and laziness in the presence of non-termination.

In the example of Figure 1, the expression $heads(repeat(coin))$ can take, under call-time choice, the values $(0, 0)$ and $(1, 1)$, but not $(0, 1)$ or $(1, 0)$. The example also illustrates a key point here: ordinary term rewriting (called *run-time choice* in Hussmann 1993) is an unsound procedure for call-time choice semantics, since a possible term rewriting derivation is:

$$heads(repeat(coin)) \to heads(coin : repeat(coin)) \to$$
$$heads(0 : repeat(coin)) \to heads(0 : coin : repeat(coin)) \to$$
$$heads(0 : 1 : repeat(coin)) \to (0, 1)$$

In operational terms, call-time choice requires to *share* the value of all copies of a given subexpression created during reduction (all the occurrences of *coin*, in

https://doi.org/10.1017/S1471068412000373 Published online by Cambridge University Press

the reduction above). In contrast, with ordinary term rewriting all copies evolve independently.

It is commonly accepted (see e.g. Hanus 2007) that call-time choice semantics combined with non-strict semantics is adequately formally expressed by the CRWL framework[2] (González-Moreno *et al.* 1996; González-Moreno *et al.* 1999), whose main component is a proof calculus that determines the semantics of programs and expressions. The flexibility and usefulness of CRWL is evidenced by the large set of extensions that have been devised for it, to cope with relevant aspects of declarative programming: higher order functions, types, constraints, constructive failure, ...; see Rodríguez-Artalejo (2001) for a survey on the CRWL approach. However, a drawback of the CRWL framework is its lack of a proper one-step reduction mechanism that could play a role similar to term rewriting with respect to equational logic. Certainly, CRWL includes operational procedures in the form of goal-solving calculi (González-Moreno *et al.* 1999; Vado-Vírseda 2003) to solve so-called *joinability* conditions, but they are too complex to be seen as a basic way to explain or understand how a reduction can proceed in the presence of non-strict non-deterministic functions with call-time choice semantics.

On the other hand, other works have been more influential on the operational side of the field, specially those based on the notion of *needed narrowing* (Antoy *et al.* 1994; Antoy *et al.* 2000), a variant of narrowing that organizes the evaluation of arguments in function calls in an adequate way (optimal, for some classes of programs). Needed narrowing became the 'official' operational procedure of functional logic languages, and has also been subject of several variations and improvements (see Escobar *et al.* 2005; Hanus 2007).

These two coexisting branches of research (one based on CRWLand the other based on classical term rewriting, mostly via needed narrowing) have remained disconnected for many years from the technical point of view, despite the fact that they both refer to what intuitively is the same programming language paradigm.

A major problem to establish the connection was that the theory underlying needed narrowing is classical term rewriting, which, as we saw above, is not valid for non-determinism with call-time choice semantics. This was not a flaw in the conception of needed narrowing, as it emerged in a time when non-deterministic functions had not yet started to play a distinctive role in the functional logic programming paradigm. The problem is overcome in practice by adding a sharing mechanism to the implementation of narrowing, using for instance standard Prolog programming techniques (Cheong and Fribourg 1993; Loogen *et al.* 1993; Antoy and Hanus 2000). But this is merely an implementation patch that cannot be used as a precise and sound technical basis for the application of results and techniques from the semantic side to the operational side and vice versa. Other works, specially Echahed and Janodet (1998) and Albert *et al.* (2005) have addressed in a more formal way the issue of sharing in functional logic programming, but they are not

[2] CRWL stands for Constructor-Based ReWriting Logic.

https://doi.org/10.1017/S1471068412000373 Published online by Cambridge University Press

good starting points to establish a relationship with the CRWL world (see 'Related work' below).

In López-Fraguas *et al.* (2007b), we aimed at establishing a bridge, by looking for a new variant of term rewriting tailored to call-time choice as realized by CRWL, trying to fulfill the following requirements:

- It should be based on a notion of rewrite step useful to follow how a computation proceeds step by step.
- It should be simple enough to be easily understandable for non-expert potential users (e.g. students or novice programmers) of functional logic languages adopting call-time choice.
- It should be provably equivalent to CRWL, as a well-established technical formulation of call-time choice.
- It should serve as a basis of subsequent notion of narrowing and evaluation strategies.

That was realized in López-Fraguas *et al.* (2007b) by means of let-rewriting, a simple modification of term rewriting using local bindings in the form of let-expressions to express sharing. Let-rewriting will be fully presented in Section 4, but its main intuitions can be summarized as follows:

(i) Do not rewrite a function call if any of its arguments is evaluable (i.e., still contains other function calls), even if there is a matching rule;
(ii) Instead, extract those evaluable arguments to outer let-bindings of the form $let\ X = e\ in\ e'$;
(iii) If after some reduction steps the *definiens* e of the let-binding becomes a constructor term t—a value—then the binding X/t can be made effective in the body e'. In this way, the values obtained for e in the reduction are shared, and therefore call-time choice is respected.

Consider, for instance, the program example of Figure 1 and the expression

$$heads(repeat(coin))$$

for which we previously performed an ordinary term rewriting reduction ending in $(0, 1)$. Now we are going to apply liberally the previous intuitive hints as a first illustration of let-rewriting. Note first that no rewrite step using a program rule can be done with the whole expression $heads(repeat(coin))$, since in this case there is no matching rule. But we can extract the argument $repeat(coin)$ to a let-binding, obtaining:

$$let\ X = repeat(coin)\ in\ heads(X)$$

Now we cannot rewrite $repeat(coin)$, even though the program rule for *repeat* matches it, because *coin* is evaluable. Again, we can create a let-binding for *coin*, that will be used to share the value selected for *coin*, if at any later step in the reduction *coin* is indeed reduced:

$$let\ Y = coin\ in\ let\ X = repeat(Y)\ in\ heads(X)$$

https://doi.org/10.1017/S1471068412000373 Published online by Cambridge University Press

At this point there is no problem with rewriting *repeat(Y)*, which gives:

$$let\ Y = coin\ in\ let\ X = Y : repeat(Y)\ in\ heads(X)$$

Rewriting *repeat(Y)* again, we have:

$$let\ Y = coin\ in\ let\ X = Y : Y : repeat(Y)\ in\ heads(X)$$

Reducing *repeat(Y)* indefinitely leads to non-termination, but, at the same time, its presence inhibits the application of the binding for X. What we can do is creating a new let-binding for the remaining *repeat(Y)*, which results in:

$$let\ Y = coin\ in\ let\ Z = repeat(Y)\ in\ let\ X = Y : Y : Z\ in\ heads(X)$$

Now, the binding for X can be performed, obtaining:

$$let\ Y = coin\ in\ let\ Z = repeat(Y)\ in\ heads(Y : Y : Z)$$

At this point, we can use the rule for *heads* to evaluate $heads(Y : Y : Z)$, because nothing evaluable remains in its argument $Y : Y : Z$, arriving at:

$$let\ Y = coin\ in\ let\ Z = repeat(Y)\ in\ (Y, Y)$$

We proceed now by reducing *coin*, for instance, to 0 (reducing it to 1 is also possible):

$$let\ Y = 0\ in\ let\ Z = repeat(Y)\ in\ (Y, Y)$$

Performing the binding for Y leads to:

$$let\ Z = repeat(0)\ in\ (0, 0)$$

Since Z does not occur in $(0, 0)$, its binding is junk that could be deleted (there will be a rule for that in the definition of let-rewriting), and the reduction is finished yielding the value

$$(0, 0)$$

It is apparent that $(1, 1)$ is another possible result, but not $(0, 1)$ nor $(1, 0)$, a behavior coherent with call-time choice.

In this example, we have tried to proceed in a more or less natural 'lazy' way. However, the previous intuitive precepts—and its complete and precise realization in Section 4—do not assume any particular strategy for organizing reductions, but only determine which are the 'legal movements' in call-time choice respectful reductions. Strategies have been left aside in the paper, not only for simplicity, but also to keep them independent of the basic rules for term rewriting with sharing (see however Section 6.2).

Let-rewriting was later on extended to cope with narrowing (López-Fraguas *et al.* 2009c) and higher order features (López-Fraguas *et al.* 2008).

This paper is a substantially revised and completed presentation of the theory of first-order let-rewriting and let-narrowing proposed in López-Fraguas *et al.* (2007b, 2009c); some contents have been also taken from López-Fraguas *et al.* (2008). Here, we unify technically those papers and develop a deeper investigation of the properties of let-rewriting and related semantics issues.

https://doi.org/10.1017/S1471068412000373 Published online by Cambridge University Press

Related work. Our let-rewriting and let-narrowing relations are not the only nor the first formal operational procedures tuned up to accomplish with the call-time choice semantics of functional logic languages. We have already mentioned the goal-solving calculi associated with the CRWL framework and its variants (González-Moreno *et al.* 1997, 1999; Vado-Vírseda 2003).

A natural option to express different levels of sharing in rewriting is given by the theory of term graph rewriting (Barendregt *et al.* 1987; Plump 2001). In Echahed and Janodet (1997, 1998), the theory of needed rewriting and narrowing was extended to the framework of so-called admissible graph rewriting systems, aiming at formally modeling the operational behavior of functional logic programs. Originally, those works considered orthogonal systems, and extra variables were not allowed. These restrictions were dropped in Antoy *et al.* (2007) (however, a formal treatment of the extension is missing).

As a matter of fact, our let-rewriting relation can be understood as a particular textual adaptation and presentation of term graph rewriting in which a shared node is made explicit in the syntax by giving it a name in a let-binding. The achievements of Echahed's works are somehow incomparable to ours, even if both are attempts to formalize sharing in constructor-based systems. They focus and succeed on adapting known optimal strategies to the graph rewriting and narrowing setting; they also take profit of the fine-grained descriptions permitted by graphs to manage aspects of data structures like cycles or pointers. However, they do not try to establish a technical relationship with other formulations of call-time choice. In contrast, proving equivalence of our operational formalisms with respect to the CRWL semantic framework has been a main motivation of our work, but we do not deal with the issue of strategies, except for a short informal discussion at the end of the paper.

It is our thought that proving equivalence with respect to CRWL of term graph rewriting as given in Echahed and Janodet (1997) would have been a task much harder than the route we follow here. We see a reason for it. The basic pieces that term rewriting and CRWL work with are purely syntactic: terms, substitutions, etc. Graph rewriting recast these notions in terms of graphs, homomorphisms, etc. In contrast, let-rewriting and let-narrowing keep the same set of basic pieces of term rewriting and CRWL. In this way, the formalisms are relatively close and moving from one to another becomes technically more natural and comfortable. This also applies to some further developments of our setting that we have made so far, such as the extension to higher order features given in López-Fraguas *et al.* (2008), the combination of semantics proposed in López-Fraguas *et al.* (2009a), or the application of let-rewriting as underlying formal notion of reduction for type systems in functional logic languages (López-Fraguas *et al.* 2010a, 2010b).

Another proposal that can be seen as reformulation of graph rewriting was given in Albert *et al.* (2005), inspired by Launchbury's natural semantics (Launchbury 1993) for lazy evaluation in functional programming. It presents two operational (natural and small-step) semantics for functional logic programs supporting sharing and residuation (a specific feature of Curry). These semantics use a flat representation of programs coming from an implicit program transformation encoding the demand

https://doi.org/10.1017/S1471068412000373 Published online by Cambridge University Press

analysis used by needed narrowing, and some kind of heaps to express bindings for variables. As in our case, let-expressions are used to express sharing. The approach is useful as a technical basis for implementation and program manipulation purposes; but we think that the approach is too low-level and close to a particular operational strategy to be a completely satisfactory choice as a basic abstract reduction mechanism for call-time choice. In López-Fraguas *et al.* (2007a) we established a technical relation of CRWL with the operational procedures of Albert *et al.* (2005). But this turned out to be a really hard task, even if it was done only for a restricted class of programs and expressions.

Our work focuses on term rewriting systems as basic formalism, as happens with the majority of papers about the foundations of functional logic programming, in particular the CRWL series. The idea of reformulating graph rewriting in a syntactic style by expressing sharing through let-bindings has also been applied to other contexts, most remarkably to λ-calculus considered as a basis of functional programming (Ariola and Arvind 1995; Ariola *et al.* 1995; Ariola and Felleisen 1997; Maraist *et al.* 1998). In a different direction, but still in relation with λ-calculus, other papers (Kutzner and Schmidt-Schauß 1998; Schmidt-Schauß and Machkasova 2008) have extended it with some kind of non-deterministic choice, an idea that comes back to McCarthy's *amb* (McCarthy 1963). As a final note, we should mention that our initial ideas about let-rewriting were somehow inspired by López-Fraguas and Sánchez-Hernández (2001) and Sánchez-Hernández (2004), where indexed unions of set expressions—a construction generalizing the idea of let-expressions—were used to express sharing in an extension of CRWL to deal with constructive failure.

The rest of the paper is organized as follows. Section 2 presents some preliminaries about term rewriting and the CRWL framework; although with them the paper becomes almost self-contained, some familiarity with the basic notions of TRS certainly help to read the paper. Section 3 contains a first discussion about failed or partial solutions to the problem of expressing non-strict call-time choice by a simple notion of rewriting. Section 4 is the central part of the paper. First, it introduces local bindings in the syntax to express sharing, defines let-rewriting as an adequate notion of rewriting for them and proves some intrinsic properties of let-rewriting. After that, in Section 4.2, we extend the CRWL logic to a new CRWL_{let} logic able to deal with lets in programs and expressions, and we investigate in depth the properties of the induced semantics, mostly through the notion of *hypersemantics*. Finally, in Section 4.3, we prove results of soundness and completeness of let-rewriting with respect to CRWL_{let}, which have as corollary the equivalence of both, and hence the equivalence of let-rewriting and CRWL for programs and expressions not containing lets, as the original CRWL ones are. Section 5 aims at showing the power of having reduction and semantics as equivalent interchangeable tools for reasoning, including a remarkable case study. In Section 6 we generalize the notion of let-rewriting to that of let-narrowing and give soundness and completeness results of the latter with respect to the former. At the end of the section, we give some hints on how computations can be organized according to known narrowing strategies. Section 7 addresses the relationship between let-rewriting and classical term rewriting, proving in particular their equivalence for semantically deterministic

https://doi.org/10.1017/S1471068412000373 Published online by Cambridge University Press

programs. Finally, Section 8 analyzes our contribution and suggests further work. For the sake of readability, most of the (fully detailed) proofs have been moved to the online appendix of the paper at http.arxiv.org/abs/1209.2617.

2 Preliminaries

2.1 Constructor-based term rewriting systems

We assume a fixed first-order signature $\Sigma = CS \cup FS$, where CS and FS are two disjoint sets of constructor and defined function symbols respectively, each of them with an associated arity. We write CS^n and FS^n for the set of constructor and function symbols of arity n respectively, and Σ^n for $CS^n \cup FS^n$. As usual notations we write $c, d, \ldots$ for constructors, $f, g, \ldots$ for functions and $X, Y, \ldots$ for variables taken from a denumerable set $\mathcal{V}$. The notation $\overline{o}$ stands for tuples of any kind of syntactic objects.

The set Exp of *expressions* is defined as $Exp \ni e ::= X \mid h(e_1, \ldots, e_n)$, where $X \in \mathcal{V}$, $h \in \Sigma^n$ and $e_1, \ldots, e_n \in Exp$. The set $CTerm$ of *constructed terms* (or *c-terms*) has the same definition of Exp, but with h restricted to CS^n (so $CTerm \subsetneq Exp$). The intended meaning is that Exp stands for evaluable expressions, i.e., expressions that can contain (user-defined) function symbols, while $CTerm$ stands for data terms representing values. We will write $e, e', \ldots$ for expressions and $t, s, p, t', \ldots$ for c-terms. The set of variables occurring in an expression e will be denoted as $var(e)$.

Contexts (with one hole) are defined by $Cntxt \ni \mathcal{C} ::= [\,] \mid h(e_1, \ldots, \mathcal{C}, \ldots, e_n)$, where $h \in \Sigma^n$. The application of a context $\mathcal{C}$ to an expression e, written as $\mathcal{C}[e]$, is defined inductively as follows:

$$[\,][e] = e$$
$$h(e_1, \ldots, \mathcal{C}, \ldots, e_n)[e] = h(e_1, \ldots, \mathcal{C}[e], \ldots, e_n)$$

Substitutions are finite mappings $\sigma : \mathcal{V} \longrightarrow Exp$ which extend naturally to $\sigma : Exp \longrightarrow Exp$. We write $e\sigma$ for the application of the substitution σ to e. The domain and variable range of a substitution σ are defined as $dom(\sigma) = \{X \in \mathcal{V} \mid X\sigma \neq X\}$ and $vran(\sigma) = \bigcup_{X \in dom(\sigma)} var(X\sigma)$. By $[X_1/e_1, \ldots, X_n/e_n]$ we denote the substitution σ such that $Y\sigma = e_i$ if $Y \equiv X_i$ for some $X_i \in \{X_1, \ldots, X_n\}$, and $Y\sigma = Y$ otherwise. Given a set of variables D, the notation $\sigma|_D$ represents the substitution σ restricted to D and $\sigma|_{\setminus D}$ is a shortcut for $\sigma|_{(\mathcal{V} \setminus D)}$. A *c-substitution* is a substitution θ such that $X\theta \in CTerm$ for all $X \in dom(\theta)$. We write $Subst$ and $CSubst$ for the sets of substitutions and c-substitutions, respectively.

A *term rewriting system* is any set of rewrite rules of the form $l \to r$ where $l, r \in Exp$ and $l \notin \mathcal{V}$. A *constructor based rewrite rule* or *program rule* has the form $f(p_1, \ldots, p_n) \to r$, where $f \in FS^n$, $r \in Exp$ and $(p_1, \ldots, p_n)$ is a linear tuple of c-terms, where linear means that no variable occurs twice in the tuple. Note that we allow r to have extra variables (i.e., variables not occurring in the left-hand side). To be precise, we say that X is an extra variable in the rule $l \to r$ iff $X \in var(r) \setminus var(l)$, and by $vExtra(R)$ we denote the set of extra variables in a rule R. Then a *constructor system*

https://doi.org/10.1017/S1471068412000373 Published online by Cambridge University Press

or *program* $\mathscr{P}$ is any set of program rules, i.e., a term rewriting system composed only of program rules.

Given a program $\mathscr{P}$, its associated *rewrite relation* $\rightarrow_{\mathscr{P}}$ is defined as $\mathscr{C}[l\sigma] \rightarrow_{\mathscr{P}} \mathscr{C}[r\sigma]$ for any context $\mathscr{C}$, rule $l \rightarrow r \in \mathscr{P}$ and $\sigma \in Subst$. There, the subexpression $l\sigma$ is called the redex used in that *rewriting step*. Note that σ can instantiate extra variables to any expression. For any binary relation $\mathscr{R}$ we write $\mathscr{R}^*$ for the reflexive and transitive closure of $\mathscr{R}$, and $\mathscr{R}^n$ for the composition of $\mathscr{R}$ with itself n times. We write $e_1 \overset{*}{\rightarrow}_{\mathscr{P}} e_2$ for a term rewriting *derivation* or *reduction* from e_1 to e_2, and $e_1 \overset{n}{\rightarrow}_{\mathscr{P}} e_2$ for an n-step reduction. e_2 is a *normal form* wrt. $\rightarrow_{\mathscr{P}}$, written as $\downarrow^{\mathscr{P}} e_2$, if there is not any e_3 such that $e_2 \rightarrow_{\mathscr{P}} e_3$; and e_2 is a normal form for e_1 wrt. $\rightarrow_{\mathscr{P}}$, written as $e_1 \downarrow^{\mathscr{P}} e_2$, iff $e_1 \overset{*}{\rightarrow}_{\mathscr{P}} e_2$ and e_2 is a normal form. When presenting derivations, we will sometimes underline the redex used at each rewriting step. In the following, we will usually omit the reference to $\mathscr{P}$ when writing $e_1 \rightarrow_{\mathscr{P}} e_2$, or denote it by $\mathscr{P} \vdash e_1 \rightarrow e_2$.

A program $\mathscr{P}$ is confluent if for any $e, e_1, e_2 \in Exp$ such that $e \rightarrow^*_{\mathscr{P}} e_1$, $e \rightarrow^*_{\mathscr{P}} e_2$ there exists $e_3 \in Exp$ such that both $e_1 \rightarrow^*_{\mathscr{P}} e_3$ and $e_2 \rightarrow^*_{\mathscr{P}} e_3$.

2.2 The CRWL framework

We present here a simplified version of the CRWL framework (González-Moreno *et al.* 1996, 1999). The original CRWL logic also considered the possible presence of *joinability* constraints as conditions in rules in order to give a better treatment of strict equality as a built-in, a subject orthogonal to the aims of this work. Furthermore, it is possible to replace conditions by the use of an *if_then* function, as has been technically proved in Sánchez-Hernández (2004) for CRWL and in Antoy (2005) for term rewriting. Therefore, we consider only unconditional program rules.

In order to deal with non-strictness at the semantic level, we enlarge Σ with a new constant (i.e., a 0-ary constructor symbol) $\bot$ that stands for the undefined value. The sets $Exp_\bot$, $CTerm_\bot$, $Subst_\bot$, $CSubst_\bot$ of partial expressions, etc., are defined naturally. Note that $\bot$ does not appear in programs. Partial expressions are ordered by the *approximation* ordering $\sqsubseteq$ defined as the least partial ordering satisfying $\bot \sqsubseteq e$ and $e \sqsubseteq e' \Rightarrow \mathscr{C}[e] \sqsubseteq \mathscr{C}[e']$ for all $e, e' \in Exp_\bot, \mathscr{C} \in Cntxt$. This partial ordering can be extended to substitutions: given $\theta, \sigma \in Subst_\bot$ we say $\theta \sqsubseteq \sigma$ if $X\theta \sqsubseteq X\sigma$ for all $X \in \mathscr{V}$.

The semantics of a program $\mathscr{P}$ is determined in CRWL by means of a proof calculus able to derive reduction statements of the form $e \twoheadrightarrow t$, with $e \in Exp_\bot$ and $t \in CTerm_\bot$, meaning informally that t is (or approximates to) a possible value of e, obtained by evaluating e using $\mathscr{P}$ under call-time choice.

The CRWL-proof calculus is presented in Figure 2. Rule **(B)** allows any expression to be undefined or not evaluated (non-strict semantics). Rule **(OR)** expresses that to evaluate a function call we must choose a compatible program rule, perform parameter passing (by means of a c-substitution θ) and then reduce the right-hand side. The use of c-substitutions in (OR) is essential to express call-time choice; note also that by the effect of θ in (OR), extra variables in the right-hand side of a rule can be replaced by any partial c-term, but not by any expression as in ordinary

https://doi.org/10.1017/S1471068412000373 Published online by Cambridge University Press

$$\textbf{(B)}\ \frac{}{e \twoheadrightarrow \bot} \qquad \textbf{(RR)}\ \frac{}{X \twoheadrightarrow X} \quad X \in \mathcal{V}$$

$$\textbf{(DC)}\ \frac{e_1 \twoheadrightarrow t_1 \ \ldots\ e_n \twoheadrightarrow t_n}{c(e_1,\ldots,e_n) \twoheadrightarrow c(t_1,\ldots,t_n)} \quad c \in CS^n$$

$$\textbf{(OR)}\ \frac{e_1 \twoheadrightarrow p_1\theta \ldots\ e_n \twoheadrightarrow p_n\theta \quad r\theta \twoheadrightarrow t}{f(e_1,\ldots,e_n) \twoheadrightarrow t} \quad \begin{array}{l}(f(p_1,\ldots,p_n) \to r) \in \mathcal{P} \\ \theta \in CSubst_\bot\end{array}$$

Fig. 2. Rules of CRWL.

$$\dfrac{\dfrac{\dfrac{\dfrac{}{\mathbf{coin} \twoheadrightarrow \mathbf{0}}\,OR}{0 \twoheadrightarrow 0}\,DC \quad \dfrac{\dfrac{}{0 \twoheadrightarrow 0}\,DC \quad \dfrac{\dfrac{\dfrac{}{0 \twoheadrightarrow 0}\,DC \quad \dfrac{}{repeat(0) \twoheadrightarrow \bot}\,B}{0 : repeat(0) \twoheadrightarrow 0 : \bot}\,DC}{repeat(0) \twoheadrightarrow 0 : \bot}\,OR}{0 : repeat(0) \twoheadrightarrow 0 : 0 : \bot}\,DC}{repeat(coin) \twoheadrightarrow 0 : 0 : \bot}\,OR \quad \dfrac{\dfrac{}{0 \twoheadrightarrow 0}\,DC \quad \dfrac{}{0 \twoheadrightarrow 0}\,DC}{(0,0) \twoheadrightarrow (0,0)}\,DC}{heads(repeat(coin)) \twoheadrightarrow (0,0)}\,OR$$

Fig. 3. A CRWL-derivation for $heads(repeat(coin)) \twoheadrightarrow (0,0)$.

term rewriting $\to_{\mathscr{P}}$. We write $\mathscr{P} \vdash_{CRWL} e \twoheadrightarrow t$ to express that $e \twoheadrightarrow t$ is derivable in the CRWL-calculus using the program $\mathscr{P}$, but in many occasions we will omit the mention to $\mathscr{P}$, writing simply $e \twoheadrightarrow t$.

Definition 1 (*CRWL-denotation*)
Given a program $\mathscr{P}$, the *CRWL-denotation* of an expression $e \in Exp_\bot$ is defined as

$$[\![e]\!]^{\mathscr{P}}_{CRWL} = \{t \in CTerm_\bot \mid \mathscr{P} \vdash_{CRWL} e \twoheadrightarrow t\}$$

We will usually omit the subscript CRWL and/or the superscript $\mathscr{P}$ when implied by the context.

As an example, Figure 3 shows a *CRWL-derivation* or *CRWL-proof* for the statement $heads(repeat(coin)) \twoheadrightarrow (0,0)$, using the program of Figure 1. Observe that in the derivation there is only one reduction statement for *coin* (namely $coin \twoheadrightarrow 0$), and the obtained value 0 is then *shared* in the whole derivation, as corresponds to call-time choice.

In alternative derivations, *coin* could be reduced to 1 (or to $\bot$). It is easy to check that:

$$[\![heads(repeat(coin))]\!] = \{(0,0),(1,1),(\bot,0),(0,\bot),(\bot,1),(1,\bot),(\bot,\bot),\bot\}$$

Note that $(1,0),(0,1) \notin [\![heads(repeat(coin))]\!]$.

We stress the fact that the CRWL-calculus *is not* an operational mechanism for executing programs, but a way of describing the logic of programs. In particular, the rule (B) is a semantic artifact to reflect in a CRWL-proof of a statement $e \twoheadrightarrow t$ the fact that, for obtaining t as value of e, one does not need to know the value of a certain subexpression e' (to which the rule (B) is applied). But the calculus comes with no indication of when to apply (B) in a successful proof. At the operational level, the CRWL framework is accompanied with various lazy narrowing-based

https://doi.org/10.1017/S1471068412000373 Published online by Cambridge University Press

goal-solving calculi (González-Moreno *et al.* 1999; Vado-Vírseda 2003) not considered in this paper.

One of the most important properties of CRWL is its compositionality, a property very close to the DET-additivity property for algebraic specifications of Hussmann (1993) or the referential transparency property of Sondergaard and Sestoft (1990). Compositionality shows that the CRWL-denotation of any expression placed in a context only depends on the CRWL-denotation of that expression. This implies that the semantics of a whole expression depends only on the semantics of its constituents, as shown by the next result, which is an adaptation of a similar one proved for the higher order case in López-Fraguas *et al.* (2008).

Theorem 1 (*Compositionality of CRWL*)
For any $\mathscr{C} \in Cntxt$, $e, e' \in Exp_\perp$

$$[\![\mathscr{C}[e]]\!] = \bigcup_{t \in [\![e]\!]} [\![\mathscr{C}[t]]\!]$$

As a consequence: $[\![e]\!] = [\![e']\!] \Leftrightarrow \forall \mathscr{C} \in Cntxt.[\![\mathscr{C}[e]]\!] = [\![\mathscr{C}[e']]\!]$

According to this result, we can express for example

$$[\![heads(repeat(coin))]\!] = \bigcup_{t \in [\![coin]\!]} \bigcup_{s \in [\![repeat(t)]\!]} [\![heads(s)]\!]$$

The right-hand side has an intuitive reading that reflects call-time choice: get a value t of *coin*, then get a value s of $repeat(t)$ and then get a value of $heads(s)$.

In Theorem 2 we give an alternative formulation to the compositionality property. Although it is essentially equivalent to Theorem 1, it is a somehow more abstract statement, based on the notion of *denotation of a context* introduced in Definition 2. Our main reason for developing such alternative is to give good insights for the compositionality results of the extension of CRWL to be presented in Section 4.3.

We will use sometimes *Den* as an alias for $\mathscr{P}(CTerm_\perp)$, i.e, for the kind of objects that are CRWL-denotations of expressions[3]. We define the denotation of a context $\mathscr{C}$ as a denotation transformer that reflects call-time choice.

Definition 2 (*Denotation of a context*)
Given $\mathscr{C} \in Cntxt$, its denotation is a function $[\![\mathscr{C}]\!] : Den \to Den$ defined as

$$[\![\mathscr{C}]\!]\delta = \bigcup_{t \in \delta} [\![\mathscr{C}[t]]\!], \ \forall \delta \in Den$$

With this notion, compositionality can be trivially re-stated as follows:

Theorem 2 (*Compositionality of CRWL, version 2*)
For any $\mathscr{C} \in Cntxt$ and $e, e' \in Exp_\perp$

$$[\![\mathscr{C}[e]]\!] = [\![\mathscr{C}]\!][\![e]\!]$$

As a consequence: $[\![e]\!] = [\![e']\!] \Leftrightarrow \forall \mathscr{C} \in Cntxt.[\![\mathscr{C}[e]]\!] = [\![\mathscr{C}[e']]\!]$

[3] *Den* is indeed a superset of the set of actual denotations, which are particular elements of $\mathscr{P}(CTerm_\perp)$, namely *cones*—see González-Moreno *et al.* (1999). But this is not relevant to the use we make of *Den*.

https://doi.org/10.1017/S1471068412000373 Published online by Cambridge University Press

The formulation of compositionality given by Theorem 2 makes even more apparent than Theorem 1 the fact that the syntactic decomposition of an expression e in the form $\mathscr{C}[e']$ has a direct semantic counterpart, in the sense that the semantics of e is determined by the semantics of its syntactic constituents $\mathscr{C}$ and e'. However, Theorems 1 and 2 are indeed of the same strength, since each of them can be easily proved from the other.

3 CRWL and rewriting: A first discussion

Before presenting let-rewriting, we find it interesting to discuss a couple of (in principle) shorter solutions to the problem of expressing non-strict call-time choice semantics by means of a simple one-step reduction relation. A first question is whether a new relation is needed at all: maybe call-time choice can be expressed by ordinary term rewriting via a suitable program transformation. The next result shows that in a certain technical sense this is not possible: due to different closedness under substitution and compositionality properties of call-time choice and term rewriting, none of them can be naturally simulated by each other.

Proposition 1
There is a program $\mathscr{P}$ for which the following two conditions hold:

(i) no term rewriting system (constructor based or not) $\mathscr{P}'$ verifies

$$\mathscr{P} \vdash_{CRWL} e \twoheadrightarrow t \text{ iff } \mathscr{P}' \vdash e \to^* t \text{ , for all } e \in Exp, t \in CTerm$$

(ii) no program $\mathscr{P}'$ verifies

$$\mathscr{P} \vdash e \to^* t \text{ iff } \mathscr{P}' \vdash_{CRWL} e \twoheadrightarrow t \text{ , for all } e \in Exp, t \in CTerm$$

Proof
The following simple program $\mathscr{P}$ suffices:

$$f(X) \to c(X,X) \qquad coin \to 0 \qquad coin \to 1$$

(*i*) We reason by contradiction. Assume that there is a term rewriting system $\mathscr{P}'$ such that: $\mathscr{P} \vdash_{CRWL} e \twoheadrightarrow t \Leftrightarrow e \to^*_{\mathscr{P}'} t$, for all e, t. Since $\mathscr{P} \vdash_{CRWL} f(X) \twoheadrightarrow c(X,X)$, we must have $f(X) \to^*_{\mathscr{P}'} c(X,X)$. Now, since $\to^*_{\mathscr{P}'}$ is closed under substitutions (Baader and Nipkow 1998), we have $f(coin) \to^*_{\mathscr{P}'} c(coin, coin)$, and therefore $f(coin) \to^*_{\mathscr{P}'} c(coin, coin) \to^*_{\mathscr{P}'} c(0,1)$. But it is easy to see that $\mathscr{P} \vdash_{CRWL} f(coin) \twoheadrightarrow c(0,1)$ does not hold.

(*ii*) Assume now that there is a program $\mathscr{P}'$ such that: $\mathscr{P} \vdash e \to^* t \Leftrightarrow \mathscr{P}' \vdash_{CRWL} e \twoheadrightarrow t$, for all e, t. Since $\mathscr{P} \vdash f(coin) \to^* c(0,1)$, we have $\mathscr{P}' \vdash_{CRWL} f(coin) \twoheadrightarrow c(0,1)$. By compositionality of call-time choice (Theorem 1), there must exist a possibly partial $t \in CTerm_\perp$ such that $\mathscr{P}' \vdash_{CRWL} coin \twoheadrightarrow t$ and $\mathscr{P}' \vdash_{CRWL} f(t) \twoheadrightarrow c(0,1)$. Now we distinguish cases on the value of t:

(a) If $t \equiv \perp$, then monotonicity of $CRWL$-derivability—see (González-Moreno *et al.* 1999) or Proposition 3 below—proves that $\mathscr{P}' \vdash_{CRWL} f(s) \twoheadrightarrow c(0,1)$ for any $s \in CTerm_\perp$, in particular $\mathscr{P}' \vdash_{CRWL} f(0) \twoheadrightarrow c(0,1)$. Then, by the assumption on $\mathscr{P}'$, it should be $\mathscr{P} \vdash f(0) \to^* c(0,1)$, but this is not true.

https://doi.org/10.1017/S1471068412000373 Published online by Cambridge University Press

$$
\begin{array}{llcll}
(\mathbf{B}^{rw}) & \mathcal{C}[e] & \rightarrowtail & \mathcal{C}[\bot] & \forall \mathcal{C} \in Cntxt, e \in Exp_{\bot} \\
(\mathbf{OR}^{rw}) & \mathcal{C}[f(t_1\theta, \ldots, t_n\theta)] & \rightarrowtail & \mathcal{C}[r\theta] & \forall \mathcal{C} \in Cntxt, f(t_1, \ldots, t_n) \rightarrow r \in \mathcal{P}, \\
 & & & & \quad \theta \in CSubst_{\bot}
\end{array}
$$

Fig. 4. A one-step reduction relation for non-strict call-time choice.

(b) If $t \equiv 0$, then $\mathscr{P}' \vdash f(0) \twoheadrightarrow c(0,1)$ as before. The cases $t \equiv 1$, $t \equiv Y$ or $t \equiv d(\overline{s})$ for a constructor d different from $0, 1$ lead to similar contradictions.

□

Note that Proposition 1 does not make any assumption about signatures: in any of (*i*) or (*ii*), no extension of the signature can lead to a simulating $\mathscr{P}'$. This does not contradict Turing completeness of term rewriting systems. Turing completeness arguments typically rely on encodings not preserving the structure of data, something not contemplated in Proposition 1.

In a second trial, requiring minimal changes over ordinary term rewriting, we impose that the substitution θ in a rewriting step must be a c-substitution, as in the rule (OR) of CRWL. This is done in the one-step rule **(OR^{rw})** in Figure 4. According to it, the step $heads(repeat(coin)) \rightarrow heads(coin : repeat(coin))$ in the introductory example of Figure 1 would not be legal anymore. However, (OR^{rw}) corresponds essentially to innermost evaluation, and is not enough to deal with non-strictness, as the following example shows.

Example 1
Consider the rules $f(X) \rightarrow 0$ and $loop \rightarrow loop$. With a non-strict semantics $f(loop)$ should be reducible to 0. But (OR^{rw}) does not allow the step $f(loop) \rightarrow 0$; only $f(loop) \rightarrow f(loop) \rightarrow \ldots$ is a valid (OR^{rw})-reduction, thus leaving $f(loop)$ semantically undefined, as would correspond to a strict semantics.

At this point, the rule (B) of CRWL is a help, since it allows to discard the evaluation of any (sub)-expression by reducing it to $\bot$. The result of this discussion is the one-step reduction relation $\rightarrowtail$ given in Figure 4.

This relation satisfies our initial goals to a partial extent, as it is not difficult to prove the following equivalence result.

Theorem 3
Let $\mathscr{P}$ be a CRWL-program, $e \in Exp_{\bot}$ and $t \in CTerm_{\bot}$. Then:

$$\mathscr{P} \vdash_{CRWL} e \twoheadrightarrow t \text{ iff } e \rightarrowtail_{\mathscr{P}}^{*} t$$

This result has an interesting reading: non-strict call-time choice can be achieved via innermost evaluation if at any step one has the possibility of reducing a subexpression to $\bot$ (then, we could speak also of *call-by-partial value*). For instance, a $\rightarrowtail$-rewrite sequence with the example of Figure 1 would be:

$$
\begin{array}{l}
heads(repeat(coin)) \rightarrowtail heads(repeat(0)) \rightarrowtail \\
heads(0 : repeat(0)) \rightarrowtail heads(0 : 0 : repeat(0)) \rightarrowtail \\
heads(0 : 0 : \bot) \rightarrowtail (0, 0)
\end{array}
$$

https://doi.org/10.1017/S1471068412000373 Published online by Cambridge University Press

This gives useful intuitions about non-strict call-time choice and can actually serve for a very easy implementation of it, but has a major drawback: in general, reduction of a subexpression e requires a *don't know* guessing between (B^{rw}) and (OR^{rw}), because at the moment of reducing e it is not known whether its value will be needed or not later on in the computation. Instead of reducing to $\bot$, let-rewriting will create a let-binding *let* $U{=}e$ *in* ..., which does not imply any guessing and keeps e for its eventual future use.

4 Rewriting with local bindings

Inspired by Ariola and Arvind (1995), Ariola *et al.* (1995), Ariola and Felleisen (1997), Maraist *et al.* (1998), Plump (1998) and Sánchez-Hernández (2004), let-rewriting extends the syntax of expressions by adding local bindings to express sharing and call-time choice. Formally the syntax for *let-expressions* is:

$$LExp \ni e ::= X \mid h(e_1,\ldots,e_n) \mid let\ X = e_1\ in\ e_2$$

where $X \in \mathcal{V}$, $h \in \Sigma^n$, and $e_1, e_2, \ldots, e_n \in LExp$. The intended behavior of $let\ X = e_1\ in\ e_2$ is that the expression e_1 will be reduced only once (at most) and then its corresponding value will be shared within e_2. For $let\ X = e_1\ in\ e_2$ we call e_1 the *definiens* of X, and e_2 the *body* of the let-expression.

The sets $FV(e)$ of *free* and $BV(e)$ *bound* variables of $e \in LExp$ are defined as:

$$\begin{array}{l} FV(X) = \{X\} \\ FV(h(\overline{e})) = \bigcup_{e_i \in \overline{e}} FV(e_i) \\ FV(let\ X = e_1\ in\ e_2) = FV(e_1) \cup (FV(e_2) \backslash \{X\}) \end{array}$$

$$\begin{array}{l} BV(X) = \emptyset \\ BV(h(\overline{e})) = \bigcup_{e_i \in \overline{e}} BV(e_i) \\ BV(let\ X = e_1\ in\ e_2) = BV(e_1) \cup BV(e_2) \cup \{X\} \end{array}$$

Note that with the given definition of $FV(let\ X = e_1\ in\ e_2)$, there are not recursive let-bindings in the language since the possible occurrences of X in e_1 are not considered bound and therefore refer to a 'different' X. For example, the expression $let\ X = f(X)\ in\ g(X)$ can be equivalently written as $let\ Y = f(X)\ in\ g(Y)$. This is similar to what is done in Ariola *et al.* (1995), Ariola and Felleisen (1997) and Maraist *et al.* (1998), but not in Launchbury (1993) and Albert *et al.* (2005). Recursive lets have their own interest but there is not a general consensus in the functional logic community about their meaning in the presence of non-determinism. We also remark that the let-bindings introduced by let-rewriting derivations to be presented in Section 4.1 are not recursive. Therefore, recursive lets are not considered in this work.

We will use the notation $let\ \overline{X = a}\ in\ e$ as a shortcut for $let\ X_1 = a_1\ in \ldots in\ let\ X_n = a_n\ in\ e$. The notion of *one-hole context* is also extended to the new syntax:

$$\mathcal{C} ::= [\,] \mid let\ X = \mathcal{C}\ in\ e \mid let\ X = e\ in\ \mathcal{C} \mid h(\ldots,\mathcal{C},\ldots)$$

By default, we will use contexts with lets from now on.

https://doi.org/10.1017/S1471068412000373 Published online by Cambridge University Press

Free variables of contexts are defined as for expressions, so that $FV(\mathscr{C}) = FV(\mathscr{C}[h])$, for any $h \in \Sigma^0$. However, the set $BV(\mathscr{C})$ of *variables bound by a context* is defined quite differently because it consists only of those let-bound variables visible from the hole of $\mathscr{C}$. Formally:

$$\begin{aligned} &BV([\,]) = \emptyset \\ &BV(h(\ldots,\mathscr{C},\ldots)) = BV(\mathscr{C}) \\ &BV(let\ X = e\ in\ \mathscr{C}) = \{X\} \cup BV(\mathscr{C}) \\ &BV(let\ X = \mathscr{C}\ in\ e) = BV(\mathscr{C}) \end{aligned}$$

As a noticeable difference with respect to López-Fraguas *et al.* (2007b), from now on we will allow to use lets in any program, so our program rules have the shape $f(p_1,\ldots,p_n) \to r$, for $f \in FS^n$, $(p_1,\ldots,p_n)$ a linear tuple of c-terms, and $r \in LExp$. Note, however, that the notion of c-term does not change: c-terms do not contain function symbols nor lets, although they can contain bound variables when put in an appropriate context as happens for example with the subexpression (X,X) in the expression $let\ X = coin\ in\ (X,X)$.

As usual with syntactical binding constructs, we assume a variable convention according to which bound variables can be consistently renamed as to ensure that the same variable symbol does not occur free and bound within an expression. Moreover, to keep simple the management of substitutions, we assume that whenever θ is applied to an expression $e \in LExp$, the necessary renamings are done in e to ensure $BV(e) \cap (dom(\theta) \cup vran(\theta)) = \emptyset$. With all these conditions the rules defining application of substitutions are simple while avoiding variable capture:

$$\begin{aligned} &X\theta = \theta(X),\ \text{for } X \in \mathscr{V} \\ &h(e_1,\ldots,e_n)\theta = h(e_1\theta,\ldots,e_n\theta),\ \text{for } h \in \Sigma^n \\ &(let\ X = e_1\ in\ e_2)\theta = let\ X = e_1\theta\ in\ e_2\theta \end{aligned}$$

The following example illustrates the use of these conventions.

$$\begin{aligned} &(let\ X = c(X)\ in\ let\ Y = z\ in\ d(X,Y))[X/c(Y)] \\ &= (let\ U = c(X)\ in\ let\ V = z\ in\ d(U,V))[X/c(Y)] \\ &= let\ U = c(c(Y))\ in\ let\ V = z\ in\ d(U,V) \end{aligned}$$

The following substitution lemma will be often a useful technical tool.

Lemma 1 (*Substitution lemma for let-expressions*)
Let $e, e' \in LExp_\perp$, $\theta \in Subst_\perp$ and $X \in \mathscr{V}$ such that $X \notin dom(\theta) \cup vran(\theta)$. Then:

$$(e[X/e'])\theta \equiv e\theta[X/e'\theta]$$

4.1 The let-rewriting relation

Let-expressions can be reduced step by step by means of the *let-rewriting* relation $\to^l$, shown in Figure 5. Rule (**Contx**) allows us to use any subexpression as redex in the derivation. (**Fapp**) performs a rewriting step in the proper sense, using a program rule. Note that only c-substitutions are allowed, to avoid copying of unevaluated expressions which would destroy sharing and call-time choice. To prevent that

https://doi.org/10.1017/S1471068412000373 Published online by Cambridge University Press

(Fapp) $f(p_1,\ldots,p_n)\theta \rightarrow^l r\theta$, if $(f(p_1,\ldots,p_n)\rightarrow r)\in\mathcal{P}, \theta\in CSubst$

(LetIn) $h(\ldots,e,\ldots)\rightarrow^l let\ X=e\ in\ h(\ldots,X,\ldots)$,
if $h\in\Sigma$, $e\equiv f(\overline{e'})$ with $f\in FS$ or $e\equiv let\ Y=e'\ in\ e''$, and X is a fresh variable

(Bind) $let\ X=t\ in\ e \rightarrow^l e[X/t]$, if $t\in CTerm$

(Elim) $let\ X=e_1\ in\ e_2\rightarrow^l e_2$, if $X\notin FV(e_2)$

(Flat) $let\ X=(let\ Y=e_1\ in\ e_2)\ in\ e_3 \rightarrow^l let\ Y=e_1\ in\ (let\ X=e_2\ in\ e_3)$
if $Y\notin FV(e_3)$

(Contx) $\mathcal{C}[e]\rightarrow^l\mathcal{C}[e']$,
if $\mathcal{C}\neq[\ ]$, $e\rightarrow^l e'$ using any of the previous rules, and in case $e\rightarrow^l e'$ is a (Fapp) step using $(f(\overline{p})\rightarrow r)\in\mathcal{P}$ and $\theta\in CSubst$, then $vran(\theta|_{\setminus var(p)})\cap BV(\mathcal{C})=\emptyset$.

Fig. 5. Rules of the let-rewriting relation $\rightarrow^l$.

the restriction of (**Fapp**) to total c-substitutions results in a strict semantics, we also provide the rule (**LetIn**) that suspends the evaluation of a subexpression by introducing a let-binding. If its value is needed later on, its evaluation can be performed by some (**Contx**) steps and the result propagated by (**Bind**). This latter rule is safe with respect to call-time choice because it only propagates c-terms, that is, either completely defined values (without any bound variable) or partially computed values with some suspension (bound variable) on it, which will be safely managed by the calculus. On the other hand, if the bound variable disappears from the body of the let-binding during evaluation, rule (**Elim**) can be used for garbage collection. This rule is useful to ensure that normal forms corresponding to values are c-terms. Finally, (**Flat**) is needed for flattening nested lets; otherwise some reductions could become wrongly blocked or forced to diverge. Consider for example the program $\{loop\rightarrow loop, g(s(X))\rightarrow 1\}$ and the expression $g(s(loop))$, which can be reduced to $let\ X=(let\ Y=loop\ in\ s(Y))\ in\ g(X)$ by applying (**LetIn**) twice. Then, without (**Flat**) we could only perform reduction steps on $loop$, thus diverging; by using (**Flat**), we can obtain $let\ Y=loop\ in\ let\ X=s(Y)\ in\ g(X)$, which can be finally reduced to 1 by applying (**Bind**), (**Fapp**) and (**Elim**). The condition $Y\notin FV(e_3)$ in (**Flat**) could be dropped by the variable convention, but we have included it to keep the rules independent of the convention. Quite different is the case of (**Elim**), where the condition $X\notin FV(e_2)$ is indeed necessary.

Note that, in contrast to CRWL or the relation $\rightarrowtail$ in Section 3, let-rewriting does not need to use the semantic value $\bot$, which does not appear in programs nor in computations.

Example 2

Consider the program of Figure 1. We can perform the following let-rewriting derivation for the expression $heads(repeat(coin))$, where in each step the

https://doi.org/10.1017/S1471068412000373 Published online by Cambridge University Press

corresponding redex has been underlined for the sake of readability.

$$
\begin{array}{ll}
\underline{heads(repeat(coin))} & (LetIn)\\
\rightarrow^{l} let\ X = \underline{repeat(coin)}\ in\ heads(X) & (LetIn)\\
\rightarrow^{l} let\ X = \underline{(let\ Y = coin\ in\ repeat(Y))}\ in\ heads(X) & (Flat)\\
\rightarrow^{l} let\ Y = coin\ in\ let\ X = \underline{repeat(Y)}\ in\ heads(X) & (Fapp)\\
\rightarrow^{l} let\ Y = coin\ in\ let\ X = \underline{Y : repeat(Y)}\ in\ heads(X) & (LetIn)\\
\rightarrow^{l} let\ Y = coin\ in\ \underline{let\ X = (let\ Z = repeat(Y)\ in\ Y : Z)}\ in\ heads(X) & (Flat)\\
\rightarrow^{l} let\ Y = coin\ in\ \underline{let\ Z = repeat(Y)\ in\ let\ X = Y : Z\ in\ heads(X)} & (Bind)\\
\rightarrow^{l} let\ Y = coin\ in\ let\ Z = \underline{repeat(Y)}\ in\ heads(Y : Z) & (Fapp)\\
\rightarrow^{l} let\ Y = coin\ in\ let\ Z = \underline{Y : repeat(Y)}\ in\ heads(Y : Z) & (LetIn)\\
\rightarrow^{l} let\ Y = coin\ in\ \underline{let\ Z = (let\ U = repeat(Y)\ in\ Y : U)}\ in\ heads(Y : Z) & (Flat)\\
\rightarrow^{l} let\ Y = coin\ in\ let\ U = repeat(Y)\ in\ \underline{let\ Z = Y : U\ in\ heads(Y : Z)} & (Bind)\\
\rightarrow^{l} let\ Y = coin\ in\ let\ U = repeat(Y)\ in\ \underline{heads(Y : Y : U)} & (Fapp)\\
\rightarrow^{l} let\ Y = coin\ in\ \underline{let\ U = repeat(Y)\ in\ (Y, Y)} & (Elim)\\
\rightarrow^{l} let\ Y = \underline{coin}\ in\ (Y, Y) & (Fapp)\\
\rightarrow^{l} \underline{let\ Y = 0\ in\ (Y, Y)} & (Bind)\\
\rightarrow^{l} (0, 0) &
\end{array}
$$

Note that there is not a unique $\rightarrow^{l}$-reduction leading to $(0,0)$. The definition of $\rightarrow^{l}$, like traditional term rewriting, does not prescribe any particular strategy. The definition of on-demand evaluation strategies for let-rewriting is out of the scope of this paper, and is only informally discussed in Section 6.2.

We study now some properties of let-rewriting that have intrinsic interest and will be useful when establishing a relation to CRWL in next sections.

The same example used in Proposition 1 to show that CRWL is not closed under general substitutions also shows that the same applies to let-rewriting. However, let-rewriting is closed under c-substitutions, as expected in a semantics for call-time choice.

Lemma 2 (*Closedness under CSubst of let-rewriting*)
For any $e, e' \in LExp$, $\theta \in CSubst$ we have that $e \rightarrow^{l\,n} e'$ implies $e\theta \rightarrow^{l\,n} e'\theta$.

Another interesting matter is the question of what happens in let-rewriting derivations in which the rule (Fapp) is not used—and as a consequence, the program is ignored.

Definition 3 (*The* $\rightarrow^{lnf}$ *relation*)
The relation $\rightarrow^{lnf}$ is defined by the rules of Figure 5 except (Fapp). As a consequence, for any program $\rightarrow^{lnf} \subseteq \rightarrow^{l}$.

We can think about any let-expression e as an expression from Exp in which some additional sharing information has been encoded using the let-construction. When we avoid the use of the rule (Fapp) in derivations, we do not make progress in the evaluation of the implicit let-less expression corresponding to e, but we change the sharing-enriched representation of that expression in the let-rewriting syntax. Following terminology from term graph rewriting—as in fact a let-expression is a textual representation of a term graph—all the rules of let-rewriting except (Fapp) move between two isomorphic term graphs (Plump 1998, 2001). The $\rightarrow^{lnf}$ relation will be used to reason about these kinds of derivations.

https://doi.org/10.1017/S1471068412000373 Published online by Cambridge University Press

The first interesting property of $\rightarrow^{lnf}$ is that it is a terminating relation.

Proposition 2 (*Termination of* $\rightarrow^{lnf}$)
For any program $\mathscr{P}$, the relation $\rightarrow^{lnf}$ is terminating. As a consequence, every $e \in LExp$ has at least one $\rightarrow^{lnf}$-normal form e' (written as $e \downarrow^{lnf} e'$).

However, for non-trivial signatures the relation $\rightarrow^{lnf}$ is not confluent (hence the relation $\rightarrow^{l}$ is not confluent either).

Example 3
Consider a signature such that $f, g \in FS^0, c \in CS^2$ and $f \not\equiv g$. Then, $c(f,g) \rightarrow^{lnf^*} let\ X = f\ in\ let\ Y = g\ in\ c(X,Y)$ and $c(f,g) \rightarrow^{lnf^*} let\ Y = g\ in\ let\ X = f\ in\ c(X,Y)$, but these expressions do not have a common reduct.

The lack of confluence of let-rewriting is alleviated by a strong semantic property of $\rightarrow^{lnf}$ which, combined with the adequacy to CRWL of let-rewriting that we will see below, may be used as a substitute for confluence in some situations. These questions will be treated in detail in Section 4.3.1.

The next result characterizes $\rightarrow^{lnf}$-normal forms. What we do in $\rightarrow^{lnf}$ derivations is exposing the computed part of e—its outer constructor part—concentrating it in the body of the resulting let, that is, the part which is not a function application whose evaluation is pending. This is why we call it *'Peeling lemma'*.

Lemma 3 (*Peeling lemma*)
For any $e, e' \in LExp$, if $e \downarrow^{lnf} e'$ then e' has the shape $e' \equiv let\ \overline{X = f(\bar{t})}\ in\ e''$ such that $e'' \in \mathscr{V}$ or $e'' \equiv h(\bar{t'})$ with $h \in \Sigma$, $\bar{f} \subseteq FS$ and $\bar{t}, \bar{t'} \subseteq CTerm$.
Moreover, if $e \equiv h(e_1, \ldots, e_n)$ with $h \in \Sigma$, then

$$e \equiv h(e_1, \ldots, e_n) \rightarrow^{lnf^*} let\ \overline{X = f(\bar{t})}\ in\ h(t_1, \ldots, t_n) \equiv e'$$

under the above conditions, and verifying also that $t_i \equiv e_i$ whenever $e_i \in CTerm$.

The next property of $\rightarrow^{l}$ and $\rightarrow^{lnf}$ uses the notion of *shell* $|e|$ of an expression e, that is the partial c-term corresponding to the outer constructor part of e. More precisely:

Definition 4 (*Shell of a let-expression*)

$$\begin{array}{lll} |X| & = X & \text{for } X \in \mathscr{V} \\ |c(e_1, \ldots, e_n)| & = c(|e_1|, \ldots, |e_n|) & \text{for } c \in CS \\ |f(e_1, \ldots, e_n)| & = \bot & \text{for } f \in FS \\ |let\ X = e_1\ in\ e_2| & = |e_2|[X/|e_1|] & \end{array}$$

Note that in the case of a let-rooted expression, the information contained in the binding is taken into account for building up the shell of the whole expression: for instance $|c(let\ X = 2\ in\ s(X))| = c(s(2))$.

During a computation, the evolution of shells reflects the progress toward a value. The next result shows that shells never decrease. Moreover, only (Fapp) may change shells. As discussed above, 'peeling' steps (i.e. $\rightarrow^{lnf}$- steps) just modify the representation of the implicit term graph corresponding to a let-expression; thus, they preserve the shell.

https://doi.org/10.1017/S1471068412000373 Published online by Cambridge University Press

Lemma 4 (*Growing of shells*)
(i) $e \rightarrow^{l^*} e'$ implies $|e| \sqsubseteq |e'|$, for any $e, e' \in LExp$
(ii) $e \rightarrow^{lnf^*} e'$ implies $|e| \equiv |e'|$, for any $e, e' \in LExp$

4.2 The $CRWL_{let}$ logic

In this section we extend the CRWL logic to deal with let-expressions, obtaining an enlarged framework that will be useful as a bridge to establish the connection between CRWL and let-rewriting.

As in the CRWL framework, we consider partial let-expressions $e \in LExp_\bot$, defined in the natural way. The approximation order $\sqsubseteq$ is also extended to $LExp_\bot$ but now using the notion of context for let-expressions, which in particular implies that $let\ X = e_1\ in\ e_2 \sqsubseteq let\ X = e_1'\ in\ e_2'$ iff $e_1 \sqsubseteq e_1'$ and $e_2 \sqsubseteq e_2'$. The CRWL_{let} logic results of adding the following rule (**Let**) to the CRWL logic of Section 2.2:

$$(\textbf{Let})\ \frac{e_1 \twoheadrightarrow t_1 \quad e_2[X/t_1] \twoheadrightarrow t}{let\ X = e_1\ in\ e_2 \twoheadrightarrow t}$$

We write $\mathscr{P} \vdash_{CRWL_{let}} e \twoheadrightarrow t$ if $e \twoheadrightarrow t$ is derivable in the CRWL_{let}-calculus using the program $\mathscr{P}$. In many occasions, we will omit $\mathscr{P}$.

Definition 5 (CRWL_{let}*-denotation*)
Given a program $\mathscr{P}$, the $CRWL_{let}$*-denotation* of $e \in LExp_\bot$ is defined as:

$$[\![e]\!]^{\mathscr{P}}_{CRWL_{let}} = \{t \in CTerm_\bot \mid \mathscr{P} \vdash_{CRWL_{let}} e \twoheadrightarrow t\}$$

We will omit the sub(super)-scripts when they are clear by the context.

There is an obvious relation between CRWL and CRWL_{let} for programs and expressions without lets:

Theorem 4 ($CRWL$ *versus* $CRWL_{let}$)
For any program $\mathscr{P}$ without lets, and any $e \in Exp_\bot$:

$$[\![e]\!]^{\mathscr{P}}_{CRWL} = [\![e]\!]^{\mathscr{P}}_{CRWL_{let}}$$

This result allows us to skip the mention to CRWL or CRWL_{let} when referring to the denotation $[\![e]\!]$ of an expression: if some let-binding occurs in e—or in the program with respect to which the denotation is considered—then $[\![e]\!]$ can be interpreted only as $[\![e]\!]_{CRWL_{let}}$; otherwise, both denotations coincide.

The CRWL_{let} logic inherits from CRWL a number of useful properties.

Lemma 5
For any program $e \in LExp_\bot$, $t, t' \in CTerm_\bot$:

(i) $t \twoheadrightarrow t'$ iff $t' \sqsubseteq t$.
(ii) $|e| \in [\![e]\!]$.
(iii) $[\![e]\!] \subseteq (|e|\uparrow)\downarrow$, where for a given $E \subseteq LExp_\bot$ its upward closure is $E\uparrow = \{e' \in LExp_\bot \mid \exists e \in E.\ e \sqsubseteq e'\}$, its downward closure is $E\downarrow = \{e' \in LExp_\bot \mid \exists e \in E.\ e' \sqsubseteq e\}$, and those operators are overloaded for let-expressions as $e\uparrow = \{e\}\uparrow$ and $e\downarrow = \{e\}\downarrow$.

https://doi.org/10.1017/S1471068412000373 Published online by Cambridge University Press

The first part of the previous result shows that c-terms can only be reduced to smaller c-terms. The other parts express that the shell of an expression represents 'stable' information contained in the expression in a similar way to Lemma 4, as the shell is in the denotation by (ii), and everything in the denotation comes from refining it by (iii).

The following results are adaptations to CRWL_{let} of properties known for CRWL (González-Moreno *et al.* 1999; Vado-Vírseda 2002). The first one states that if we can compute a value for an expression then from greater expressions we can reach smaller values. The second one says that CRWL_{let}-derivability is closed for partial c-substitutions.

Proposition 3 (*Polarity of* $CRWL_{let}$)
For any $e, e' \in LExp_\bot$, $t, t' \in CTerm_\bot$, if $e \sqsubseteq e'$ and $t' \sqsubseteq t$ then $e \twoheadrightarrow t$ implies $e' \twoheadrightarrow t'$ with a proof of the same size or smaller—where the size of a CRWL_{let}-proof is measured as the number of rules of the calculus used in the proof.

Proposition 4 (*Closedness under c-substitutions of* $CRWL_{let}$)
For any $e \in LExp_\bot$, $t \in CTerm_\bot$, $\theta \in CSubst_\bot$, $t \in [\![e]\!]$ implies $t\theta \in [\![e\theta]\!]$.

Compositionality is a more delicate issue. Theorem 1 does not hold for CRWL_{let}, as shown by the following example: consider the program $\{f(0) \to 1\}$, the expression $e \equiv f(X)$ and the context $\mathscr{C} \equiv let\ X = 0\ in\ []$. $\mathscr{C}[e]$ can produce the value 1. However, $f(X)$ can only be reduced to $\bot$, and $\mathscr{C}[\bot]$ cannot reach the value 1. The point in that example is that the subexpression e needs some information from the context to produce a value that is then used by the context to compute the value for the whole expression $\mathscr{C}[e]$. This information may only be the definientia of some variables of e that get bound when put in $\mathscr{C}$; with this idea in mind, we can state the following weak compositionality result for CRWL_{let}.

Theorem 5 (*Weak Compositionality of* $CRWL_{let}$)
For any $\mathscr{C} \in Cntxt$, $e \in LExp_\bot$

$$[\![\mathscr{C}[e]]\!] = \bigcup_{t \in [\![e]\!]} [\![\mathscr{C}[t]]\!] \qquad \text{if } BV(\mathscr{C}) \cap FV(e) = \emptyset$$

As a consequence, $[\![let\ X = e_1\ in\ e_2]\!] = \bigcup_{t_1 \in [\![e_1]\!]} [\![e_2[X/t_1]]\!]$.

In spite of not being a fully general compositionality result, Theorem 5 can be used to prove new properties of CRWL_{let}, like the following monotonicity property related to substitutions, that will be used later on. It is formulated for the partial order $\sqsubseteq$ over $LSubst_\bot$ (defined naturally as it happened for $Susbt_\bot$) and the preorder $\trianglelefteq$ over $LSubst_\bot$, defined by $\sigma \trianglelefteq \sigma'$ iff $\forall X \in \mathscr{V}, [\![\sigma(X)]\!] \subseteq [\![\sigma'(X)]\!]$.

Proposition 5 (*Monotonicity for substitutions of* $CRWL_{let}$)
If $\sigma \sqsubseteq \sigma'$ or $\sigma \trianglelefteq \sigma'$ then $[\![e\sigma]\!] \subseteq [\![e\sigma']\!]$, for any $e \in LExp_\bot$ and $\sigma, \sigma' \in LSubst_\bot$.

The limitations of Theorem 5 make us yearn for another semantic notion for let-expressions with a better compositional behavior. We have already seen that the problem with CRWL_{let} is the possible loss of definientia when extracting an expression from its context. But in fact what bound variables need is access to the

https://doi.org/10.1017/S1471068412000373 Published online by Cambridge University Press

values of their corresponding definientia, as it is done in the rule (Let) where the value of the definiens is transmitted to the body of the let-binding by applying a c-substitution replacing the bound variable by that value. With these ideas in mind, we define the stronger notion of *hyperdenotation* (sometimes we say *hypersemantics*), which gives a more active role to variables in expressions: in contrast to the denotation of an expression e, which is a set of c-terms, its hyperdenotation $[\![e]\!]$ is a function mapping c-substitutions to denotations, i.e., to sets of c-terms.

Definition 6 (*Hyperdenotation*)
The hyperdenotation of an expression $e \in LExp_\bot$ under a program $\mathscr{P}$ is a function $[\![e]\!]^{\mathscr{P}} : CSubst_\bot \to Den$ defined by $[\![e]\!]^{\mathscr{P}}\ \theta = [\![e\theta]\!]^{\mathscr{P}}$.

As usual, in most cases we will omit the mention to $\mathscr{P}$. We will use sometimes HD as an alias for $CSubst_\bot \to Den$, i.e., for the kind of objects that are hyperdenotations of expressions.

The notion of hyperdenotation is strictly more powerful than the notion of CRWL_{let} denotation. Equality of hyperdenotations implies equality of denotations—because if $[\![e]\!] = [\![e']\!]$ then $[\![e]\!] = [\![e\epsilon]\!] = [\![e]\!]\epsilon = [\![e']\!]\epsilon = [\![e'\epsilon]\!] = [\![e']\!]$—but the opposite does not hold: consider the program $\{f(0) \to 1\}$ and the expressions $f(X)$ and $\bot$; they have the same denotation (the set $\{\bot\}$) but different hyperdenotations, as $[\![\bot]\!][X/0] \not\ni 1 \in [\![f(X)]\!][X/0]$. Hypersemantics are useful to characterize the meaning of expressions present in a context in which some of its variables may get bound, like in the body of a let-binding or in the right-hand side of a program rule. Therefore, these are useful to reason about expressions put in arbitrary contexts, in which let-bindings may freely appear.

Most remarkably, hyperdenotations allow us to recover strong compositionality results for let-expressions similar to Theorems 1 and 2. We find it more intuitive to start the analog to the latter. Semantics of contexts were defined as denotation transformers (Definition 2). Analogously, the hypersemantics $[\![\mathscr{C}]\!]$ of a context $\mathscr{C}$ is a hyperdenotation transformer defined as follows.

Definition 7 (*Hypersemantics of a context*)
Given $\mathscr{C} \in Cntxt$, its hyperdenotation is a function $[\![\mathscr{C}]\!] : HD \to HD$ defined by induction over the structure of $\mathscr{C}$ as follows:

- $[\![[\,]]\!]\varphi\theta = \varphi\theta$
- $[\![h(e_1, \ldots, \mathscr{C}, \ldots, e_n)]\!]\varphi\theta = \bigcup_{t \in [\![\mathscr{C}]\!]\varphi\theta} [\![h(e_1\theta, \ldots, t, \ldots, e_n\theta)]\!]$
- $[\![let\ X = \mathscr{C}\ in\ e]\!]\varphi\theta = \bigcup_{t \in [\![\mathscr{C}]\!]\varphi\theta} [\![let\ X = t\ in\ e\theta]\!]$
- $[\![let\ X = e\ in\ \mathscr{C}]\!]\varphi\theta = \bigcup_{t \in [\![e]\!]\theta} [\![\mathscr{C}]\!]\varphi(\theta[X/t])$

With this notion, our first version of strong compositionality for hypersemantics looks like Theorem 2.

Theorem 6 (*Compositionality of hypersemantics*)
For all $\mathscr{C} \in Cntxt$, $e \in LExp_\bot$

$$[\![\mathscr{C}[e]]\!] = [\![\mathscr{C}]\!]\,[\![e]\!]$$

https://doi.org/10.1017/S1471068412000373 Published online by Cambridge University Press

As a consequence: $[\![e]\!] = [\![e']\!] \Leftrightarrow \forall \mathscr{C} \in Cntxt.[\![\mathscr{C}[e]]\!] = [\![\mathscr{C}[e']]\!]$.

This result implies that, in any context, we can replace any subexpression by another one having the same hypersemantics (and therefore also the same semantics) without changing the hypersemantics (hence the semantics) of the global expression.

In Theorems 2 and 6 the role of call-time choice is hidden in the definition of semantics and hypersemantics of a context, respectively. To obtain a version of strong compositionalty of hypersemantics closer to Theorems 1 and 5, we need some more notions and notations about hyperdenotations or, more generally, about functions in HD. Since they are set-valued functions, many usual set operations and relations can be lifted naturally in a pointwise manner to HD. The precise definitions become indeed clearer if we give them for general sets, abstracting away the details about HD. We also introduce some notions about decomposing set-valued functions that will be useful for hyperdenotations. We freely use λ-notation to write down a function in the mathematical sense; we may write $\lambda x \in A$ to indicate its domain A, if it is not clear by the context.

Definition 8 (*Operations and relations for set-valued functions*)
Let A, B be two sets, $\mathscr{F}$ the set of functions $A \to \mathscr{P}(B)$, and $f, g \in \mathscr{F}$. Then:

(i) The *hyperunion* of f, g is defined as $f \Cup g = \lambda x \in A.f(x) \cup g(x)$.
(ii) More generally, the *hyperunion of a family* $\mathscr{I} \subseteq \mathscr{F}$, written indistinctly as $\Cup \mathscr{I}$ or $\Cup_{f \in \mathscr{I}} f$, is defined as

$$\Cup \mathscr{I} \equiv \Cup_{f \in \mathscr{I}} f \ =_{def} \ \lambda x \in A. \bigcup_{f \in \mathscr{I}} f(x)$$

Note that $f \Cup g = \Cup \{f, g\}$.
(iii) We say that f is *hyperincluded* in g, written $f \Subset g$, iff $\forall x \in A.f(x) \subseteq g(x)$.
(iv) A *decomposition* of f is any $\mathscr{I} \subseteq \mathscr{F}$ such that $\Cup \mathscr{I} = f$.
(v) The *elemental decomposition* of f is the following set of functions of $\mathscr{F}$:

$$\Delta f = \{\lambda x \in A. \begin{cases} \{b\} \text{ if } x = a \\ \emptyset \text{ otherwise} \end{cases} \mid a \in A, b \in f(a)\}$$

Or, using the abbreviation $\hat{\lambda} a.\{b\}$ as a shorthand for $\lambda x. \begin{cases} \{b\} \text{ if } x = a \\ \emptyset \text{ otherwise} \end{cases}$,

$$\Delta f = \{\hat{\lambda} a.\{b\} \mid a \in A, b \in f(a)\}$$

Decompositions are used to split set-valued functions into smaller pieces; elemental decompositions do it with minimal ones. For instance, if $f : \{a, b\} \to \mathscr{P}(\{0, 1, 2\})$ is given by $f(a) = \{0, 2\}$ and $f(b) = \{1, 2\}$, then $\Delta f = \{\hat{\lambda} a.\{0\}, \hat{\lambda} a.\{2\}, \hat{\lambda} b.\{1\}, \hat{\lambda} b.\{2\}\}$.

Hyperinclusion and hyperunion share many properties of standard set inclusion and union. Some of them are collected in the next result, which also refer to decompositions:

Proposition 6
Consider two sets A, B, and let $\mathscr{F}$ be the set of functions $A \to \mathscr{P}(B)$. Then:

https://doi.org/10.1017/S1471068412000373 Published online by Cambridge University Press

(i) $\Subset$ is indeed a partial order on $\mathscr{F}$, and Δf is indeed a decomposition of $f \in \mathscr{F}$, i.e., $\bigcup\!\!\bigcup (\Delta f) = f$.

(ii) Monotonicity of hyperunion with respect to inclusion: for any $\mathscr{I}_1, \mathscr{I}_2 \subseteq \mathscr{F}$

$$\mathscr{I}_1 \subseteq \mathscr{I}_2 \text{ implies } \bigcup\!\!\bigcup \mathscr{I}_1 \Subset \bigcup\!\!\bigcup \mathscr{I}_2$$

(iii) Distribution of unions: for any $\mathscr{I}_1, \mathscr{I}_2 \subseteq \mathscr{F}$

$$\bigcup\!\!\bigcup (\mathscr{I}_1 \cup \mathscr{I}_2) = \left(\bigcup\!\!\bigcup \mathscr{I}_1\right) \cup\!\!\cup \left(\bigcup\!\!\bigcup \mathscr{I}_2\right)$$

(iv) Monotonicity of decomposition with respect to hyperinclusion: for any $f_1, f_2 \in \mathscr{F}$

$$f_1 \Subset f_2 \text{ implies } \Delta f_1 \subseteq \Delta f_2$$

We will apply all these notions, notations and properties to the case when $A \equiv CSubst_\perp$ and $B \equiv CTerm_\perp$ (i.e. $\mathscr{P}(B) \equiv Den$ and therefore $\mathscr{F} \equiv HD$). Therefore, we can speak of the hyperunion of two hyperdenotations, or of a family of them, we can elementarily decompose a hyperdenotation, etc.

Proposition 7 (*Distributivity under context of hypersemantics unions*)

$$[\![\mathscr{C}]\!]\left(\bigcup\!\!\bigcup H\right) = \bigcup_{\varphi \in H} [\![\mathscr{C}]\!]\varphi$$

With this result, we can easily prove our desired new version of a strong compositionality result for hypersemantics, with a style closer to the formulations of Theorems 1 and 5. This new form of compositionality will be used in Section 5.1 for building a straightforward proof of the adequacy of a transformation that otherwise becomes highly involved by using other techniques.

Theorem 7 (*Compositionality of hypersemantics, version 2*)
For any $\mathscr{C} \in Cntxt$, $e \in LExp_\perp$:

$$[\![\mathscr{C}[e]]\!] = \bigcup_{\varphi \in H} [\![\mathscr{C}]\!]\varphi, \text{ for any decomposition } H \text{ of } [\![e]\!]$$

In particular: $[\![\mathscr{C}[e]]\!] = \bigcup_{\varphi \in \Delta[\![e]\!]} [\![\mathscr{C}]\!]\varphi$.

As a consequence: $[\![e]\!] = [\![e']\!] \Leftrightarrow \forall \mathscr{C} \in Cntxt. [\![\mathscr{C}[e]]\!] = [\![\mathscr{C}[e']]\!]$.

Proof

$$\begin{aligned}
&[\![\mathscr{C}[e]]\!] = [\![\mathscr{C}]\!][\![e]\!] && \text{by compositionality } v.1 \text{ (Theorem 6)}\\
&= [\![\mathscr{C}]\!]\left(\bigcup\!\!\bigcup H\right) && \text{by definition of decomposition (Definition 8(iv))}\\
&= \bigcup_{\varphi \in H} [\![\mathscr{C}]\!]\varphi && \text{by distributivity (Proposition 7)}
\end{aligned}$$

□

As happened with Theorems 1 and 2 with respect to denotations, Theorems 6 and 7 are different aspects of the same property, which shows that the hypersemantics of a whole let-expression depends only on the hypersemantics of its constituents;

https://doi.org/10.1017/S1471068412000373 Published online by Cambridge University Press

it also allows us to interchange in a context any pair of expressions with the same hypersemantics. This is reflected on the fact that we have attached $[\![e]\!] = [\![e']\!] \Leftrightarrow \forall \mathscr{C} \in Cntxt.[\![\mathscr{C}[e]]\!] = [\![\mathscr{C}[e']]\!]$ as a trivial consequence both in Theorems 6 and 7. Moreover, Theorem 6 can also be proved by a combination of Theorem 7 and Propositions 6(i) and 7, in a similar way to the proof for Theorem 7 above.

$$\begin{aligned}
[\![\mathscr{C}[e]]\!] &= \bigcup_{\varphi \in H} [\![\mathscr{C}]\!]\varphi && \text{by compositionality } v.2 \text{ (Theorem 7)} \\
&= [\![\mathscr{C}]\!]\left(\bigcup (\Delta[\![e]\!])\right) && \text{by distributivity (Proposition 7)} \\
&= [\![\mathscr{C}]\!][\![e]\!] && \text{because } \Delta[\![e]\!] \text{ decomposes } e \text{ (Proposition 6(i))}
\end{aligned}$$

Therefore, Theorems 6 and 7 are results with the same strength, two sides of the same coin that will be useful tools for reasoning with hypersemantics.

To conclude, we present the following monotonicity property under contexts of hypersemantics, which will be useful in the next section.

Lemma 6 (*Monotonicity under contexts of hypersemantics*)
For any $\mathscr{C} \in Cntxt, \varphi_1, \varphi_2 \in HD$:

$$\varphi_1 \Subset \varphi_2 \text{ implies that } [\![\mathscr{C}]\!]\varphi_1 \Subset [\![\mathscr{C}]\!]\varphi_2$$

Proof
Assume $\varphi_1 \Subset \varphi_2$. Then:

$$\begin{aligned}
[\![\mathscr{C}]\!]\varphi_1 &= [\![\mathscr{C}]\!](\bigcup (\Delta\varphi_1)) && \text{by Proposition 6(i)} \\
&= [\![\mathscr{C}]\!]\left(\bigcup \{\hat{\lambda}\mu.\{t\} \mid \mu \in CSubst_\perp, t \in \varphi_1\mu\}\right) && \text{by definition of } \Delta \\
&\Subset [\![\mathscr{C}]\!]\left(\bigcup \{\hat{\lambda}\mu.\{t\} \mid \mu \in CSubst_\perp, t \in \varphi_2\mu\}\right) && \text{by Proposition 6(ii)} \\
&= [\![\mathscr{C}]\!]\left(\bigcup (\Delta\varphi_2)\right) && \text{by definition of } \Delta \\
&= [\![\mathscr{C}]\!]\varphi_2 && \text{by Proposition 6(i)}
\end{aligned}$$

□

We have now the tools needed to tackle the task of formally relating CRWL and let-rewriting.

4.3 Equivalence of let-rewriting to CRWL and $CRWL_{let}$

In this section, we prove soundness and completeness results of let-rewriting with respect to CRWL_{let} and CRWL.

4.3.1 Soundness

Concerning soundness we want to prove that $\rightarrow^l$-steps do not create new CRWL-semantic values. More precisely:

Theorem 8 (*Soundness of let-rewriting*)
For all $e, e' \in LExp$, if $e \rightarrow^{l^*} e'$ then $[\![e']\!] \subseteq [\![e]\!]$.

https://doi.org/10.1017/S1471068412000373 Published online by Cambridge University Press

Note that because of non-determinism $\subseteq$ cannot be replaced by $=$ in this theorem. For example, with the program $\mathscr{P} = \{coin \to 0, coin \to 1\}$ we can perform the step $coin \to^l 0$, for which $[\![0]\!] = \{0, \bot\}$, $[\![coin]\!] = \{0, 1, \bot\}$.

It is interesting to explain why a direct reasoning with denotations fails to prove Theorem 8.

A proof could proceed straightforwardly by a case distinction on the rules for $\to^l$ to prove the soundness of a single $\to^l$ step. The problem is that the case for a (Contx) step would need the following monotonicity property under context of CRWL_{let} denotations:

$$[\![e]\!] \subseteq [\![e']\!] \text{ implies } [\![\mathscr{C}[e]]\!] \subseteq [\![\mathscr{C}[e']]\!]$$

Unfortunately, the property is false, for the same reasons as already explained the weakness of Theorem 5: the possible capture of variables when switching from e to $\mathscr{C}[e]$.

Counterexample 1
Consider the program $\mathscr{P} = \{f(0) \to 1\}$. We have $[\![f(X)]\!] = \{\bot\} \subseteq \{\bot, 0\} = [\![0]\!]$, but when these expressions are placed within the context $let\ X = 0\ in\ [\ ]$ we obtain $[\![let\ X = 0\ in\ f(X)]\!] = \{\bot, 1\} \not\subseteq \{\bot, 0\} = [\![let\ X = 0\ in\ 0]\!]$.

The good thing is that we can overcome these problems by using hypersemantics. Theorem 8 will be indeed an easy corollary of the following generalization to hypersemantics.

Theorem 9 (*Hyper-Soundness of let-rewriting*)
For all $e, e' \in LExp$, if $e \to^{l^*} e'$ then $[\![\![e']\!]\!] \Subset [\![\![e]\!]\!]$.

And, in order to prove this generalized theorem, we also devise a generalization of the faulty monotonicity property of CRWL_{let} denotations mentioned above. That generalization is an easy consequence of the compositionality and monotonicity under contexts of hypersemantics.

Lemma 7
For all $e, e' \in LExp_\bot$ and $\mathscr{C} \in Cntxt$, if $[\![\![e]\!]\!] \Subset [\![\![e']\!]\!]$ then $[\![\![\mathscr{C}[e]]\!]\!] \Subset [\![\![\mathscr{C}[e']]\!]\!]$.

Proof

$$\begin{array}{ll} [\![\![\mathscr{C}[e]]\!]\!] = [\![\![\mathscr{C}]\!]\!]\,[\![\![e]\!]\!] & \text{by Theorem 6} \\ \Subset [\![\![\mathscr{C}]\!]\!]\,[\![\![e']\!]\!] & \text{by Lemma 6, as } [\![\![e]\!]\!] \Subset [\![\![e']\!]\!] \\ = [\![\![\mathscr{C}[e']]\!]\!] & \text{by Theorem 6} \end{array}$$

□

With the help of Lemma 7, we can now prove Theorem 9 by a simple case distinction on the rules for $\to^l$ and a trivial induction on the length of the derivation. Now, Theorem 8 follows as an easy consequence.

https://doi.org/10.1017/S1471068412000373 Published online by Cambridge University Press

Proof for Theorem 8
Assume $e \rightarrow^{l^*} e'$. By Theorem 9 we have $[\![e']\!] \Subset [\![e]\!]$, and therefore $[\![e'\theta]\!] \subseteq [\![e\theta]\!]$ for every $\theta \in CSubst_\perp$. Choosing $\theta = \epsilon$ (the empty substitution) we obtain $[\![e']\!] \subseteq [\![e]\!]$ as desired. □

The moral then is that *when reasoning about the semantics of expressions and programs with lets it is usually better to lift the problem to the hypersemantic world*, and then particularize to semantics the obtained result. This is done, for instance, in the following result.

Proposition 8 (*The* $\rightarrow^{lnf}$ *relation preserves hyperdenotation*)
For all $e, e' \in LExp$, if $e \rightarrow^{lnf^*} e'$ then $[\![e]\!] = [\![e']\!]$—and therefore $[\![e]\!] = [\![e']\!]$.

This result mirrors semantically the fact that $\rightarrow^{lnf}$ performs transitions between let-expressions corresponding to the same implicit term graph. Proposition 8 in some sense lessens the importance of the lack of confluence for the $\rightarrow^{lnf}$ relation seen in Section 4.1. Preservation of hyperdenotation may be used in some situations as a substitute for confluence, specially taking into account that let-rewriting and CRWL_{let} enjoy a really strong equivalence, as it is shown in this section.

Finally, we combine the previous results in order to get our main result concerning the soundness of let-rewriting with respect to the CRWL_{let} calculus:

Theorem 10 (*Soundness of let-rewriting*)
For any program $\mathscr{P}$ and $e \in LExp$ we have:

(i) $e \rightarrow^{l^*} e'$ implies $\mathscr{P} \vdash_{CRWL_{let}} e \twoheadrightarrow |e'|$, for any $e' \in LExp$.
(ii) $e \rightarrow^{l^*} t$ implies $\mathscr{P} \vdash_{CRWL_{let}} e \twoheadrightarrow t$, for any $t \in CTerm$.

Furthermore, if neither $\mathscr{P}$ nor e have lets then we also have:

(iii) $e \rightarrow^{l^*} e'$ implies $\mathscr{P} \vdash_{CRWL} e \twoheadrightarrow |e'|$, for any $e' \in LExp$.
(iv) $e \rightarrow^{l^*} t$ implies $\mathscr{P} \vdash_{CRWL} e \twoheadrightarrow t$, for any $t \in CTerm$.

Proof
(i) Assume $e \rightarrow^{l^*} e'$. Then, by Theorem 8 we have $[\![e']\!]_{CRWL_{let}} \subseteq [\![e]\!]_{CRWL_{let}}$. Since $|e'| \in [\![e']\!]_{CRWL_{let}}$ by Lemma 5, we get $|e'| \in [\![e]\!]_{CRWL_{let}}$, which means $e \twoheadrightarrow |e'|$.
(ii) Trivial by (i), since $|t| = t$ for any $t \in CTerm$.
(iii) Just combining (i) and Theorem 4.
(iv) Just combining (ii) and Theorem 4.
□

4.3.2 Completeness

Now we look for the reverse implication of Theorem 10, that is, the completeness of let-rewriting as its ability to compute, for any given expression, any value that can been computed by the CRWL-calculi. With the aid of the Peeling Lemma 3 we can prove the following strong completeness result for let-rewriting, which still has a certain technical nature.

https://doi.org/10.1017/S1471068412000373 Published online by Cambridge University Press

Lemma 8 (*Completeness lemma for let-rewriting*)
For any $e \in LExp$ and $t \in CTerm_\bot$ such that $t \not\equiv \bot$,

$$e \twoheadrightarrow t \text{ implies } e \rightarrow^{l^*} let\ \overline{X = a}\ in\ t'$$

for some $t' \in CTerm$ and $\overline{a} \subseteq LExp$ such that $t \sqsubseteq |let\ \overline{X = a}\ in\ t'|$ and $|a_i| = \bot$ for every $a_i \in \overline{a}$. As a consequence, $t \sqsubseteq t'[\overline{X/\bot}]$.

Note the condition $t \not\equiv \bot$ is essential for this lemma to be true, as we can see by taking $\mathscr{P} = \{loop \rightarrow loop\}$ and $e \equiv loop$: while $loop \twoheadrightarrow \bot$, the only $LExp$ reachable from $loop$ is $loop$ itself.

Our main result concerning completeness of let-rewriting follows easily from Lemma 8. It shows that any c-term computed by CRWL or $CRWL_{let}$ for an expression can be refined by a let-rewriting derivation; moreover, if the c-term is total, then it can be exactly reached by let-rewriting.

Theorem 11 (*Completeness of let-rewriting*)
For any program $\mathscr{P}$, $e \in LExp$, and $t \in CTerm_\bot$ we have:

(i) $\mathscr{P} \vdash_{CRWL_{let}} e \twoheadrightarrow t$ implies $e \rightarrow^{l^*} e'$ for some $e' \in LExp$ such that $t \sqsubseteq |e'|$
(ii) Besides, if $t \in CTerm$ then $\mathscr{P} \vdash_{CRWL_{let}} e \twoheadrightarrow t$ implies $e \rightarrow^{l^*} t$

Furthermore, if neither $\mathscr{P}$ nor e have lets then we also have

(iii) $\mathscr{P} \vdash_{CRWL} e \twoheadrightarrow t$ implies $e \rightarrow^{l^*} e'$ for some $e' \in LExp$ such that $t \sqsubseteq |e'|$
(iv) Besides, if $t \in CTerm$ then $\mathscr{P} \vdash_{CRWL} e \twoheadrightarrow t$ implies $e \rightarrow^{l^*} t$

Proof
Regarding part (i), if $t \equiv \bot$ then we are done with $e \rightarrow^{l^0} e$ as $\forall e, \bot \sqsubseteq |e|$. On the other hand, if $t \not\equiv \bot$ then by Lemma 8 we have $e \rightarrow^{l^*} let\ \overline{X} = \overline{a}\ in\ t'$ such that $t \sqsubseteq |let\ \overline{X} = \overline{a}\ in\ t'|$.

To prove part (ii), assume $\mathscr{P} \vdash_{CRWL_{let}} e \twoheadrightarrow t$. Then, by Lemma 8, we get $e \rightarrow^{l^*} let\ \overline{X} = \overline{a}\ in\ t'$ such that $t \sqsubseteq |let\ \overline{X} = \overline{a}\ in\ t| \equiv t'[\overline{X/\bot}]$, for some $t' \in CTerm, \overline{a} \subseteq LExp$. As $t \in CTerm$ then t is maximal with respect to $\sqsubseteq$, so $t \sqsubseteq t'[\overline{X/\bot}]$ implies $t'[\overline{X/\bot}] \equiv t$, but then $t'[\overline{X/\bot}] \in CTerm$ so it must happen that $FV(t') \cap \overline{X} = \emptyset$ and therefore $t' \equiv t'[\overline{X/\bot}] \equiv t$. But then $let\ \overline{X} = \overline{a}\ in\ t' \rightarrow^{l^*} t' \equiv t$ by zero or more steps of (Elim), so $e \rightarrow^{l^*} let\ \overline{X} = \overline{a}\ in\ t' \rightarrow^{l^*} t$, that is $e \rightarrow^{l^*} t$.

Finally, parts (ii) and (iv) follow from (ii), (iii) and Theorem 4. □

As an immediate corollary of this completeness result and soundness (Theorem 10), we obtain the following result relating let-rewriting to CRWL and $CRWL_{let}$ for total c-terms, which gives a clean and easy way to understand the formulation of the adequacy of let-rewriting.

Corollary 1 (*Equivalence of $CRWL_{let}$ and let-rewriting for total values*)
For any program $\mathscr{P}$, $e \in LExp$, and $t \in CTerm$ we have

$$\mathscr{P} \vdash_{CRWL_{let}} e \twoheadrightarrow t \text{ iff } e \rightarrow^{l^*} t.$$

Besides, if neither $\mathscr{P}$ nor e have lets then we also have

https://doi.org/10.1017/S1471068412000373 Published online by Cambridge University Press

$$\mathscr{P} \vdash_{CRWL} e \twoheadrightarrow t \text{ iff } e \rightarrow^{l\,*} t.$$

As a final consequence of Theorems 10 and 11, we obtain another strong equivalence result for both formalisms, this time expressed in terms of semantics and hypersemantics.

Theorem 12 (*Equivalence of $CRWL_{let}$ and let-rewriting*)
For any program $\mathscr{P}$ and $e \in LExp$:

(i) $[\![e]\!] = \{|e'| \mid e \rightarrow^{l^*} e'\}\downarrow$
(ii) $[\![[e]]\!] = \lambda\theta \in CSubst_{\perp}.(\{|e'| \mid e \rightarrow^{l^*} e'\}\downarrow)$

where $\downarrow$ is the downward closure operator defined in Lemma 5.

Proof

(i) We prove both inclusions. Regarding $[\![e]\!] \subseteq \{|e'| \mid e \rightarrow^{l^*} e'\}\downarrow$, assume $t \in [\![e]\!]$. By Theorem 11 there must exist some $e' \in LExp$ such that $e \rightarrow^{l^*} e'$ and $t \sqsubseteq |e'|$, therefore $|e'| \in \{|e'| \mid e \rightarrow^{l^*} e'\}$. But this, combined with $t \sqsubseteq |e'|$, results in $t \in \{|e'| \mid e \rightarrow^{l^*} e'\}\downarrow$.
Regarding the other inclusion, consider some $t \in \{|e'| \mid e \rightarrow^{l^*} e'\}\downarrow$. By definition of the $\downarrow$ operator, there must exist some $e' \in LExp$ such that $t \sqsubseteq |e'|$ and $e \rightarrow^{l^*} e'$. But that implies $|e'| \in [\![e]\!]$, by Theorem 10, which combined with $t \sqsubseteq |e'|$ and the polarity property (Proposition 3) gives us that $t \in [\![e]\!]$.
(ii) Trivial by applying the previous item and the definition of hypersemantics of an expression.
□

5 Semantic reasoning

Having equivalent notions of semantics and reduction allows to reason interchangeably at the rewriting and semantic levels. In this section, we show the power of such technique in different situations. We start with a concrete example, adapted from López-Fraguas *et al.* (2009b), where semantic reasoning leads easily to conclusions non-trivially achievable when thinking directly in operational terms.

Example 4
Imagine a program using constructors $a, b \in CS^0, c \in CS^1, d \in CS^2$ and defining a function $f \in FS^1$ for which we know that $f(a)$ can be let-rewritten to $c(a)$ and $c(b)$ but no other c-terms. Consider also an expression e having $f(a)$ as subexpression, i.e., e has the shape $\mathscr{C}[f(a)]$. We are interested now in the following question: can we safely replace in e the subexpression $f(a)$ by any other ground expression e' let-reducible to the same set of values[4]? By safely we mean not changing the values reachable from e.

The question is less trivial than it could appear. For instance, if reductions were made with term rewriting instead of let-rewriting—i.e., considering run-time instead

[4] More precisely, to the same set of shells in the sense of Theorem 12 part *i*).

https://doi.org/10.1017/S1471068412000373 Published online by Cambridge University Press

of call-time choice—the answer is negative (López-Fraguas *et al.* 2009b). To see that, consider the program

$$f(a) \to c(a)\ g \to a\ h(c(X)) \to d(X,X)$$
$$f(a) \to c(b)\ g \to b$$

and the expressions $e \equiv h(f(a))$ and $e' \equiv c(g)$. All this is compatible with the assumptions of our problem. However, e is reducible by term rewriting only to $d(a,a)$ and $d(b,b)$, while replacing $f(a)$ by e' in e gives $h(c(g))$, which is reducible by term rewriting to two additional values, $d(a,b)$ and $d(b,a)$; thus, the replacement of $f(a)$ by e' has been unsafe.

However, the answer to our question is affirmative in general for let-rewriting, as it is very easily proved by a semantic reasoning using compositionality of CRWL_{let}: the assumption on $f(a)$ and e' means that they have the same denotation $[\![f(a)]\!] = [\![e']\!] = \{c(a), c(b)\}\downarrow$ and, since they are ground, the same hyperdenotation $[\![\![f(a)]\!]\!] = [\![\![e']\!]\!] = \lambda\theta.\{c(a), c(b)\}\downarrow$. By compositionality of hypersemantics, $\mathscr{C}[f(a)]$ and $\mathscr{C}[e']$ have the same (hyper)denotation, too. By equivalence of CRWL_{let} and let-rewriting this implies that both expressions reach the same value by let-rewriting.

Despite its simplicity, the example raises naturally interesting questions about replaceability, for which semantic methods could be simpler than direct reasonings about reduction sequences. This is connected to the *full abstraction* problem that we have investigated for run-time and call-time choice in López-Fraguas *et al.* (2009b) and López-Fraguas and Rodríguez-Hortalá (2010).

Semantic methods can also be used to prove the correctness of new operational rules not directly provided by our set of let-rewriting rules. Such rules can be useful for different purposes: to make computations simpler, for program transformations, to obtain new properties of the framework, etc. Consider for instance the following generalization of the (LetIn) rule in Figure 5:

(CLetIn) $\mathscr{C}[e] \to^l let\ X = e\ in\ \mathscr{C}[X]$, if $BV(\mathscr{C}) \cap FV(e) = \emptyset$ and X is fresh

This rule allows to create let-bindings in more situations and to put them in outer positions than the original (LetIn) rule. If we have not considered it in the definition of let-rewriting it is because it would destroy the strong termination property of Proposition 2, as it is easy to see. However, this rule may shorten derivations. For instance, the derivation in Example 2 could be shortened to:

$$\begin{array}{ll}
\underline{heads(repeat(coin))} & (CLetIn)\\
\to^l let\ C = coin\ in\ heads(\underline{repeat(C)}) & (Fapp)\\
\to^l let\ C = coin\ in\ heads(\underline{C : repeat(C)}) & (Fapp)\\
\to^l let\ C = coin\ in\ \underline{heads(C : C : repeat(C))} & (CLetIn)\\
\to^l let\ C = coin\ in\ \underline{let\ X = repeat(C)\ in\ heads(C : C : X)} & (Fapp)\\
\to^l let\ C = coin\ in\ \underline{let\ X = repeat(C)\ in\ (C,C)} & (Elim)\\
\to^l let\ C = \underline{coin}\ in\ (C,C) & (Fapp)\\
\to^l \underline{let\ C = 0\ in\ (C,C)} & (Bind)\\
\to^l (0,0) &
\end{array}$$

Reasoning the correctness of (CLetIn) rule is not difficult by means of semantic methods. We only need to prove that the rule preserves hypersemantics.

https://doi.org/10.1017/S1471068412000373 Published online by Cambridge University Press

Lemma 9
If $BV(\mathscr{C}) \cap FV(e) = \emptyset$ and X is fresh, then $[\![\mathscr{C}[e]]\!] = [\![let\ X = e\ in\ \mathscr{C}[X]]\!]$.

Proof
Assume an arbitrary $\theta \in CSubst_\perp$:

$$\begin{aligned}
&[\![let\ X = e\ in\ \mathscr{C}[X]]\!]\theta = [\![(let\ X = e\ in\ \mathscr{C}[X])\theta]\!] \\
&= [\![let\ X = e\theta\ in\ \mathscr{C}\theta[X]]\!] && \text{as } X \text{ is fresh} \\
&= \bigcup_{t \in [\![e\theta]\!]} [\![(\mathscr{C}\theta[X])[X/t]]\!] && \text{by Theorem 5} \\
&= \bigcup_{t \in [\![e\theta]\!]} [\![\mathscr{C}\theta[t]]\!] && \text{as } X \text{ is fresh} \\
&= [\![\mathscr{C}\theta[e\theta]]\!] && \text{by Theorem 5} \\
&= [\![(\mathscr{C}[e])\theta]\!] = [\![\mathscr{C}[e]]\!]\theta
\end{aligned}$$

□

The rule (CLetIn) is indeed used in some of the proofs in the online appendix, together with another derived rule:

$$\textbf{(Dist)}\quad \mathscr{C}[let\ X = e_1\ in\ e_2] \to^l let\ X = e_1\ in\ \mathscr{C}[e_2], \quad \text{if } BV(\mathscr{C}) \cap FV(e_1) = \emptyset \text{ and } X \notin FV(\mathscr{C})$$

which also preserves hypersemantics.

Lemma 10
If $BV(\mathscr{C}) \cap FV(e_1) = \emptyset$ and $X \notin FV(\mathscr{C})$ then $[\![\mathscr{C}[let\ X = e_1\ in\ e_2]]\!] = [\![let\ X = e_1\ in\ \mathscr{C}[e_2]]\!]$.

These ideas can be made more general. Consider the equivalence relation $e_1 \asymp e_2$ iff $[\![e_1]\!] = [\![e_2]\!]$. This relation is especially relevant because $e_1 \asymp e_2$ iff $\forall \mathscr{C} \in Cntxt.[\![\mathscr{C}[e]]\!] = [\![\mathscr{C}[e']]\!]$, by Theorem 6. We can contemplate $\asymp$ as an abstract, although non-effective, reduction relation, of which the relations $\to^{lnf}$ of Section 4 and the rules (CLetIn) and (Dist) are particular subrelations. It is trivial to check that, by construction, the combined relation $\to^l \cup \asymp$ is sound and complete with respect ot CRWL_{let}. We can use that relation to reason about the meaning or equivalence of let-expressions and programs. We could also employ it in the definition of on-demand evaluation strategies for let-rewriting. As any subrelation of $\to^l \cup \asymp$ is sound with respect to CRWL_{let}, an approach to strategies for let-rewriting could consist in defining a suitable operationally effective subrelation of $\to^l \cup \asymp$ and then proving its completeness and optimality (if it is the case).

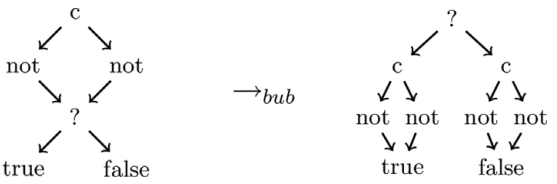
5.1 A case study: Correctness of bubbling

We develop here another nice application of the 'semantic route', where let-rewriting provides a good level of abstraction to formulate a new operational rule—*bubbling*—while the semantic point of view is appropriate for proving its correctness.

Bubbling, proposed in Antoy *et al.* (2007), is an operational rule devised to improve the efficiency of functional logic computations. Its correctness was formally studied in Antoy *et al.* (2006) in the framework of a variant (Echahed and Janodet 1998) of

https://doi.org/10.1017/S1471068412000373 Published online by Cambridge University Press

term graph rewriting. The idea of bubbling is to concentrate all non-determinism of a system into a *choice* operation ? defined by the rules $X\ ?\ Y \to X$ and $X\ ?\ Y \to Y$, and to lift applications of ? out of their surrounding context, as illustrated by the following graph transformation taken from Antoy *et al.* (2006):



As it is shown in Antoy *et al.* (2007), bubbling can be implemented in such a way that many functional logic programs become more efficient, but we will not deal with these issues here.

Due to the technical particularities of term graph rewriting, not only the proof of correctness, but even the definition of bubbling in Antoy *et al.* (2006, 2007) are involved and need subtle care concerning the appropriate contexts over which choices can be bubbled. In contrast, bubbling can be expressed within our framework in a remarkably easy and abstract way as a new rewriting rule:

$$\textbf{(Bub)}\quad \mathscr{C}[e_1\,?\,e_2] \to^{bub} \mathscr{C}[e_1]?\mathscr{C}[e_2],\ \text{for } e_1, e_2 \in LExp$$

With this rule, the bubbling step corresponding to the graph transformation of the example above is:

$$\begin{array}{l} let\ X = true\ ?\ false\ in\ c(not(X), not(X)) \to^{bub} \\ let\ X = true\ in\ c(not(X), not(X))\ ?\ let\ X = false\ in\ c(not(X), not(X)) \end{array}$$

Note that the effect of this bubbling step is not a shortening of any existing let-rewriting derivation; bubbling is indeed a genuine new rule, the correctness of which must therefore be subject of proof. Call-time choice is essential, since bubbling is not correct with respect to ordinary term rewriting, i.e., run-time choice.

Counterexample 2 (*Incorrectness of bubbling for run-time choice*)
Consider a function *pair* defined by the rule $pair(X) \to c(X, X)$ and the expression $pair(0\ ?\ 1)$ for $c \in CS^2$ and $0, 1 \in CS^0$. Under term rewriting/run-time choice the derivation

$$pair(0\ ?\ 1) \to c(0\ ?\ 1, 0\ ?\ 1) \to c(0, 0\ ?\ 1) \to c(0, 1)$$

is valid. But if we performed the bubbling step

$$pair(0\ ?\ 1) \to^{bub} pair(0)\ ?\ pair(1)$$

then the c-term $c(0, 1)$ would not be reachable anymore by term rewriting from $pair(0)\ ?\ pair(1)$.

Formulating and proving the correctness of bubbling for call-time choice becomes easy by using semantics. As we did before, we simply prove that bubbling steps preserve hypersemantics. We need first a basic property of the (hyper)semantics of binary choice ?. Its proof stems almost immediately from the rules for ? and the definition of CRWL-(hyper)denotation.

https://doi.org/10.1017/S1471068412000373 Published online by Cambridge University Press

Proposition 9 (*(Hyper)semantic properties of* ?)
For any $e_1, e_2 \in LExp_\perp$

(i) $[\![e_1 \ ?\ e_2]\!] = [\![e_1]\!] \cup [\![e_2]\!]$
(ii) $[\![\![e_1 \ ?\ e_2]\!]\!] = [\![\![e_1]\!]\!] \cup\!\!\cup\ [\![\![e_2]\!]\!]$

Combining this property with some of the powerful hypersemantic results from Section 4.2 leads to an appealing proof of the correctness of bubbling.

Theorem 13 (*Correctness of bubbling for call-time choice*)
If $e \rightarrow^{bub} e'$ then $[\![\![e]\!]\!] = [\![\![e']\!]\!]$, for any $e, e' \in LExp$.

Proof
If $e \rightarrow^{bub} e'$ then $e = \mathscr{C}[e_1 \ ?\ e_2]$ and $e' = \mathscr{C}[e_1] \ ?\ \mathscr{C}[e_2]$, for some e_1, e_2. Then:

$$
\begin{array}{ll}
[\![\![\mathscr{C}[e_1 \ ?\ e_2]]\!]\!] = [\![\![\mathscr{C}]\!]\!]\,[\![\![e_1 \ ?\ e_2]\!]\!] & \text{by Theorem 6} \\
= [\![\![\mathscr{C}]\!]\!]([\![\![e_1]\!]\!] \cup\!\!\cup\ [\![\![e_2]\!]\!]) & \text{by Proposition 9(ii)} \\
= [\![\![\mathscr{C}]\!]\!]\,[\![\![e_1]\!]\!] \cup\!\!\cup\ [\![\![\mathscr{C}]\!]\!]\,[\![\![e_2]\!]\!] & \text{by Proposition 7} \\
= [\![\![\mathscr{C}[e_1]]\!]\!] \cup\!\!\cup\ [\![\![\mathscr{C}[e_2]]\!]\!] & \text{by Theorem 6} \\
= [\![\![\mathscr{C}[e_1] \ ?\ \mathscr{C}[e_2]]\!]\!] & \text{by Proposition 9(ii)}
\end{array}
$$

□

This property was also proved for the HO case in López-Fraguas *et al.* (2008). But the proof given here is much more elegant thanks to the new semantic tools developed in Section 4.2.

6 Let-narrowing

It is well known that there are situations in functional logic computations where rewriting is not enough and must be lifted to some kind of *narrowing*, because the expression being reduced contains variables for which different bindings might produce different evaluation results. This might happen either because variables are already present in the initial expression to reduce, or due to the presence of extra variables in the program rules. In the latter case, let-rewriting certainly works, but not in an effective way, since the parameter passing substitution in the rule (Fapp) of Figure 5 (page 180) 'magically' guesses the appropriate values for those extra variables (see Example 6 below). Some works (Antoy and Hanus 2006; Braßel and Huch 2007; Dios-Castro and López-Fraguas 2007) have proved that guessing can be replaced by a systematic non-deterministic generation of all (ground) possible values. However, this does not cover all aspects of narrowing, which is able to produce non-ground answers, while generator functions are not. In this section, we present *let-narrowing*, a natural lifting of let-rewriting devised to effectively deal with free and extra variables.

Using the notation of contexts, the standard definition of narrowing as a lifting of term rewriting in ordinary TRS's is the following: $\mathscr{C}[f(\bar{t})] \leadsto_\theta \mathscr{C}\theta[r\theta]$, if θ is a mgu of $f(\bar{t})$ and $f(\bar{s})$, where $f(\bar{s}) \rightarrow r$ is a fresh variant of a rule of the TRS. The requirement that the binding substitution θ is a mgu can be relaxed to accomplish

https://doi.org/10.1017/S1471068412000373 Published online by Cambridge University Press

with certain narrowing strategies like needed narrowing (Antoy *et al.* 2000), which use unifiers but not necessarily most general ones.

This definition of narrowing cannot be directly translated as it is to the case of let-rewriting, for two reasons. First, binding substitutions must be c-substitutions, as for the case of let-rewriting. Second, let-bound variables should not be narrowed, but their values should be rather obtained by evaluation of their binding expressions. The following example illustrates some of the points above.

Example 5
Consider the following program over Peano natural numbers:

$$\begin{array}{ll} 0 + Y \to Y & even(X) \to if\ (Y + Y == X)\ then\ true \\ s(X) + Y \to s(X + Y) & if\ true\ then\ Y \to Y \\ 0 == 0 \to true & s(X) == s(Y) \to X == Y \\ 0 == s(Y) \to false & s(X) == 0 \to false \\ coin \to 0 & coin \to s(0) \end{array}$$

Note the extra variable Y in the rule for *even*. The evaluation of *even(coin)* by let-rewriting could start as follows:

$$\begin{array}{l} even(coin) \to^l let\ X = coin\ in\ even(X) \\ \to^l let\ X = coin\ in\ if\ (Y + Y == X)\ then\ true \\ \to^{l\,*} let\ X = coin\ in\ let\ U = Y + Y\ in\ let\ V = (U == X)\ in\ if\ V\ then\ true \\ \to^{l\,*} let\ U = Y + Y\ in\ let\ V = (U == 0)\ in\ if\ V\ then\ true \end{array}$$

Now, because all function applications involve variables, the evaluation cannot continue merely by rewriting, and therefore narrowing is required instead. We should not perform standard narrowing steps that bind already let-bound variables; otherwise, the syntax of let-expressions can be lost. For instance, narrowing at *if V then true* generates the binding $[V/true]$ that, if applied naively to the surrounding context, results in the syntactically illegal expression:

$$let\ U{=}Y{+}Y\ in\ let\ true{=}(U{==}0)\ in\ true$$

What is harmless is to perform narrowing at $Y + Y$ (Y is a free variable). This gives the substitution $[Y/0]$ and the result 0 for the subexpression $Y + Y$. Placing it in its surrounding context, the derivation continues as follows:

$$\begin{array}{l} let\ U = 0\ in\ let\ V = (U == 0)\ in\ if\ V\ then\ true \\ \to^l let\ V = (0 == 0)\ in\ if\ V\ then\ true \\ \to^l let\ V = true\ in\ if\ V\ then\ true \\ \to^l if\ true\ then\ true \to^l true \end{array}$$

The previous example shows that let-narrowing *must protect bound variables* against substitutions, which is the key observation for defining narrowing in the presence of let-bindings.

The one-step let-narrowing relation $e \leadsto^l_\theta e'$ (assuming a given program $\mathscr{P}$) is defined in Figure 6.

- The rule **(X)** collects *(Elim), (Bind), (Flat), (LetIn)* of $\to^l$, that remain the same in $\leadsto^l$, except for the decoration with the empty substitution ϵ.

https://doi.org/10.1017/S1471068412000373 Published online by Cambridge University Press

(X) $e \rightsquigarrow^{l}_{\epsilon} e'$ if $e \rightarrow^{l} e'$ using $\boldsymbol{X} \in \{LetIn, Bind, Elim, Flat\}$ in Figure 5 (page 16).

(Narr) $f(\overline{t}) \rightsquigarrow^{l}_{\theta} r\theta$, for any fresh variant $(f(\overline{p}) \rightarrow r) \in \mathcal{P}$ and $\theta \in CSubst$ such that $f(\overline{t})\theta \equiv f(\overline{p})\theta$.

(Contx) $\mathcal{C}[e] \rightsquigarrow^{l}_{\theta} \mathcal{C}\theta[e']$, for $\mathcal{C} \neq []$, if $e \rightsquigarrow^{l}_{\theta} e'$ by any of the previous rules, and if the step is (Narr) using $(f(\overline{p}) \rightarrow r) \in \mathcal{P}$, then:

(i) $dom(\theta) \cap BV(\mathcal{C}) = \emptyset$

(ii) $vRan(\theta|_{\setminus var(\overline{p})}) \cap BV(\mathcal{C}) = \emptyset$

Fig. 6. Rules of the let-narrowing relation $\rightsquigarrow^{l}$.

- The rule **(Narr)** performs a narrowing step in a proper sense. To avoid unnecessary loss of generality or applicability of our approach, we do not impose θ to be a mgu. For the sake of readability, we will sometimes decorate (Narr) steps with $\theta|_{FV(f(\overline{t}))}$ instead of θ, i.e., with the projection over the variables in the narrowed expression.
- The rule **(Contx)** indicates how to use the previous rules in inner positions. The condition $\mathscr{C} \neq [\,]$ simply avoids trivial overlappings of (Contx) with the previous rules. The rest of the conditions are set to ensure that the combination of (Contx) with (Narr) makes a proper treatment of bound variables:
  - — (i) expresses the protection of bound variables against narrowing justified in Example 5.
  - — (ii) is a rather technical condition needed to prevent undesired situations when the narrowing step has used a program rule with extra variables and a unifier θ which is not a mgu. Concretely, the condition states that the bindings created by θ for the extra variables in the program rule do not introduce variables that are bound by the surrounding context $\mathscr{C}$. To see the problems that can arise without (ii), consider for instance the program rules $f \rightarrow Y$ and $loop \rightarrow loop$ and the expression $let\ X = loop\ in\ f$. A legal reduction for this expression, respecting condition (ii), could be the following:

    $$let\ X = loop\ in\ f \rightsquigarrow^{l}_{\epsilon} let\ X = loop\ in\ Z$$

    by applying (Narr) to f with $\theta = \epsilon$ taking the fresh variant rule $f \rightarrow Z$, and using (Contx) for the whole expression. However, if we drop condition (ii), we could perform a similar derivation using the same fresh variant of the rule for f, but now using the substitution $\theta = [Z/X]$:

    $$let\ X = loop\ in\ f \rightsquigarrow^{l}_{\epsilon} let\ X = loop\ in\ X$$

    which is certainly not intended because the free variable Z in the previous derivation now appears as a bound variable, i.e., we get an undesired capture of variables.

    We remark that if the substitution θ in (Narr) is chosen to be a standard mgu[5] of $f(\overline{t})$ and $f(\overline{p})$ (which is always possible) then the condition (ii) is always fulfilled.

[5] By standard mgu of t, s we mean an idempotent mgu θ with $dom(\theta) \cup ran(\theta) \subseteq var(t) \cup var(s)$.

https://doi.org/10.1017/S1471068412000373 Published online by Cambridge University Press

The one-step relation $\leadsto^{l}_{\theta}$ is extended in the natural way to the multiple-steps narrowing relation $\leadsto^{l^*}_{\theta}$, which is defined as the least relation verifying:

$$e \leadsto^{l^*}_{\epsilon} e \quad e \leadsto^{l}_{\theta_1} e_1 \leadsto^{l}_{\theta_2} \ldots e_n \leadsto^{l}_{\theta_n} e' \Rightarrow e \leadsto^{l^*}_{\theta_1 \ldots \theta_n} e'$$

We write $e \leadsto^{l^n}_{\theta} e'$ for a n-steps narrowing sequence.

Example 6

Example 5 essentially contains already a narrowing derivation. For the sake of clarity, we repeat it here making explicit the rule of let-narrowing used at each step (maybe in combination with (Contx), which is not written). Besides, if the step uses (Narr), the narrowed expression is underlined.

$$\begin{array}{ll}
even(coin) \leadsto^{l}_{\epsilon} & (LetIn) \\
let\ X = coin\ in\ \underline{even(X)} \leadsto^{l}_{\epsilon} & (Narr) \\
let\ X = coin\ in\ if\ Y + Y == X\ then\ true \leadsto^{l^3}_{\epsilon} & (LetIn^2, Flat) \\
let\ X = \underline{coin}\ in\ let\ U = Y + Y\ in & \\
\quad let\ V = (U == X)\ in\ if\ V\ then\ true \leadsto^{l}_{\epsilon} & (Narr) \\
let\ X = 0\ in\ let\ U = Y + Y\ in & \\
\quad let\ V = (U == X)\ in\ if\ V\ then\ true \leadsto^{l}_{\epsilon} & (Bind) \\
let\ U = \underline{Y + Y}\ in\ let\ V = (U == 0)\ in\ if\ V\ then\ true \leadsto^{l}_{[Y/0]} & (Narr) \\
let\ U = 0\ in\ let\ V = (U == 0)\ in\ if\ V\ then\ true \leadsto^{l}_{\epsilon} & (Bind) \\
let\ V = \underline{(0 == 0)}\ in\ if\ V\ then\ true \leadsto^{l}_{\epsilon} & (Narr) \\
let\ V = true\ in\ if\ V\ then\ true \leadsto^{l}_{\epsilon} & (Bind) \\
if\ true\ then\ true \leadsto^{l}_{\epsilon} & (Narr) \\
true &
\end{array}$$

Note that all (Narr) steps in the derivation except one have ϵ as narrowing substitution (because of the projection over the variables of the narrowed expression), so they are really rewriting steps. An additional remark that could help to further explain the relationship between the let-narrowing relation $\leadsto^{l}$ and the let-rewriting relation $\rightarrow^{l}$ is the following: since we have $even(coin) \leadsto^{l}_{\theta} true$ for some θ, but $even(coin)$ is ground, Theorem 14 in the next section ensures that there must also be a successful let-rewriting derivation $even(coin) \rightarrow^{l^*} true$. This derivation could have the form:

$$\begin{array}{ll}
even(coin) \rightarrow^{l} & (LetIn) \\
let\ X = coin\ in\ even(X) \rightarrow^{l} & (Fapp) \\
let\ X = coin\ in\ if\ (0 + 0 == X)\ then\ true \rightarrow^{l} & \\
\ldots\ldots\ldots \rightarrow^{l} true &
\end{array}$$

The indicated (Fapp)-step in this let-rewriting derivation has used the substitution $[Y/0]$, thus anticipating and 'magically guessing' the correct value of the extra variable Y of the rule of *even*. In contrast, in the let-narrowing derivation, the binding for Y is not done while reducing $even(X)$ but in a later (Narr)-step over $Y + Y$. This corresponds closely to the behavior of narrowing-based systems like Toy or Curry.

https://doi.org/10.1017/S1471068412000373 Published online by Cambridge University Press

6.1 Soundness and completeness of the let-narrowing relation $\leadsto^l$

In this section, we show the adequacy of let-narrowing with respect to let-rewriting. From now on we assume a fixed program $\mathscr{P}$.

As usual with narrowing relations, soundness results are not difficult to formulate and prove. The following *soundness* result for $\leadsto^l$ states that we can mimic any $\leadsto^l$ derivation with $\to^l$ by applying over the starting expression the substitution computed by the original let-narrowing derivation.

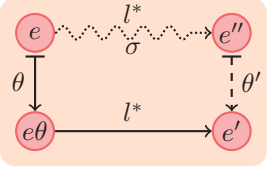
Theorem 14 (*Soundness of the let-narrowing relation* $\leadsto^l$)
For any $e, e' \in LExp$, $e \leadsto^{l^*}_{\theta} e'$ implies $e\theta \to^{l\,*} e'$.

Completeness is more complicated to prove. The key result is a generalization to let-rewriting of Hullot's *lifting lemma* (Hullot 1980) for classical term rewriting and narrowing. It states that any rewrite sequence for a particular instance of an expression can be generalized by a narrowing derivation.

Lemma 11 (*Lifting lemma for the let-rewriting relation* $\to^l$)
Let $e, e' \in LExp$ such that $e\theta \to^{l^*} e'$ for some $\theta \in CSubst$, and let $\mathscr{W}, \mathscr{B} \subseteq \mathscr{V}$ with $dom(\theta) \cup FV(e) \subseteq \mathscr{W}$, $BV(e) \subseteq \mathscr{B}$ and $(dom(\theta) \cup vran(\theta)) \cap \mathscr{B} = \emptyset$, and for each (Fapp) step of $e\theta \to^{l^*} e'$ using a rule $R \in \mathscr{P}$ and a substitution $\gamma \in CSubst$ then $vran(\gamma|_{vExtra(R)}) \cap \mathscr{B} = \emptyset$. Then, there exist a derivation $e \leadsto^{l^*}_{\sigma} e''$ and $\theta' \in CSubst$ such that:

(i) $e''\theta' = e'$ (ii) $\sigma\theta' = \theta[\mathscr{W}]$ (iii) $(dom(\theta') \cup vran(\theta')) \cap \mathscr{B} = \emptyset$

Besides, the let-narrowing derivation can be chosen to use mgu at each (Narr) step. Graphically:



With the aid of this lemma, we are now ready to state and prove the following strong completeness result for $\leadsto^l$.

Theorem 15 (*Completeness of the let-narrowing relation* $\leadsto^l$)
Let $e, e' \in LExp$ and $\theta \in CSubst$. If $e\theta \to^{l\,*} e'$, then there exist a let-narrowing derivation $e \leadsto^{l^*}_{\sigma} e''$ and $\theta' \in CSubst$ such that $e''\theta' \equiv e'$ and $\sigma\theta' = \theta[FV(e)]$.

Proof
Applying Lemma 11 to $e\theta|_{FV(e)} \to^{l\,*} e'$ with $\mathscr{W} = FV(e)$ and $\mathscr{B} = BV(e)$, as $e\theta|_{FV(e)} \equiv e\theta$ and the additional conditions over $\mathscr{B}$ hold by the variable convention. □

Finally, by combining Theorems 14 and 15, we obtain a strong adequacy theorem for let-narrowing with respect to let-rewriting.

Theorem 16 (*Adequacy of the let-narrowing relation* $\leadsto^l$ *wrt.* $\to^l$)

https://doi.org/10.1017/S1471068412000373 Published online by Cambridge University Press

Let $e, e_1 \in LExp$ and $\theta \in CSubst$, then:

$$e\theta \rightarrow^{l\,*} e_1 \;\Leftrightarrow\; \begin{array}{l}\text{there exist a let-narrowing derivation } e \leadsto^{l^*}_{\sigma} e_2 \text{ and}\\ \text{some } \theta' \in CSubst \text{ such that } \sigma\theta' = \theta[FV(e)],\ e_2\theta' \equiv e_1\end{array}$$

Proof

($\Rightarrow$) Assume $e\theta \rightarrow^{l^*} e_1$. As $e\theta|_{FV(e)} \equiv e\theta$ then trivially $e\theta|_{FV(e)} \rightarrow^{l\,*} e_1$. We can apply Lemma 11 taking $\mathcal{W} = FV(e)$ to get $e \leadsto^{l^*}_{\sigma} e_2$ such that there exists $\theta' \in CSubst$ with $\sigma\theta' = \theta|_{FV(e)}[\mathcal{W}]$ and $e_2\theta' \equiv e_1$. But as $\mathcal{W} = FV(e)$ then $\sigma\theta' = \theta|_{FV(e)}[\mathcal{W}]$ implies $\sigma\theta' = \theta[FV(e)]$.

We remark that the lifting lemma ensures that the narrowing derivation can be chosen to use mgu at each (Narr) step.

($\Leftarrow$) Assume $e \leadsto^{l^*}_{\sigma} e_2$ and θ' under the conditions above. Then by Theorem 14 we have $e\sigma \rightarrow^{l\,*} e_2$. As $\rightarrow^{l}$ is closed under c-substitutions (Lemma 2) then $e\sigma\theta' \rightarrow^{l\,*} e_2\theta'$. But as $\sigma\theta' = \theta[FV(e)]$, then $e\theta \equiv e\sigma\theta' \rightarrow^{l\,*} e_2\theta' \equiv e_1$.

□

6.2 Organizing computations

Deliberately, in this paper we have kept the definitions of let-rewriting and narrowing apart from any particular computation strategy. In this section, we explain rather informally how the ideas of some known strategies for functional logic programming (Antoy 2005) can also be adapted to our formal setting. For the sake of brevity, we focus only on let-narrowing computations. As a running example, consider the program

$$\begin{array}{ll} leq(0, Y) \rightarrow true & f(0) \rightarrow 0\\ leq(s(X), 0) \rightarrow false & \\ leq(s(X), s(Y)) \rightarrow leq(X, Y) & \end{array}$$

and the initial expression $leq(X, f(Y))$ to be let-narrowed using it.

As a first remark, when designing a strategy, one can freely use 'peeling' steps in a *don't care* manner using the relation $\rightarrow^{lnf}$ (Definition 3), since it is terminating and (hyper-)semantics-preserving. In our case, one step suffices: $leq(X, f(Y)) \leadsto^{l}_{\epsilon} let\ U = f(Y)\ in\ leq(X, U)$. After a peeling (multi-)step, a (Narr) step must be done. Where? Certainly, the body $leq(\ldots)$ must be narrowed at some point. One *don't know* possibility is narrowing at $leq(X, U)$ using the first rule for leq that does not bind U: $let\ U = f(Y)\ in\ leq(X, U) \leadsto^{l}_{[X/0]} let\ U = f(Y)\ in\ true$. A new peeling step leads to a first final result *true*, with computed substitution $[X/0]$.

The second and third rules for leq could lead to more results. Those rules have non-variable patterns as second arguments, and then the bound variable U in $leq(X, U)$ inhibits a direct (Narr) step in that position. Typically it is said that U is *demanded* by those leq rules. Therefore, we narrow $f(Y)$ to get values for U, and then we 'peel':

$$let\ U = f(Y)\ in\ leq(X, U) \leadsto^{l}_{[Y/0]} let\ U = 0\ in\ leq(X, U) \leadsto^{l}_{\epsilon} leq(X, 0) \qquad (1)$$

https://doi.org/10.1017/S1471068412000373 Published online by Cambridge University Press

The computation proceeds now by two don't know choices using the rules for *leq*, leading to two more solutions $(true, [Y/0, X/0])$ and $(false, [Y/0, X/s(Z)])$.

This implicitly applied strategy can be seen as a translation to let-narrowing of *lazy narrowing* (Moreno-Navarro and Rodríguez-Artalejo 1992; Alpuente *et al.* 2003). As a known drawback of lazy narrowing, note that the second solution $(true, [Y/0, X/0])$ is redundant, since it is less general than the first one $(true, [X/0])$. Redundancy is explained because we have narrowed the expression $f(Y)$ whose evaluation was demanded only by some of the rules for the outer function application $leq(X, f(Y))$, but after that we have used the rules not demanding the evaluation (the first rule for *leq*). This problem is tackled successfully by *needed narrowing* (Antoy *et al.* 2000) which takes into account, when narrowing an inner expression, what are the rules for an outer function application demanding such evaluation. A needed narrowing step 'anticipates' the substitution that will perform these rules when they are to be applied. The ideas of needed narrowing can be adapted to our setting. In our example, we get the following derivation instead of (1):

$$\begin{array}{l} let\ U = f(Y)\ in\ leq(X,U) \leadsto^{l}_{[X/s(Z),Y/0]} let\ U = 0\ in\ leq(s(Z),U) \leadsto^{l}_{\epsilon} \\ leq(s(Z),0) \leadsto^{l}_{\epsilon} false \end{array} \quad (1')$$

The first step does not use a mgu. This is a typical feature of needed narrowing, and is also allowed by let-narrowing steps. Needed narrowing steps rely on *definitional trees* that structure demandness information from the rules of a given function. This information can also be embedded into a program transformation. There are simple transformations for which the transformed program, under a lazy narrowing regime using mgu, obtains the same solutions than the original program (Zartmann 1997), although it is not guaranteed that the number of steps is also preserved. In our example, the definition of *leq* can be transformed as follows:

$$\begin{array}{ll} leq(0,Y) \to true & leqS(X,0) \to false \\ leq(s(X),Y) \to leqS(X,Y) & leqS(X,s(Y)) \to leq(X,Y) \end{array}$$

As happened with $(1')$, the derivation

$$\begin{array}{l} let\ U = f(Y)\ in\ leq(X,U) \leadsto^{l}_{[X/s(Z)]} let\ U = f(Y)\ in\ leqS(Z,U) \leadsto^{l}_{[Y/0]} \\ let\ U = 0\ in\ leqS(Z,U) \leadsto^{l}_{\epsilon} leqS(Z,0) \leadsto^{l}_{\epsilon} false \end{array}$$

gets rid of redundant solutions.

To which extent do our results guarantee the adequateness of the adaptation to let-narrowing of these strategies or others that could be defined? Certainly any strategy is *sound* for call-time choice semantics, because unrestricted $\leadsto^{l}$ is already sound (Theorem 14). This will also be true if the strategy uses derived rules in the sense of Section 5. With respect to completeness, we know that the space of let-narrowing derivations is complete with respect to let-rewriting (Theorem 15). But this does not imply the completeness of the strategy, which in general will determine a smaller narrowing space. Therefore, the completeness of the strategy must be proved independently. Such a proof may use semantic methods (i.e., prove completeness with respect to CRWL-semantics) or operational methods (i.e., prove completeness with respect to $\to^{l}$-derivations). We will not go deeper into the issue of strategies.

https://doi.org/10.1017/S1471068412000373 Published online by Cambridge University Press

7 Let-rewriting versus classical term rewriting

In this section, we examine the relationship between let-rewriting and ordinary term rewriting, with the focus put in the set of c-terms reachable by rewriting with each of these relations. As term rewriting is not able to handle expressions with let-bindings, in this section we assume that all considered programs do not have let-bindings in the right-hand side of its rules.

We will first prove in Section 7.1 that let-rewriting is sound with respect to term rewriting, in the sense that any c-term that can be reached by a let-rewriting derivation from a given expression can also be reached by a term rewriting derivation starting from the same expression. As we know, completeness does not hold in general because run-time choice computes more values than call-time choice for arbitrary programs. However, we will be able to prove completeness of let-rewriting with respect to term rewriting for the class of *deterministic* programs, a notion close to confluence that will be defined in Section 7.2. Finally, we will conclude in Section 7.3 with a comparison between let-narrowing and narrowing, which will follow easily from the results in previous subsections and the adequacy of let-narrowing to let-rewriting.

Thanks to the strong equivalence between CRWL and let-rewriting, we can choose the most appropriate point of view for each of the two goals (soundness and completeness): we will use let-rewriting for proving soundness, and CRWL for defining the property of determinism and proving that, under determinism, completeness of let-rewriting with respect to term rewriting also holds.

7.1 Soundness of let-rewriting with respect to classical term rewriting

In order to relate let-rewriting to term rewriting, we first need to find a way for term rewriting to cope with let-bindings, which are not supported by its syntax, that is only able to handle expressions from Exp. Therefore, we define the following syntactic transformation from $LExp$ into Exp that takes care of removing the let constructions, thus losing the sharing information they provide.

Definition 9 (*Let-binding elimination transformation*)
Given $e \in LExp$ we define its transformation into a let-free expression $\widehat{e} \in Exp$ as:

$$\begin{aligned} \widehat{X} &=_{def} X \\ \widehat{h(e_1,\ldots,e_n)} &=_{def} h(\widehat{e_1},\ldots,\widehat{e_n}) \\ \widehat{let\ X = e_1\ in\ e_2} &=_{def} \widehat{e_2}[X/\widehat{e_1}] \end{aligned}$$

Note that $\widehat{e} \equiv e$ for any $e \in Exp$.

We will also need the following auxiliary lemma showing the interaction between term rewriting derivations and substitution application.

Lemma 12 (*Copy lemma*)
For all $e, e_1, e_2 \in Exp$, $X \in \mathcal{V}$:

(i) $e_1 \to e_2$ implies $e[X/e_1] \to^* e[X/e_2]$.

https://doi.org/10.1017/S1471068412000373 Published online by Cambridge University Press

(ii) $e_1 \rightarrow^* e_2$ implies $e[X/e_1] \rightarrow^* e[X/e_2]$.

Note how in (i), each of the different copies of e_1 introduced in e by the substitution has to be reduced to e_2 in a different term rewriting step in order to reach the expression $e[X/e_2]$.

Using this lemma we can get a first soundness result stating that the result of one let-rewriting step can also be obtained in zero or more steps of ordinary rewriting, after erasing the sharing information by means of the let-binding elimination transformation.

Lemma 13 (*One-Step Soundness of let-rewriting with respect to term rewriting*)
For all $e, e' \in LExp$ we have that $e \rightarrow^l e'$ implies $\widehat{e} \rightarrow^* \widehat{e'}$.

The remaining soundness results follow easily from this lemma. The first one shows how we can mimic let-rewriting with term rewriting through the let-binding elimination transformation. But then, as $\widehat{e} \equiv e$ for any $e \in Exp$, we conclude that for let-free expressions let-rewriting is a subrelation of term rewriting.

Theorem 17 (*Soundness of let-rewriting with respect to term rewriting*)
For any $e, e' \in LExp$ we have that $e \rightarrow^{l^*} e'$ implies $\widehat{e} \rightarrow^* \widehat{e'}$. As a consequence, if $e, e' \in Exp$ then $e \rightarrow^{l^*} e'$ implies $e \rightarrow^* e'$, i.e., $(\rightarrow^{l^*} \cap (Exp \times Exp)) \subseteq \rightarrow^*$.

Proof
The first part follows from an immediate induction on the length of the let-derivation, using Lemma 13 for the inductive step. The rest is obvious taking into account that $e \equiv \widehat{e}$ and $e' \equiv \widehat{e'}$ when $e, e' \in Exp$. □

To conclude this part, we can combine this last result with the equivalence of CRWL and let-rewriting, thus getting the following soundness result for CRWL with respect to term rewriting.

Theorem 18 (*Soundness of CRWL with respect to term rewriting*)
For any $e \in Exp$, $t \in CTerm_\perp$, if $e \twoheadrightarrow t$ then there exists $e' \in Exp$ such that $e \rightarrow^* e'$ and $t \sqsubseteq |e'|$.

Proof
Assume $e \twoheadrightarrow t$. By Theorem 11, there exists $e'' \in LExp$ such that $e \rightarrow^{l^*} e''$ and $t \sqsubseteq |e''|$. Then, by Theorem 17, we have $\widehat{e} \rightarrow^* \widehat{e''}$. As $e \in Exp$, we have $e \equiv \widehat{e}$ and we can choose $e' \equiv \widehat{e''} \in Exp$ so we get $e \rightarrow^* e'$. It is easy to check that $|e''| = |\widehat{e''}|$ and then we have $t \sqsubseteq |e''| = |\widehat{e''}| = |e'|$. □

7.2 Completeness of CRWL with respect to classical term rewriting

We prove here the completeness of the CRWL framework with respect to term rewriting for the class of CRWL-deterministic programs, which are defined as follows.

Definition 10 (*CRWL-deterministic program*)

https://doi.org/10.1017/S1471068412000373 Published online by Cambridge University Press

A program $\mathscr{P}$ is *CRWL-deterministic* iff for any expression $e \in Exp_\perp$ its denotation $[\![e]\!]^{\mathscr{P}}$ is a directed set. In other words, iff for all $e \in Exp_\perp$ and $t_1, t_2 \in [\![e]\!]^{\mathscr{P}}$, there exists $t_3 \in [\![e]\!]^{\mathscr{P}}$ with $t_1 \sqsubseteq t_3$ and $t_2 \sqsubseteq t_3$.

Thanks to the equivalence of CRWL and let-rewriting, it is also easy to characterize CRWL-determinism in terms of let-rewriting derivations.

Lemma 14
A program $\mathscr{P}$ is CRWL-deterministic iff for any $e \in Exp$, $e', e'' \in LExp$ with $\mathscr{P} \vdash e \to^{l^*} e'$ and $\mathscr{P} \vdash e \to^{l^*} e''$ there exists $e''' \in LExp$ such that $\mathscr{P} \vdash e \to^{l^*} e'''$ and $|e'''| \sqsupseteq |e'|, |e'''| \sqsupseteq |e''|$.

Proof
For the left to right implication, assume a CRWL-deterministic program $\mathscr{P}$ and $e \in Exp$, $e', e'' \in LExp$ with $e \to^{l^*} e'$ and $e \to^{l^*} e''$. By part (*iii*) of Theorem 10 we have $|e'|, |e''| \in [\![e]\!]$ and then by Definition 10 there exists $t \in [\![e]\!]$ such that $|e'|, |e''| \sqsubseteq t$. Now, by part (*iii*) of Theorem 11 there exists $e''' \in LExp$ such that $e \to^{l^*} e'''$ and $t \sqsubseteq |e'''|$, so we have $|e'|, |e''| \sqsubseteq t \sqsubseteq |e'''|$ as expected.

Regarding the converse implication, assume $e \in Exp$ with $t_1, t_2 \in [\![e]\!]$. By part (*iii*) of Theorem 11 there exist $e', e'' \in LExp$ such that $e \to^{l^*} e'$, $e \to^{l^*} e''$ and $t_1 \sqsubseteq |e'|$, $t_2 \sqsubseteq |e''|$. Then by hypothesis there exists $e''' \in LExp$ such that $e \to^{l^*} e'''$ and $|e'|, |e''| \sqsubseteq |e'''|$. Now, by part (*iii*) of Theorem 10 we have $|e'''| \in [\![e]\!]$ and this $|e'''|$ is the t_3 of Definition 10 we are looking for, i.e., $t_3 \in [\![e]\!]$ and $t_1, t_2 \sqsubseteq t_3$. □

CRWL-determinism is intuitively close to confluence of term rewriting, but these two properties are not equivalent, as shown by the following example of a CRWL-deterministic but not confluent program.

Example 7
Consider the program $\mathscr{P}$ given by the rules

$$f \to a \qquad f \to loop \qquad loop \to loop$$

where a is a constructor. It is clear that $\to_{\mathscr{P}}$ is not confluent (f can be reduced to a and *loop*, which cannot be joined into a common reduct), but it is CRWL-deterministic, since $[\![f]\!]^{\mathscr{P}} = \{\perp, a\}$, $[\![loop]\!]^{\mathscr{P}} = \{\perp\}$ and $[\![a]\!]^{\mathscr{P}} = \{\perp, a\}$, which are all directed sets.

We conjecture that the reverse implication is true, i.e., that confluence of term rewriting implies CRWL-determinism. Nevertheless, a precise proof for this fact seems surprisingly complicated and we have not yet completed it.

A key ingredient in our completeness proof is the notion of CRWL-denotation of a substitution, which is the set of c-substitutions whose range can be obtained by CRWL-reduction over the range of the starting expression.

Definition 11 (*CRWL-denotation for a substitution*)
Given a program $\mathscr{P}$, the CRWL-denotation of a $\sigma \in Subst_\perp$ is defined as:

$$[\![\sigma]\!]^{\mathscr{P}}_{CRWL} = \{\theta \in CSubst_\perp \mid \forall X \in \mathscr{V},\ \mathscr{P} \vdash_{CRWL} \sigma(X) \twoheadrightarrow \theta(X)\}$$

We will usually omit the subscript CRWL and/or the superscript $\mathscr{P}$ when implied by the context.

https://doi.org/10.1017/S1471068412000373 Published online by Cambridge University Press

Any substitution θ in the denotation of some substitution σ contains less information than σ, because it only holds in its range a finite part of the possibly infinite denotation of the expressions in the range of σ. We formalize this property in the following result.

Proposition 10
For all $\sigma \in Subst_\perp$, $\theta \in [\![\sigma]\!]$, we have that $\theta \trianglelefteq \sigma$.

Besides, we will use the notion of deterministic substitution, which is a substitution with only deterministic expressions in its range.

Definition 12 (*Deterministic substitution*)
The set $DSubst_\perp$ of *deterministic substitutions* for a given program $\mathscr{P}$ is defined as

$$DSubst_\perp = \{\sigma \in Subst_\perp \mid \forall X \in dom(\sigma).[\![\sigma(X)]\!] \text{ is a directed set}\}$$

Then $CSubst_\perp \subseteq DSubst_\perp$, and under any program $\forall \sigma \in Subst_\perp.[\![\sigma]\!] \subseteq CSubst_\perp \subseteq DSubst_\perp$. Note that the determinism of substitutions depends on the program, which gives meaning to the functions in its range. Obviously if a program is deterministic then $Subst_\perp = DSubst_\perp$.

A good thing about deterministic substitutions is that their denotation is always a directed set.

Proposition 11
For all $\sigma \in DSusbt_\perp$, $[\![\sigma]\!]$ is a directed set.

But the fundamental property of deterministic substitutions is that, for any CRWL-statement starting from an instance of an expression that has been constructed using a deterministic substitution, there is another CRWL-statement to the same value from another instance of the same expression that now has been built using a c-substitution taken from the denotation of the starting substitution. This property is a direct consequence of Proposition 11.

Lemma 15
For all $\sigma \in DSusbt_\perp$, $e \in Exp_\perp, t \in CTerm_\perp$,

$$\text{if } e\sigma \twoheadrightarrow t \text{ then } \exists \theta \in [\![\sigma]\!] \text{ such that } e\theta \twoheadrightarrow t$$

Proof (sketch)
We proceed by a case distinction over e. If $e \equiv X \in dom(\sigma)$ then we have $e\sigma \equiv \sigma(X) \twoheadrightarrow t$, and we can define

$$\theta(Y) = \begin{cases} t & \text{if } Y \equiv X \\ \perp & \text{if } Y \in dom(\sigma) \setminus \{X\} \\ Y & \text{otherwise} \end{cases}$$

Then it is easy to check that $\theta \in [\![\sigma]\!]$ and besides $e\theta \equiv \theta(X) \equiv t \twoheadrightarrow t$ by Lemma 5, so we are done. If $e \equiv X \in \mathscr{V} \setminus dom(\sigma)$ then we have $e\sigma \equiv \sigma(X) \equiv X \twoheadrightarrow t$, and given $\overline{Y} = dom(\sigma)$ it is easy to check that $[\overline{Y / \perp}] \in [\![\sigma]\!]$, and besides $e[\overline{Y / \perp}] \equiv X \twoheadrightarrow t$ by hypothesis.

https://doi.org/10.1017/S1471068412000373 Published online by Cambridge University Press

Finally, if $e \notin \mathcal{V}$ we proceed by induction on the structure of the proof for $e\sigma \twoheadrightarrow t$. The interesting cases are those for (DC) and (OR) where we use that $\sigma \in DSusbt_\perp$, so by Proposition 11 its denotation is directed. Then there must exist some $\theta \in [\![\sigma]\!]$ which is greater than each of the θ_i obtained by induction hypothesis over the premises of the starting CRWL-proof for $e\sigma \twoheadrightarrow t$. Using the monotonicity of Proposition 5 we can prove $e\theta \twoheadrightarrow t$, which also holds for CRWL, by Theorem 4; see the online appendix. □

Now we are finally ready to prove our first completeness result of CRWL with respect to term rewriting, for deterministic programs.

Lemma 16 (*Completeness lemma for CRWL with respect to term rewriting*)
Let $\mathcal{P}$ be a CRWL-deterministic program, and $e, e' \in Exp$. Then:

$$e \to^* e' \text{ implies } [\![e']\!] \subseteq [\![e]\!]$$

Proof
We can just prove this result for $e \to e'$, then its extension for an arbitrary number of term rewriting steps holds by a simple induction on the length of the term rewriting derivation, using transitivity of $\subseteq$.

Assume $e \to e'$, then the step must be of the shape $e \equiv \mathcal{C}[f(\overline{p})\sigma] \to \mathcal{C}[r\sigma] \equiv e'$ for some program rule $(f(\overline{p}) \to r) \in \mathcal{P}$, $\sigma \in Subst$. First, let us focus on the case for $\mathcal{C} = [\ ]$, and then assume some $t \in CTerm_\perp$ such that $\mathcal{P} \vdash_{CRWL} e' \equiv r\sigma \twoheadrightarrow t$. As $\mathcal{P}$ is deterministic then $\sigma \in DSubst_\perp$; therefore, by Lemma 15 there must exist some $\theta \in [\![\sigma]\!]$ such that $\mathcal{P} \vdash_{CRWL} r\theta \twoheadrightarrow t$. But then we can use θ to build the following CRWL-proof.

$$\frac{\ldots\ p_i\theta \twoheadrightarrow p_i\theta \ldots\ r\theta \twoheadrightarrow t}{f(\overline{p})\theta \twoheadrightarrow t}\ OR$$

where for each $p_i \in \overline{p}$ we have $\mathcal{P} \vdash_{CRWL} p_i\theta \twoheadrightarrow p_i\theta$ by Lemma 5, as $p_i \in CTerm$ because $\mathcal{P}$ is a constructor system, and so $p_i\theta \in CTerm_\perp$, as $\theta \in [\![\sigma]\!] \subseteq CSubst_\perp$. But we also have $\theta \trianglelefteq \sigma$ by Proposition 10; therefore, by applying the monotonicity for substitutions from Proposition 5—which also holds for CRWL, by Theorem 4—we get $\mathcal{P} \vdash_{CRWL} e \equiv f(\overline{p})\sigma \twoheadrightarrow t$. Hence, $[\![e']\!] = [\![r\sigma]\!] \subseteq [\![f(\overline{p})\sigma]\!] = [\![e]\!]$.

Finally, we can generalize this result to arbitrary contexts by using the compositionality of CRWL from Theorem 1. Given a term rewriting step $e \equiv \mathcal{C}[f(\overline{p})\sigma] \to \mathcal{C}[r\sigma] \equiv e'$ then by the proof for $\mathcal{C} = [\ ]$ we get $[\![r\sigma]\!] \subseteq [\![f(\overline{p})\sigma]\!]$, but then

$$\begin{aligned}
&[\![e']\!] = [\![\mathcal{C}[r\sigma]]\!] \\
&= \bigcup_{t \in [\![r\sigma]\!]} [\![\mathcal{C}[t]]\!] && \text{by Theorem 1} \\
&\subseteq \bigcup_{t \in [\![f(\overline{p})\sigma]\!]} [\![\mathcal{C}[t]]\!] && \text{as } [\![r\sigma]\!] \subseteq [\![f(\overline{p})\sigma]\!] \\
&= [\![\mathcal{C}[f(\overline{p})\sigma]]\!] = [\![e]\!] && \text{by Theorem 1}
\end{aligned}$$

□

The previous lemma, together with the equivalence of CRWL and let-rewriting given by Theorems 12 and 4, allows us to obtain a strong relationship between term rewriting, let-rewriting and CRWL, for the class of CRWL-deterministic programs.

https://doi.org/10.1017/S1471068412000373 Published online by Cambridge University Press

Theorem 19
Let $\mathscr{P}$ be a CRWL-deterministic program, and $e, e' \in Exp, t \in CTerm$. Then:

(a) $e \rightarrow^* e'$ implies $e \rightarrow^{l^*} e''$ for some $e'' \in LExp$ with $|e''| \sqsupseteq |e'|$.
(b) $e \rightarrow^* t$ iff $e \rightarrow^{l^*} t$ iff $\mathscr{P} \vdash_{CRWL} e \twoheadrightarrow t$.

Note that in part (a) we cannot ensure $e \rightarrow^* e'$ implies $e \rightarrow^{l^*} e'$, because term rewriting can reach some intermediate expressions not reachable by let-rewriting. For instance, given the deterministic program with the rules $g \rightarrow a$ and $f(x) \rightarrow c(x, x)$, we have $f(g) \rightarrow^* c(g, a)$, but $f(g) \not\rightarrow^{l^*} c(g, a)$. Still, part (a) is a strong completeness result for let-rewriting with respect to term rewriting for deterministic programs, since it says that the outer constructed part obtained in a rewriting derivation can also be obtained or even refined in a let-rewriting derivation. Combined with Theorem 17, part (a) expresses a kind of equivalence between let-rewriting and term rewriting, valid for general derivations, even non-terminating ones. For derivations reaching a constructor term (not further reducible), part (b) gives an even stronger equivalence result.

7.3 Let-narrowing versus narrowing for deterministic systems

Joining the results of the previous section with the adequacy of let-narrowing to let-rewriting, we can easily establish some relationships between let-narrowing and ordinary term rewriting/narrowing, summarized in the following result.

Theorem 20
For any program $\mathscr{P}$, $e \in Exp, \theta \in CSubst$ and $t \in CTerm$:

(a) If $e \leadsto^{l^*}_{\theta} t$ then $e\theta \rightarrow^* t$.
(b) If in addition $\mathscr{P}$ is CRWL-deterministic, then:

(b_1) If $e\theta \rightarrow^* t$ then $\exists t' \in CTerm,\ \sigma, \theta' \in CSubst$ such that $e \leadsto^{l^*}_{\sigma} t'$, $t'\theta' \equiv t$ and $\sigma\theta' = \theta[var(e)]$.
(b_2) If $e \leadsto^{*}_{\theta} t$, the same conclusion of (b_1) holds.

Part (a) expresses soundness of $\leadsto^{l}$ with respect to term rewriting, and part (b) is a completeness result for $\leadsto^{l}$ with respect to term rewriting/narrowing, for the class of deterministic programs.

Proof
Part (a) follows from soundness of let-narrowing with respect to let-rewriting (Theorem 14) and soundness of let-rewriting with respect to term rewriting of Theorem 19.

For part (b_1), for let-narrowing, assume $e\theta \rightarrow^* t$. By the completeness of let-rewriting with respect to term rewriting for deterministic programs (Theorem 19), we have $e\theta \rightarrow^{l^*} t$, and then by the completeness of let-narrowing with respect to let-rewriting (Theorem 15), there exists a narrowing derivation $e \leadsto^{l^*}_{\sigma} t'$ with $t'\theta' = t$ and $\sigma\theta' = \theta[FV(e)]$. But note that for $e \in Exp$, the sets $FV(e)$ and $var(e)$ coincide, and the proof is finished.

Finally, (b_2) follows simply from soundness of (ordinary) narrowing with respect to term rewriting and b_1). □

https://doi.org/10.1017/S1471068412000373 Published online by Cambridge University Press

8 Conclusions

This paper contains a thorough presentation of the theory of first order let-rewriting and let-narrowing for constructor-based term rewriting systems. These two relations are simple notions of one-step reduction that express sharing as it is required by the call-time choice semantics of non-determinism adopted in the functional logic programming paradigm. In a broad sense, let-rewriting and let-narrowing can be seen as particular syntactical presentations of term graph rewriting and narrowing. However, keeping our formalisms very close to the syntax and basic notions of term rewriting systems (terms, substitutions, syntactic unification, etc.) has been an essential aid in establishing strong equivalence results with respect to the CRWL-framework—a well-established realization of call-time choice semantics—which was one of the main aims of the paper.

Along the way of proving such equivalence, we have developed powerful semantic tools that are interesting in themselves. Most remarkably, the CRWL_{let}-logic, a conservative extension of CRWL that deals with let-bindings, and the notion of hypersemantics of expressions and contexts, for which we prove deep compositionality results not easily achievable by thinking directly in terms of reduction sequences.

We have shown in several places the methodological power of having provably equivalent reduction-based and logic-based semantics. In some occasions, we have used the properties of the CRWL-semantics to investigate interesting aspects of reductions, as replaceability conditions or derived operational rules, like bubbling. In others, we have followed the converse way. For instance, by transforming let-rewriting reductions into ordinary term rewriting reductions, we easily concluded that let-rewriting (call-time choice) provides less computed values than term rewriting (run-time choice). By using again semantic methods, we proved the opposite inclusion for deterministic programs, obtaining for such programs an equivalence result of let-rewriting and term rewriting.

In our opinion, the different pieces of this work can be used separately for different purposes. The CRWL_{let}-logic provides a denotational semantics reflecting call-time choice for programs making use of local bindings. The let-rewriting and let-narrowing relations provide clear and abstract descriptions of how computations respecting call-time choice can proceed. They can be useful to explain basic operational aspects of functional logic languages to students or novice programmers, for instance. They have also been used as underlying formalisms to investigate other aspects of functional logic programming that need a clear notion of reduction; for instance, when proving essential properties of type systems, like subject reduction or progress. In addition, all the pieces are interconnected by strong theoretical results, which may be useful depending on the pursued goal.

Just like classical term rewriting and narrowing, the let-rewriting and narrowing relations define too broad computation spaces as to be adopted directly as concrete operational procedures of a programming language. To that purpose, they should be accompanied by a strategy that selects only certain computations. In this paper, we have only given an example-driven discussion of strategies. We are quite confident that some known on-demand evaluation strategies, like lazy, needed or natural

https://doi.org/10.1017/S1471068412000373 Published online by Cambridge University Press

rewriting/narrowing, can be adapted to our formal setting. In Riesco and Rodríguez-Hortalá (2010) and Sánchez-Hernández (2011), we work out in more detail two concrete on-demand strategies for slight variants of let-rewriting and narrowing formalisms.

A subject of future work that might be of interest to the functional logic community is that of completing the comparison of different formalisms proposed in the field to capture call-time choice semantics: CRWL, admissible term graph rewriting/narrowing, natural semantics *à la* Launchbury, and let-rewriting/narrowing. Proving their equivalence would greatly enrich the set of tools available to the functional logic programming theoretician, since any known or future result obtained for one of the approaches could be applied to the rest on a sound technical basis.

References

Albert, E., Hanus, M., Huch, F., Oliver, J. and Vidal, G. 2005. Operational semantics for declarative multi-paradigm languages. *Journal of Symbolic Computation 40*, 1, 795–829.

Alpuente, M., Falaschi, M., Iranzo, P. J. and Vidal, G. 2003. Uniform lazy narrowing. *Journal of Logic and Computation 13*, 2, 287–312.

Antoy, S. 2005. Evaluation strategies for functional logic programming. *Journal of Symbolic Computation 40*, 1, 875–903.

Antoy, S., Brown, D. and Chiang, S. 2006. On the correctness of bubbling. In *17th International Conference on Rewriting Techniques and Applications (RTA'06)*, F. Pfenning, Ed. Lecture Notes in Computer Science, vol. 4098. Springer, Berlin, 35–49.

Antoy, S., Brown, D. and Chiang, S. 2007. Lazy context cloning for non-deterministic graph rewriting. *Electronic Notes in Theoretical Computer Science 176*, 1, 3–23.

Antoy, S., Echahed, R. and Hanus, M. 1994. A needed narrowing strategy. In *21st ACM SIGPLAN-SIGACT Symposium on Principles of Programming Languages (POPL'94)*, H.-J. Boehm, B. Lang and D. M. Yellin, Eds. ACM, New York, 268–279.

Antoy, S., Echahed, R. and Hanus, M. 2000. A needed narrowing strategy. *Journal of the ACM 47*, 4, 776–822.

Antoy, S. and Hanus, M. 2000. Compiling multi-paradigm declarative programs into prolog. In *3rd International Workshop on Frontiers of Combining Systems (FroCoS'00)*, H. Kirchner and C. Ringeissen, Eds. Lecture Notes in Computer Science, vol. 1794. Springer, Berlin, 171–185.

Antoy, S. and Hanus, M. 2006. Overlapping rules and logic variables in functional logic programs. In *22nd International Conference on Logic Programming (ICLP'06)*, S. Etalle and M. Truszczynski, Eds. Lecture Notes in Computer Science, vol. 4079. Springer, Berlin, 87–101.

Ariola, Z. M. and Arvind. 1995. Properties of a first-order functional language with sharing. *Theoretical Computer Science 146*, *1&2*, 69–108.

Ariola, Z. M. and Felleisen, M. 1997. The call-by-need lambda calculus. *Journal of Functional Programming 7*, 3, 265–301.

Ariola, Z. M., Felleisen, M., Maraist, J., Odersky, M. and Wadler, P. 1995. The call-by-need lambda calculus. In *22nd ACM SIGPLAN-SIGACT Symposium on Principles of Programming Languages (POPL'95)*, R. K. Cytron and P. Lee, Eds. ACM, New York, 233–246.

Baader, F. and Nipkow, T. 1998. *Term Rewriting and All That*. Cambridge University Press, Cambridge, England, UK.

https://doi.org/10.1017/S1471068412000373 Published online by Cambridge University Press

Barendregt, H. P., Eekelen, M. C. J. D., Glauert, J. R. W., Kennaway, J. R., Plasmeijer, M. J. and Sleep, M. R. 1987. Term graph rewriting. In *1st Parallel Architectures and Languages Europe (PARLE'87)*, vol. II, J. W. de Bakker, A. J. Nijman and P. C. Treleaven, Eds. Lecture Notes in Computer Science, vol. 259. Springer, Berlin, 141–158.

Brassel, B. and Huch, F. 2007. On a tighter integration of functional and logic programming. In *5th Asian Symposium on Programming Languages and Systems (APLAS'07)*, Z. Shao, Ed. Lecture Notes in Computer Science, vol. 4807. Springer, Berlin, 122–138.

Caballero, R. and Sánchez, J., Eds. 2006. *TOY: A Multiparadigm Declarative Language*, version 2.2.3. Technical report, Universidad Complutense de Madrid.

Cheong, P. and Fribourg, L. 1993. Implementation of narrowing: The Prolog-based approach. In *Logic Programming Languages: Constraints, Functions, and Objects*. MIT Press, Cambridge, MA, USA, 1–20.

DeGroot, D. and Lindstrom, G. Eds. 1986. *Logic Programming, Functions, Relations, and Equations*. Prentice Hall, Upper Saddle River, NJ, USA.

Dios-Castro, J. and López-Fraguas, F. J. 2007. Extra variables can be eliminated from functional logic programs. *Electronic Notes in Theoretical Computer Science 188*, 3–19.

Echahed, R. and Janodet, J.-C. 1997. *On Constructor-Based Graph Rewriting Systems*. Research Report 985-I, IMAG.

Echahed, R. and Janodet, J.-C. 1998. Admissible graph rewriting and narrowing. In *Joint International Conference and Symposium on Logic Programming (JICSLP'98)*. MIT Press, Cambridge, MA, USA, 325–340.

Escobar, S., Meseguer, J. and Thati, P. 2005. Natural narrowing for general term rewriting systems. In *16th International Conference on Rewriting Techniques and Applications (RTA'05)*, J. Giesl, Ed. Lecture Notes in Computer Science, vol. 3467. Springer, Berlin, 279–293.

González-Moreno, J. C., Hortalá-González, T., López-Fraguas, F. J. and Rodríguez-Artalejo, M. 1996. A rewriting logic for declarative programming. In *6th European Symposium on Programming (ESOP'96)*, H. R. Nielson, Ed. Lecture Notes in Computer Science, vol. 1058. Springer, Berlin, 156–172.

González-Moreno, J. C., Hortalá-González, T., López-Fraguas, F. J. and Rodríguez-Artalejo, M. 1999. An approach to declarative programming based on a rewriting logic. *Journal of Logic Programming 40*, 1, 47–87.

González-Moreno, J. C., Hortalá-González, T. and Rodríguez-Artalejo, M. 1997. A higher order rewriting logic for functional logic programming. In *14th International Conference on Logic Programming (ICLP'97)*. MIT Press, 153–167.

Hanus, M. 1994. The integration of functions into logic programming: From theory to practice. *Journal of Logic Programming 19–20*, 583–628.

Hanus, M., Ed. 2012. Curry: An Integrated Functional Logic Language (version 0.8.3), September 11, 2012 [online]. http://www.informatik.uni-kiel.de/~curry/report.html

Hanus, M. 2007. Multi-paradigm declarative languages. In *23rd International Conference on Logic Programming (ICLP'07)*, V. Dahl and I. Niemelä, Eds. Lecture Notes in Computer Science, vol. 4670. Springer, Berlin, 45–75.

Hanus, M., Kuchen, H. and Moreno-Navarro, J. J. 1995. Curry: A truly functional logic language. In *Workshop on Visions for the Future of Logic Programming (ILPS'95)*, 95–107.

Hullot, J. 1980. Canonical forms and unification. In *5th Conference on Automated Deduction (CADE'80)*, W. Bibel and R. A. Kowalski, Eds. Lecture Notes in Computer Science, vol. 87. Springer, Berlin, 318–334.

Hussmann, H. 1993. *Non-Determinism in Algebraic Specifications and Algebraic Programs*. Birkhäuser Verlag, Berlin.

https://doi.org/10.1017/S1471068412000373 Published online by Cambridge University Press

Kutzner, A. and Schmidt-Schauss, M. 1998. A non-deterministic call-by-need lambda calculus. In *3rd ACM SIGPLAN International Conference on Functional Programming (ICFP'98)*, M. Felleisen and P. Hudak, Eds. *ACM SIGPLAN Notices 34*, 1, 324–335.

Launchbury, J. 1993. A natural semantics for lazy evaluation. In *20th ACM SIGPLAN-SIGACT Symposium on Principles of Programming Languages (POPL'93)*, M. S. Van Deusen and B. Lang, Eds. ACM, New York, 144–154.

Loogen, R., López-Fraguas, F. J. and Rodríguez-Artalejo, M. 1993. A demand-driven computation strategy for lazy narrowing. In *5th International Symposium on Programming Language Implementation and Logic Programming (PLILP'93)*, M. Bruynooghe and J. Penjam, Eds. Lecture Notes in Computer Science, vol. 714. Springer, Berlin, 184–200.

López-Fraguas, F. J., Martin-Martin, E. and Rodríguez-Hortalá, J. 2010a. Liberal typing for functional logic programs. In *8th Asian Symposium on Programming Languages and Systems (APLAS'10)*, K. Ueda, Ed. Lecture Notes in Computer Science, vol. 6461. Springer, Berlin, 80–96.

López-Fraguas, F. J., Martin-Martin, E. and Rodríguez-Hortalá, J. 2010b. New results on type systems for functional logic programming. In *18th International Workshop on Functional and (Constraint) Logic Programming (WFLP'09)*, S. Escobar, Ed. Revised Selected Papers. Lecture Notes in Computer Science, vol. 5979. Springer, Berlin, 128–144.

López-Fraguas, F. J. and Rodrí-Hortalá, J. 2010. The full abstraction problem for higher order functional-logic programs. In *19th Workshop on Logic-based methods in Programming Environments (WLPE'09)*, R. Caballero and J. Gallagher, Eds. CoRR, arXiv:1002:1833.

López-Fraguas, F. J., Rodríguez-Hortalá, J. and Sánchez-Hernández, J. 2007a. Equivalence of two formal semantics for functional logic programs. *Electronic Notes in Theoretical Computer Science 188*, 117–142.

López-Fraguas, F. J., Rodríguez-Hortalá, J. and Sánchez-Hernández, J. 2007b. A simple rewrite notion for call-time choice semantics. In *9th International Conference on Principles and Practice of Declarative Programming (PPDP'07)*, M. Leuschel and A. Podelski, Eds. ACM, New York, 197–208.

López-Fraguas, F. J., Rodríguez-Hortalá, J. and Sánchez-Hernández, J. 2008. Rewriting and call-time choice: The HO case. In *9th International Symposium on Functional and Logic Programming (FLOPS'08)*, J. Garrigue and M. V. Hermenegildo, Eds. Lecture Notes in Computer Science, vol. 4989. Springer, Berlin, 147–162.

López-Fraguas, F. J., Rodríguez-Hortalá, J. and Sánchez-Hernández, J. 2009a. A flexible framework for programming with non-deterministic functions. In *2009 ACM SIGPLAN Symposium on Partial Evaluation and Program Manipulation (PEPM'09)*, G. Puebla and G. Vidal, Eds. ACM, New York, 91–100.

López-Fraguas, F. J., Rodríguez-Hortalá, J. and Sánchez-Hernández, J. 2009b. A fully abstract semantics for constructor based term rewriting systems. In *20th International Conference on Rewriting Techniques and Applications (RTA'09)*, R. Treinen, Ed. Lecture Notes in Computer Science, vol. 5595. Springer, Berlin, 320–334.

López-Fraguas, F. J., Rodríguez-Hortalá, J. and Sánchez-Hernández, J. 2009c. Narrowing for first order functional logic programs with call-time choice semantics. In *17th International Conference on Applications of Declarative Programming and Knowledge Management (INAP'07) and 21st Workshop on (Constraint) Logic Programming (WLP'07)*, D. Seipel, M. Hanus and A. Wolf, Eds. Revised Selected Papers. Lecture Notes in Artificial Intelligence, vol. 5437, Springer, Berlin, 206–222.

López-Fraguas, F. J. and Sánchez-Hernández, J. 1999. $\mathcal{TOY}$: A multiparadigm declarative system. In *10th International Conference on Rewriting Techniques and Applications (RTA'99)*, P. Narendran and M. Rusinowitch, Eds. Lecture Notes in Computer Science, vol. 1631. Springer, Berlin, 244–247.

https://doi.org/10.1017/S1471068412000373 Published online by Cambridge University Press

López-Fraguas, F. J. and Sánchez-Hernández, J. 2001. Functional logic programming with failure: A set-oriented view. In *8th International Conference on Logic for Programming and Automated Reasoning (LPAR'01)*, R. Nieuwenhuis and A. Voronkov, Eds. Lecture Notes in Artificial Intelligence, vol. 2250. Springer, Berlin, 455–469.

Maraist, J., Odersky, M. and Wadler, P. 1998. The call-by-need lambda calculus. *Journal of Functional Programming 8*, 3, 275–317.

McCarthy, J. 1963. A basis for a mathematical theory of computation. In *Computer Programming and Formal Systems*. North-Holland, the Netherlands, 33–70.

Moreno-Navarro, J. J. and Rodríguez-Artalejo, M. 1992. Logic programming with functions and predicates: The language Babel. *Journal of Logic Programming 12*, 189–223.

Plump, D. 1998. *Term Graph Rewriting*. Report CSI-R9822, Computing Science Institute, University of Nijmegen.

Plump, D. 2001. Essentials of term graph rewriting. *Electronic Notes Theoretical Computer Science 51*, 277–289.

Riesco, A. and Rodríguez-Hortalá, J. 2010. Programming with singular and plural non-deterministic functions. In *ACM SIGPLAN Workshop on Partial Evaluation and Program Manipulation (PEPM'10)*, J. P. Gallagher and J. Voigtländer, Eds. ACM, New York, 83–92.

Rodríguez-Artalejo, M. 2001. Functional and constraint logic programming. In *Revised Lectures of the International Summer School CCL'99*, H. Comon and C. Marché and R. Treinen, Eds. Lecture Notes in Computer Science, vol. 2002. Springer, Berlin, 202–270.

Sánchez-Hernández, J. 2004. *Una aproximación al fallo constructivo en programación declarativa multiparadigma*. PhD thesis, Departamento Sistemas Informáticos y Programación, Universidad Complutense de Madrid.

Sánchez-Hernández, J. 2011. Reduction strategies for rewriting with call-time choice. In *11th Jornadas sobre Programación y Lenguajes (PROLE'11)*, P. Arenas, V. Gulías and P. Nogueira, Eds. A Coruña, Spain

Schmidt-Schauss, M. and Machkasova, E. 2008. A finite simulation method in a non-deterministic call-by-need lambda-calculus with letrec, constructors, and case. In *19th International Conference on Rewriting Techniques and Applications (RTA'08)*, A. Voronkov, Ed. Lecture Notes in Computer Science, vol. 5117. Springer, Berlin, 321–335.

Sondergaard, H. and Sestoft, P. 1990. Referential transparency, definiteness and unfoldability. *Acta Informatica 27*, 6, 505–517.

Søndergaard, H. and Sestoft, P. 1992. Non-determinism in functional languages. *The Computer Journal 35*, 5, 514–523.

Vado-Vírseda, R. D. 2002. Estrategias de estrechamiento perezoso. Trabajo de Investigación de Tercer Ciclo, Dpto. de Sistemas Informáticos y Programación, Universidad Complutense de Madrid.

Vado-Vírseda, R. d. 2003. A demand-driven narrowing calculus with overlapping definitional trees. In *5th ACM SIGPLAN Conference on Principles and Practice of Declarative Programming (PPDP'03)*. ACM, New York, 213–227.

Zartmann, F. 1997. Denotational abstract interpretation of functional logic programs. In *4th International Symposium on Static Analysis (SAS'97)*, P. Van Hentenryck, Ed. Lecture Notes in Computer Science, vol. 1302. Springer, Berlin, 141–159.

https://doi.org/10.1017/S1471068412000373 Published online by Cambridge University Press