ON MAGNETO-THERMO-ELASTIC PLANE WAVES

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ABSTRACT. The propagation of plane waves in a thermo-elastic body placed in a magnetic field has been investigated. It is shown that the problem can be reduced to one of pure thermo-elasticity, provided that the longitudinal elastic wave velocity is suitably increased in the case of infinite electrical conductivity. For finite electrical conductivity and low frequency, however, the thermo-elastic coupling factor is to be increased.

1. *Introduction.* The theory of magneto-elasticity which deals with the effect of a magnetic field on the elastic deformation of solids has been the subject of investigation only in recent years, although its counterpart, magneto-fluid-dynamics, is much developed. A complete bibliography on the former subject can be found in a paper by Paria (1). If again the solid is subjected to a differential temperature distribution, there arises the problem of interaction between the three fields, namely, the magnetic field, the thermal field and the elastic field. Each of these contributes to the total deformation of the solid body. It will be seen in § 2 that these fields are not however independent, but are coupled with one another. The electro-magnetic field generated by the application of the primary magnetic force is described by the Maxwell equations, the thermal field by the modified heat conduction equation (2) and the elastic field by the Hooke–Duhamel–Neumann law. It may be mentioned in this connexion that the problem of interaction between thermal and elastic deformations has been formulated afresh by Biot (3) and applied to special problems by Paria (2) and others (4).

In the present paper the propagation of plane waves in a thermo-elastic solid placed in a magnetic field has been investigated. In § 2 the fundamental equations of magneto-thermo-elasticity are presented, while equations governing the plane wave motion are deduced in § 3. It is shown in § 4 that, in the special case when the electrical conductivity is large (infinite), the plane wave motion is the same as in the purely thermo-elastic body, provided that the magnitude of the longitudinal elastic wave velocity is suitably increased. § 5 is concerned with the determination of the wave-velocity equation for finite electrical conductivity. It is again shown there that, if the frequency is sufficiently smaller than a certain critical frequency, the attenuation factor as well as the phase velocity retain the same characteristics as in a purely thermo-elastic body, provided the thermo-elastic coupling factor is suitably increased.

2. *Fundamental equations.* The electromagnetic field is governed by the Maxwell equations

$$\left.\begin{aligned} \operatorname{curl} \mathbf{H} = \mathbf{j}, \quad \operatorname{curl} \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}, \\ \operatorname{div} \mathbf{B} = 0, \quad \mathbf{B} = \mu_e \mathbf{H}, \end{aligned}\right\} \tag{1}$$

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if the displacement current is neglected. The generalized Ohm's law in the deformable medium is (1)

$$\mathbf{j} = \sigma\left[\mathbf{E} + \frac{\partial \mathbf{u}}{\partial t} \times \mathbf{B}\right], \tag{2}$$

where $\mathbf{u}$ is the displacement vector in the strained solid. The stress-strain-temperature relation in a thermo-elastic solid is given by (2):

$$\tau_{ij} = 2\mu e_{ij} + (\lambda e - \beta T)\,\delta_{ij}, \tag{3}$$

where

$$\left.\begin{aligned} e_{ij} &= \tfrac{1}{2}(u_{i,j} + u_{j,i}), \\ e &= \operatorname{div} \mathbf{u}. \end{aligned}\right\} \tag{4}$$

The equation of motion is

$$\rho \frac{\partial^2 u_i}{\partial t^2} = \frac{\partial \tau_{ik}}{\partial x_k} + (\mathbf{j} \times \mathbf{B})_i. \tag{5}$$

The thermal field is described by the equation

$$\kappa \nabla^2 T = \rho c_v \frac{\partial T}{\partial t} + T_0 \beta \frac{\partial e}{\partial t}, \tag{6}$$

if the Joule heat is neglected. The symbols used in the above equations have their usual meanings as explained in the references quoted. It is seen from equations (2) and (6) that the electro-magnetic field and the thermal field are in interaction with the elastic field due to the presence of the terms $\partial \mathbf{u}/\partial t$ and $\partial e/\partial t$ respectively in them. The equations from (1) to (6) are the field equations in the theory of magneto-thermo-elasticity. They are to be solved under prescribed initial and boundary conditions which are electro-magnetic and thermal as well as mechanical in character. The non-linear terms occurring in any equation will be linearized by taking $\mathbf{H} = \mathbf{H}_0 + \mathbf{h}$, where $\mathbf{H}_0$ is the primary field, and the perturbation field $\mathbf{h}$ is so small that the products of $\mathbf{h}$ and $\mathbf{u}$ and their derivatives can be neglected.

If (3) and (4) are used, the equation (5) gives

$$\mu \nabla^2 \mathbf{u} + (\lambda + \mu) \operatorname{grad} e - \beta \operatorname{grad} T + (\mathbf{j} \times \mathbf{B}) = \rho \frac{\partial^2 \mathbf{u}}{\partial t^2}. \tag{7}$$

3. *One-dimensional deformation.* Let all the vectors be functions of x and t only, and independent of the y and z coordinates. We also assume that the displacement vector $\mathbf{u}$ has the components $(u, 0, 0)$ parallel to coordinate axes. The first relation of (1) then gives

$$j_x = 0, \quad j_y = -\frac{\partial H_z}{\partial x}, \quad j_z = \frac{\partial H_y}{\partial x}, \tag{8}$$

while the second and fourth relations simplify to

$$\frac{\partial H_x}{\partial t} = 0, \quad \frac{\partial E_z}{\partial x} = \mu_e \frac{\partial H_y}{\partial t}, \quad \frac{\partial E_y}{\partial x} = -\mu_e \frac{\partial H_z}{\partial t}. \tag{9}$$

The first relation of (9) implies that

$$H_x = 0 \tag{10}$$

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if initially no magnetic field was applied parallel to the x-axis. The third relation of (1) is then identically satisfied. The vector equation (2) has the components

$$\left.\begin{aligned} j_x &= \sigma E_x, \\ j_y &= \sigma\left[E_y - \mu_e H_z \frac{\partial u}{\partial t}\right], \\ j_z &= \sigma\left[E_z + \mu_e H_y \frac{\partial u}{\partial t}\right], \end{aligned}\right\} \tag{11}$$

so that, by (8),

$$E_x = 0. \tag{12}$$

By eliminating with the help of (8) and (9), the last two equations of (11) yield

$$\frac{\partial H_y}{\partial t} = \nu_H \frac{\partial^2 H_y}{\partial x^2} - \frac{\partial}{\partial x}\left(H_y \frac{\partial u}{\partial t}\right), \tag{13}$$

$$\frac{\partial H_z}{\partial t} = \nu_H \frac{\partial^2 H_z}{\partial x^2} - \frac{\partial}{\partial x}\left(H_z \frac{\partial u}{\partial t}\right), \tag{14}$$

where $\nu_H = (\mu_e \sigma)^{-1}$ is the magnetic viscosity (5). Equation (7) with the help of (8) reduces to

$$(\lambda + 2\mu)\frac{\partial^2 u}{\partial x^2} - \beta \frac{\partial T}{\partial x} - \frac{\partial}{\partial x}[\tfrac{1}{2}\mu_e(H_y^2 + H_z^2)] = \rho \frac{\partial^2 u}{\partial t^2}, \tag{15}$$

while (6) takes the form

$$\kappa \frac{\partial^2 T}{\partial x^2} = \rho c_v \frac{\partial T}{\partial t} + T_0 \beta \frac{\partial^2 u}{\partial x \partial t}. \tag{16}$$

If we put

$$H_y = h_y, \quad H_z = H_3 + h_z, \tag{17}$$

where h_y and h_z are small, then equations (13), (14) and (15) linearize to

$$\frac{\partial h_y}{\partial t} = \nu_H \frac{\partial^2 h_y}{\partial x^2}, \tag{18}$$

$$\frac{\partial h_z}{\partial t} = \nu_H \frac{\partial^2 h_z}{\partial x^2} - H_3 \frac{\partial^2 u}{\partial x \partial t}, \tag{19}$$

$$(\lambda + 2\mu)\frac{\partial^2 u}{\partial x^2} - \beta \frac{\partial T}{\partial x} - \mu_e H_3 \frac{\partial h_z}{\partial x} = \rho \frac{\partial^2 u}{\partial t^2}. \tag{20}$$

4. *Infinite conductivity.* In the special case when $\sigma \to \infty$, that is, $\nu_H = 0$, equation (18) implies that $h_y = 0$, while (19) gives

$$h_z = -H_3 \frac{\partial u}{\partial x}. \tag{21}$$

Substituting this value in (20) we obtain

$$c_1^2(1 + R_H)\frac{\partial^2 u}{\partial x^2} - \frac{\beta}{\rho}\frac{\partial T}{\partial x} = \frac{\partial^2 u}{\partial t^2}, \tag{22}$$

where $c_1 = [(\lambda + 2\mu)/\rho]^{\frac{1}{2}}$ is the longitudinal elastic wave velocity, and $R_H = (\mu_e H_3^2)/(\rho c_1^2)$ is the magnetic pressure number (5). In the absence of the magnetic field we have $H_3 = 0$, that is, $R_H = 0$, and in that case equation (22), together with equation (16), constitute the fundamental equations of pure thermo-elasticity. Thus, the effect of

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the applied magnetic field is to increase the elastic wave velocity from c_1 to $c_1(1+R_H)^{\frac{1}{2}}$, or, in other words, to increase the solidity of the solid body. This effect is in conformity with the 'frozen-in field' of magneto-hydrodynamics ((6), p. 81).

5. *Finite conductivity.* When σ is finite, we take

$$u = u' \exp[i(\gamma x - \omega t)],$$
$$h_z = h' \exp[i(\gamma x - \omega t)],$$
$$T = T' \exp[i(\gamma x - \omega t)],$$

where ω is real. Then equations (19), (20) and (16) yield

$$\left.\begin{aligned} H_3\gamma\omega u' + (\nu_H\gamma^2 - i\omega)\,h' &= 0,\\ (c_1^2\gamma^2 - \omega^2)\,u' + i(\mu_e/\rho)\,H_3\gamma h' + i(\beta/\rho)\,\gamma T' &= 0,\\ T_0\beta\gamma\omega u' + (\kappa\gamma^2 - i\rho c_v\omega)\,T' &= 0. \end{aligned}\right\} \tag{23}$$

Elimination of u', h' and T' from the above equations gives the wave-velocity equation

$$\begin{vmatrix} H_3\gamma\omega & \nu_H\gamma^2 - i\omega & 0\\ c_1^2\gamma^2 - \omega^2 & i(\mu_e/\rho)\,H_3\gamma & i(\beta/\rho)\,\gamma\\ T_0\beta\gamma\omega & 0 & \kappa\gamma^2 - i\rho c_v\omega \end{vmatrix} = 0. \tag{24}$$

If we define the characteristic frequency

$$\omega^* = \rho c_v c_1^2/\kappa,$$

and introduce the dimensionless quantities

$$\left.\begin{aligned} \chi = \omega/\omega^*, \quad \xi = \gamma c_1/\omega^*,\\ \epsilon_T = (\beta^2 T_0)/(\rho^2 c_v c_1^2), \quad \epsilon_H = (\nu_H\omega^*)/c^2, \end{aligned}\right\} \tag{25}$$

the determinantal equation (24) simplifies to

$$R_H\xi^2\chi(\chi + i\xi^2) + (\chi + i\epsilon_H\xi^2)\,[(\xi^2 - \chi^2)\,(\chi + i\xi^2) + \epsilon_T\xi^2\chi] = 0. \tag{26}$$

By putting $R_H = 0$, $\epsilon_H = 0$, we obtain the purely thermo-elastic equation corresponding to (26) as (4)

$$(\xi^2 - \chi^2)\,(\chi + i\xi^2) + \epsilon_T\xi^2\chi = 0. \tag{26'}$$

Equation (26) can be written in the form

$$\epsilon_H\xi^6 - \xi^4\chi[\epsilon_H\chi + i(1 + R_H + \epsilon_H + \epsilon_H\epsilon_T)] - \xi^2\chi^2[1 + R_H + \epsilon_T - i\chi(1 + \epsilon_H)] + \chi^4 = 0. \tag{27}$$

This is a cubic equation in ξ^2 with complex coefficients.

6. *Low frequency.* The characteristic frequency ω^* for some common metals at temperature 20 °C are given below (4).

	Aluminium	Copper	Iron	Lead
ω^*/sec.	$4{\cdot}66 \times 10^{11}$	$1{\cdot}73 \times 10^{11}$	$1{\cdot}75 \times 10^{12}$	$1{\cdot}91 \times 10^{11}$

Let it be assumed therefore that the frequency of wave ω is much less than the characteristic frequency ω^*, that is, $x \ll 1$. Consequently, the square and higher powers of χ can be neglected, and the equation (27) in that case gives

$$\xi = \pm(1+i)\,(\chi/2)^{\frac{1}{2}}\,[1 + \epsilon_T + (1 + R_H)/\epsilon_H]^{\frac{1}{2}}. \tag{28}$$

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With the same approximation the equation (26′) for a purely thermo-elastic case gives

$$\xi = \pm (1+i)(\chi/2)^{\frac{1}{2}}(1+\epsilon_T)^{\frac{1}{2}}. \tag{28′}$$

It is seen from (28) and (28′) that the effect of the magnetic field is to increase the thermo-elastic coupling factor ϵ_T by the amount $(1+R_H)/\epsilon_H$. With this modification the attenuation factor and the phase velocity retain the same characteristic as in purely thermo-elastic waves.

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