$$\begin{aligned} A_1 + A_2 &= (A_2 + A_4 + A_7) + (A_1 + A_3 + A_6) - (A_3 + A_4 + A_6 + A_7) \\ &= (A_5 + A_8) + \tfrac{1}{2}a^{r+1} - \tfrac{1}{2}c^{r+1} \\ &= (A_5 + A_8) + \tfrac{1}{2}c^{r+1} - \tfrac{1}{2}b^{r+1} \\ &= (A_5 + A_8) + (A_3 + A_4 + A_6 + A_7) - (A_6 + A_7 + A_8) \\ &= A_3 + A_4 + A_5. \end{aligned}$$

Hence, the c value is the same for these two cases. As a final comment regarding Figure 8, the reader may find it interesting to determine the values of r for which

$$(A_2 + A_4 + A_7) + (A_5 + A_8) = (A_1 + A_2) + (A_3 + A_4 + A_5)$$

that is, when the entire shaded region in Figure 2 equals the entire shaded region in Figure 6. (The reasoning is similar when $0 < r < 1$ and $r > 1$, but the number of regions differs in each case.)

With some thought and imagination, it is possible to create interesting and accessible problems for undergraduate students using the area and volume applications considered here. If nothing else, the examples presented in this Note provide some geometric motivation for power means associated with two positive numbers. We should mention that power means are defined for finite sets of non-negative numbers; see [1] for an extensive discussion of the properties of such means. (An Internet search also reveals a great deal of information about these generalised means.)

Reference

1. G. H. Hardy, J. E. Littlewood, and G. Pólya, *Inequalities* (2nd edn.), Cambridge University Press (1952).

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103.21 Closed form evaluation of a class of improper integrals

In the paper [1] which is part of a series of papers in a project to prove all the integrals in the book [2], the following integral is calculated ($n \geqslant 0$):

$$I = \int_0^{+\infty} \frac{\ln t}{(1 + t^2)^{n+1}}\, dt. \tag{1}$$

The calculation is based on properties of the gamma and digamma functions, and Feynman's trick of differentiating under the integral sign. In this Note we use a more straightforward method, based on properties of self-reciprocal polynomials, a topic that I first came across when still at school, and which is not very well-known.

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The main result in this paper relies on a closed form evaluation of the following indefinite integral:

$$\int \frac{t^{2n} - 1}{(1 + t^2)^{n+1}}\, dt. \tag{2}$$

This integral pops up if we rewrite (1) as:

$$\begin{aligned} I &= \int_0^1 \frac{\ln t}{(1 + t^2)^{n+1}}\, dt + \int_1^{+\infty} \frac{\ln t}{(1 + t^2)^{n+1}}\, dt \\ &= -\int_0^1 \frac{t^{2n} - 1}{(1 + t^2)^{n+1}} \ln t\, dt \end{aligned}$$

where we have used the substitution $t \to 1/t$ in the second integral in the sum. Note that both integrals in the sum converge; the first one is bounded by $\int_0^1 \ln t\, dt$, the second one by $\int_1^{+\infty} \frac{\ln t}{t^2}\, dt$.

We can write the result as the double integral

$$I = \int_0^1 \frac{t^{2n} - 1}{(1 + t^2)^{n+1}} \left(\int_t^1 \frac{1}{y}\, dy \right) dt$$

and if we change the order of integration we get a form with essentially the integral (2):

$$I = \int_0^1 \frac{1}{v} \left(\int_0^v \frac{t^{2n} - 1}{(1 + t^2)^{n+1}}\, dt \right) dv.$$

To calculate the indefinite integral (2), we start by factorising the polynomial in the numerator:

$$t^{2n} - 1 = \left(t^2 - 1\right)\left(t^{2n-2} + t^{2n-4} + \ldots + t^2 + 1\right).$$

The second factor is what we call a self-reciprocal polynomial (see [3]). A polynomial $p(t) = \sum_{i=0}^{m} a_i t^i$ is called *self-reciprocal* or *palindromic* if the coefficients satisfy $a_i = a_{m-i}$ for all i. If $m = 2k$ is even, such a polynomial can be written in the form $p(t) = t^k q\left(t + \frac{1}{t}\right)$ with q a polynomial of degree k in $t + \frac{1}{t}$.

In our case we have

$$t^{2n-2} + t^{2n-4} + \ldots + t^2 + 1 = t^{n-1} F_{n-1}(t)$$

with $F_{n-1}(t) = t^{n-1} + t^{n-3} + \ldots + \frac{1}{t^{n-3}} + \frac{1}{t^{n-1}}$, and the following theorem tells us how we can write $F_{n-1}(t)$ as a polynomial in $t + \frac{1}{t}$.

Theorem 1: For $n \geqslant 1$ we have

$$F_{n-1}(t) = \sum_{i=1}^{n} \binom{n - i}{n - 2i + 1} (-1)^{i-1} \left(t + \frac{1}{t}\right)^{n-2i+1}. \tag{3}$$

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Proof: Note that we assume that $\binom{m}{k} = 0$ if $k < 0$. This theorem is an immediate consequence of the relationship between $f_{m-1}(t)$, $f_m(t)$ and $f_{m+1}(t)$ with

$$f_k(t) = t^k + \frac{1}{t^k}.$$

It is easy to see that for $m \geqslant 1$

$$f_{m+1}(t) = \left(t + \frac{1}{t}\right) f_m(t) - f_{m-1}(t).$$

As a consequence we have that for $m \geqslant 1$

$$F_{m+1}(t) = \left(t + \frac{1}{t}\right) F_m(t) - F_{m-1}(t).$$

To prove this, use the following table:

$$\begin{aligned}
F_0(t) &= 1 \\
F_1(t) &= f_1(t) \\
F_2(t) &= 1 + f_2(t) \\
F_3(t) &= f_1(t) + f_3(t) \\
F_4(t) &= 1 + f_2(t) + f_4(t) \\
F_5(t) &= f_1(t) + f_3(t) + f_5(t) \\
F_6(t) &= 1 + f_2(t) + f_4(t) + f_6(t) \\
&\vdots
\end{aligned}$$

and the fact that $1 + f_2(t) = \left(t + \frac{1}{t}\right) f_1(t) - 1$.

Note that

$$F_0(t) = 1 = \sum_{i=1}^{1} \binom{1-i}{1-2i+1} (-1)^{i-1} \left(t + \frac{1}{t}\right)^{1-2i+1}$$

and

$$F_1(t) = t + \frac{1}{t} = \sum_{i=1}^{2} \binom{2-i}{2-2i+1} (-1)^{i-1} \left(t + \frac{1}{t}\right)^{2-2i+1}.$$

Hence (3) is satisfied for $n = 1$ and $n = 2$. Using induction and a well-known identity for the binomial coefficients, the theorem follows easily.

We use this theorem to rewrite (2) as an integral with variable of integration $u = t + \frac{1}{t}$:

$$\begin{aligned}
\int \frac{t^{2n} - 1}{(1 + t^2)^{n+1}}\, dt &= \int \frac{(t^2 - 1)\, t^{n-1} F_{n-1}(t)}{(1 + t^2)^{n+1}}\, dt \\
&= \int \frac{F_{n-1}(t)}{\left(t + \frac{1}{t}\right)^{n+1}} \left(1 - \frac{1}{t^2}\right) dt
\end{aligned}$$

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$$= \int \frac{F_{n-1}(t)}{\left(t + \frac{1}{t}\right)^{n+1}} d\left(t + \frac{1}{t}\right)$$

$$= \sum_{i=1}^{n} \binom{n-i}{n-2i+1} (-1)^{i-1} \int u^{-2i}\, du$$

leading immediately to

Theorem 2: For $n \geqslant 1$ we have

$$\int_0^v \frac{t^{2n}-1}{(1+t^2)^{n+1}}\, dt = \sum_{i=1}^{n} \binom{n-i}{n-2i+1} \frac{(-1)^i}{2i-1} \left(\frac{v}{v^2+1}\right)^{2i-1}.$$

Note that this theorem also holds for $n = 0$.

The same method can be applied to prove more generally

Theorem 3: For $n \geqslant k \geqslant 0$ we have

$$\int_0^v \frac{t^{2n-k}-t^k}{(1+t^2)^{n+1}}\, dt = \sum_{i=1}^{n-k} \binom{n-k-i}{n-k-2i+1} \frac{(-1)^i}{k+2i-1} \left(\frac{v}{v^2+1}\right)^{k+2i-1}.$$

We are now ready to calculate the integral we started off with. We have rewritten it as a double integral, and now use Theorem 2:

$$I = \int_0^1 \frac{1}{v} \left(\int_0^v \frac{t^{2n}-1}{(1+t^2)^{n+1}}\, dt \right) dv = \sum_{i=1}^{n} \binom{n-i}{n-2i+1} \frac{(-1)^i}{2i-1} \int_0^1 \frac{v^{2i-2}}{(v^2+1)^{2i-1}}\, dv.$$

This new integral is easy to calculate using the substitution $v = \tan z$:

$$\int_0^1 \frac{v^{2i-2}}{(v^2+1)^{2i-1}}\, dv = \int_0^{\frac{\pi}{4}} (\sin z \cos z)^{2i-2}\, dz = \frac{1}{2^{2i-1}} \int_0^{\frac{\pi}{2}} \sin^{2i-2} w\, dw$$

leading to a Wallis integral, for which we know that

$$\int_0^{\frac{\pi}{2}} \sin^{2m} w\, dw = \frac{1}{2^{2m}} \binom{2m}{m} \frac{\pi}{2}.$$

Combining all this, we get our final result.

Theorem 4: For $n \geqslant 0$ we have

$$\frac{2}{\pi} \int_0^{+\infty} \frac{\ln t}{(1+t^2)^{n+1}}\, dt = \sum_{i=1}^{n} \frac{(-1)^i}{2i-1} \binom{n-i}{n-2i+1} \binom{2i-2}{i-1} \frac{1}{2^{4i-3}}.$$

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Some remarks

1. If we compare this result with the one found in [1], we get the following combinatorial identity:

$$(A_n =) \sum_{i=1}^{n} \frac{(-1)^{i-1}}{2i-1} \binom{n-i}{n-2i+1} \binom{2i-2}{i-1} \frac{1}{2^{4i-3}} = \frac{\binom{2n}{n}}{2^{2n}} \sum_{i=1}^{n} \frac{1}{2i-1}$$

which I have not been able to prove directly. But I have tried: see the next remark.

2. If you want to simplify expressions like the one above, it is always a good idea to consult Sloane's On-Line Encyclopedia of Integer Sequences (OEIS) [4]. If we look for the sequence of the numerators of A_n, which starts with 1, 1, 23, 11, 563, 1627, …, we immediately find Sloane's sequence A002549, and as it turns out, the value of the integral in Theorem 4 is precisely the coefficient of x^n in the Maclaurin series of the function $f(x) = \dfrac{\ln(1-x)}{2\sqrt{1-x}}$. Note that this series can be found by multiplying the series for $\ln(1-x)$ with the binomial series for $\dfrac{1}{\sqrt{1-x}}$. The result is

$$\frac{\ln(1-x)}{2\sqrt{1-x}} = \sum_{n=1}^{\infty} a_n x^n \text{ with } a_n = -\sum_{i=1}^{n} \binom{2i-2}{i-1} \frac{1}{2^{2i-1}(n-i+1)}.$$

This does not help much but leads to the conjecture that A_n can be written in yet another way:

$$A_n = \sum_{i-1}^{n} \binom{2i-2}{i-1} \frac{1}{2^{2i-1}(n-i+1)}.$$

3. It is possible to prove the equality of these three expressions using the sophisticated machinery of Zeilberger's algorithm, which is explained in the very nice (and freely downloadable) book A=B [5] and which is implemented in most computer algebra systems. Zeilberger's algorithm finds a linear recurrence relation for a given sequence. It turns out that for the three sequences above the recurrence is essentially the same.

References

1. T. Amdeberhan, V. H. Moll, J. Rosenberg, A. Straub and P. Whitworth, *The integrals in Gradshteyn and Ryzhik. Part 9: combinations of logarithms, rational and trigonometric functions*, available at https://arxiv.org/pdf/0707.2124.pdf
2. I. S. Gradshteyn and I. M. Ryzhik, *Table of integrals, series, and products* (7th edn.), Academic Press (Elsevier) (2007).
3. D. Kalman, *Polynomia and related realms. Uncommon mathematical excursions*, Dolciani Mathematical Expositions #35, Mathematical Association of America (2009).

https://doi.org/10.1017/mag.2019.68 Published online by Cambridge University Press

4. N. J. A. Sloane, *The On-Line Encyclopedia of Integer Sequences*, http://oeis.org
5. M. Petkovšek, H. Wilf and D. Zeilberger, *A=B*, A K Peters, Ltd. (1996). See http://www.math.upenn.edu/~wilf/AeqB.html.

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103.22 On a problem of Sastry and a theorem of Serenus

Let ABC be a triangle with side lengths a, b and c, and with angles A, B and C in the standard order. Let Ω be the circumcircle of ABC, as shown in Figure 1. Define $\overset{\frown}{BC}$ to be the arc BC that does *not* contain A, and define $\overset{\frown}{CA}$ and $\overset{\frown}{AB}$ similarly. Unless otherwise stated, these arcs are taken to be *closed* arcs.

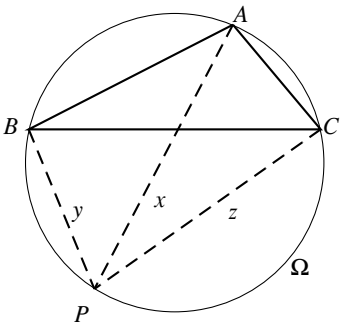


FIGURE 1

In [1], Sastry raises the question whether there exists a point P on the *open* arc $\overset{\frown}{BC}$ for which

$$PB + PC = PA. \tag{1}$$

He proves that if ABC is not equilateral, then such a point exists if, and only if, a lies (strictly) between b and c, and that in this case P is unique. He also proves that P bisects $\overset{\frown}{BC}$ if, and only if, a is the arithmetic mean of b and c. The problem is motivated by the fact that for an equilateral triangle ABC, (1) holds for every $P \in \overset{\frown}{BC}$. This is an immediate consequence of Ptolemy's theorem

$$(PA)(BC) = (PB)(CA) + (PC)(AB).$$

In this Note, we give a shorter treatment of Sastry's problem, and we also give a new proof of a theorem of Serenus.

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