J. Plasma Phys. (2017), *vol.* 83, 725830601 © Cambridge University Press 2017
doi:10.1017/S0022377817000885

Longitudinal sound waves in a collisionless, quasineutral plasma

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(Received 18 October 2017; revised 2 November 2017; accepted 2 November 2017)

The time evolution of slow sound waves in a homogeneous, collisionless and quasineutral plasma, in particular their Landau damping, is investigated using the kinetic-magnetohydrodynamics formulation of Ramos (*J. Plasma Phys.* vol. 81, 2015 p. 905810325; vol. 82, 2016 p. 905820607). In this approach, the electric field is eliminated from a closed, hybrid fluid-kinetic system that ensures automatically the fulfilment of the charge neutrality condition. Considering the time dependence of a spatial-Fourier-mode linear perturbation with wavevector parallel to the equilibrium magnetic field, this can be cast as a second-order self-adjoint problem with a continuum spectrum of real and positive squared frequencies. Therefore, a conventional resolution of the identity with a continuum basis of singular normal modes is guaranteed, which simplifies significantly a Van Kampen-like treatment of the Landau damping. The explicit form of such singular normal modes is obtained, along with their orthogonality relations. These are used to derive the damped time evolution of the fluid moments of solutions of initial-value problems, for the most general kinds of initial conditions. The non-zero parallel electric field is not used explicitly in this analysis, but it is calculated from any given solution after the later has been obtained.

Key words: plasma waves

1. Introduction

The linear theory of collisionless plasma waves and their Landau damping is a well-known area of plasma physics that is covered in familiar textbooks such as Stix (1962), Krall & Trivelpiece (1973), Goldston & Rutherford (2000), Hazeltine & Waelbroeck (2004), Bers (2016). The standard approach is based on the Vlasov–Maxwell system (Vlasov–Poisson for electrostatic problems) and on the plasma dielectric tensor and associated 'effective dispersion relation' derived following Landau's analytic continuation and complex contour integration for the Laplace-transformed solution of an initial-value problem (Landau 1946). Among the thus obtained plasma waves is the slow sound wave, more often called ion-acoustic wave in kinetic contexts. This wave does not perturb the magnetic field and is the low-frequency counterpart of the Langmuir plasma wave (Kovrizhnykh 1960; Kutsenko & Stepanov 1960; Barnes 1966). It involves perturbations of the densities,

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https://doi.org/10.1017/S0022377817000885 Published online by Cambridge University Press

the pressures and the components of the electric field and the plasma motion in the direction of the propagation wavevector. The standard analysis of this slow sound (ion-acoustic) wave, as described in the above quoted textbooks, yields the result that it is strongly Landau damped unless the electron-to-ion mass ratio and equilibrium temperature ratio are $m_e/m_i \ll 1$ and $T_{e0}/T_{i0} \gg 1$, so that a propagation phase velocity much greater than the ion thermal velocity but much smaller than the electron thermal velocity can exist. In this case, a weakly damped solution is found for a wave that propagates with phase velocity

$$\frac{\omega_0}{k_\parallel} = \left(\frac{T_{e0}}{m_i}\right)^{1/2} \tag{1.1}$$

and small damping rate

$$\gamma_0 = -|\omega_0| \left(\frac{\pi}{8}\right)^{1/2} \left[\left(\frac{m_e}{m_i}\right)^{1/2} + \left(\frac{T_{e0}}{T_{i0}}\right)^{3/2} \exp\left(-\frac{T_{e0}}{2T_{i0}}\right)\right]. \tag{1.2}$$

This result has been simplified by taking the quasineutral limit of interest ($k_\parallel \lambda_D \to 0$ where $k_\parallel$ is the wavenumber and λ_D is the Debye length) in the more general Vlasov–Poisson solution. It applies both to unmagnetized plasmas and to magnetized plasmas provided the propagation wavevector is parallel to the magnetic field.

The present work revisits the linear theory of longitudinal sound waves in a magnetized, collisionless and quasineutral plasma with a homogeneous and Maxwellian equilibrium background. Instead of being based on the Vlasov–Poisson system, it will use as starting foundation the kinetic-magnetohydrodynamics (KMHD) model, already reduced for wavelengths much larger than the Debye length and the Larmor radii, and for frequency scales much lower than the plasma and cyclotron frequencies. Most importantly, the analysis will be based on the continuum of singular normal modes of the system, i.e. its Van Kampen modes. The Van Kampen normal-mode approach is less popular but has a number of advantages over Landau's approach. It makes clear the nature of the Landau damping, as the result of the mixing of the phases of different spectral components of the perturbation, not normal modes with complex frequency. It also makes obvious and not paradoxical the time-reversal invariance of the damped solutions, as should be the case in the considered dissipation-free, collisionless theory. Finally, it allows to take into account the most general initial conditions, whereas it should be borne in mind that the standard result (1.1, 1.2) applies only to a subset of initial conditions.

In its original formulation (Van Kampen 1955; Case 1959), the normal-mode approach had the drawback of being mathematically awkward because the considered normal modes were not eigenfunctions of a self-adjoint operator. This necessitated the consideration of two dual bases of normal modes (left and right bases) and the explicit demonstration of their completeness (resolution of the identity theorem). On the contrary, the formulation to be presented here yields a basis of normal modes made of eigenfunctions of a self-adjoint operator. Therefore, its completeness is guaranteed by the spectral theorem without the need for any additional proof and a transparent solution of the most general initial-value problems is readily obtained, much like in ordinary quantum mechanics. In particular, it is straightforward to prove that, for any functional dependence of the macroscopic variables versus time that is consistent with the fluid moment equations and has a sufficiently regular Fourier

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transform, one can construct an initial condition for the distribution functions whose dynamical evolution yields such prescribed time dependence of the corresponding fluid moments. Such a result was first observed in a nonlinear solution for high-frequency plasma waves (Bernstein, Greene & Kruskal 1957), but it will appear in the present work as a simple corollary of the linear analysis of low-frequency sound waves.

2. Kinetic magnetohydrodynamics and its linear wave equations

Kinetic magnetohydrodynamics is a hybrid fluid-kinetic model of a collisionless plasma (made of electrons and one ion species of unit charge for simplicity), applicable to the quasineutral and strong magnetization limits where the space and time scales of interest are much greater than those associated with the plasma Langmuir oscillations and the particle Larmor gyrations (Chandrasekhar, Kaufman & Watson 1957; Kruskal & Oberman 1958; Rosenbluth & Rostoker 1959; Kurlsrud 1962). The formulation to be adopted here is that of Ramos (2015, 2016) which uses as dynamical variables the magnetic field $\boldsymbol{B}(\boldsymbol{x}, t)$, the quasineutral density $n(\boldsymbol{x}, t)$, the single-fluid macroscopic velocity $\boldsymbol{u}(\boldsymbol{x}, t)$ and the zero-Larmor-radius ion and electron distribution functions considered as functions of space, time and the random velocity in the reference frame of the macroscopic flow, $\boldsymbol{w} = \boldsymbol{v} - \boldsymbol{u}$, independent of its gyrophase angle, $f_s(\boldsymbol{x}, t, w_\parallel, w_\perp)$:

$$\frac{\partial \boldsymbol{B}}{\partial t} = \boldsymbol{\nabla} \times (\boldsymbol{u} \times \boldsymbol{B}), \quad \boldsymbol{j} = \boldsymbol{\nabla} \times \boldsymbol{B} \tag{2.1a,b}$$

$$\frac{\partial n}{\partial t} + \boldsymbol{\nabla} \cdot (n\boldsymbol{u}) = 0, \quad \rho = (m_i + m_e)n \tag{2.2a,b}$$

$$\rho \left[\frac{\partial \boldsymbol{u}}{\partial t} + (\boldsymbol{u} \cdot \boldsymbol{\nabla})\boldsymbol{u} \right] - \boldsymbol{j} \times \boldsymbol{B} + \sum_{s=i,e} \boldsymbol{\nabla} \cdot \boldsymbol{P}_s = 0 \tag{2.3}$$

$$\frac{\partial f_s}{\partial t} + \left(\boldsymbol{u} + w_\parallel \boldsymbol{b}\right) \cdot \left. \frac{\partial f_s}{\partial \boldsymbol{x}} \right|_{w_\parallel, w_\perp} + \frac{w_\perp}{2} \left[(\boldsymbol{bb} - \boldsymbol{I}) : (\boldsymbol{\nabla u}) + w_\parallel \boldsymbol{b} \cdot \boldsymbol{\nabla} \ln B \right] \frac{\partial f_s}{\partial w_\perp}$$
$$+ \left[\frac{\boldsymbol{b} \cdot (\boldsymbol{\nabla} \cdot \boldsymbol{P}_s)}{m_s n} - w_\parallel (\boldsymbol{bb}) : (\boldsymbol{\nabla u}) - \frac{w_\perp^2}{2} \boldsymbol{b} \cdot \boldsymbol{\nabla} \ln B \right] \frac{\partial f_s}{\partial w_\parallel} = 0, \tag{2.4}$$

where the ion- and electron-pressure tensors are

$$\boldsymbol{P}_s = p_{s\perp} \boldsymbol{I} + (p_{s\parallel} - p_{s\perp}) \boldsymbol{bb} \tag{2.5}$$

with $\boldsymbol{b} = \boldsymbol{B}/B$ denoting the magnetic unit vector, $\boldsymbol{I}$ denoting the identity tensor, and the parallel and perpendicular pressures being the corresponding moments of the distribution functions of each species

$$p_{s\parallel} = m_s \int \mathrm{d}^3\boldsymbol{w}\, w_\parallel^2 f_s, \quad p_{s\perp} = \frac{m_s}{2} \int \mathrm{d}^3\boldsymbol{w}\, w_\perp^2 f_s. \tag{2.6a,b}$$

The electric field has been eliminated algebraically and does not appear explicitly in this closed system (2.1)–(2.6) of KMHD equations.

A linear wave analysis for small-amplitude perturbations about a stable, homogenous and Maxwellian equilibrium without flow is to be carried out, with the subscripts 0 and 1 to be used to denote respectively equilibrium and perturbation quantities. Thus,

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the equilibrium magnetic field, density and temperatures ($\boldsymbol{B}_0, n_0, T_{s0}$) are constant, the equilibrium fluid velocity is $\boldsymbol{u}_0 = 0$ and the equilibrium distribution functions are

$$f_{Ms0} = \left(\frac{m_s}{2\pi}\right)^{3/2} \frac{n_0}{T_{s0}^{3/2}} \exp\left(-\frac{m_s w^2}{2T_{s0}}\right). \tag{2.7}$$

Since the equilibrium is spatially homogeneous, the perturbation can be analysed as a superposition of independent Fourier modes. Considering one such spatial-Fourier-mode with wavevector $\boldsymbol{k}$, one has $\partial/\partial \boldsymbol{x} = \mathrm{i}\boldsymbol{k}$. Then, introducing the fluid displacement $\boldsymbol{\xi}$ such that the perturbed fluid velocity is $\boldsymbol{u}_1 = \partial \boldsymbol{\xi}/\partial t$, the linearized KMHD system yields

$$\boldsymbol{B}_1 = \mathrm{i}\boldsymbol{k} \times (\boldsymbol{\xi}_\perp \times \boldsymbol{B}_0), \quad \boldsymbol{j}_1 = \mathrm{i}\boldsymbol{k} \times \boldsymbol{B}_1 \tag{2.8a,b}$$

$$n_1 = -\mathrm{i} n_0 \boldsymbol{k} \cdot \boldsymbol{\xi} \tag{2.9}$$

$$\rho_0 \frac{\partial^2 \boldsymbol{\xi}}{\partial t^2} = \boldsymbol{F}_\perp^F[\boldsymbol{\xi}_\perp] - \mathrm{i} \sum_{s=i,e} (p_{s\parallel}[f_{s1}] k_\parallel \boldsymbol{b}_0 + p_{s\perp}[f_{s1}] \boldsymbol{k}_\perp), \tag{2.10}$$

where $\boldsymbol{F}_\perp^F$ is the perpendicular ideal-MHD force operator

$$\boldsymbol{F}_\perp^F[\boldsymbol{\xi}_\perp] = \boldsymbol{j}_1 \times \boldsymbol{B}_0 = -B_0^2 \left[k_\parallel^2 \boldsymbol{\xi}_\perp + (\boldsymbol{k}_\perp \cdot \boldsymbol{\xi}_\perp)\boldsymbol{k}_\perp\right] \tag{2.11}$$

and the perturbed pressures are

$$p_{s\parallel}[f_{s1}] = m_s \int \mathrm{d}^3\boldsymbol{w}\, w_\parallel^2 f_{s1}, \quad p_{s\perp}[f_{s1}] = \frac{m_s}{2} \int \mathrm{d}^3\boldsymbol{w}\, w_\perp^2 f_{s1}. \tag{2.12a,b}$$

The perturbed distribution functions are expressed as sums of their even and odd parts with respect to $w_\parallel$ and the linearized drift-kinetic equation for $f_{s1} = f_{s1}^{\mathrm{even}} + f_{s1}^{\mathrm{odd}}$ splits into the system

$$\frac{\partial f_{s1}^{\mathrm{even}}}{\partial t} + \mathrm{i} k_\parallel w_\parallel f_{s1}^{\mathrm{odd}} + \frac{\mathrm{i} m_s}{T_{s0}} \left(w_\parallel^2 k_\parallel \frac{\partial \xi_\parallel}{\partial t} + \frac{w_\perp^2}{2} \boldsymbol{k}_\perp \cdot \frac{\partial \boldsymbol{\xi}_\perp}{\partial t}\right) f_{Ms0} = 0 \tag{2.13}$$

$$\frac{\partial f_{s1}^{\mathrm{odd}}}{\partial t} + \mathrm{i} k_\parallel w_\parallel f_{s1}^{\mathrm{even}} - \frac{\mathrm{i} k_\parallel w_\parallel}{n_0 T_{s0}} p_{s\parallel}[f_{s1}^{\mathrm{even}}] f_{Ms0} = 0. \tag{2.14}$$

This system satisfies the constraints $\int \mathrm{d}^3\boldsymbol{w}\, w_\parallel f_{s1}^{\mathrm{odd}} = 0$ and $\int \mathrm{d}^3\boldsymbol{w}\, f_{s1}^{\mathrm{even}} = -\mathrm{i} n_0 \boldsymbol{k} \cdot \boldsymbol{\xi}$ at all times once they are imposed on the initial condition, therefore the consistency with quasineutrality and fluid continuity (2.9) is guaranteed automatically.

After eliminating $\xi_\parallel$ and f_{s1}^{odd}, the spatial-Fourier-mode system becomes a second-order linear problem with respect to time for $\boldsymbol{\xi}_\perp(t)$ and $f_{s1}^{\mathrm{even}}(t, w_\parallel, w_\perp)$:

$$-\frac{\partial^2}{\partial t^2} \begin{pmatrix} \boldsymbol{\xi}_\perp \\ f_{s1}^{\mathrm{even}} \end{pmatrix} = \mathcal{L} \begin{pmatrix} \boldsymbol{\xi}_\perp \\ f_{s'1}^{\mathrm{even}} \end{pmatrix}, \tag{2.15}$$

where the linear operator $\mathcal{L}$ is

$$\mathcal{L} = \begin{pmatrix} -\rho_0^{-1} \boldsymbol{F}_\perp^F & \mathrm{i}\rho_0^{-1} \boldsymbol{k}_\perp \sum_{s'=i,e} p_{s'\perp} \\ -\mathrm{i} B_0^2 \Phi_{s0} k^2 \boldsymbol{k}_\perp \cdot & H_s + \Phi_{s0} k_\perp^2 \sum_{s'=i,e} p_{s'\perp} \end{pmatrix} \tag{2.16}$$

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with

$$\Phi_{s0} \equiv \frac{m_s w_\perp^2}{2\rho_0 T_{s0}} f_{Ms0} \tag{2.17}$$

and

$$H_s\left[(f_1^{\text{even}})\right] = k_\parallel^2 w_\parallel^2 f_{s1}^{\text{even}} - k_\parallel^2 w_\parallel^2 \frac{f_{Ms0}}{\rho_0 T_{s0}} \left(m_{s'} p_{s\parallel}[f_{s1}^{\text{even}}] - m_s p_{s'\parallel}[f_{s'1}^{\text{even}}]\right)_{s'\neq s}. \tag{2.18}$$

For $\boldsymbol{k}_\perp = 0$ the operator $\mathcal{L}$ becomes block diagonal, with a fluid block that describes the degenerate ideal-MHD shear-Alfvén and compressional-Alfvén waves characterized by $f_{s1}^{\text{even}} = 0$ and $-\partial^2 \boldsymbol{\xi}_\perp / \partial t^2 = -\rho_0^{-1} \boldsymbol{F}_\perp^F[\boldsymbol{\xi}_\perp] = \rho_0^{-1} B_0^2 k_\parallel^2 \boldsymbol{\xi}_\perp \neq 0$, decoupled from a kinetic block that describes the longitudinal sound waves characterized by $\boldsymbol{\xi}_\perp = 0$ and $-\partial^2 f_{1s}^{\text{even}} / \partial t^2 = H_s\left[(f_1^{\text{even}})\right] \neq 0$.

3. Longitudinal sound waves

From this point on, the analysis will concentrate on the main subject of this work, namely the purely kinetic problem of longitudinal sound waves characterized by $\boldsymbol{k}_\perp = \boldsymbol{\xi}_\perp = \boldsymbol{B}_1 = 0$ and $f_{s1}^{\text{even}} \neq 0$. After integration over $w_\perp$, a system is obtained which involves only the functions of $w_\parallel^2$

$$g_{s1}(t, w_\parallel^2) = 2\pi \int_0^\infty \mathrm{d}w_\perp w_\perp f_{s1}^{\text{even}}(t, w_\parallel, w_\perp) \tag{3.1}$$

and

$$g_{Ms0}(w_\parallel^2) = 2\pi \int_0^\infty \mathrm{d}w_\perp w_\perp f_{Ms0}(w^2) = \left(\frac{m_s}{2\pi}\right)^{1/2} \frac{n_0}{T_{s0}^{1/2}} \exp\left(-\frac{m_s w_\parallel^2}{2T_{s0}}\right). \tag{3.2}$$

Then, calling $\zeta \equiv w_\parallel^2$, this second-order system for $g_{s1}(t, w_\parallel^2)$ can be written as

$$-\frac{1}{k_\parallel^2}\frac{\partial^2}{\partial t^2}\begin{pmatrix} g_{i1} \\ g_{e1} \end{pmatrix} = \mathcal{H}\begin{pmatrix} g_{i1} \\ g_{e1} \end{pmatrix}, \tag{3.3}$$

where the linear operator $\mathcal{H}$ is

$$\mathcal{H} = \begin{pmatrix} \zeta\left[1 - \rho_0^{-1} T_{i0}^{-1} g_{Mi0}(\zeta)\ m_e p_{i\parallel}\right] & \rho_0^{-1} T_{i0}^{-1} \zeta\, g_{Mi0}(\zeta) m_i p_{e\parallel} \\ \rho_0^{-1} T_{e0}^{-1} \zeta\, g_{Me0}(\zeta) m_e p_{i\parallel} & \zeta\left[1 - \rho_0^{-1} T_{e0}^{-1} g_{Me0}(\zeta)\ m_i p_{e\parallel}\right] \end{pmatrix} \tag{3.4}$$

with

$$p_{s\parallel}[g_{s1}] = m_s \int_0^\infty \mathrm{d}\zeta\, \zeta^{1/2} g_{s1}. \tag{3.5}$$

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The operator $\mathcal{H}$ is self-adjoint. Defining the following scalar product in the space of pairs of ion and electron distribution functions:

$$\langle (h)|(g)\rangle = \sum_{s=i,e} \int_0^\infty \frac{\mathrm{d}\zeta}{\zeta^{1/2}} \frac{T_{s0}}{g_{Ms0}(\zeta)} h_s^*(\zeta) g_s(\zeta), \tag{3.6}$$

the scalar product $\langle (h)|\mathcal{H}[(g)]\rangle$ can be cast in the Hermite-symmetric form

$$\langle (h)|\mathcal{H}[(g)]\rangle = \sum_{s=i,e} \int_0^\infty \mathrm{d}\zeta\, \zeta^{1/2} \frac{T_{s0}}{g_{Ms0}(\zeta)} h_s^*(\zeta) g_s(\zeta)$$
$$- \frac{m_i m_e}{\rho_0} \left(\frac{1}{m_i} p_{i\|}[h_i^*] - \frac{1}{m_e} p_{e\|}[h_e^*] \right) \left(\frac{1}{m_i} p_{i\|}[g_i] - \frac{1}{m_e} p_{e\|}[g_e] \right) = \langle \mathcal{H}[(h)]|(g)\rangle. \tag{3.7}$$

Moreover, setting $(h) = (g)$ and using the Cauchy–Schwarz inequality

$$\int_0^\infty \mathrm{d}\zeta\, \zeta^{1/2} \frac{T_{s0}}{g_{Ms0}(\zeta)} |g_s(\zeta)|^2 \geqslant \frac{m_s}{n_0} \left| \int_0^\infty \mathrm{d}\zeta\, \zeta^{1/2} g_s(\zeta) \right|^2 = \frac{1}{m_s n_0} \left| p_{s\|}[g_s] \right|^2, \tag{3.8}$$

one gets

$$\langle (g)|\mathcal{H}[(g)]\rangle \geqslant \frac{1}{\rho_0} \left| p_{i\|}[g_i] + p_{e\|}[g_e] \right|^2 \tag{3.9}$$

so $\mathcal{H}$ is a positive operator.

4. Continuum of singular normal modes

The normal modes of the longitudinal sound wave system (3.3)–(3.5) are separable solutions of the form

$$g_{s1}(t, \zeta) = g_s^\lambda(\zeta) \exp(-\mathrm{i}\omega t), \tag{4.1}$$

where the label λ is the squared phase velocity, $\lambda \equiv \omega^2/k_\|^2$, so that (g^λ) is an eigenfunction of the operator $\mathcal{H}$ with eigenvalue λ

$$\mathcal{H} \begin{pmatrix} g_i^\lambda \\ g_e^\lambda \end{pmatrix} = \lambda \begin{pmatrix} g_i^\lambda \\ g_e^\lambda \end{pmatrix}. \tag{4.2}$$

Since $\mathcal{H}$ is self-adjoint and positive, the λ spectrum is real and positive, therefore the normal-mode frequencies ω are real.

The normal-mode eigenvalue equations (4.2) can be expressed as

$$(\zeta - \lambda) g_s^\lambda(\zeta) = G_s^\lambda(\zeta) \tag{4.3}$$

with

$$G_s^\lambda(\zeta) \equiv \frac{\zeta}{\rho_0 T_{s0}} g_{Ms0}(\zeta) \left(m_{s'} p_{s\|}[g_s^\lambda] - m_s p_{s'\|}[g_{s'}^\lambda] \right)_{s' \neq s} \tag{4.4}$$

and, for any $\lambda > 0$, this has the singular function solution (a distribution in the mathematical sense)

$$g_s^\lambda(\zeta) = \mathcal{P} \frac{G_s^\lambda(\zeta)}{(\zeta - \lambda)} + \Lambda_s(\lambda) \lambda^{-1/2} \delta(\zeta - \lambda), \tag{4.5}$$

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where $\mathcal{P}$ stands for the Cauchy principal value and δ is the Dirac distribution. The coefficient $\Lambda_s(\lambda)$ can be specified by the condition that the parallel pressure moment of g_s^λ be equal to some arbitrarily chosen value, $p_{s\parallel}^\lambda$:

$$p_{s\parallel}[g_s^\lambda] = m_s \int_0^\infty \mathrm{d}\zeta\, \zeta^{1/2} g_s^\lambda(\zeta) = p_{s\parallel}^\lambda. \tag{4.6}$$

This yields

$$m_s \Lambda_s(\lambda) = p_{s\parallel}^\lambda - \frac{(m_{s'} p_{s\parallel}^\lambda - m_s p_{s'\parallel}^\lambda)_{s'\neq s}}{m_i + m_e} \left[1 + 2\hat{\lambda}_s W(\hat{\lambda}_s)\right], \tag{4.7}$$

where

$$\hat{\lambda}_s \equiv \frac{m_s \lambda}{2T_{s0}} \tag{4.8}$$

is the ratio of the squared phase velocity to the squared thermal velocity of the species s, and

$$W(\hat{\lambda}) \equiv \pi^{-1/2} \int_0^\infty \mathrm{d}\hat{\zeta}\, \hat{\zeta}^{1/2} \mathcal{P} \frac{\exp(-\hat{\zeta})}{\hat{\zeta} - \hat{\lambda}} \tag{4.9}$$

which is related to the plasma dispersion function Z by $W(\hat{\lambda}) = 1 + \hat{\lambda}^{1/2} \mathrm{Re} Z(\hat{\lambda}^{1/2})$ and has the asymptotic behaviours

$$W(\hat{\lambda} \to 0) = 1 - 2\hat{\lambda} + \cdots, \quad W(\hat{\lambda} \to \infty) = -\frac{1}{2\hat{\lambda}} - \frac{3}{4\hat{\lambda}^2} + \cdots. \tag{4.10a,b}$$

Bringing this result to (4.5), one gets the expression of the general normal-mode solution

$$\begin{aligned} g_s^\lambda(\zeta) \;=\; & \frac{p_{i\parallel}^\lambda + p_{e\parallel}^\lambda}{m_i + m_e} \lambda^{-1/2} \delta(\zeta - \lambda) \\ & + \frac{(m_{s'} p_{s\parallel}^\lambda - m_s p_{s'\parallel}^\lambda)_{s'\neq s}}{T_{s0}(m_i + m_e)} \left[\frac{g_{Ms0}(\zeta)}{n_0} \mathcal{P} \frac{\zeta}{\zeta - \lambda} - W(\hat{\lambda}_s)\; \lambda^{1/2} \delta(\zeta - \lambda)\right]. \end{aligned} \tag{4.11}$$

This solution is identically consistent with quasineutrality, parallel momentum conservation and continuity. Taking its density moment, one gets

$$n_s[g_s^\lambda] = \int_0^\infty \mathrm{d}\zeta\, \zeta^{-1/2} g_s^\lambda(\zeta) = \frac{p_{i\parallel}^\lambda + p_{e\parallel}^\lambda}{\lambda(m_i + m_e)} = \frac{n_0 k_\parallel^2 (p_{i\parallel}^\lambda + p_{e\parallel}^\lambda)}{\rho_0\; \omega^2} \tag{4.12}$$

which is the same for either species s. Besides, the fluid parallel momentum conservation equation (the parallel component of (2.10) and the fluid continuity equation (2.9), in the $\boldsymbol{k}_\perp = 0$ case under consideration, yield

$$\mathrm{i} k_\parallel (p_{i\parallel}^\lambda + p_{e\parallel}^\lambda) = \rho_0 \omega^2 \xi_\parallel^\lambda = \frac{\mathrm{i} \rho_0 \omega^2 n_1^\lambda}{n_0 k_\parallel} \tag{4.13}$$

which means that the kinetically defined perturbed densities $n_s[g_s^\lambda]$ are equal to the fluid-defined perturbed density n_1^λ.

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For any $\lambda > 0$ and any choice of the parameters $p^{\lambda}_{i\|}$ and $p^{\lambda}_{e\|}$, there is a normal-mode solution given by (4.11). Thus, the operator $\mathcal{H}$ has a doubly degenerate continuum spectrum, extending over the real and positive λ semiaxis. The general normal-mode solution (4.11) can be written as a linear combination of two independent normal modes for each λ: an ion-pressure mode, to be denoted by $g^{i,\lambda}_s(\zeta)$ and defined by setting $p^{\lambda}_{e\|} = 0$ (zero electron parallel pressure); and an electron-pressure mode to be denoted by $g^{e,\lambda}_s(\zeta)$ and defined by setting $p^{\lambda}_{i\|} = 0$ (zero ion parallel pressure). It is convenient to normalize them to unit density, which results in the expression

$$g^{\sigma,\lambda}_s(\zeta) = \lambda^{1/2}\delta(\zeta-\lambda) + \frac{m_i m_e e_s \lambda}{m_\sigma e_\sigma T_{s0}}\left[\frac{g_{Ms0}(\zeta)}{n_0}\mathcal{P}\frac{\zeta}{\zeta-\lambda} - W(\hat{\lambda}_s)\lambda^{1/2}\delta(\zeta-\lambda)\right], \tag{4.14}$$

where e_s is the species electric charge ($e_i = -e_e = e$). In this notation, the upper Greek indices (the continuum index $\lambda > 0$ and the discrete index $\sigma = i, e$) label the mode, which has two distribution function components labelled by the lower discrete index $s = i, e$. These distribution functions are real and have the moments

$$n_s[g^{\sigma,\lambda}_s] = 1, \quad p_{s\|}[g^{\sigma,\lambda}_s] = \lambda(m_i + m_e)\delta^{\sigma}_s, \tag{4.15a,b}$$

where δ^{σ}_s is the Kronecker delta. The last piece of information needed is the orthogonality relations among these normal modes. The calculation is detailed in appendix A and yields

$$\left\langle (g^{\sigma,\lambda}) \middle| (g^{\sigma',\lambda'}) \right\rangle = \sum_{s=i,e}\int_0^{\infty}\frac{\mathrm{d}\zeta}{\zeta^{1/2}}\frac{T_{s0}}{g_{Ms0}(\zeta)}g^{\sigma,\lambda}_s(\zeta)g^{\sigma',\lambda'}_s(\zeta) = \gamma^{\sigma\sigma'}\delta(\lambda-\lambda'), \tag{4.16}$$

where

$$\begin{aligned}\gamma^{\sigma\sigma'} \;=\; & \lambda^{1/2}\sum_{s=i,e}\frac{T_{s0}}{g_{Ms0}(\lambda)}\left\{\pi^2\frac{m_i^2 m_e^2 e^2}{m_\sigma m_{\sigma'} e_\sigma e_{\sigma'}}\lambda^3\frac{g_{Ms0}(\lambda)^2}{n_0^2 T_{s0}^2}\right. \\ & \left. + \left[1 - \frac{2m_i m_e e_s}{m_s m_\sigma e_\sigma}\hat{\lambda}_s W(\hat{\lambda}_s)\right]\left[1 - \frac{2m_i m_e e_s}{m_s m_{\sigma'} e_{\sigma'}}\hat{\lambda}_s W(\hat{\lambda}_s)\right]\right\}.\end{aligned} \tag{4.17}$$

Thus, normal modes with different λ eigenvalues are orthogonal as expected from the self-adjointness of $\mathcal{H}$. Within the two-dimensional subspaces spanned by eigenfunctions with the same λ, the two linearly independent normal modes selected are not orthogonal. It is possible to find a linear combination of them that is orthogonal, but it is more advantageous to work with the choice of non-orthogonal modes (4.14) that have the simpler fluid moments (4.15). The metric matrix $\gamma^{\sigma\sigma'}$ is real and symmetric, and its determinant $D(\lambda) = \gamma^{ii}\gamma^{ee} - \gamma^{ie}\gamma^{ei}$ is positive:

$$\begin{aligned}D(\lambda) \;=\; & \frac{\lambda(m_i+m_e)^2 T_{i0}T_{e0}}{g_{Mi0}(\lambda)g_{Me0}(\lambda)}\left\{\pi^2\lambda^3\left[\frac{g_{Mi0}(\lambda)}{n_0 T_{i0}} + \frac{g_{Me0}(\lambda)}{n_0 T_{e0}}\right]^2\right. \\ & \left. + 4\left[\frac{\hat{\lambda}_i}{m_i}W(\hat{\lambda}_i) + \frac{\hat{\lambda}_e}{m_e}W(\hat{\lambda}_e)\right]^2\right\}.\end{aligned} \tag{4.18}$$

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5. Initial-value problems

Once the normal modes (4.14) of the longitudinal sound wave system (3.3)–(3.5) have been obtained, one can readily solve for any initial-value problem. The normal modes $g_s^{\sigma,\lambda}(\zeta)$ constitute a complete continuum basis in the space of square-integrable functions with the scalar product (3.6), because they are singular eigenfunctions of a self-adjoint operator. Therefore, any initial conditions for distribution functions g_{s1} belonging to such Hilbert space can be expanded as

$$g_{s1}(0,\zeta)=\sum_{\sigma=i,e}\int_0^\infty \mathrm{d}\lambda\,\lambda^{-1/2}C_\sigma(\lambda)g_s^{\sigma,\lambda}(\zeta) \tag{5.1}$$

$$\frac{\partial g_{s1}(0,\zeta)}{\partial t}=k_\parallel\sum_{\sigma=i,e}\int_0^\infty \mathrm{d}\lambda\,S_\sigma(\lambda)g_s^{\sigma,\lambda}(\zeta). \tag{5.2}$$

Then, the solution of the corresponding initial-value problem is

$$g_{s1}(t,\zeta)=\sum_{\sigma=i,e}\int_0^\infty \mathrm{d}\lambda\,\lambda^{-1/2}g_s^{\sigma,\lambda}(\zeta)\left[C_\sigma(\lambda)\cos(\lambda^{1/2}k_\parallel t)+S_\sigma(\lambda)\sin(\lambda^{1/2}k_\parallel t)\right] \tag{5.3}$$

and, using the results of (4.15), its macroscopic fluid moments are

$$n_{s1}(t)=\sum_{\sigma=i,e}\int_0^\infty \mathrm{d}\lambda\,\lambda^{-1/2}\left[C_\sigma(\lambda)\cos(\lambda^{1/2}k_\parallel t)+S_\sigma(\lambda)\sin(\lambda^{1/2}k_\parallel t)\right] \tag{5.4}$$

$$p_{s\parallel 1}(t)=(m_i+m_e)\int_0^\infty \mathrm{d}\lambda\,\lambda^{1/2}\left[C_s(\lambda)\cos(\lambda^{1/2}k_\parallel t)+S_s(\lambda)\sin(\lambda^{1/2}k_\parallel t)\right]. \tag{5.5}$$

This solution is manifestly invariant under the time reversal

$$t\to -t,\quad g_{s1}(0,\zeta)\to g_{s1}(0,\zeta),\quad \partial g_{s1}(0,\zeta)/\partial t\to -\partial g_{s1}(0,\zeta)/\partial t \tag{5.6a–c}$$

and, for sufficiently well behaved $C_\sigma(\lambda)$, $S_\sigma(\lambda)$, the moments (5.4) and (5.5) decay to zero as $t\to\pm\infty$ by virtue of the Riemann–Lebesgue lemma. This demonstrates the Landau damping of the macroscopic variables in their time-reversal-invariant dynamical evolution, as the consequence of the superposition of a continuum of spectral components with rapidly varying phases.

Again, it can be verified that the initial-value solution (5.4) and (5.5) is identically consistent with quasineutrality, continuity and parallel momentum conservation. The perturbed density (5.4) is the same for either species s and its second time derivative is related to the perturbed pressures (5.5) by

$$(m_i+m_e)\frac{\partial^2 n_1}{\partial t^2}=-k_\parallel^2(p_{i\parallel 1}+p_{e\parallel 1}). \tag{5.7}$$

This becomes the parallel momentum conservation equation, $\rho_0\partial^2\xi_\parallel/\partial t^2=-\mathrm{i}k_\parallel(p_{i\parallel 1}+p_{e\parallel 1})$, after substituting the continuity equation, $n_1=-\mathrm{i}n_0k_\parallel\xi_\parallel$.

Rewriting (5.4) and (5.5) in terms of $\omega=k_\parallel\lambda^{1/2}$, one obtains the relations

$$n_1(t)=\left(\frac{2}{\pi}\right)^{1/2}\int_0^\infty \mathrm{d}\omega\left[\tilde{n}_C(\omega)\cos(\omega t)+\tilde{n}_S(\omega)\sin(\omega t)\right] \tag{5.8}$$

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$$p_{s\|1}(t) = \left(\frac{2}{\pi}\right)^{1/2} \int_0^\infty \mathrm{d}\omega \left[\tilde{p}_{s\|C}(\omega)\cos(\omega t) + \tilde{p}_{s\|S}(\omega)\sin(\omega t)\right], \tag{5.9}$$

where

$$\tilde{n}_C(\omega) = \frac{(2\pi)^{1/2}}{k_\|}\left[C_i\left(\frac{\omega^2}{k_\|^2}\right) + C_e\left(\frac{\omega^2}{k_\|^2}\right)\right], \quad \tilde{n}_S(\omega) = \frac{(2\pi)^{1/2}}{k_\|}\left[S_i\left(\frac{\omega^2}{k_\|^2}\right) + S_e\left(\frac{\omega^2}{k_\|^2}\right)\right] \tag{5.10a,b}$$

$$\tilde{p}_{s\|C}(\omega) = \frac{(2\pi)^{1/2}(m_i+m_e)}{k_\|^3}\omega^2 C_s\left(\frac{\omega^2}{k_\|^2}\right), \quad \tilde{p}_{s\|S}(\omega) = \frac{(2\pi)^{1/2}(m_i+m_e)}{k_\|^3}\omega^2 S_s\left(\frac{\omega^2}{k_\|^2}\right). \tag{5.11a,b}$$

So, up to the multiplicative constants specified in (5.10) and (5.11), $C_i(\omega^2/k_\|^2) + C_e(\omega^2/k_\|^2)$ is the cosine Fourier transform of $n_1(t)$ and $\omega^2 C_s(\omega^2/k_\|^2)$ is the cosine Fourier transform of $p_{s\|1}(t)$; and similarly for $S_\sigma(\omega^2/k_\|^2)$ and the sine Fourier transforms. This implies that there is an invertible correspondence between the initial conditions $[g_{s1}(0, w_\|^2), \partial g_{s1}(0, w_\|^2)/\partial t]$ and the functions of time $p_{s\|1}(t)$ whose Fourier transforms exist and are such that the functions $C_\sigma(\lambda)$, $S_\sigma(\lambda)$ defined by (5.11) yield square-integrable distribution functions when substituted in (5.1) and (5.2). The fluid continuity and parallel momentum conservation equations relate $n_1(t)$ and $\xi_\|(t)$ to $p_{s\|1}(t)$ and, as shown in appendix B, the parallel electric field $E_\|(t)$ is also determined by $p_{s\|1}(t)$. Therefore, for any functional dependence of the macroscopic variables versus time (consistent with the fluid moment equations and subject to the above mathematical requirements) one can construct an initial condition for the distribution functions whose dynamical evolution yields such prescribed time dependence of the corresponding macroscopic moments. This property was first observed in a nonlinear solution for high-frequency plasma waves (Bernstein *et al.* 1957) and appears naturally in the present linear analysis of the quasineutral, low-frequency sound waves.

It is useful to classify the initial conditions into two categories. The first one, that will be called 'standard initial conditions', corresponds to the cases when the functions $C_\sigma(\lambda)$, $S_\sigma(\lambda)$ are inversely proportional to the normal-mode metric determinant $D(\lambda)$ (4.18) in such a way that, if $D(\lambda)$ became close to zero for a narrow interval around some λ_0, the Fourier transforms of the fluid moments would have a sharp peak for frequencies around $k_\|\lambda_0^{1/2}$. Then, $n_1(t)$, $\xi_\|(t)$, $p_{s\|1}(t)$ would oscillate with a well-defined frequency $\omega_0 = k_\|\lambda_0^{1/2}$ and a small damping rate. The second category, that will be called 'special initial conditions', corresponds to the cases when $C_\sigma(\lambda)$, $S_\sigma(\lambda)$ are not inversely proportional to $D(\lambda)$. For these, anything consistent with quasineutrality and the fluid equations is possible as shown above. Examples of these two different kinds of initial conditions are given next.

As an example of standard initial conditions, consider

$$g_{s1}(0, \zeta) = \frac{n_1(0)}{n_0} g_{Ms0}(\zeta) \tag{5.12}$$

and

$$\frac{\partial g_{s1}(0, \zeta)}{\partial t} = 0, \tag{5.13}$$

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i.e. an initial perturbation of the distribution functions proportional to the equilibrium Maxwellians, with just a perturbed density, and with zero initial time derivative. Then, $S_\sigma(\lambda) = 0$ and the projection of the expression (5.1) and (5.12) for $g_{s1}(0, \zeta)$ onto the normal-mode basis, making use of the orthogonality relations (4.16) and (4.17), yields

$$n_0 \lambda^{-1/2} \left[\gamma^{ii} C_i(\lambda) + \gamma^{ie} C_e(\lambda)\right] = n_1(0)(T_{i0} + T_{e0}) \tag{5.14}$$

$$n_0 \lambda^{-1/2} \left[\gamma^{ie} C_i(\lambda) + \gamma^{ee} C_e(\lambda)\right] = n_1(0)(T_{i0} + T_{e0}). \tag{5.15}$$

Solving for the combination that gives the Fourier transform of the perturbed density, $C_i(\lambda) + C_e(\lambda)$ is inversely proportional to $D(\lambda)$:

$$C_i(\lambda) + C_e(\lambda) = \frac{n_1(0)(T_{i0} + T_{e0})\lambda^{1/2}(\gamma^{ii} - 2\gamma^{ie} + \gamma^{ee})}{n_0\ D(\lambda)}. \tag{5.16}$$

The denominator $D(\lambda)$ can be near zero for real λ if and only if $m_e/m_i \ll 1$ and $T_{i0}/T_{e0} \ll 1$, so that $[\hat{\lambda}_i W(\hat{\lambda}_i)/m_i + \hat{\lambda}_e W(\hat{\lambda}_e)/m_e]^2$ can have a zero with $\hat{\lambda}_i \gg 1$ and $\hat{\lambda}_e \ll 1$, for which the other squared term in the expression of $D(\lambda)$ (4.18) is small. Then, using the asymptotic forms (4.10), one can approximate

$$\frac{\hat{\lambda}_i}{m_i} W(\hat{\lambda}_i) + \frac{\hat{\lambda}_e}{m_e} W(\hat{\lambda}_e) \simeq \frac{\lambda}{2T_{e0}} - \frac{1}{2m_i} \tag{5.17}$$

which has a zero at $\lambda_0 = T_{e0}/m_i \equiv c_S^2$. Substituting the approximation (5.17) and setting $\lambda = \lambda_0$ in the remaining terms of (5.16), one obtains the approximation for the Fourier transform of the density

$$\frac{\tilde{n}_C(\omega)}{n_1(0)} \simeq \left(\frac{8}{\pi}\right)^{1/2} \frac{\eta_0\ \omega_0^3}{(\omega^2 - \omega_0^2)^2 + 4\eta_0^2\omega_0^4}, \tag{5.18}$$

where $\omega_0 = k_\parallel (T_{e0}/m_i)^{1/2} = k_\parallel c_S$ and

$$\eta_0 = \left(\frac{\pi}{8}\right)^{1/2} \left[\left(\frac{m_e}{m_i}\right)^{1/2} + \left(\frac{T_{e0}}{T_{i0}}\right)^{3/2} \exp\left(-\frac{T_{e0}}{2T_{i0}}\right)\right] \ll 1. \tag{5.19}$$

Its inverse Fourier transform yields

$$\frac{n_1(t)}{n_1(0)} \simeq \exp(-\eta_0 c_S |k_\parallel t|) \left[\cos(c_S k_\parallel t) + \eta_0 \sin(c_S |k_\parallel t|)\right], \tag{5.20}$$

in agreement with the standard textbook result (1.1) and (1.2) for the quasineutral ion-acoustic wave. Consistent with the time-reversal invariance and the assumed zero-time-derivative initial condition, $n_1(t)$ is an even function of time with the same decaying behaviour as $t \to \pm\infty$.

As an example of special initial conditions, consider now the following:

$$g_{s1}(0, \zeta) = \int_0^\infty \mathrm{d}\lambda\, \lambda^{-1/2} C_i(\lambda) g_s^{i,\lambda}(\zeta), \quad g_{i1}(0, \zeta) = \frac{n_1(0)}{n_0} g_{Mi0}(\zeta) \tag{5.21a,b}$$

and

$$\frac{\partial g_{s1}(0, \zeta)}{\partial t} = 0. \tag{5.22}$$

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This initial perturbation has a zero time derivative and an ion distribution function proportional to the equilibrium Maxwellian with just the perturbation of the density, like in the standard initial condition example discussed above. The difference is that now the initial electron distribution function is adjusted so that the overall perturbation is in the space of the ion-pressure modes, with zero perturbed electron parallel pressure. In this case, $C_e(\lambda) = S_\sigma(\lambda) = 0$. The projection of such initial perturbation onto the ion-pressure normal modes yields

$$C_i(\lambda) = \frac{n_1(0) g_{Mi0}(\lambda)}{n_0} \left\{ \left[1 - \frac{2m_e}{m_i} \hat{\lambda}_i W(\hat{\lambda}_i) \right]^2 + \frac{\pi^2 m_e^2 \lambda^3 g_{Mi0}(\lambda)^2}{T_{i0}^2 n_0^2} \right\}^{-1} \tag{5.23}$$

which is not inversely proportional to $D(\lambda)$ and does not have any sharp peak if $m_e/m_i \leqslant 1$ (the equality applying to an electron–positron plasma). Therefore, the macroscopic fluid variables have a broad frequency spectrum and, as functions of time, they decay without oscillating. In particular, for $m_e/m_i \ll 1$ the Fourier transform of the density reduces to

$$\frac{\tilde{n}_C(\omega)}{n_1(0)} \simeq \left(\frac{m_i}{T_{i0} k_\parallel^2} \right)^{1/2} \exp\left(-\frac{m_i \omega^2}{2 T_{i0} k_\parallel^2} \right) \tag{5.24}$$

hence

$$\frac{n_1(t)}{n_1(0)} \simeq \exp\left(-\frac{T_{i0} k_\parallel^2 t^2}{2 m_i} \right). \tag{5.25}$$

Finally, consider a second example of special initial conditions, similar to the previous one, but with the roles of the ion and electron species reversed:

$$g_{s1}(0, \zeta) = \int_0^\infty \mathrm{d}\lambda \, \lambda^{-1/2} C_e(\lambda) g_s^{e,\lambda}(\zeta), \quad g_{e1}(0, \zeta) = \frac{n_1(0)}{n_0} g_{Me0}(\zeta) \tag{5.26a,b}$$

and

$$\frac{\partial g_{s1}(0, \zeta)}{\partial t} = 0. \tag{5.27}$$

Now $C_i(\lambda) = S_\sigma(\lambda) = 0$, and the projection of the initial perturbation onto the electron-pressure normal modes yields

$$C_e(\lambda) = \frac{n_1(0) g_{Me0}(\lambda)}{n_0} \left\{ \left[1 - \frac{2m_i}{m_e} \hat{\lambda}_e W(\hat{\lambda}_e) \right]^2 + \frac{\pi^2 m_i^2 \lambda^3 g_{Me0}(\lambda)^2}{T_{e0}^2 n_0^2} \right\}^{-1}. \tag{5.28}$$

This function is not inversely proportional to $D(\lambda)$, but it has a sharp peak if $m_e/m_i \ll 1$ and $\hat{\lambda}_e \ll 1$, independent of the temperature ratio T_{i0}/T_{e0}. Then, using the small-argument asymptotic form (4.10), one can approximate

$$1 - \frac{2m_i}{m_e} \hat{\lambda}_e W(\hat{\lambda}_e) \simeq 1 - \frac{m_i \lambda}{T_{e0}} \tag{5.29}$$

which, again, has a zero at $\lambda_0 = T_{e0}/m_i \equiv c_S^2$. Substituting the approximation (5.29) and setting $\lambda = \lambda_0$ in the remaining terms of (5.28), one obtains an approximation for

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the Fourier transform of the density that has the same form as (5.18), but with the different resonance width parameter

$$\eta_0' = \left(\frac{\pi m_e}{8m_i}\right)^{1/2} \ll 1 \tag{5.30}$$

instead of η_0. The corresponding time evolution of the density perturbation is

$$\frac{n_1(t)}{n_1(0)} \simeq \exp(-\eta_0' c_S|k_\parallel t|)\left[\cos(c_S k_\parallel t) + \eta_0' \sin(c_S|k_\parallel t|)\right]. \tag{5.31}$$

This is a weakly damped wave that propagates with the standard phase velocity $\omega_0/k_\parallel = (T_{e0}/m_i)^{1/2}$, and whose small damping rate is equal to the zero-ion-temperature limit of the standard one (1.2) and (5.19), but is applicable to any value of T_{i0}/T_{e0} not necessarily small.

6. Summary

Kinetic magnetohydrodynamics provides a natural framework to study the collisionless plasma waves in their low-frequency, quasineutral limit. This work has carried out a linear analysis of such longitudinal sound (or ion-acoustic) waves about a homogeneous and Maxwellian equilibrium, based on a KMHD formulation that eliminates the electric field and is identically consistent with quasineutrality and the fluid continuity and parallel momentum conservation equations. This approach has also the desirable feature that the dynamical evolution of the sound wave perturbation is described by a self-adjoint system. The corresponding complete basis of singular (Van Kampen) normal modes was derived and it was used in a straightforward treatment of the initial-value problem and the Landau damping, for the most general kinds of initial conditions. Considering several explicit examples of initial conditions, some non-conventional solutions in addition to the standard 'ion-acoustic effective dispersion relation' (1.1) and (1.2) were obtained. The invariance under time reversal and the property that, provided it is consistent with the fluid moment equations, any sufficiently well-behaved dependence of the macroscopic variables as functions of time can be realized with an appropriate choice of distribution function initial conditions, appear also naturally in this KMHD-based analysis.

Acknowledgements

The author thanks J. Freidberg and A. Ram for their interest and helpful discussions. This work was sponsored by the US Department of Energy under grant no. DEFG02-91ER54109 at the Massachusetts Institute of Technology.

Appendix A. Derivation of the normal-mode orthogonality relations

This appendix outlines the derivation of the orthogonality relations (4.16) and (4.17) among the longitudinal sound wave normal modes. Substituting the normal-mode expression (4.14) and carrying out the trivial integration of Dirac deltas, one obtains

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$$\int_0^\infty \frac{\mathrm{d}\zeta}{\zeta^{1/2}} \frac{T_{s0}}{g_{Ms0}(\zeta)} g_s^{\sigma,\lambda}(\zeta) g_s^{\sigma',\lambda'}(\zeta) = \frac{m_i^2 m_e^2 e^2 \lambda\lambda'}{m_\sigma m_{\sigma'} e_\sigma e_{\sigma'}} \int_0^\infty \mathrm{d}\zeta\, \zeta^{3/2} \frac{g_{Ms0}(\zeta)}{n_0^2 T_{s0}} \mathcal{P}\frac{1}{\zeta-\lambda}\mathcal{P}\frac{1}{\zeta-\lambda'}$$
$$+ \frac{\lambda\lambda'}{n_0}\left\{\frac{m_i m_e e_s}{m_{\sigma'} e_{\sigma'}}\left[1-\frac{m_i m_e e_s}{T_{s0} m_\sigma e_\sigma}\lambda W(\hat{\lambda}_s)\right] - \frac{m_i m_e e_s}{m_\sigma e_\sigma}\left[1-\frac{m_i m_e e_s}{T_{s0} m_{\sigma'} e_{\sigma'}}\lambda' W(\hat{\lambda}'_s)\right]\right\}\mathcal{P}\frac{1}{\lambda-\lambda'}$$
$$+ \frac{\lambda^{1/2} T_{s0}}{g_{Ms0}(\lambda)}\left[1-\frac{m_i m_e e_s}{T_{s0} m_\sigma e_\sigma}\lambda W(\hat{\lambda}_s)\right]\left[1-\frac{m_i m_e e_s}{T_{s0} m_{\sigma'} e_{\sigma'}}\lambda W(\hat{\lambda}_s)\right]\delta(\lambda-\lambda'). \tag{A 1}$$

The integral in the first term of this expression is calculated with the help of the identity

$$\mathcal{P}\frac{1}{\zeta-\lambda}\mathcal{P}\frac{1}{\zeta-\lambda'} = \mathcal{P}\frac{1}{\lambda-\lambda'}\left(\mathcal{P}\frac{1}{\zeta-\lambda} - \mathcal{P}\frac{1}{\zeta-\lambda'}\right) + \pi^2\ \delta(\lambda-\lambda')\delta(\zeta-\lambda). \tag{A 2}$$

Recalling also the definition of the W function (4.9), the result is

$$\int_0^\infty \mathrm{d}\zeta\, \zeta^{3/2} g_{Ms0}(\zeta)\mathcal{P}\frac{1}{\zeta-\lambda}\mathcal{P}\frac{1}{\zeta-\lambda'}$$
$$= n_0\left[\lambda W(\hat{\lambda}_s) - \lambda' W(\hat{\lambda}'_s)\right]\mathcal{P}\frac{1}{\lambda-\lambda'} + \pi^2\lambda^{3/2} g_{Ms0}(\lambda)\delta(\lambda-\lambda'), \tag{A 3}$$

after which (A 1) reduces to

$$\int_0^\infty \frac{\mathrm{d}\zeta}{\zeta^{1/2}} \frac{T_{s0}}{g_{Ms0}(\zeta)} g_s^{\sigma,\lambda}(\zeta)\ g_s^{\sigma',\lambda'}(\zeta) = \frac{m_i m_e e_s \lambda\lambda'}{n_0}\left(\frac{1}{m_{\sigma'} e_{\sigma'}} - \frac{1}{m_\sigma e_\sigma}\right)\mathcal{P}\frac{1}{\lambda-\lambda'}$$
$$+ \left\{\frac{\lambda^{1/2} T_{s0}}{g_{Ms0}(\lambda)}\left[1-\frac{m_i m_e e_s}{T_{s0} m_\sigma e_\sigma}\lambda W(\hat{\lambda}_s)\right]\left[1-\frac{m_i m_e e_s}{T_{s0} m_{\sigma'} e_{\sigma'}}\lambda W(\hat{\lambda}_s)\right]\right.$$
$$\left. + \frac{\pi^2 m_i^2 m_e^2 e^2 \lambda^{7/2} g_{Ms0}(\lambda)}{n_0^2 T_{s0} m_\sigma m_{\sigma'} e_\sigma e_{\sigma'}}\right\}\delta(\lambda-\lambda'). \tag{A 4}$$

Since $e_e = -e_i$, the term proportional to $\mathcal{P}(\lambda-\lambda')^{-1}$ cancels out when summing over the species index s and the result of (4.16) and (4.17) follows.

Appendix B. The parallel electric field

The longitudinal sound wave perturbation has a small but non-zero parallel electric field, although the complete theory formulated in this work was developed without any explicit consideration of it. Nevertheless, the parallel electric field can be determined once a solution for the macroscopic fluid moments has been obtained. In the considered quasineutral ($n_i = n_e$) and single-fluid ($\boldsymbol{u}_i = \boldsymbol{u}_e = \partial\boldsymbol{\xi}/\partial t$) model, the linearized parallel momentum conservation equation for each species is

$$m_s n_0 \frac{\partial^2 \xi_\parallel}{\partial t^2} = -\mathrm{i}k_\parallel p_{s\parallel 1} + e_s n_0 E_\parallel. \tag{B 1}$$

One linear combination of these two equations (their sum) was used in the fluid system:

$$\rho_0 \frac{\partial^2 \xi_\parallel}{\partial t^2} = -\mathrm{i}k_\parallel (p_{i\parallel 1} + p_{e\parallel 1}). \tag{B 2}$$

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The other independent linear combination that eliminates $\xi_\parallel$ determines the parallel electric field:

$$eE_\parallel = \frac{\mathrm{i}k_\parallel}{\rho_0}(m_e p_{i\parallel 1} - m_i p_{e\parallel 1}). \tag{B 3}$$

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