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ASYMPTOTICS OF SPECTRAL DENSITY ESTIMATES

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We consider nonparametric estimation of spectral densities of stationary processes, a fundamental problem in spectral analysis of time series. Under natural and easily verifiable conditions, we obtain consistency and asymptotic normality of spectral density estimates. Asymptotic distribution of maximum deviations of the spectral density estimates is also derived. The latter result sheds new light on the classical problem of tests of white noises.

1. INTRODUCTION

A fundamental problem in spectral analysis of time series is the estimation of spectral density functions. Let $X_k, k \in \mathbb{Z}$, be a stationary process with mean zero and finite covariance function $\gamma_k = \mathbb{E}(X_0 X_k)$. Assume that

$$\sum_{k\in\mathbb{Z}} |\gamma_k| < \infty. \tag{1.1}$$

Let $\iota = \sqrt{-1}$ denote the imaginary unit. Under (1.1), the spectral density function

$$f(\theta) = \frac{1}{2\pi}\sum_{k\in\mathbb{Z}} \gamma_k e^{\iota k\theta} = \frac{1}{2\pi}\sum_{k\in\mathbb{Z}} \gamma_k \cos(k\theta), \qquad 0 \le \theta < 2\pi, \tag{1.2}$$

exists and is continuous. The primary goal of the paper is to consider asymptotic properties of estimates of f. Based on observations $X_1, \ldots, X_n$, let the sample covariances

$$\hat{\gamma}_k = \frac{1}{n}\sum_{i=|k|+1}^{n} X_i X_{i-|k|}, \qquad 1-n \le k \le n-1. \tag{1.3}$$

It is well known that the periodogram

$$I_n(\theta) = \frac{1}{2\pi n}|S_n(\theta)|^2 = \frac{1}{2\pi n}\sum_{k=1-n}^{n-1} \hat{\gamma}_k e^{\iota k\theta}, \qquad \text{where } S_n(\theta) = \sum_{k=1}^{n} X_k e^{\iota k\theta},$$

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is an asymptotically unbiased but inconsistent estimate of $f(\theta)$. In the paper we consider the *lag-window estimate*

$$f_n(\theta) = \frac{1}{2\pi} \sum_{k=1-n}^{n-1} K(k/B_n)\hat{\gamma}_k e^{\iota k\theta}, \tag{1.4}$$

where $b_n = B_n^{-1}$ is the bandwidth satisfying $b_n \to 0$ and $nb_n \to \infty$ and the kernel K is symmetric and bounded, $K(0) = 1$, and K is continuous at zero. If K has bounded support, because $nb_n \to \infty$, the summands for large k in (1.4) are zero. The function $K(\cdot/B_n)$ becomes more concentrated at the origin for bigger B_n. Let

$$\check{\gamma}_k = \frac{1}{n} \sum_{i=|k|+1}^{n} (X_i - \bar{X}_n)(X_{i-|k|} - \bar{X}_n), \quad \text{where } \bar{X}_n = \frac{\sum_{i=1}^n X_i}{n}.$$

A variant of (1.4) that allows unknown mean $\mu = \mathbb{E}X_j$ is the following estimate:

$$\check{f}_n(\theta) = \frac{1}{2\pi} \sum_{k=1-n}^{n-1} K(k/B_n)\check{\gamma}_k e^{\iota k\theta}. \tag{1.5}$$

Properties of spectral density estimates have been explored in many classical textbooks on time series; see, for example, Anderson (1971), Brillinger (1975), Brockwell and Davis (1991), Grenander and Rosenblatt (1957), Priestley (1981), and Rosenblatt (1985), among others. See Shao and Wu (2007) for further references. It seems that many of the previous results require restrictive conditions on the underlying processes such as linear processes or strong mixing processes. Other contributions can be found in Phillips, Sun, and Jin (2006, 2007) and Velasco and Robinson (2001). In this paper we shall present an asymptotic theory for $f_n(\theta)$ under very mild and natural conditions, thus substantially extending the applicability of spectral analysis to nonlinear and/or non–strong mixing processes. Some open problems are solved under our dependence framework (2.1).

The rest of the paper is structured as follows. Main results are presented in Section 2 and proved in the Appendix. Section 3 provides bounds for approximations by m-dependent random variables, and Section 4 presents inequalities for m-dependent processes. Section 5 proves a very general central limit theorem (CLT) for quadratic forms of stationary processes that is of independent interest. Many classical results are special cases of Theorem 6 in Section 5.

We now introduce some notation. We say that a random variable $X \in \mathcal{L}^p$, $p > 0$, if $\|X\|_p := [\mathbb{E}(|X|^p)]^{1/p} < \infty$. Write $\|\cdot\| = \|\cdot\|_2$. For $u, v \in \mathbb{R}$, let $\lfloor u \rfloor = \max\{k \in \mathbb{Z} : k \le u\}$, $u \vee v = \max(u, v)$, and $u \wedge v = \min(u, v)$. Let $\mathbb{C}$ be the set of complex numbers. Denote by C_p a constant that only depends on p and denote by C an absolute constant. Their values may vary from display to display. For two positive sequences (a_n) and (b_n), write $a_n \sim b_n$ if $\lim_{n\to\infty} a_n/b_n = 1$ and $a_n \asymp b_n$ if, for some $c > 0$, $c \le a_n/b_n \le c^{-1}$ holds for all sufficiently large n. Define $\bar{\omega}(u) = 2$ if $u/\pi \in \mathbb{Z}$ and $\bar{\omega}(u) = 1$ if $u/\pi \notin \mathbb{Z}$.

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2. MAIN RESULTS

Assume throughout the paper that ε_j, $j \in \mathbb{Z}$, are independent and identically distributed (i.i.d.) random variables and R is a measurable function such that

$$X_j = R(\ldots, \varepsilon_{j-1}, \varepsilon_j) = R(\mathcal{F}_j), \quad \text{where } \mathcal{F}_j = (\ldots, \varepsilon_{j-1}, \varepsilon_j), \tag{2.1}$$

is well defined. The class of processes under the framework (2.1) is huge; see Wiener (1958), Priestley (1988), Tong (1990), and Wu (2005), among others. Shao and Wu (2007) provide examples of nonlinear time series that are of form (2.1).

To develop an asymptotic theory for the spectral density estimate $f_n(\cdot)$, we need to introduce appropriate dependence measures. Following Wu (2005), we shall apply the idea of coupling and use a *physical dependence measure*. Let $\{\varepsilon_j, \varepsilon_k' : j, k \in \mathbb{Z}\}$ be i.i.d. random variables. For a set $T \subset \mathbb{Z}$, let $\varepsilon_{j,T} = \varepsilon_j'$ if $j \in T$, and $\varepsilon_{j,T} = \varepsilon_j$ if $j \notin T$. Let $\mathcal{F}_{j,k,T} = (\varepsilon_{l,T},\ j \le l \le k)$. If a random variable W is a function of $\mathcal{F}_{-\infty,\infty}$, say, $W = w(\mathcal{F}_{-\infty,\infty})$, write $W_T = w(\mathcal{F}_{-\infty,\infty,T})$. For $X_j \in \mathcal{L}^p$, $p > 0$, and $T = \{0\}$, define the physical dependence measure

$$\delta_{j,p} = \|X_j - X_{j,\{0\}}\|_p. \tag{2.2}$$

Here, by our convention, $X_{j,\{0\}} = R(\mathcal{F}_{j,\{0\}})$, where $\mathcal{F}_{j,\{0\}} = (\ldots, \varepsilon_{-1}, \varepsilon_0', \varepsilon_1, \ldots, \varepsilon_j)$. Namely, $X_{j,\{0\}}$ is obtained by replacing ε_0 in X_j by ε_0'. If $j < 0$, $\delta_{j,p} = 0$. If we view $\mathcal{F}_j$ as input and X_j as output of a physical system, then $\delta_{j,p}$ measures the dependence of X_j on the input ε_0 via coupling. In many situations it is easy to work with $\delta_{j,p}$, which is directly related to the underlying data-generating mechanism (Wu, 2005).

Example 1

Let $X_j = g(\sum_{l=0}^{\infty} a_l \varepsilon_{j-l})$, where a_l are real coefficients, ε_l are i.i.d. with $\varepsilon_l \in \mathcal{L}^p$, $p > 0$, and g is a Lipschitz continuous function. For $j \ge 0$, we have $X_{j,\{0\}} = g(\sum_{l=0}^{j-1} a_l \varepsilon_{j-l} + a_j \varepsilon_0' + \sum_{l=1+j}^{\infty} a_l \varepsilon_{j-l})$. Hence $\delta_{j,p} = \|X_j - X_{j,\{0\}}\|_p = O(|a_j| \|\varepsilon_0 - \varepsilon_0'\|_p) = O(|a_j|)$. In the special case in which $g(x) = x$, $a_l = 2^{-l}$ and ε_l are i.i.d. with $\mathbb{P}(\varepsilon_l = 1) = \mathbb{P}(\varepsilon_l = -1) = \frac{1}{2}$, the process (X_j) is not strong mixing (Andrews, 1984).

Example 2

Let (X_j) be a nonlinear time series recursively defined by $X_j = g(X_{j-1}, \varepsilon_j)$, where ε_j are i.i.d. and g is a measurable function. Assume that there exist $p > 0$ and $x_0 \in \mathbb{R}$ such that $g(x_0, \varepsilon_0) \in \mathcal{L}^p$ and $\mathbb{E}(L_{\varepsilon_0}^p) < 1$, where $L_{\varepsilon_0} = \sup_{x \ne x'} |g(x, \varepsilon_0) - g(x', \varepsilon_0)|/|x - x'|$. Then (X_j) has a unique stationary solution of the form (2.1), and $\delta_{j,p} = O(\rho^j)$ for some $\rho \in (0, 1)$ (Wu, 2005). Shao and Wu (2007) showed that the latter holds for a variety of processes including autoregressive–autoregressive conditionally heteroskedastic processes, amplitude-dependent exponential autoregressive processes, asymmetric generalized autoregressive conditionally heteroskedastic processes, and signed volatility models.

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Sections 2.1, 2.2, and 2.3 concern consistency, asymptotic normality, and maximum deviations of $f_n(\cdot)$, respectively. Our results are all based on $\delta_{j,p}$. Define

$$\Theta_{m,p} = \sum_{j=m}^{\infty} \delta_{j,p} \text{ and } \Psi_{m,p} = \left(\sum_{j=m}^{\infty} \delta_{j,p}^{p'} \right)^{1/p'}, \quad \text{where } p' = \min(2, p). \tag{2.3}$$

Theorems 1 and 2 in Sections 2.1 and 2.2 require the short-range dependence condition $\Theta_{0,p} < \infty$; namely, the cumulative dependence of ε_0 on the future values $(X_j)_{j \geq 0}$ is finite. A careful check of the proofs of Theorems 1–6 indicates that analogous results also hold for the two-sided process $X_j = R(\ldots, \varepsilon_{j-1}, \varepsilon_j, \varepsilon_{j+1}, \ldots)$ because our main tool is the m-dependence approximation; see Section 3. For two-sided processes, similar approximations hold. Details are omitted because this does not involve essential extra difficulties.

2.1. Consistency

To state our consistency result, we need some regularity conditions on the kernel K. Slightly different forms are needed for asymptotic normality and maximum deviations; see Conditions 2 and 3 in Sections 2.2 and 2.3. All those conditions on K are mild, and they are satisfied for Parzen, triangle, Tukey, and many other commonly used windows (Priestley, 1988).

Condition 1. Assume that K is a bounded, absolutely integrable, even function, $K(0) = 1$ and K is continuous. Further assume that $\lim_{w \to 0} w \sum_{k \in \mathbb{Z}} K^2(kw) = \int_{-\infty}^{\infty} K^2(u)\,du =: \kappa < \infty$ and its Fourier transform $\hat{K}(x) = \int_{-\infty}^{\infty} K(u) e^{\imath x u}\,du$ satisfies $\int_{-\infty}^{\infty} |\hat{K}(x)|\,dx < \infty$.

THEOREM 1. *Let Condition 1 be satisfied. Assume that* $\mathbb{E}(X_k) = 0$, $X_k \in \mathcal{L}^p$, $p \geq 2$, *and* $\Theta_{0,p} = \sum_{j=0}^{\infty} \delta_{j,p} < \infty$. *Let* $B_n \to \infty$ *and* $B_n = o(n)$ *as* $n \to \infty$. *Then*

$$\sup_{\theta \in \mathbb{R}} \| f_n(\theta) - f(\theta) \|_{p/2} \to 0. \tag{2.4}$$

Because f_n and f are even and have period 2π, $\sup_{\theta \in \mathbb{R}}$ in (2.4) is equivalent to $\sup_{\theta \in [0,\pi]}$.

Remark 1. By Theorem 2 in Wu (2005), under $\Theta_{0,p} < \infty$, we have $\|X_1 + \cdots + X_j\|_p = O(\sqrt{j})$. Hence, for $0 \leq k \leq n-1$, $\|\bar{X}_n \sum_{i=1+k}^{n} X_{i-k}\|_{p/2} \leq \|\bar{X}_n\|_p \|\sum_{i=1+k}^{n} X_{i-k}\|_p = O(1)$, from which, by elementary calculations, we obtain $\max_{|k| \leq n-1} \|\hat{\gamma}_k - \check{\gamma}_k\|_{p/2} = O(n^{-1})$. Assume

$$\sum_{k=1-n}^{n-1} |K(k/B_n)| = O(B_n). \tag{2.5}$$

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Then (2.4) also holds if f_n therein is replaced by $\check{f}_n$ in view of

$$\left\| \sup_{\theta\in\mathbb{R}} |f_n(\theta) - \check{f}_n(\theta)| \right\|_{p/2} \le \frac{1}{2\pi} \sum_{k=1-n}^{n-1} |K(k/B_n)| \max_{|k|\le n} \|\hat{\gamma}_k - \check{\gamma}_k\|_{p/2} = O(B_n/n). \tag{2.6}$$

With (2.6), Theorems 2–5 in Sections 2.2 and 2.3 also hold if f_n therein is replaced by $\check{f}_n$. Condition (2.5) holds for Epanechnikov, triangle, Parzen, and many other commonly seen kernels.

Theorem 1 imposes very mild conditions. Clearly we need $B_n \to \infty$ and $B_n = o(n)$ to ensure consistency. The short-range dependence condition $\Theta_{0,2} < \infty$ (Wu, 2005) implies (1.1) and hence entails the existence of the spectral density function. If $\Theta_{0,2} = \infty$, then f may not exist. Consider, for example, the linear process $X_j = \sum_{l=0}^{\infty} a_l \varepsilon_{j-l}$, where ε_l are i.i.d. with mean zero and variance 1. Then $\delta_{j,2} = |a_j| \|\varepsilon_0 - \varepsilon_0'\| = |a_j|\sqrt{2}$ and $2\pi f(\theta) = |\sum_{j=0}^{\infty} a_j e^{\iota j\theta}|^2$. If $\sum_{j=0}^{\infty} a_j = \infty$, for example, $a_j = j^{-\beta}$, $j \in \mathbb{N}$, $\beta \in (\frac{1}{2}, 1)$, then f has a pole at $\theta = 0$, and the left-hand side of (2.4) is ∞. In this case $\sum_{k\in\mathbb{Z}} \gamma_k = \infty$, and the process (X_j) is long-range dependent.

Davidson and de Jong (2002) considered a closely related problem of estimating the variance $s_n^2 = \mathrm{var}(X_1 + \cdots + X_n)$, in which X_i are mean zero random variables. They proved that, for the process $X_i = \sum_{j=0}^{\infty} a_j \eta_{i-j}$, where $\sum_{j=0}^{\infty} |a_j| < \infty$ and η_n is L^2-near epoch dependent (NED) of size $-\frac{1}{2}$, $\hat{s}_n^2 = \sum_{k=1}^{n} X_k^2 + 2\sum_{k=1}^{B_n} K(k/B_n) n\hat{\gamma}_k$ satisfies $\hat{s}_n^2/s_n^2 \to 1$ in probability. Their result and our Theorem 1 have different ranges of applicability. Consider the case that both results are applicable: $X_i = \sum_{j=0}^{\infty} a_j \eta_{i-j}$, where $\eta_i = (\sum_{j=0}^{\infty} b_j \varepsilon_{i-j})^2 - \sum_{j=0}^{\infty} b_j^2$ and ε_i are i.i.d. $N(0, 1)$. Their condition of L^2 NED of size $-\frac{1}{2}$ requires that $\|\eta_n - \mathbb{E}(\eta_n|\varepsilon_1, \ldots, \varepsilon_n)\| = O(n^{-q})$ for some $q > \frac{1}{2}$. With elementary calculations, the latter condition is reduced to $\sum_{j=n}^{\infty} b_j^2 = O(n^{-2q})$, which implies that $\sum_{j=2}^{\infty} |b_j| \le \sum_{k=1}^{\infty} \big(\sum_{j=2^k}^{2^{k+1}-1} b_j^2\big)^{\frac{1}{2}} 2^{k/2} = \sum_{k=1}^{\infty} 2^{k/2} O(2^{-qk}) < \infty$. We now apply our Theorem 1 with $p = 2$. Elementary calculations show that physical dependence measure $\delta_{n,2} = \sum_{i=0}^{n} O(|a_i b_{n-i}|)$. Hence our condition $\Theta_{0,2} < \infty$ only requires $\sum_{j=0}^{\infty} |a_j| < \infty$ and $\sum_{j=0}^{\infty} |b_j| < \infty$. Hence in this example their NED-based condition is slightly stronger. Jansson (2002) considered the consistency of covariance matrix estimation for linear process with η_n being $\mathbb{R}^d$-valued martingale differences. In the special case $d = 1$, Jansson's result requires $B_n = o(\sqrt{n})$, whereas our result permits $B_n = o(n)$.

2.2. Asymptotic Normality

A classical problem in spectral analysis of time series is to develop an asymptotic distributional theory for the spectral density estimate $f_n(\theta)$. With the latter results one can perform statistical inference such as hypothesis testing and construction of confidence intervals. However, it turns out that the central limit problem for

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$f_n(\theta)$ is highly nontrivial. Earlier results require stringent conditions. The case of linear processes has been dealt with in Anderson (1971). Rosenblatt (1984) obtained a central limit theorem (CLT) under strong mixing and cumulant summability conditions. More restrictive cumulant conditions are used in Brillinger (1969). Bentkus and Rudzkis (1982) dealt with Gaussian processes. Shao and Wu (2007) required the condition that $\delta_{i,p}$ converges to zero geometrically fast.

Here we present a CLT for $f_n(\theta)$ under very mild and natural conditions, and it allows a wide class of nonlinear processes. In Theorem 2, the condition on dependence $\Theta_{0,4} < \infty$ is natural, because otherwise the process (X_j) may be long-range dependent and the spectral density function may not be well defined. In Rosenblatt (1985), a summability condition of eighth-order joint cumulants is required. Rosenblatt asked whether the eighth-order summability condition can be weakened to the fourth order. The latter conjecture is solved in Theorem 2 in the sense that it imposes a summability condition of fourth-order physical dependence measures, which in many applications has the additional advantage that it is easier to work with than conditions on joint cumulants. The bandwidth condition $B_n \to \infty$ and $B_n = o(n)$ is also natural; see our consistency result, Theorem 1. Recall that $\bar{\omega}(u) = 2$ if $u/\pi \in \mathbb{Z}$ and $\bar{\omega}(u) = 1$ if $u/\pi \notin \mathbb{Z}$. Theorem 2 is proved in the Appendix.

Condition 2. K is symmetric and bounded, $\lim_{u\to 0} K(u) = K(0) = 1$, and $\kappa := \int_{-\infty}^{\infty} K^2(x)\,dx < \infty$. Further assume that K is continuous at all but a finite number of points and suppose that $\sup_{0<w\le 1} w \sum_{j\ge c/w} K^2(jw) \to 0$ as $c \to \infty$.

THEOREM 2. *Assume* $\mathbb{E}(X_k) = 0$, $\mathbb{E}(X_k^4) < \infty$, *and* $\Theta_{0,4} < \infty$. *Let* $B_n \to \infty$ *and* $B_n = o(n)$ *as* $n \to \infty$. *Then under Condition 2, for any fixed* $0 \le \theta \le \pi$,

$$\sqrt{\frac{n}{B_n}}\{f_n(\theta) - \mathbb{E}[f_n(\theta)]\} \Rightarrow N[0, \sigma^2(\theta)], \quad \text{where } \sigma^2(\theta) = \bar{\omega}(\theta) f^2(\theta)\kappa. \tag{2.7}$$

Remark 2. Theorem 2 is applicable for a very wide range of bandwidths. In practice, one can use the bandwidth selector in Bühlmann and Künsch (1999), Politis, Romano, and Wolf (1999), or Song and Schmeiser (1995).

Remark 3. The bias $\mathbb{E}[f_n(\theta)] - f(\theta)$ can be calculated by some standard arguments; see Anderson (1971) or Priestley (1981).

2.3. Maximum Deviations

Theorem 2 provides a CLT for $f_n(\theta) - \mathbb{E}[f_n(\theta)]$. In the inference of spectra, one often needs to know the asymptotic distribution of the maximum deviation $\sup_{0\le\theta\le\pi} |f_n(\theta) - f(\theta)|$. Such a result can be used to construct simultaneous confidence bands for $f(\theta)$ over $\theta \in [0, \pi]$ and to conduct a parametric specification test for f. For example, if a constant function can be embedded into the band, then we can accept the hypothesis that (X_k) is a white noise sequence. However, the maximum deviation problem is extremely difficult. In 1967, Woodroofe and

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Van Ness considered linear processes and obtained an asymptotic theory for maximum deviations. Rudzkis (1985) considered the special Gaussian processes. Over the past 40 years, however, it seems that there has been no significant progress on generalizing their results to nonlinear processes.

Shao and Wu (2007) posed an open problem whether an asymptotic distributional theory for maximum deviations can be obtained for a wide class of nonlinear time series satisfying the geometric-moment contraction condition $\delta_{n,p} = O(\rho^n)$. Theorem 5 solves the conjecture by considering $\max_{i\le B_n} |f_n(\lambda_i^*) - \mathbb{E}f_n(\lambda_i^*)|/f(\lambda_i^*)$, where $\lambda_i^* = \pi|i|/B_n$. Theorems 3 and 4 present similar results under weaker dependence conditions.

Condition 3. K is an even, bounded function with bounded support $[-1,1]$, $\lim_{u\to 0} K(u) = K(0) = 1, \kappa := \int_{-1}^{1} K^2(u)\,du < \infty$, and $\sum_{j\in\mathbb{Z}} \sup_{|s-j|\le 1} |K(j\omega) - K(s\omega)| = O(1)$ as $\omega \to 0$.

Let the total variation $V_a^b(K) = \sup\{\sum_{i=1}^{l} |K(x_i) - K(x_{i-1})| : a = x_0 < x_1 < \cdots < x_l = b\}$. Then $\sup_{|s-j|\le 1} |K(j\omega) - K(s\omega)| \le V_{j\omega-\omega}^{j\omega+\omega}(K)$. If K has bounded variation, namely, $V_{-\infty}^{\infty}(K) < \infty$, then $\sum_{j\in\mathbb{Z}} \sup_{|s-j|\le 1} |K(j\omega) - K(s\omega)| = O(1)$. Popular choices such as triangle, Epanechnikov, quartic, and other kernels all have bounded variations.

Condition 4. There exists $0 < \underline{\delta} < \delta < 1$ and $c_1, c_2 > 0$ such that, for all large n, $c_1 n^{\underline{\delta}} \le B_n \le c_2 n^{\delta}$ holds.

Condition 5.

(a) Let $d_{m,q} = \sum_{t=0}^{\infty} \min(\delta_{t,q}, \Psi_{m+1,q})$. Assume $d_{n,p} = O(n^{-T_1})$ with $T_1 > \max\left[\frac{1}{2} - (p-4)/(2p\delta), 2\delta/p\right]$.

(b) $\Theta_{n,p} = O(n^{-T_2})$, $T_2 > \max[0, 1 - (p-4)/(2p\delta)]$.

THEOREM 3. *Assume $X_0 \in \mathcal{L}^p$, $p > \max(4, 2/(1-\delta))$, and $\mathbb{E}X_0 = 0$. Further assume Conditions 3, 4, 5(a), and 5(b). Let $\lambda_i^* = \pi|i|/B_n$. Then, for all $x \in \mathbb{R}$,*

$$\mathbb{P}\left[\max_{0\le i\le B_n} \frac{n}{B_n} \frac{|f_n(\lambda_i^*) - \mathbb{E}f_n(\lambda_i^*)|^2}{f^2(\lambda_i^*)\kappa} - 2\log B_n + \log(\pi \log B_n) \le x\right] \to e^{-e^{-x/2}}. \tag{2.8}$$

Theorem 3 requires the moment condition $X_i \in \mathcal{L}^p$ with $p > \max(4, 2/(1-\delta)) \to \infty$ as $\delta \to 1$. Theorems 4 and 5 aim to weaken the latter moment condition.

THEOREM 4. *Assume $\mathbb{E}X_0 = 0$, $X_0 \in \mathcal{L}^p$, $p > 4$, and Conditions 3, 4, and 5(a). Further assume that K is continuous and $\hat{K}(x) := \int_{-\infty}^{\infty} e^{-i\lambda x} K(\lambda)\,d\lambda$ satisfies $\int_{-\infty}^{\infty} |\hat{K}(x)|\,dx < \infty$. Then (2.8) holds.*

THEOREM 5. *Assume Conditions 3 and 4 and $\mathbb{E}X_0 = 0$, $X_0 \in \mathcal{L}^p$, $p > 4$. Further assume that $\delta_{n,p} = O(\rho^n)$ for some $0 < \rho < 1$. Then (2.8) holds.*

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Remark 4. Theorems 3–5 allow nonlinear processes. When they are applied to linear processes, our conditions are weaker than the classical one in Woodroofe and Van Ness (1967). To derive (2.8), the latter paper requires $\delta < \frac{2}{5}$, $\underline{\delta} \geq \frac{1}{4}$, $\varepsilon_0 \in \mathcal{L}^8$, and $|a_k| = O(k^{-1-\beta})$ with $\beta > \frac{1}{5}$. If $p = 8$ and $\delta < \frac{2}{5}$, our Theorem 4 only requires the weaker condition $|a_k| = O(k^{-1-\beta})$ with $\beta > (\sqrt{14}-2)/10$. We also note that the requirement on β becomes weaker for smaller δ. Additionally, we allow smaller p with $4 < p < 8$.

Remark 5. If $K(x) - 1 = O(x)$ as $x \to 0$ and $\sum_{k\geq 1} k\delta_{k,2} < \infty$, then $\mathbb{E}f_n(\theta) - f(\theta) = O(B_n^{-1})$. Hence we can replace $\mathbb{E}f_n(\lambda_i^*)$ in (2.8) by $f(\lambda_i^*)$ if $n \log n = o(B_n^3)$.

3. APPROXIMATIONS BY *m*-DEPENDENT PROCESSES

With the physical dependence measure $\delta_{j,p}$ in (2.2), we are able to provide explicit error bounds for approximating functionals of X_k by functionals of the m-dependent process

$$\tilde{X}_t := X_{t,m} = \mathbb{E}(X_t|\varepsilon_{t-m}, \ldots, \varepsilon_t) = \mathbb{E}(X_t|\mathcal{F}_{t-m,t}), \qquad m \geq 0, \tag{3.1}$$

where $\mathcal{F}_{t-m,t} = \sigma(\varepsilon_{t-m}, \ldots, \varepsilon_t)$. Define the projection operator $\mathcal{P}_k$ by

$$\mathcal{P}_k \cdot = \mathbb{E}(\cdot|\mathcal{F}_k) - \mathbb{E}(\cdot|\mathcal{F}_{k-1}), \qquad k \in \mathbb{Z}.$$

Lemma 1, which follows, concerns linear forms, whereas Proposition 1 in this section is for quadratic forms. Proposition 2 in this section gives a martingale approximation for quadratic forms of m-dependent processes.

LEMMA 1. *Assume $X_i \in \mathcal{L}^p$, $p > 1$, and $\mathbb{E}(X_k) = 0$. Let $C_p = 18p^{3/2}(p-1)^{-1/2}$ and $p' = \min(2, p)$. Let $\alpha_1, \alpha_2, \ldots, \in \mathbb{C}$. Then*

$$\left\| \sum_{k=1}^n \alpha_k (X_k - \tilde{X}_k) \right\|_p \leq C_p A_n \Theta_{m+1,p}, \quad \text{where } A_n = \left(\sum_{k=1}^n |\alpha_k|^{p'} \right)^{1/p'}. \tag{3.2}$$

Also, we have (i) $\| \sum_{k=1}^n \alpha_k X_k \|_p \leq C_p A_n \Theta_{0,p}$ and (ii) $\| \sum_{k=1}^n \alpha_k \tilde{X}_k \|_p \leq C_p A_n \Theta_{0,p}$.

Proof. Let $D_{k,j} := \mathbb{E}(X_k|\mathcal{F}_{k-j,k}) - \mathbb{E}(X_k|\mathcal{F}_{k-j+1,k})$. Then $D_{k,j}, k = n, \ldots, 1$, form martingale differences with respect to $\mathcal{F}_{k-j,\infty}$, and $\|D_{k,j}\|_p \leq \delta_{j,p}$. By Minkowski's and Burkholder's inequalities (see Wu and Shao, 2007, Lem. 1), we have

$$\left\| \sum_{k=1}^n \alpha_k D_{k,j} \right\|_p^{p'} \leq C_p^{p'} \sum_{k=1}^n \|\alpha_k D_{k,j}\|_p^{p'} \leq C_p^{p'} \sum_{k=1}^n |\alpha_k|^{p'} \delta_{j,p}^{p'}.$$

Because $X_k - \tilde{X}_k = \sum_{j=1+m}^{\infty} D_{k,j}$, (3.2) follows.

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Because $X_k = \sum_{j=0}^{\infty} D_{k,j}$, the preceding argument implies (i). By Jensen's inequality,

$$\tilde{\delta}_{k,p} := \|\tilde{X}_k - \tilde{X}_{k,\{0\}}\|_p = \|\mathbb{E}(X_k|\mathcal{F}_{k-m,k}) - \mathbb{E}(X_{k,\{0\}}|\mathcal{F}_{k-m,k,\{0\}})\|_p$$
$$= \|\mathbb{E}(X_k - X_{k,\{0\}}|\mathcal{F}_{k-m,k}, \varepsilon_0')\|_p \le \delta_{k,p}. \tag{3.3}$$

So (ii) follows from (i). ■

PROPOSITION 1. *Assume* $\mathbb{E}X_0 = 0$, $\mathbb{E}|X_0|^{2p} < \infty$, $p \ge 2$, *and* $\Theta_{0,2p} < \infty$. *Let*

$$L_n = \sum_{1\le j<j'\le n} \alpha_{j'-j} X_j X_{j'} \quad \text{and} \quad \tilde{L}_n = \sum_{1\le j<j'\le n} \alpha_{j'-j} \tilde{X}_j \tilde{X}_{j'},$$

where $\alpha_1, \alpha_2, \ldots, \in \mathbb{C}$ *and* $\tilde{X}_t$ *is defined in (3.1). Let* $A_n = (\sum_{s=1}^{n-1} |\alpha_s|^2)^{1/2}$. *Then*

$$\frac{\|L_n - \mathbb{E}L_n - (\tilde{L}_n - \mathbb{E}\tilde{L}_n)\|_p}{n^{\frac{1}{2}} A_n \Theta_{0,2p}} \le C_p d_{m,2p}, \quad \text{where } d_{m,q} = \sum_{t=0}^{\infty} \min(\delta_{t,q}, \Psi_{m+1,q}). \tag{3.4}$$

Proof. Let $Z_{t-1} = \sum_{j=1}^{t-1} \alpha_{t-j} X_j$, $\tilde{Z}_{t-1} = \sum_{j=1}^{t-1} \alpha_{t-j} \tilde{X}_j$, $Y_t = X_t Z_{t-1}$, $\tilde{Y}_t = \tilde{X}_t \tilde{Z}_{t-1}$, and

$$L_n^\star = \sum_{1\le j<j'\le n} \alpha_{j'-j} \tilde{X}_j X_{j'} = \sum_{t=2}^{n} X_t \tilde{Z}_{t-1}.$$

Recall that the notation $X_{j,\{k\}}$ represents a coupled version of $X_i = R(\mathcal{F}_i)$ by replacing ε_k in $\mathcal{F}_i$ by ε_k'. If $k > j$, then $X_{j,\{k\}} = X_j$. We similarly define $Z_{t-1,\{k\}}$ and $\tilde{Z}_{t-1,\{k\}}$. So

$$\|\mathcal{P}_k(L_n - L_n^\star)\|_p \le \left\| \sum_{t=2}^{n} X_t (Z_{t-1} - \tilde{Z}_{t-1}) - X_{t,\{k\}} (Z_{t-1,\{k\}} - \tilde{Z}_{t-1,\{k\}}) \right\|_p$$
$$\le \left\| \sum_{t=2}^{n} X_{t,\{k\}} [(Z_{t-1} - \tilde{Z}_{t-1}) - (Z_{t-1,\{k\}} - \tilde{Z}_{t-1,\{k\}})] \right\|_p$$
$$+ \sum_{t=2}^{n} \|(X_t - X_{t,\{k\}})(Z_{t-1} - \tilde{Z}_{t-1})\|_p =: I_k + II_k. \tag{3.5}$$

By (3.3), $\|\tilde{X}_j - \tilde{X}_{j,\{k\}}\|_{2p} \le \delta_{j-k,2p}$. Because $\|X_j - \tilde{X}_j\|_{2p} \le \Psi_{m+1,2p}$, we have $\|X_j - \tilde{X}_j - X_{j,\{k\}} + \tilde{X}_{j,\{k\}}\|_{2p} \le 2\min(\delta_{j-k,2p}, \Psi_{m+1,2p})$. By Lemma 1,

$$I_k = \left\| \sum_{t=2}^{n} X_{t,\{k\}} \sum_{j=1}^{t-1} \alpha_{t-j} (X_j - \tilde{X}_j - X_{j,\{k\}} + \tilde{X}_{j,\{k\}}) \right\|_p$$

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$$= \left\| \sum_{j=1}^{n-1} (X_j - \tilde{X}_j - X_{j,\{k\}} + \tilde{X}_{j,\{k\}}) \sum_{t=1+j}^{n} \alpha_{t-j} X_{t,\{k\}} \right\|_p$$

$$\le 2 \sum_{j=1}^{n-1} \min(\delta_{j-k,2p}, \Psi_{m+1,2p}) C_{2p} A_n \Theta_{0,2p}.$$

By Lemma 1, $\|Z_{t-1} - \tilde{Z}_{t-1}\|_{2p} \le C_{2p} A_n \Theta_{m+1,2p}$. Because $\sum_{t=2}^{n} \delta_{t-k,2p} \le \Theta_{0,2p}$ and $\|(X_t - X_{t,\{k\}})(Z_{t-1} - \tilde{Z}_{t-1})\|_p \le \|X_t - X_{t,\{k\}}\|_{2p} \|Z_{t-1} - \tilde{Z}_{t-1}\|_{2p}$,

$$\sum_{k=-\infty}^{n} II_k^2 \le C_p A_n^2 \Theta_{m+1,2p}^2 \sum_{k=-\infty}^{n} \Theta_{0,2p} \sum_{j=1}^{n-1} \delta_{j-k,2p} \le C_p A_n^2 n \Theta_{m+1,2p}^2 \Theta_{0,2p}^2,$$

$$\sum_{k=-\infty}^{n} I_k^2 \le C_p A_n^2 \Theta_{0,2p}^2 \sum_{k=-\infty}^{n} \Theta_{0,2p} \sum_{j=1}^{n-1} \min(\delta_{j-k,2p}, \Psi_{m+1,2p})$$

$$\le C_p A_n^2 \Theta_{0,2p}^2 n \Theta_{0,2p} d_{m,2p}.$$

Hence, by (3.5), because $\Theta_{m+1,2p} \le d_{m,2p}$,

$$\|L_n - \mathbb{E}L_n - (L_n^\star - \mathbb{E}L_n^\star)\|_p^2 \le C_p \sum_{k=-\infty}^{n} \|\mathcal{P}_k(L_n - L_n^\star)\|_p^2 \le C_p n A_n^2 \Theta_{0,2p}^2 d_{m,2p}^2,$$

which, by a similar inequality for $\|L_n^\star - \mathbb{E}L_n^\star - (\tilde{L}_n - \mathrm{E}\tilde{L}_n)\|_p^2$, implies (3.4). ■

PROPOSITION 2. *Assume* $\mathbb{E}X_0 = 0$, $X_0 \in \mathcal{L}^4$, *and* $\Theta_{0,4} < \infty$. *Let* $\alpha_j = \beta_j e^{\iota j\lambda}$, *where* $\lambda \in \mathbb{R}$, $\beta_j \in \mathbb{R}$, $1-n \le j \le -1$, $m \in \mathbb{N}$, *and* $\tilde{L}_n = \sum_{1 \le j < t \le n} \alpha_{j-t} \tilde{X}_j \tilde{X}_t$. *Define*

$$D_k = A_k - \mathbb{E}(A_k|\mathcal{F}_{k-1}), \quad \text{where } A_k = \sum_{t=0}^{\infty} \mathbb{E}(\tilde{X}_{t+k}|\mathcal{F}_k) e^{\iota t\lambda}, \tag{3.6}$$

and $M_n = \sum_{t=1}^{n} \bar{D}_t \sum_{j=1}^{t-1} \alpha_{j-t} D_j$, *where* $\bar{\cdot}$ *denotes complex conjugate. Then*

$$\frac{\|\tilde{L}_n - \mathbb{E}\tilde{L}_n - M_n\|}{m^{\frac{3}{2}} n^{1/2} \|X_0\|_4^2} \le C V_m^{1/2}(\beta), \quad \text{where } V_m(\beta)$$

$$= \max_{1-n \le i \le -1} \beta_i^2 + m \sum_{j=-1}^{-n-1} |\beta_j - \beta_{j-1}|^2.$$

Proof. Note that $A_k = \sum_{t=0}^{m} \mathbb{E}(\tilde{X}_{t+k}|\mathcal{F}_k) e^{\iota t\lambda}$ because $\mathbb{E}(\tilde{X}_{t+k}|\mathcal{F}_k) = 0$ if $t > m$. Also D_k are m-dependent, and they form martingale differences with respect to $\mathcal{F}_k$. Let $\varrho = e^{\iota\lambda}$, $U_j = \varrho^{j-t} \mathbb{E}(A_j|\mathcal{F}_{j-1})$, $c_4 = \|X_0\|_4$, and $\beta_j = 0$ if $j \ge 0$ or $j \le -n$. Observe that $\tilde{X}_k = A_k - \mathbb{E}(A_{k+1}|\mathcal{F}_k)\varrho$ and $\|A_1\|_4 \le 2mc_4$. Then

$$\left\| \sum_{j=1}^{t-8m} \alpha_{j-t} (\tilde{X}_j - D_j) \right\| = \left\| \sum_{j=1}^{t-8m} \beta_{j-t} (U_j - U_{j+1}) \right\|$$

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$$\le Cmc_4 \max_j |\beta_j| + \left\| \sum_{j=1}^{t-8m} (\beta_{j-t} - \beta_{j-1-t}) U_j \right\|. \tag{3.7}$$

Note that $U_j = \sum_{l=1}^{m} \mathcal{P}_{j-l} U_j$ and $\mathcal{P}_{j-l} U_j$, $j \in \mathbb{Z}$, are martingale differences,

$$\left\| \sum_{j=1}^{t-8m} (\beta_{j-t} - \beta_{j-1-t}) \mathcal{P}_{j-l} U_j \right\|^2 = \sum_{j=1}^{t-8m} (\beta_{j-t} - \beta_{j-1-t})^2 \|\mathcal{P}_{-l} U_0\|^2. \tag{3.8}$$

Because $\sum_{l=1}^{m} \|\mathcal{P}_{-l} U_0\| \le m^{1/2} (\sum_{l=1}^{m} \|\mathcal{P}_{-l} U_0\|^2)^{1/2} \le Cm^{3/2} c_4$, by (3.7),

$$\left\| \sum_{j=1}^{t-8m} \alpha_{j-t} (\tilde{X}_j - D_j) \right\| \le C V_m^{1/2}(\beta) mc_4. \tag{3.9}$$

Let $Q_j = \varrho^{j-t} \mathbb{E}(\bar{A}_j | \mathcal{F}_{j-1})$. Using $\tilde{X}_k = \bar{A}_k - \mathbb{E}(\bar{A}_{k+1} | \mathcal{F}_k) \bar{\varrho}$, similarly, we have

$$\left\| \sum_{t=j+8m}^{n} \alpha_{j-t} (\tilde{X}_t - \bar{D}_t) \right\| = \left\| \sum_{t=j+8m}^{n} \beta_{j-t} (Q_j - Q_{j+1}) \right\| \le C V_m^{1/2}(\beta) mc_4. \tag{3.10}$$

Let $W_{1,t} = \tilde{X}_t \sum_{j=1}^{t-8m} \beta_{j-t} \varrho^{j-t} (\tilde{X}_j - D_j)$. Then $W_{1,t}, W_{1,t+4m}, W_{1,t+8m}, \ldots$ are martingale differences. By (3.9), $\|W_{1,t}\| \le C V_m^{1/2}(\beta) mc_4^2$. Write $\Delta = V_m^{1/2}(\beta) m^{3/2} n^{1/2} c_4^2$. So

$$\left\| \sum_{t=1}^{n} W_{1,t} \right\| \le \sum_{i=1}^{4m-1} \left\| \sum_{l=0}^{\lfloor (n-i)/(4m) \rfloor} W_{1,i+4ml} \right\| \le C\Delta. \tag{3.11}$$

Let $W_t = W_{1,t} + W_{2,t}$, where $W_{2,t} = \tilde{X}_t \sum_{j=t-8m+1}^{t-1} \beta_{j-t} \varrho^{j-t} (\tilde{X}_j - D_j)$ are $12m$-dependent. As in (3.9), $\|W_{2,t}\| \le C V_m^{1/2}(\beta) mc_4^2$. Similarly as (3.11), we have $\|\sum_{t=1}^{n} (W_{2,t} - \mathbb{E} W_{2,t})\| \le C\Delta$. By (3.11), $\|\sum_{t=1}^{n} (W_t - \mathbb{E} W_t)\| \le C\Delta$. Similarly, using (3.10), we have $\|\sum_{t=1}^{n} (W_t^\circ - \mathbb{E} W_t^\circ)\| \le C\Delta$ for $W_t^\circ = (\tilde{X}_t - \bar{D}_t) \sum_{j=1}^{t-1} \alpha_{j-t} D_j$. Hence Proposition 2 follows. ∎

4. INEQUALITIES FOR *m*-DEPENDENT PROCESSES

As argued in Section 3, quadratic forms of (X_k) can be approximated by those of m-dependent random variables. So probability inequalities under m-dependence are useful for the asymptotic spectral estimation problem. Lemma 2, which follows, is an easy consequence of Corollary 1.6 in Nagaev (1979) via a simple blocking argument. We omit its proof. Proposition 3 is a Fuk–Nagaev-type inequality for quadratic forms of m-dependent random variables. It is useful for proving the maximum deviation results in Section 2.3.

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LEMMA 2. *Let (X_k) be m-dependent with $\mathbb{E}X_k = 0$ and $X_k \in \mathcal{L}^p$, $p \geq 2$. Let $W_n = \sum_{i=1}^n X_i$. Then for any $Q > 0$, there exists $C_1, C_2 > 0$ only depending on Q such that*

$$\mathbb{P}(|W_n| \geq x) \leq C_1 \left(\frac{m}{x^2}\mathbb{E}W_n^2\right)^Q + C_1 \min\left[\frac{m^{p-1}}{x^p}\sum_{k=1}^n \|X_k\|_p^p, \sum_{k=1}^n \mathbb{P}\left(|X_k| \geq C_2\frac{x}{m}\right)\right].$$

PROPOSITION 3. *Let (X_t) be m-dependent with $\mathbb{E}X_t = 0$, $|X_t| \leq M$ a.s., $m \leq n$, and $M \geq 1$. Let $S_{k,l} = \sum_{t=l+1}^{l+k} X_t \sum_{s=1}^{t-1} a_{n,t-s}X_s$, where $l \geq 0$, $l + k \leq n$ and assume $\max_{1\leq t\leq n}|a_{n,t}| \leq K_0$, $\max_{1\leq t\leq n}\mathbb{E}X_t^2 \leq K_0$, $\max_{1\leq t\leq n}\mathbb{E}X_t^4 \leq K_0$ for some $K_0 > 0$. Then for any $x \geq 1$, $y \geq 1$, and $Q > 0$,*

$$\mathbb{P}(|S_{k,l} - \mathbb{E}S_{k,l}| \geq x) \leq 2e^{-y/4} + C_1 n^3 M^2 \left(x^{-2}y^2m^3(M^2+k)\sum_{s=1}^n a_{n,s}^2\right)^Q + C_1 n^4 M^2 \mathbb{P}\left(|X_0| \geq \frac{C_2 x}{ym^2(M + k^{\frac{1}{2}})}\right),$$

where $C_1, C_2 > 0$ are constants depending only on Q and K_0.

Proof. Without loss of generality, we assume $l = 0$. Let $\widehat{S}_{k,0} = \sum_{t=1}^k X_t \sum_{s=1}^{t-2m} a_{n,t-s}X_s$. Split the interval $[1, k]$ into consecutive blocks $H_1, \ldots, H_{k_n}$ with same size m. Here, for convenience, we assume $k_n = k/m \in \mathbb{N}$. Let $\widehat{S}_j = \sum_{t\in H_j} X_t \sum_{s=1}^{t-2m} a_{n,t-s}X_s$, $1 \leq j \leq k_n$. Then $\{\widehat{S}_j; j = 1, 3, \ldots\}$ and $\{\widehat{S}_j; j = 2, 4, \ldots\}$ are two sets of martingale differences. Let $\mathcal{G}_j = \sigma(\widehat{S}_i; 1 \leq i \leq j)$. By Freedman's inequality (see Freedman, 1975), we have

$$\mathbb{P}(|\widehat{S}_{k,0}| \geq 2x) \leq 2e^{-y/4} + \sum_{j=1}^{k_n}\mathbb{P}(|\widehat{S}_j| \geq x/y) + \mathbb{P}\left(\left|\sum_{j=1}^{k_n}\mathbb{E}\left[\widehat{S}_j I\{\widehat{S}_j \geq x/y\}|\mathcal{G}_{j-2}\right]\right| \geq x\right) + \mathbb{P}\left(\sum_{j=1}^{k_n}\mathbb{E}[\widehat{S}_j^2|\mathcal{G}_{j-2}] \geq x^2/y\right) =: 2e^{-y/4} + I_n + II_n + III_n.$$

Because $|\widehat{S}_j| \leq CnmM^2$, we have $II_n \leq Cx^{-1}nmM^2\sum_{j=1}^{k_n}\mathbb{P}(|\widehat{S}_j| \geq x/y)$. By Lemma 2,

$$\mathbb{P}\left(|\widehat{S}_j| \geq x/y\right) \leq m \max_{1\leq t\leq n}\mathbb{P}\left(\left|\sum_{s=1}^t a_{n,t-s}X_s\right| \geq \frac{x}{ymM}\right)$$

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$$\le C_1 m \left(x^{-2} y^2 m^3 M^2 \sum_{s=1}^{n} a_{n,s}^2 \right)^Q + C_1 mn \mathbb{P}\left(|X_0| \ge \frac{C_2 x}{ym^2 M} \right).$$

Because $\mathbb{E}(\widehat{S}_j^2 | \mathcal{G}_{j-2}) = \sum_{t_1,t_2 \in H_j} \mathbb{E}(X_{t_1} X_{t_2}) \sum_{s=1}^{t_1-2m} a_{n,t-s} X_s \sum_{s=1}^{t_2-2m} a_{n,t-s} X_s$, we have

$$\begin{aligned} III_n &\le \sum_{j=1}^{k_n} \mathbb{P}\left(\mathbb{E}[\widehat{S}_j^2 | \mathcal{G}_{j-2}] \ge mx^2/(yk) \right) \\ &\le \sum_{j=1}^{k_n} \sum_{t_1,t_2 \in H_j} \mathbb{P}\left(\left| \sum_{s=1}^{t_1-2m} a_{n,t_1-s} X_s \sum_{s=1}^{t_2-2m} a_{n,t_2-s} X_s \right| \ge \frac{x^2}{ymk} \right) \\ &\le 2km \max_{1 \le t \le n} \mathbb{P}\left(\left| \sum_{s=1}^{t} a_{n,t-s} X_s \right| \ge \frac{x}{(ymk)^{\frac{1}{2}}} \right) \\ &\le C_1 km \left(x^{-2} ym^2 k \sum_{s=1}^{n} a_{n,s}^2 \right)^Q + C_1 kmn \mathbb{P}\left(|X_0| \ge \frac{C_2 x}{(ym^3 k)^{\frac{1}{2}}} \right). \end{aligned}$$

It remains to consider $S_{k,0}^{\diamond} := \sum_{t=1}^{k} X_t \sum_{s=t-2m+1}^{t-1} a_{n,t-s} X_s$. By Lemma 2, we have

$$\mathbb{P}\left(|S_{k,0}^{\diamond} - \mathbb{E} S_{k,0}^{\diamond}| \ge x \right) \le C_1 \left(x^{-2} m^2 \sum_{s=1}^{n} a_t^2 \right)^Q + C_1 km \mathbb{P}\left(|X_0| \ge \frac{C_2 x}{m^2 M} \right).$$

The proof is now complete. ■

5. A GENERAL CLT FOR QUADRATIC FORMS

In this section we establish a very general CLT for quadratic forms of stationary processes, which can be used for proving Theorem 2. For quadratic forms of independent random variables see de Jong (1987), Mikosch (1991), ten Vregelaar (1991), and Götze and Tikhomirov (1999), among others. For more references see Wu and Shao (2007), which gives a CLT for quadratic forms of martingale differences. Theorem 6 imposes the very mild dependence condition $\Theta_{0,4} < \infty$, and it allows a wide class of weights $a_{n,j}$. Recall that $\bar{\omega}(u) = 2$ if $u/\pi \in \mathbb{Z}$ and $\bar{\omega}(u) = 1$ if $u/\pi \notin \mathbb{Z}$.

THEOREM 6. *Let $a_{n,j} = b_{n,j} e^{\iota j \lambda}$, where $\lambda \in \mathbb{R}$, $b_{n,j} \in \mathbb{R}$ with $b_{n,j} = b_{n,-j}$, and*

$$T_n = \sum_{1 \le j,j' \le n} a_{n,j-j'} X_j X_{j'} \quad \text{and} \quad \sigma_n^2 = \bar{\omega}(\lambda) \sum_{k=1}^{n} \sum_{t=1}^{n} b_{n,t-k}^2. \tag{5.1}$$

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Assume that $\mathbb{E}X_0 = 0$, $\mathbb{E}|X_0|^4 < \infty$, $\Theta_{0,4} < \infty$, *and*

$$\max_{0\le t\le n} b_{n,t}^2 = o(\varsigma_n^2), \quad \text{where } \varsigma_n^2 = \sum_{t=1}^n b_{n,t}^2; \tag{5.2}$$

$$n\varsigma_n^2 = O(\sigma_n^2); \tag{5.3}$$

$$\sum_{k=1}^n \sum_{t=1}^{k-1} \left| \sum_{j=1+k}^n a_{n,k-j} a_{n,t-j} \right|^2 = o(\sigma_n^4); \tag{5.4}$$

$$\sum_{k=1}^n |b_{n,k} - b_{n,k-1}|^2 = o(\varsigma_n^2). \tag{5.5}$$

Then for $0 \le \lambda < 2\pi$, $\sigma_n^{-1}(T_n - \mathbb{E}T_n) \Rightarrow N(0, 4\pi^2 f^2(\lambda))$.

Proof. We shall apply Propositions 1 and 2 with $\alpha_j = a_{n,j} = b_{n,j} e^{\iota j\lambda}$. Recall Propositions 1 and 2 for $L_n, \tilde{L}_n, D_k$, and M_n. Let $\bar{L}_n$ and $\bar{M}_n$ be the complex conjugates of L_n and M_n. Note that $T_n = L_n + \bar{L}_n + a_{n,0}\sum_{j=1}^n X_j^2$. Because $\mathbb{E}|X_0|^4 < \infty$ and $\Theta_{0,4} < \infty$, we have $\|\sum_{j=1}^n X_j^2 - n\gamma_0\| \le C\sqrt{n}\|X_0\|_4\Theta_{0,4} = O(\sqrt{n})$. Therefore, because $d_{m,2p} \to 0$ as $m \to \infty$, by (5.2), (5.3), and Propositions 1 and 2, it suffices to show that

$$\frac{M_n + \bar{M}_n}{\sigma_n} \Rightarrow N(0, 4\pi^2 \tilde{f}^2(\lambda)), \quad \text{where } 2\pi\tilde{f}(\lambda) = \sum_{j=-m}^m \tilde{\gamma}_j \cos(j\lambda), \tag{5.6}$$

holds for every m. Here $\tilde{\gamma}_j = \mathbb{E}(\tilde{X}_0\tilde{X}_j)$. Let $U_t^\diamond = \sum_{j=(t-4m+1)\vee 1}^{t-1} a_{n,j-t} D_j$. By (5.2) and (5.3), we have $\|\sum_{t=1}^n \bar{D}_t U_t^\diamond\| = O(\sqrt{n}) \max_{1\le t\le n} |b_{n,t}|$. So it remains to show that

$$\frac{1}{\sigma_n} \sum_{t=1+4m}^n (\bar{D}_t U_t + D_t \bar{U}_t) \Rightarrow N(0, 4\pi^2 \tilde{f}^2(\lambda)), \quad \text{where } U_t = \sum_{j=1}^{t-4m} a_{n,j-t} D_j. \tag{5.7}$$

Because $\sum_{t=1+4m}^n \|\bar{D}_t U_t\|_4^4 = O(n)\varsigma_n^4 = o(\sigma_n^4)$, the Lindeberg condition easily follows. By the martingale CLT (see Hall and Heyde, 1980), (5.7) holds if

$$\frac{1}{\sigma_n^2} \sum_{t=1+4m}^n \mathbb{E}[(\bar{D}_t U_t + D_t \bar{U}_t)^2 | \mathcal{F}_{t-1}] \to 4\pi^2 \tilde{f}^2(\lambda) \quad \text{in probability} \tag{5.8}$$

For the rest of the proof, we shall verify (5.8). Let $-m \le l \le m-1$. Then

$$\left\| \sum_{t=1+4m}^n \mathcal{P}_{t+l}(\bar{D}_t U_t + D_t \bar{U}_t)^2 \right\|^2 = \sum_{t=1+4m}^n \|\mathcal{P}_{t+l}(\bar{D}_t U_t + D_t \bar{U}_t)^2\|^2$$

$$\le 4 \sum_{t=1+4m}^n \|D_t\|_4^4 \|U_t\|_4^4.$$

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Hence $\|\sum_{t=1+4m}^{n}\mathcal{P}_{t+l}(\bar{D}_t U_t + D_t\bar{U}_t)^2\| = o(\sigma_n^2)$. Because D_t is $\mathcal{F}_{t-m,t}$ measurable and U_t is $\mathcal{F}_{t-3m}$ measurable, $\mathbb{E}(\bar{D}_t^2 U_t^2|\mathcal{F}_{t-m-1}) = U_t^2\mathbb{E}(\bar{D}_t^2)$, (5.8) is then reduced to

$$\frac{1}{\sigma_n^2}\sum_{t=1+4m}^{n}[U_t^2\mathbb{E}(\bar{D}_t^2) + \bar{U}_t^2\mathbb{E}(D_t^2) + 2|U_t|^2\mathbb{E}(|D_t^2|)] \to \|D_0\|^4 \quad \text{in probability} \tag{5.9}$$

by noting that $\|D_0\|^2 = \sum_{j=-m}^{m}\tilde{\gamma}_j e^{\imath j\lambda} = 2\pi\tilde{f}(\lambda)$. For the rest of the proof we shall only deal with the case of $\lambda \neq 0, \pi$ because the case of $\lambda = 0, \pi$ can be similarly proved. Using the argument of Lemma 3 in Wu and Shao (2007), under (5.2)–(5.4), we can prove that $\|\sum_{t=1+4m}^{n}(U_t^2 - \mathbb{E}U_t^2)\| = o(\sigma_n^2)$. So (5.9) is further reduced to

$$\frac{1}{\sigma_n^2}\sum_{t=1+4m}^{n}[\mathbb{E}(U_t^2)\mathbb{E}(\bar{D}_t^2) + \mathbb{E}(\bar{U}_t^2)\mathbb{E}(D_t^2) + 2\mathbb{E}(|U_t|^2)\mathbb{E}(|D_t^2|)] \to \|D_0\|^4. \tag{5.10}$$

Clearly $\mathbb{E}(U_t^2) = \sum_{j=1}^{t-4m} a_{n,j-t}^2\mathbb{E}D_j^2$. By summation by parts and (5.5), because $|\sum_{l=1}^{j} e^{2\imath l\lambda}| \le 1/|\sin\lambda|$, we have $\max_{t\le n}|\sum_{j=1}^{t} a_{n,j-t}^2| = o(\varsigma_n^2)$ and $\sum_{t=1+4m}^{n}|\mathbb{E}(U_t^2)| = o(\sigma_n^2)$. So (5.10) follows from (5.2) and (5.3) because $\mathbb{E}(|U_t|^2) = \sum_{j=1}^{t-4m} b_{n,j-t}^2\mathbb{E}(|D_t^2|)$. ■

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TECHNICAL APPENDIX AND PROOFS

A.1. Proofs of the Results in Sections 2.1–2.3.

Proof of Theorem 1. Recall (3.1) for $\tilde{X}_t = X_{t,m}$, which are m-dependent. Let $a_{n,t} = K(t/B_n)\cos(t\theta)$. For $4m+1 \le t \le n$ let $Y_t = \tilde{X}_t \sum_{s=1}^{t-4m} a_{n,t-s}\tilde{X}_s$ and $R_m = \sum_{t=1+4m}^{n} Y_t$. By Condition 1, $\sum_{s\in\mathbb{Z}} a_{n,s}^2 = O(B_n)$. By independence and Lemma 1,

$$\|Y_t\|_p = \|\tilde{X}_t\|_p \left\| \sum_{s=1}^{t-4m} a_{n,t-s}\tilde{X}_s \right\|_p \le \|X_0\|_p C_p \Theta_{0,p} \left(\sum_{s=1}^{t-4m} a_{n,t-s}^2 \right)^{1/2} = O(B_n^{1/2}). \quad \textbf{(A.1)}$$

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Let $J_l = \lfloor (n-l)/(4m) \rfloor$. Because $Y_t, Y_{t+4m}, Y_{t+8m}, \ldots$, are $\mathcal{L}^p$ martingale differences,

$$\|R_n\|_p \le \sum_{l=1}^{4m} \left\| \sum_{j=1}^{J_l} Y_{l+4mj} \right\|_p = \sum_{l=1}^{4m} J_l^{1/2} O(B_n^{1/2}) = O[(mnB_n)^{1/2}). \tag{A.2}$$

Let $\tilde{\gamma}_k = \mathbb{E}(\tilde{X}_0 \tilde{X}_k)$, $\tilde{g}_n(\theta) = 2\pi \tilde{f}_n(\theta)$ and

$$\tilde{g}_n(\theta) = \frac{1}{n} \sum_{t=1}^{n} \tilde{X}_t^2 + \frac{2}{n} \sum_{t=2}^{n} \tilde{X}_t \sum_{s=1}^{t-1} a_{n,t-s} \tilde{X}_s$$

$$= \frac{1}{n} \sum_{t=1}^{n} \tilde{X}_t^2 + \frac{2}{n} \sum_{t=2}^{n} \tilde{X}_t \sum_{s=\max(1,t-4m+1)}^{t-1} a_{n,t-s} \tilde{X}_s + \frac{2R_n}{n}. \tag{A.3}$$

By the ergodic theorem, for $1 \le l \le 4m$, $\|n^{-1} \sum_{t=1}^{n} \tilde{X}_t \tilde{X}_{t+l} - \tilde{\gamma}_l\|_{p/2} \to 0$. So

$$\lim_{n\to\infty} \|\tilde{g}_n(\theta) - 2n^{-1} R_n - \mathbb{E}[\tilde{g}_n(\theta) - 2n^{-1} R_n]\|_{p/2} = 0. \tag{A.4}$$

Let $\tilde{I}_n(u) = (2\pi n)^{-1} |\tilde{S}_n(u)|^2$, where $\tilde{S}_n(u) = \sum_{k=1}^{n} \tilde{X}_k e^{\iota k u}$. By Lemma 1, $\|S_n(u) - \tilde{S}_n(u)\|_p = O(n^{1/2}) \Theta_{m,p}$, $\|S_n(u)\|_p + \|\tilde{S}_n(u)\|_p = O(n^{1/2})$. Hence $\|I_n(u) - \tilde{I}_n(u)\|_{p/2} = O(1)\Theta_{m,p}$. Because $\tilde{f}_n(\theta) = \int_{\mathbb{R}} \hat{K}(u) \tilde{I}_n(B_n^{-1} u + \theta)\, du$, $p/2 \ge 1$, and $\int_{\mathbb{R}} |\hat{K}(u)|\, du < \infty$,

$$\|f_n(\theta) - \tilde{f}_n(\theta)\|_{p/2} \le \int_{\mathbb{R}} |\hat{K}(u)| \|I_n(B_n^{-1} u + \theta) - \tilde{I}_n\left(B_n^{-1} u + \theta\right)\|_{p/2} du = O(1)\Theta_{m,p}.$$

So $|\mathbb{E}[f_n(\theta) - \tilde{f}_n(\theta)]| = O(\Theta_{m,p})$. By (A.2) and (A.4), because $B_n = o(n)$, $\lim_{n\to\infty} \|\tilde{f}_n(\theta) - \mathbb{E}[\tilde{f}_n(\theta)]\|_{p/2} = 0$. Hence $\lim_{n\to\infty} \|f_n(\theta) - \mathbb{E} f_n(\theta)\|_{p/2} = 0$ in view of

$$\|f_n(\theta) - \mathbb{E} f_n(\theta)\|_{p/2} \le \|f_n(\theta) - \tilde{f}_n(\theta)\|_{p/2} + |\mathbb{E}[f_n(\theta) - \tilde{f}_n(\theta)]| + \|\tilde{f}_n(\theta) - \mathbb{E}[\tilde{f}_n(\theta)]\|_{p/2}$$

and $\Theta_{m,p} \to 0$ as $m \to \infty$. It is well known in time series analysis that, under (1.1), $B_n \to \infty$, and $\lim_{u\to 0} K(u) = 1$, the bias $\mathbb{E} f_n(\theta) - f(\theta) \to 0$. So (2.4) follows. ■

Proof of Theorem 2. We shall apply Theorem 6 to $b_{n,j} = K(j/B_n)$ and $a_{n,j} = b_{n,j} e^{\iota j \lambda}$. By Condition 2, easy calculations show that $2\sum_{k=1}^{n} K^2(k/B_n) \sim B_n \kappa$ and $\sum_{k=1}^{n} \sum_{t=1}^{n} b_{n,t-k}^2 \sim n B_n \kappa$. So (5.2) and (5.3) hold. Let M be a positive integer that will be specialized later. Because $B_n = o(n)$, by Schwarz's inequality,

$$\sum_{k=1}^{n} \sum_{t=1 \vee (k - MB_n)}^{k-1} \left(\sum_{j=1+k}^{n} b_{n,k-j} b_{n,t-j} \right)^2 = O(nB_n^3) = o(\sigma_n^4). \tag{A.5}$$

Using Schwarz's inequality again, we can get

$$\sum_{k=1}^{n} \sum_{t=1}^{k-MB_n} \left(\sum_{j=1+k}^{n} b_{n,k-j} b_{n,t-j} \right)^2 \le C n^2 B_n \sum_{k=MB_n}^{n} K^2(k/B_n) \le \tau_M n^2 B_n^2 \tag{A.6}$$

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with $\tau_M \to 0$ as $M \to \infty$. Combining (A.5) and (A.6), we have (5.4). It remains to prove (5.5). Because K is continuous at all but a finite number of points, it can be easily obtained that $|K(k/B_n) - K((k-1)/B_n)| = o(1)$ uniformly for $|k| \le MB_n$, except for εB_n points $k \in [-MB_n, MB_n]$, where $\varepsilon > 0$ is an arbitrary number. Thus, $\sum_{k=1}^{MB_n}\{K(k/B_n) - K[(k-1)/B_n]\}^2 = o(B_n)$. Moreover, it is easy to see that $\sum_{k=MB_n}^{n}\{K(k/B_n) - K[(k-1)/B_n]\}^2 \le \tau_M B_n$. Hence (5.5) is proved. ■

Remark A.1. It is easily seen that Theorem 2 also holds if the requirement that K is continuous at all but a finite number of points in Condition 2 is replaced by the following one: K has bounded variation. The two conditions have different ranges of applicability.

A.2. Proofs of Theorems 3–5. Let $a_{n,t} = K(t/B_n)\cos(t\lambda)$. Recall (5.1) for T_n and define $T_{n,m}$ by replacing $\{X_t\}$ in (5.1) by $\{X_{t,m}\}$. Let $g_n(\lambda) = T_n - \mathbb{E}T_n - \sum_{k=1}^{n}(X_k^2 - \mathbb{E}X_k^2)$ and $g_{n,m}(\lambda) = T_{n,m} - \mathbb{E}T_{n,m} - \sum_{k=1}^{n}(X_{k,m}^2 - \mathbb{E}X_{k,m}^2)$. Then $2\pi n\{f_n(\lambda) - \mathbb{E}f_n(\lambda)\} = g_n(\lambda) + \sum_{k=1}^{n}(X_k^2 - \mathbb{E}X_k^2)$. The proofs of Theorems 3–5 are quite complicated, and they are based on a series of lemmas. Let $\tau_n = \sqrt{nB_n}/\log B_n$.

LEMMA A.1. *Suppose that the conditions of Theorem 3 hold. Then for any* $0 < \mathcal{C} < 1$ *there exists* $\gamma \in (0, \mathcal{C})$ *such that, for* $m = \lfloor n^{\gamma} \rfloor$,

$$\max_{0\le i\le B_n} |g_n(\lambda_i^*) - g_{n,m}(\lambda_i^*)| = o_{\mathbb{P}}\left(\sqrt{nB_n/\log B_n}\right).$$

Remark A.2. By Proposition 1, Lemma A.1 also holds under conditions of Theorem 5.

Proof. Let $\varrho \in (0,1)$ be fixed and be sufficiently close to 1. Let $s_l = \lfloor n^{\varrho^l} \rfloor$, $1 \le l \le r$, and $r \in \mathbb{N}$ be such that $0 < \varrho^r < \mathcal{C}$. Let $r_0(n) \in \mathbb{N}$ satisfy $1 \le r_0(n) \le r$ and $s_{r_0(n)} < B_n \le s_{r_0(n)-1}$. By Markov's inequality and Proposition 1 and because $p > 4$,

$$\begin{aligned}\mathbb{P}\left(\max_{0\le i\le B_n} |g_n(\lambda_i^*) - g_{n,s_1}(\lambda_i^*)| \ge \tau_n\right) &\le (1+B_n)\max_{0\le i\le B_n}\frac{\mathbb{E}|g_n(\lambda_i^*) - g_{n,s_1}(\lambda_i^*)|^{p/2}}{(nB_n)^{p/4}(\log B_n)^{-p/2}}\\ &\le C B_n d_{s_1,p}^{p/2}(\log B_n)^{p/2}\\ &\le C n^{\delta - T_1\varrho p/2}(\log n)^{p/4-2} = o(1).\end{aligned}$$

So we only need to show that, for every $1 \le l \le r-1$, $\max_{0\le i\le B_n}|g_{n,s_l}(\lambda_i^*) - g_{n,s_{l+1}}(\lambda_i^*)| = o_{\mathbb{P}}\left(\sqrt{nB_n/\log B_n}\right)$. Let $Y_{t,m}(\lambda) = X_{t,m}\sum_{s=1}^{t-1} X_{s,m}a_{n,t-s}$ and note that, for any $1 \le l \le r$, $Y_{t,s_l}(\lambda)$, $1 \le t \le n$, are $(B_n + s_l)$-dependent. Split the interval $[1,n]$ into consecutive blocks $H_1, H_2, \ldots, H_{t_n}$ with equal length $B_n + s_l$, where the number of intervals $t_n \sim n/(B_n + s_l)$ and the last interval may be incomplete. For convenience we assume that the length of the last interval is also $B_n + s_l$. Define

$$\breve{u}_j(\lambda) = \sum_{t\in H_j}(Y_{t,s_l}(\lambda) - Y_{t,s_{l+1}}(\lambda)), \qquad \overline{u}_j(\lambda) = \breve{u}_j(\lambda) - \mathbb{E}\breve{u}_j(\lambda), \qquad 1 \le j \le t_n. \quad \textbf{(A.7)}$$

Then $\overline{u}_1(\lambda), \ldots, \overline{u}_{t_n}(\lambda)$ are 1-dependent and $g_{n,s_l}(\lambda) - g_{n,s_{l+1}}(\lambda) = 2\sum_{j=1}^{t_n}\overline{u}_j(\lambda)$. By Lemma 2, for any large Q and $1 \le l \le r-1$,

$$\mathbb{P}\left(\max_{0\le i\le B_n} |g_{n,s_l}(\lambda_i^*) - g_{n,s_{l+1}}(\lambda_i^*)| \ge \tau_n\right)$$

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$$\le C\sum_{i=0}^{B_n}\left(\frac{\sum_{j=1}^{t_n}\mathbb{E}|\overline{u}_j(\lambda_i^*)|^2}{\tau_n^2}\right)^Q + C\sum_{i=0}^{B_n}\sum_{j=1}^{t_n}\mathbb{P}\left(|\overline{u}_j(\lambda_i^*)|\ge C_Q\tau_n\right). \tag{A.8}$$

Because $d_{n,p}=O(n^{-T_1})$, by Proposition 1, $\max_{\lambda\in\mathbb{R}}\max_{1\le j\le t_n}\mathbb{E}|\overline{u}_j(\lambda)|^2 = O\left(B_n(B_n+s_l)s_{l+1}^{-2T_1}\right)$. Let $Q\in\mathbb{N}$ be sufficiently large. Then the first term in the preceding expression is $o(1)$. It remains to show that the second one is also $o(1)$. We first deal with the case $1\le l\le r_0(n)-1$. Because $s_l\sim s_{l+1}n^{\rho^l(1-\rho)}$ and $s_{r_0(n)-1}\ge B_n$, we have $t_n\asymp n/s_l$. By Proposition 1 and Markov's inequality,

$$\sum_{i=0}^{B_n}\sum_{j=1}^{t_n}\mathbb{P}\left(|\overline{u}_j(\lambda_i^*)|\ge C_Q\tau_n\right) \le CB_n n s_l^{-1}s_l^{p/4}s_{l+1}^{-pT_1/2}n^{-p/4}(\log B_n)^{p/2}$$

$$\le CB_n s_l^{p/4-1-pT_1/2}n^{1-p/4+\varepsilon_\varrho}(\log B_n)^{p/2} =: F_1, \tag{A.9}$$

where $\varepsilon_\varrho\to 0$ as $\varrho\to 1$. If $4<p\le 4+4\delta$, then $p/4-1-pT_1/2\le 0$, and hence

$$F_1 \le CB_n^{p/4-pT_1/2}n^{1-p/4+\varepsilon_\varrho}(\log B_n)^{p/2} \le Cn^{0\vee\{\delta(p/4-pT_1/2)\}+1-p/4+\varepsilon_\varrho}(\log B_n)^{p/2}$$

$$= o(1),$$

where we have used $s_l\ge B_n$ and Condition 5(a). If $p>4+4\delta$ and $p/4-1-pT_1/2\le 0$, then $F_1=o(1)$. Finally, if $p>4+4\delta$ and $p/4-1-pT_1/2>0$, then because $s_l\le n$, we have $F_1\le Cn^{\delta-pT_1/2+\varepsilon_\varrho}(\log B_n)^{p/2}=o(1)$. Hence $F_1=o(1)$ when $1\le l\le r_0(n)-1$. We now deal with the case $r_0\le l\le r-1$. For $r_0(n)\le j\le r$, $r_0(n)\le l\le r-1$, let

$$U_{t,j}=\sum_{s=1}^{t-s_l-1}a_{n,t-s}X_{s,s_j}, \qquad \check{Y}_{t,s_l}=X_{t,s_l}U_{t,l}-X_{t,s_{l+1}}U_{t,l+1}.$$

Let $j_1=\min\{k:k\in H_j\}$ and $j_2=\max\{k:k\in H_j\}$. Because $X_{t,i}=\sum_{k=t-s_l}^{t}\mathcal{P}_kX_{t,i}$ for $i=s_l,s_{l+1}$, we have

$$u_j'(\lambda):=\sum_{t\in H_j}\check{Y}_{t,s_l}=\sum_{k=j_1-s_l}^{j_2}\overline{W}_{k,l}, \quad \text{where } \overline{W}_{k,l}$$

$$=\sum_{t=k\vee j_1}^{(k+s_l)\wedge j_2}(\mathcal{P}_kX_{t,s_l}U_{t,l}-\mathcal{P}_kX_{t,s_{l+1}}U_{t,l+1}).$$

Note that $\overline{W}_{k,l}$, $j_1-s_l\le k\le j_2$, are martingale differences; by Lemma 1,

$$\max_{\lambda\in\mathbb{R}}\|\overline{W}_{k,l}\|_p \le \max_{\lambda\in\mathbb{R}}\sum_{t=k\vee j_1}^{(k+s_l)\wedge j_2}\|\mathcal{P}_kX_{t,s_l}\|_p\|U_{t,l}-U_{t,l+1}\|_p$$

$$+\max_{\lambda\in\mathbb{R}}\sum_{t=k\vee j_1}^{(k+s_l)\wedge j_2}\|\mathcal{P}_k(X_{t,s_l}-X_{t,s_{l+1}})\|_p\|U_{t,l+1}\|_p = O(B_n^{1/2}).$$

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By Markov's inequality, because $p > 2/(1-\delta)$, we have

$$B_n \max_{0\le i\le B_n}\sum_{j=1}^{t_n}\mathbb{P}\left(|u_j'(\lambda_i^*)|\ge\tau_n\right)\le Cn^{1-p/2}B_n^{-1}(\log B_n)^{2p}\sum_{k=j_1-s_l}^{j_2}\max_{\lambda\in\mathbb{R}}\mathbb{E}|\overline{W}_{k,l}|^p$$

$$\le Cn^{1-p/2}B_n^{p/2}(\log B_n)^{2p}=o(1). \tag{A.10}$$

Putting $\widehat{Y}_{t,s_l}=Y_{t,s_l}(\lambda)-Y_{t,s_{l+1}}(\lambda)-\check{Y}_{t,s_l}$, then $\overline{u}_j(\lambda)=u_j'(\lambda)+\sum_{t\in H_j}(\widehat{Y}_{t,s_l}-\mathbb{E}\widehat{Y}_{t,s_l})$. By Lemma 1, we have $\mathbb{E}|\widehat{Y}_{t,s_l}|^{p/2}\le Cs_l^{p/4}s_{l+1}^{-T_2p/2}$. Because $\widehat{Y}_{t,s_l}$, $1\le t\le n$, are s_l-dependent, by Lemma 2, for Q large enough,

$$B_n\max_{0\le i\le B_n}\sum_{j=1}^{t_n}\mathbb{P}\left(|\overline{u}_j(\lambda_i^*)-u_j'(\lambda_i^*)|\ge\tau_n\right)$$

$$\le CB_n\max_{0\le i\le B_n}\sum_{j=1}^{t_n}\left(\frac{s_l\sum_{t\in H_j}\mathbb{E}|\widehat{Y}_{t,s_l}|^2}{nB_n(\log B_n)^{-2}}\right)^Q$$

$$+CB_n(nB_n)^{-p/4}(\log B_n)^{p/2}s_l^{p/2-1}\max_{0\le i\le B_n}\sum_{j=1}^{t_n}\sum_{t\in H_j}\mathbb{E}|\widehat{Y}_{t,s_l}|^{p/2}$$

$$\le Cs_l^{3p/4-1}s_{l+1}^{-pT_2/2}(nB_n)^{-p/4+1}(\log B_n)^{p/2}=o(1), \tag{A.11}$$

where the last relation is due to $s_l<B_n$, $s_l\sim s_{l+1}n^{\rho^l(1-\rho)}$ and Condition 5(b). By (A.8)–(A.11), Lemma A.1 follows. ■

LEMMA A.1*. *Under the conditions of Theorem 4, the conclusion in Lemma A.1 holds.*

Proof. By the argument in the proof of Lemma A.1, we only need to show that, for every $r_0(n)\le l\le r-1$, $\max_{0\le i\le B_n}|g_{n,s_l}(\lambda_i^*)-g_{n,s_{l+1}}(\lambda_i^*)|=o_{\mathbb{P}}\left(\sqrt{nB_n/\log B_n}\right)$. Recall (A.7) for $\overline{u}_j(\lambda)$. Let $\overline{u}_j'(\lambda)=\overline{u}_j(\lambda)I\left\{|\overline{u}_j(\lambda)|\le\sqrt{nB_n}/(\log B_n)^3\right\}$. Then

$$\mathbb{P}\left(\max_{0\le i\le B_n}\left|\sum_{j=1}^{t_n}\overline{u}_j(\lambda_i^*)\right|\ge\tau_n\right)\le\mathbb{P}(G_n\ge\tau_n)+\sum_{j=1}^{t_n}\mathbb{P}\left(P_j\ge\frac{\sqrt{nB_n}}{(\log B_n)^3}\right),$$

where $G_n=\max_{0\le i\le B_n}|\sum_{j=1}^{t_n}\overline{u}_j'(\lambda_i^*)|$ and $P_j=\max_{0\le i\le B_n}|\overline{u}_j(\lambda_i^*)|$. By Proposition 1, $\sum_{j=1}^{t_n}\mathbb{E}|\overline{u}_j'(\lambda)|^2\le CnB_nd_{s_l,4}^2$, and

$$\sum_{j=1}^{t_n}|\mathbb{E}\overline{u}_j'(\lambda)|\le\sum_{j=1}^{t_n}\mathbb{E}|\overline{u}_j(\lambda)|^2(nB_n)^{-1/2}(\log B_n)^3$$

$$\le C(nB_n)^{1/2}d_{s_l,4}^2(\log B_n)^3=o\left(\sqrt{nB_n}/\log B_n\right),$$

we have, by Bernstein's inequality, $\mathbb{P}(G_n\ge\tau_n)\le CB_n\exp(-C(\log B_n)^2)=o(1)$. To finish the proof of Lemma A.1*, we only need to show that

$$\Delta_n:=\sum_{j=1}^{t_n}\mathbb{P}\left(P_j\ge\sqrt{nB_n}/(\log B_n)^3\right)=o(1).$$

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Recall $Y_{t,m}(\lambda) = X_{t,m} \sum_{s=1}^{t-1} X_{s,m} a_{n,t-s}$. Because for $2 \le j \le t_n$, $\overline{u}_j(\lambda) \overset{D}{=} \overline{u}_2(\lambda)$ and

$$\breve{u}_2(\lambda) = \sum_{t=1}^{2B_n+2s_l} (Y_{t,s_l}(\lambda) - Y_{t,s_{l+1}}(\lambda)) - \sum_{t=1}^{B_n+s_l} (Y_{t,s_l}(\lambda) - Y_{t,s_{l+1}}(\lambda)).$$

Let $S_{n,1}(u) = \sum_{k=1}^{2B_n+2s_l} X_{k,s_l} e^{\iota k u}$ and $S_{n,2}(u) = \sum_{k=1}^{2B_n+2s_l} X_{k,s_{l+1}} e^{\iota k u}$, and similarly define $S'_{n,1}(u)$, $S'_{n,2}(u)$, by replacing $2B_n + 2s_l$ in $S_{n,1}(u)$ and $S_{n,2}(u)$ by $B_n + s_l$. Then

$$\begin{aligned} 2\breve{u}_2(\lambda) = {} & B_n \int_{-\infty}^{\infty} \hat{K}(B_n(u-\lambda)) \left[|S_{n,1}(u)|^2 - |S_{n,2}(u)|^2 \right] du \\ & - B_n \int_{-\infty}^{\infty} \hat{K}(B_n(u-\lambda)) \left[|S'_{n,1}(u)|^2 - |S'_{n,2}(u)|^2 \right] du \\ & - \sum_{t=B_n+s_l+1}^{2B_n+2s_l} X_{t,s_l}^2 + \sum_{t=B_n+s_l+1}^{2B_n+2s_l} X_{t,s_{l+1}}^2. \end{aligned}$$

Let $Q_{n,k}(u) = |S_{n,k}(u)|^2 - \mathbb{E}|S_{n,k}(u)|^2$, $Q'_{n,k}(u) = |S'_{n,k}(u)|^2 - \mathbb{E}|S'_{n,k}(u)|^2$, $k = 1, 2$, $W_n = \max_{u\in\mathbb{R}} |Q_{n,1}(u) - Q_{n,2}(u)|$, and $W'_n = \max_{u\in\mathbb{R}} |Q'_{n,1}(u) - Q'_{n,2}(u)|$. Then

$$\begin{aligned} 2 \max_{\lambda\in\mathbb{R}} |\overline{u}_2(\lambda)| \le {} & (W_n + W'_n) \int_{-\infty}^{\infty} |\hat{K}(u)| du \\ & + \left| \sum_{t=B_n+s_l+1}^{2B_n+2s_l} (X_{t,s_l}^2 - \mathbb{E}X_{t,s_l}^2) \right| + \left| \sum_{t=B_n+s_l+1}^{2B_n+2s_l} (X_{t,s_{l+1}}^2 - \mathbb{E}X_{t,s_{l+1}}^2) \right|. \end{aligned}$$

Define $\mu_n = \sqrt{nB_n}/(\log B_n)^3$ and $\nu_n = (nB_n)^{1/4}/(\log B_n)^3$. Then

$$\begin{aligned} \Delta_n \le {} & \sum_{j=1}^{t_n} \mathbb{P}(W_n \ge 2^{-1}\mu_n) + \sum_{j=1}^{t_n} \mathbb{P}(W'_n \ge 2^{-1}\mu_n) + o(1) \\ \le {} & \sum_{j=1}^{t_n} \mathbb{P}\left(\max_{0\le u\le 2\pi} (|S_{n,1}(u)| + |S_{n,2}(u)|) \ge \nu_n \right) \\ & + \sum_{j=1}^{t_n} \mathbb{P}\left(\max_{0\le u\le 2\pi} (|S'_{n,1}(u)| + |S'_{n,2}(u)|) \ge \nu_n \right) + o(1). \end{aligned}$$

Let $l_i = i/n^2$, $0 \le i \le 2\pi n^2$. Then

$$\max_{0\le u\le 2\pi} |S_{n,1}(u)| \le \max_{0\le i\le 2\pi n^2} |S_{n,1}(l_i)| + Cn^{-2} B_n \sum_{k=1}^{B_n+s_l} |X_{k,s_l}|.$$

So it suffices to show that $\sum_{j=1}^{t_n} \mathbb{P}(\max_{0\le i\le 2\pi n^2} |S_{n,1}(l_i)| \ge \nu_n) = o(1)$. Note that

$$S_{n,1}(u) = \sum_{j=1-s_l}^{2B_n+2s_l} \sum_{k=1\vee j}^{(2B_n+2s_l)\wedge(j+s_l)} \mathcal{P}_j(X_{k,s_l}) e^{\iota k u} =: \sum_{j=1-s_l}^{2B_n+2s_l} R_{n,j}(u).$$

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Set $\widetilde{R}_{n,j}(u) = R_{n,j}(u) I\left\{|R_{n,j}(u)| \le (nB_n)^{1/4}/(\log B_n)^5\right\}$. Then

$$\sum_{j=1}^{t_n} \mathbb{P}\left(\max_{0\le u\le 2\pi}\left|\sum_{j=1-s_l}^{2B_n+2s_l} R_{n,j}(u) - \overline{R}_{n,j}(u)\right| \ge \nu_n\right)$$

$$\le C\frac{n(\log B_n)^{18}}{B_n(nB_n)} \sum_{j=1-s_l}^{2B_n+2s_l} \left\| \sum_{k=1\vee j}^{(B_n+s_l)\wedge(j+s_l)} |\mathcal{P}_j(X_{k,s_l})| \right\|_4^4 \le C\frac{(\log B_n)^{18}}{B_n} = o(1),$$

where $\overline{R}_{n,j}(u) = \widetilde{R}_{n,j}(u) - \mathbb{E}(\widetilde{R}_{n,j}(u)|\mathcal{F}_{j-1})$. Also,

$$\left\| \max_{u\in\mathbb{R}} \sum_{j=1-s_l}^{2B_n+2s_l} \mathbb{E}(|\overline{R}_{n,j}(u)|^2|\mathcal{F}_{j-1}) \right\|_{p/2} \le \left\| \max_{u\in\mathbb{R}} \sum_{j=1-s_l}^{2B_n+2s_l} \mathbb{E}(|R_{n,j}(u)|^2|\mathcal{F}_{j-1}) \right\|_{p/2}$$

$$\le \sum_{j=1-s_l}^{2B_n+2s_l} \left(\sum_{k=1\vee j}^{(2B_n+2s_l)\wedge(j+s_l)} \delta_{k-j,p} \right)^2 \le CB_n.$$

We can get, by Freedman's inequality and because $p > 4$,

$$\sum_{j=1}^{t_n} \mathbb{P}\left(\max_{0\le i\le 2\pi n^2}\left|\sum_{j=1-s_l}^{B_n+s_l} \overline{R}_{n,j}(l_i)\right| \ge \nu_n\right)$$

$$\le Cn^3 B_n^{-1} \exp(-C(\log B_n)^2) + nB_n^{-1} B_n^{p/2} (nB_n)^{-p/4} (\log B_n)^{5p/2} = o(1).$$

The proof is now complete. ∎

For Lemmas A.2–A.4, we need to introduce truncation. Let $\alpha < \frac{1}{4}$ be close to $\frac{1}{4}$ sufficiently and $m = \lfloor n^\gamma \rfloor$, where γ is small enough. Define

$$X'_{t,m} = X_{t,m} I\{|X_{t,m}| \le (nB_n)^\alpha\}, \qquad \overline{X}_{t,m} = X'_{t,m} - \mathbb{E}X'_{t,m},$$

$$\overline{g}_{n,m}(\lambda) = 2\sum_{t=2}^{n} \overline{X}_{t,m} \sum_{s=1}^{t-1} a_{n,t-s} \overline{X}_{s,m} - 2\mathbb{E}\sum_{t=2}^{n} \overline{X}_{t,m} \sum_{s=1}^{t-1} a_{n,t-s} \overline{X}_{s,m}.$$

Let $p_n = \lfloor B_n^{1+\beta} \rfloor$, $q_n = B_n + m$, and $k_n = \lfloor n/(p_n+q_n) \rfloor$, where $\beta > 0$ is sufficiently close to zero. Split the interval $[1, n]$ into alternating big and small blocks H_j and I_j by

$$H_j = [(j-1)(p_n+q_n)+1, jp_n+(j-1)q_n];$$

$$I_j = [jp_n+(j-1)q_n+1, j(p_n+q_n)];$$

$$1 \le j \le k_n, \quad I_{k_n+1} = [k_n(p_n+q_n)+1, n].$$

Set $\overline{Y}_{t,m}(\lambda) = \overline{X}_{t,m} \sum_{s=1}^{t-1} a_{n,t-s} \overline{X}_{s,m}$. For $1 \le j \le k_n+1$ let

$$u_j(\lambda) = \sum_{t\in H_j} (\overline{Y}_{t,m}(\lambda) - \mathbb{E}\overline{Y}_{t,m}(\lambda) \quad \text{and} \quad v_j(\lambda) = \sum_{t\in I_j} (\overline{Y}_{t,m}(\lambda) - \mathbb{E}\overline{Y}_{t,m}(\lambda)). \quad \textbf{(A.12)}$$

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Because $K(\cdot)$ is bounded, we have

$$\mathbb{E}\max_{\lambda\in\mathbb{R}}\left|g_{n,m}(\lambda)-\overline{g}_{n,m}(\lambda)\right| \le C\mathbb{E}\sum_{t=2}^{n}|X_{t,m}|\sum_{s=1\vee(t-B_n)}^{t-1}|X_{s,m}-\overline{X}_{s,m}| + C\mathbb{E}\sum_{t=2}^{n}|X_{t,m}-\overline{X}_{t,m}|\sum_{s=1\vee(t-B_n)}^{t-1}|\overline{X}_{s,m}|. \tag{A.13}$$

Recall $\tau_n=\sqrt{nB_n}/\log B_n$. By independence and because $X_0\in\mathcal{L}^p,\ p>4$,

$$\begin{aligned}&\mathbb{E}\left(\sum_{t=m+1}^{n}|X_{t,m}|\sum_{s=1\vee(t-B_n)}^{t-1}|X_{s,m}-\overline{X}_{s,m}|\right) \\ &\quad\le \mathbb{E}\left(\sum_{t=m+1}^{n}|X_{t,m}|\sum_{s=1\vee(t-B_n)}^{t-m}|X_{s,m}-\overline{X}_{s,m}|\right) \\ &\qquad+\mathbb{E}\left(\sum_{t=2}^{n}|X_{t,m}|\sum_{s=(t-m+1)\vee1}^{t-1}|X_{s,m}-\overline{X}_{s,m}|\right) \\ &\quad\le C(nB_n)^{1-(p-1)\alpha}+Cnm(nB_n)^{-(p-2)\alpha}=o(\tau_n).\end{aligned} \tag{A.14}$$

Similar arguments yield to

$$\mathbb{E}\left(\sum_{t=2}^{n}|X_{t,m}-\overline{X}_{t,m}|\sum_{s=1\vee(t-B_n)}^{t-1}|\overline{X}_{s,m}|\right)=o(\tau_n). \tag{A.15}$$

Combining (A.13)–(A.15), we get

$$\mathbb{E}\max_{\lambda\in\mathbb{R}}\left|g_{n,m}(\lambda)-\overline{g}_{n,m}(\lambda)\right|=o(\tau_n). \tag{A.16}$$

LEMMA A.2. *Assume* $\mathbb{E}X_0=0$ *and* $\mathbb{E}X_0^4<\infty$. *Recall* $\tau_n=\sqrt{nB_n}/\log B_n$. *We have*

$$\max_{|i|\le B_n}|\Gamma_i|=O_{\mathbb{P}}(\tau_n),\quad \textit{where } \Gamma_i=\sum_{j=1}^{k_n+1}v_j(\lambda_i^*).$$

Proof. Because $v_j(\lambda_i^*)$, $1\le j\le k_n+1$, are independent, by Lemma 2, for all large Q,

$$\mathbb{P}(|\Gamma_i|\ge\tau_n)\le C\left(\frac{\sum_{j=1}^{k_n+1}\mathbb{E}v_j^2(\lambda_i^*)}{nB_n(\log B_n)^{-2}}\right)^{Q}+C\sum_{j=1}^{k_n+1}\mathbb{P}\left(|v_j(\lambda_i^*)|\ge C_Q\tau_n\right).$$

In Proposition 3 we let $x=C_Q\tau_n$, $M=(nB_n)^{\alpha}$, $k=B_n+m$, $m=\lfloor n^{\gamma}\rfloor$, and $y=(\log B_n)^2$. Then for all $c>0$, $\mathbb{P}(|v_j(\lambda_i^*)|\ge C_Q\tau_n)=O(n^{-c})$. Because $\mathbb{P}(\max_{|i|\le B_n}|\Gamma_i|\le\tau_n)\le\sum_{|i|\le B_n}\mathbb{P}(|\Gamma_i|\ge\tau_n)$, by elementary manipulations, the lemma follows. ■

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LEMMA A.3. *Assume that* $\mathbb{E}X_0 = 0$ *and* $\mathbb{E}X_0^4 < \infty$. *For* $1 \le j \le k_n$ *let*

$$\widehat{u}_j(\lambda) = u_j(\lambda) I\left\{ \left|u_j(\lambda)\right| \le \frac{\sqrt{nB_n}}{(\log B_n)^4} \right\} - \mathbb{E}u_j(\lambda) I\left\{ \left|u_j(\lambda)\right| \le \frac{\sqrt{nB_n}}{(\log B_n)^4} \right\} \tag{A.17}$$

be the truncated version of $u_j(\lambda)$. *Then*

$$\mathbb{P}\left(\max_{0 \le i \le B_n} \left| \sum_{j=1}^{k_n} (u_j(\lambda_i^*) - \widehat{u}_j(\lambda_i^*)) \right| \ge \sqrt{nB_n} (\log B_n)^{-1} \right) = o(1).$$

Proof. As in the proof of Lemma A.2, we can get

$$\mathbb{P}\left(|u_j(\lambda_i^*)| \ge C_Q \frac{\sqrt{nB_n}}{(\log B_n)^4} \right) = O(n^{-c}) \tag{A.18}$$

for any large $c > 0$. The lemma immediately follows. ■

LEMMA A.4. *Assume that* $\mathbb{E}X_0 = 0$ *and* $\mathbb{E}X_0^4 < \infty$. *We have for any* $x > 0$,

$$\mathbb{P}\left(\max_{i \in \mathbb{B}} \left| \sum_{j=1}^{k_n} \widehat{u}_j(\lambda_i^*) \right| \ge x\sqrt{nB_n \log B_n} \right) = o(1),$$

where $\mathbb{B} = \{|i| \le (\log B_n)^2\} \cup \{B_n - (\log B_n)^2 \le |i| \le B_n\}$.

Proof. The lemma easily follows from Bernstein's inequality. ■

LEMMA A.5. *Suppose that* $\mathbb{E}X_0 = 0$, $\mathbb{E}X_0^4 < \infty$, *and* $d_{n,4} = O((\log n)^{-2})$.

(i) We have

$$|\mathbb{E}[g_n(\lambda_1) - \mathbb{E}g_n(\lambda_1)][g_n(\lambda_2) - \mathbb{E}g_n(\lambda_2)]| \le CnB_n(\log B_n)^{-2}$$

uniformly on $\{(\lambda_1, \lambda_2) : 0 \le \lambda_k \le \pi - B_n^{-1}(\log B_n)^2,\ k = 1, 2,$ *and* $|\lambda_1 - \lambda_2| \ge B_n^{-1}(\log B_n)^2\}$.

(ii) For $\alpha_n > 0$ *with* $\limsup_{n\to\infty} \alpha_n < 1$, *we have, uniformly on* $\{(\lambda_1, \lambda_2) : B_n^{-1}(\log B_n)^2 \le \lambda_k \le \pi - B_n^{-1}(\log B_n)^2,\ k = 1, 2,$ *and* $|\lambda_1 - \lambda_2| \ge B_n^{-1}\}$, *that*

$$|\mathbb{E}[g_n(\lambda_1) - \mathbb{E}g_n(\lambda_1)][g_n(\lambda_2) - \mathbb{E}g_n(\lambda_2)]| \le 4\pi^2 \alpha_n nB_n f(\lambda_1) f(\lambda_2)\kappa.$$

(iii) We have uniformly on $\{B_n^{-1}(\log B_n)^2 \le \lambda \le \pi - B_n^{-1}(\log B_n)^2\}$ *that*

$$|\mathbb{E}[g_n(\lambda) - \mathbb{E}g_n(\lambda)]^2 - 4\pi^2 nB_n f^2(\lambda)\kappa| \le CnB_n(\log B_n)^{-2}.$$

Proof.

(i) Let $m = \lfloor n^\epsilon \rfloor$ with $\epsilon > 0$ being small enough. Then $d_{m,4} = O((\log B_n)^{-2})$. Let $a_{n,j} = K(j/B_n)e^{\iota j\lambda}$. Denote D_k in (3.6) by $\bar{D}_{k,\lambda}$. Let $M_n(\lambda) = \sum_{t=1}^n \bar{D}_{t,\lambda} \sum_{j=1}^{t-1} a_{n,j-t} D_{j,\lambda}$. By Propositions 1 and 2, it suffices to verify that

$$r_{n,\lambda_1,\lambda_2} := |\mathbb{E}[M_n(\lambda_1) + \bar{M}_n(\lambda_1)][M_n(\lambda_2) + \bar{M}_n(\lambda_2)]| \le C\frac{nB_n}{(\log B_n)^2}.$$

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Let $N_n(\lambda) = \sum_{t=1}^n \bar{D}_{t,\lambda} \sum_{j=1}^{t-m-1} a_{n,j-t} D_{j,\lambda}$. Because $D_{t,\lambda}, t \geq 1$ are martingale differences, elementary manipulations of trigonometric identities show that

$$\bar{r}_{n,\lambda_1,\lambda_2} = 2|\mathbb{E}\bar{D}_{0,\lambda_1}\bar{D}_{0,\lambda_2}|^2 \sum_{t=1}^n \sum_{s=1}^{t-m-1} K^2((t-s)/B_n)\cos((t-s)(\lambda_1+\lambda_2))$$

$$+2|\mathbb{E}\overline{D}_{0,\lambda_1} D_{0,\lambda_2}|^2 \sum_{t=1}^n \sum_{s=1}^{t-m-1} K^2((t-s)/B_n)$$

$$\times \cos((t-s)(\lambda_1-\lambda_2)). \tag{A.19}$$

Using the identity $1+2\sum_{k=1}^n \cos(k\lambda) = \sin((n+1)\lambda/2)/\sin(\lambda/2)$, by the summation by parts formula and Condition 3, it follows that

$$\left|\sum_{t=1}^n \sum_{s=1}^{t-m-1} K^2((t-s)/B_n)\cos((t-s)(\lambda_1 \pm \lambda_2))\right|$$

$$\leq Cnm + CB_n^2 + \sum_{t=B_n+m+1}^n \left|\sum_{s=1}^{B_n} K^2(s/B_n)\cos s(\lambda_1 \pm \lambda_2)\right|$$

$$\leq Cnm + CB_n^2 + CnB_n/(\log B_n)^2,$$

which by (A.19) implies $\bar{r}_{n,\lambda_1,\lambda_2} = O(nB_n/(\log B_n)^2)$. By orthogonality, $\|M_n(\lambda) - N_n(\lambda)\| = O(\sqrt{nm})$, we have $|r_{n,\lambda_1,\lambda_2}| \leq |\bar{r}_{n,\lambda_1,\lambda_2}| + O(n\sqrt{B_n m} + \sqrt{nm B_n}) = O(nB_n/(\log B_n)^2)$, and hence (i) holds.

(ii) As in the proof of Lemma 3.2(ii) in Woodroofe and Van Ness (1967), using Condition 3, we can show that

$$\limsup_{n\to\infty} 2(nB_n)^{-1} \sum_{t=1}^n \sum_{s=1}^{t-m-1} K^2((t-s)/B_n)\cos((t-s)(\lambda_1-\lambda_2)) < \kappa.$$

Hence (ii) follows.

(iii) Recall that $\|D_{0,\lambda}\|^2 = \sum_{j=-m}^m \mathbb{E}(X_{0,m}X_{j,m})e^{\iota j\lambda}$. From the proof of (i) and Condition 3, we see that for $B_n^{-1}(\log B_n)^2 \leq \lambda \leq \pi - B_n^{-1}(\log B_n)^2$,

$$\bar{r}_n(\lambda,\lambda) = O(nB_n/(\log B_n)^2) + \|D_{0,\lambda}\|_2^4 n \sum_{s=-B_n}^{B_n} K^2(s/B_n)$$

$$= O(nB_n/(\log B_n)^2) + 4\pi^2 f^2(\lambda) n B_n \kappa.$$

Hence (iii) holds. ■

LEMMA A.6. *Recall (A.17) for $\widehat{u}_j(\lambda)$ and set $E_n = B_n - (\log B_n)^2$. Under the conditions of Theorem 3 or 4 or 5, we have*

$$\mathbb{P}\left(\max_{(\log B_n)^2 \leq i \leq \mathrm{E}_n} \frac{|\sum_{j=1}^{k_n} \widehat{u}_j(\lambda_i^*)|^2}{4\pi^2 n B_n f^2(\lambda_i^*)\kappa} - 2\log(B_n) + \log(\pi \log B_n) \leq x\right) \to e^{-e^{-x/2}}.$$

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Proof. For convenience we assume $\kappa = \int_{-1}^{1} K^2(t)dt = 1$. Let $\mathbf{V}_n = \sum_{j=1}^{k_n} V_j$, where

$$V_j = (f^{-1}(\lambda_{i_1}^*)\widehat{u}_j(\lambda_{i_1}^*), \ldots, f^{-1}(\lambda_{i_d}^*)\widehat{u}_j(\lambda_{i_d}^*)), \qquad 1 \le j \le k_n,$$

$(\log B_n)^2 \le i_1 < \cdots < i_d \le \mathrm{E}_n$.

By Fact 2.2 in Einmahl and Mason (1997), there exist independent centered normal random vectors $N_1, \ldots, N_{k_n}$ with $\mathrm{Cov}(V_j) = \mathrm{Cov}(N_j)$, $1 \le j \le k_n$, such that

$$\mathbb{P}(|\mathbf{V}_n - \mathbf{N}_n| \ge \tau_n) = O\left(e^{-(\log B_n)^3}\right),$$

where $\mathbf{N}_n = \sum_{j=1}^{k_n} N_j$. For $z = (z_1, \ldots, z_d)$ define $|z|_d = \min\{z_i : 1 \le i \le d\}$. Then

$$\mathbb{P}\left(|\mathbf{V}_n|_d \ge y_n\sqrt{nB_n}\right) \le \mathbb{P}\left(|\mathbf{N}_n|_d \ge y_n\sqrt{nB_n} - \tau_n\right) + O\left(e^{-(\log B_n)^3}\right). \tag{A.20}$$

Let $\mathbf{I}_d$ be a $d \times d$ identity matrix. We claim that

$$|\mathrm{Cov}(\mathbf{N}_n) - 4\pi^2 nB_n\mathbf{I}_d| = O(nB_n/(\log B_n)^2). \tag{A.21}$$

To prove (A.21), we first show that for $|i_k - i_l| \ge (\log B_n)^2/B_n$,

$$\left|\mathbb{E}\sum_{j=1}^{k_n}\widehat{u}_j(\lambda_{i_k}^*)\sum_{j=1}^{k_n}\widehat{u}_j(\lambda_{i_l}^*)\right| = O(nB_n/(\log B_n)^2). \tag{A.22}$$

In fact, by (A.18), we have $\max_{i_k}\mathbb{E}|\sum_{j=1}^{k_n}(\widehat{u}_j(\lambda_{i_k}^*) - u_j(\lambda_{i_k}^*))|^2 = O(n^{-c})$ for any $c > 0$. Recall (A.12) for $\upsilon_j(\lambda)$. By independence, $\|\sum_{j=1}^{k_n+1}\upsilon_j(\lambda)\|^2 = \sum_{j=1}^{k_n+1}\|\upsilon_j(\lambda)\|^2$. We now estimate $\|\upsilon_j(\lambda)\|^2$. Let $\acute{Y}_{t,m}(\lambda) = \overline{X}_{t,m}\sum_{s=t-4m+1}^{t-1} a_{n,t-s}\overline{X}_{s,m}$, $Y_{t,m}^*(\lambda) = \overline{X}_{t,m}\sum_{s=1}^{t-4m} a_{n,t-s}\overline{X}_{s,m}$. As in the proof of (A.2), routine calculations yield that, for $1 \le j \le k_n$,

$$\begin{aligned}\max_{\lambda}\|\upsilon_j(\lambda)\|^2 &\le 2\max_{\lambda}\left\|\sum_{t\in I_j}Y_{t,m}^*(\lambda)\right\|^2 + 2\max_{\lambda}\left\|\sum_{t\in I_j}(\acute{Y}_{t,m}(\lambda) - \mathbb{E}\acute{Y}_{t,m}(\lambda))\right\|^2 \\ &= O(mB_n^2) + m\sum_{t\in I_j}\|\acute{Y}_{t,m}(\lambda)\|^2 = O(mB_n^2)\end{aligned}$$

and $\max_\lambda\|\upsilon_{k_n+1}(\lambda)\|^2 = O\left(mB_n^{2+2\beta}\right)$, which, together with the fact that β, γ are sufficiently small numbers, imply $\max_\lambda\mathbb{E}|\sum_{j=1}^{k_n+1}\upsilon_j(\lambda)|^2 = O(nB_n^{1-\varepsilon})$ for some $\varepsilon > 0$. So we have

$$\max_{i_k}\mathbb{E}|\textstyle\sum_{j=1}^{k_n}\widehat{u}_j(\lambda_{i_k}^*) - \overline{g}_{n,m}(\lambda_{i_k}^*)|^2 = O(nB_n^{1-\varepsilon/2}).$$

We next prove that

$$\max_{\lambda}\mathbb{E}|g_{n,m}(\lambda) - \overline{g}_{n,m}(\lambda)|^2 = O(nB_n/(\log B_n)^2). \tag{A.23}$$

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Let $I_t = \sum_{s=1}^{t-m} a_{n,t-s}(X_{s,m} - \overline{X}_{s,m})$. As in the proof of (A.2),

$$\max_{\lambda} \left\| \sum_{t=1}^{n} X_{t,m} I_t \right\|^2 \le Cm \max_{\lambda} \sum_{t=1}^{n} \|X_{t,m}\|^2 \|I_t\|^2$$

$$= O(m^2 n B_n) \mathbb{E}(X_{s,m} - \overline{X}_{s,m})^2 = O(m^2 (nB_n)^{1-2\alpha})$$

and, for $J_t = \sum_{s=t-m+1}^{t-1} a_{n,t-s}(X_{s,m} - \overline{X}_{s,m})$,

$$\max_{\lambda} \left\| \sum_{t=1}^{n} X_{t,m} J_t \right\|^2 \le Cm^3 n \|X_{s,m} - \overline{X}_{s,m}\|^2 = O(m^3 n (nB_n)^{-2\alpha}).$$

Hence $\max_\lambda \|\sum_{t=1}^n X_{t,m}(I_t + J_t)\| = O(\tau_n)$. Similarly, we can also have $\|\sum_{t=1}^n (X_{t,m} - \overline{X}_{t,m}) \sum_{s=1}^{t-1} a_{n,t-s} \overline{X}_{s,m}\| = O(\tau_n)$. Thus (A.23) is proved. By Proposition 1, $\max_\lambda \|g_n(\lambda) - g_{n,m}(\lambda)\| = O(\tau_n)$. So to prove (A.22) it suffices to show that

$$|\mathrm{Cov}(g_n(\lambda_{i_k}^*), g_n(\lambda_{i_l}^*))| = O(\tau_n^2),$$

which follows from Lemma A.5(i) immediately. Similarly, from Lemma A.5(iii) we have

$$\left| \mathbb{E} \left[\sum_{j=1}^{k_n} \widehat{u}_j(\lambda_{i_k}^*) \right]^2 - 4\pi^2 n B_n f^2(\lambda_{i_l}^*) \right| = O(\tau_n^2).$$

This together with (A.22) yields (A.21), and hence

$$\left| \mathrm{Cov}^{1/2}(\mathbf{N}_n) - 2\pi \sqrt{nB_n} \mathbf{I}_d \right| = O\left(\sqrt{nB_n} / (\log B_n)^2 \right). \tag{A.24}$$

Let N be a standard normal R^d-valued random vector. By virtue of (A.24) it follows from the tail probabilities of a normal variable that

$$\mathbb{P}\left(\left| \mathrm{Cov}^{1/2}(\mathbf{N}_n) - 2\pi \sqrt{nB_n} \mathbf{I}_d || N \right| \ge \tau_n \right) = O(e^{-(\log B_n)^2/4}),$$

which, together with (A.20), yields that

$$\mathbb{P}\left(|\mathbf{V}_n|_d \ge y_n \sqrt{nB_n} \right) \le \mathbb{P}\left(2\pi \sqrt{nB_n} |N|_d \ge y_n \sqrt{nB_n} - 2\tau_n \right) + O\left(e^{-(\log B_n)^2/4} \right)$$

$$= (1 + o(1)) \left(\sqrt{8\pi}\, y_n^{-1} \exp\left(-\frac{y_n^2}{8\pi^2} \right) \right)^d. \tag{A.25}$$

Similarly, for (A.25) the reverse direction with $\ge$ also holds. Hence

$$\mathbb{P}\left(|\mathbf{V}_n|_d \ge y_n \sqrt{nB_n} \right) = (1 + o(1)) \left(\sqrt{8\pi}\, y_n^{-1} \exp\left(-\frac{y_n^2}{8\pi^2} \right) \right)^d \tag{A.26}$$

uniformly on $\{(\lambda_{i_1}^*, \ldots, \lambda_{i_d}^*), (\log B_n)^2 \le i_1 < \cdots < i_d \le \mathrm{E}_n, |\lambda_{i_j}^* - \lambda_{i_k}^*| \ge (\log B_n)^2 / B_n, k \ne j\}$.

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By Lemma A.5(ii) and Lemma 2 in Berman (1962), similarly to (A.25) we have

$$\mathbb{P}\left(\left|\sum_{j=1}^{k_n}\widehat{u}_j(\lambda_{i_k}^*)\right| \ge y_n\sqrt{nB_n}f(\lambda_{i_k}^*),\ k=1,\dots,d\right)$$

$$\le C\left(\sqrt{8\pi}\,y_n^{-1}\exp\left(-\frac{y_n^2}{8\pi^2}\right)\right)^{d-2}\times y_n^{-2}\exp\left(-\frac{y_n^2}{8\pi^2}(1+\delta)\right) \tag{A.27}$$

for some $\delta>0$, uniformly on $\{\lambda_{i_2}^*-\lambda_{i_1}^*\ge B_n^{-1},\lambda_{i_k}^*-\lambda_{i_{k-1}}^*\ge B_n^{-1}(\log B_n)^2;k=3,\dots,d\}$. The details of the derivation are omitted. Let $t_n=2\log B_n-\log(\pi\log B_n)+x$. Define

$$A_i=\left\{\frac{|\sum_{j=1}^{k_n}\widehat{u}_j(\lambda_i^*)|^2}{4\pi^2nB_nf^2(\lambda_i^*)}\ge t_n\right\}\quad\text{and}\quad A=\bigcup_{(\log B_n)^2\le i\le \mathrm{E}_n}A_i.$$

By Bonferroni's inequality, we have for every fixed k that

$$\sum_{t=1}^{2k}(-1)^{t-1}P_t\le\mathbb{P}(A)\le\sum_{t=1}^{2k-1}(-1)^{t-1}P_t,$$

where $P_t=\sum_{(\log B_n)^2\le i_1<\cdots<i_t\le \mathrm{E}_n}\mathbb{P}(A_{i_1}\cap\cdots\cap A_{i_t})$. By (A.26) and (A.27), it follows as in Woodroofe and Van Ness (1967) and Watson (1954) that $P_t\to\exp(-tx/2)/t!$ as $n\to\infty$. Thus $\mathbb{P}(A)\to1-e^{-e^{-x/2}}$, and the proof is complete. ■

Proof of Theorems 3–5. From Lemmas A.1 and A.1*, Remark A.2, and (A.16), we only need to prove

$$\mathbb{P}\left(\max_{0\le i\le B_n}\frac{|\overline{g}_{n,m}(\lambda_i^*)-\mathbb{E}\overline{g}_{n,m}(\lambda_i^*)|^2}{4\pi^2nB_nf^2(\lambda_i^*)\kappa}-2\log B_n+\log(\pi\log B_n)\le x\right)\to e^{-e^{-x/2}}.$$

By Lemmas A.2 and A.3, we see that

$$\max_{0\le i\le B_n}\left|\overline{g}_{n,m}(\lambda_i^*)-\mathbb{E}\overline{g}_{n,m}(\lambda_i^*)-\sum_{j=1}^{k_n}\widehat{u}_j(\lambda_i^*)\right|=o_{\mathbb{P}}\left(\sqrt{\frac{nB_n}{\log B_n}}\right).$$

This together with Lemmas A.4 and A.6 implies the theorems. ■

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