THE STRENGTH OF METALS

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1. When crystals of a metal are highly perfect, their elastic limit is low, plastic flow taking place when the applied shear stress is very small. The elastic limit reaches higher values as the perfect structure is progressively broken up by cold-work. A steady state is finally reached where further distortion of the metal does not increase the elastic limit. The stress at which a metal passes beyond the elastic limit and begins to yield is ill-defined, as in general the crystal begins to 'creep' appreciably near this point and the time element enters into the definition of the stress-strain curve. Nevertheless, there is a fairly well-defined stress beyond which a metal, which has been brought into a steady state by being saturated with cold-work, ceases to behave elastically and undergoes permanent deformation. An attempt is made in this paper to derive an expression for the ultimate elastic limit or yield stress of a cold-worked metal, in terms of its structure and elastic constants.

2. The manner in which a single crystal of a metal deforms by slip has been intensively studied, and in a qualitative way we can see why it hardens as the perfect structure is broken up. Slip generally takes place along definite crystallographic planes, and always parallel to densely packed rows of atoms. Slip is to be pictured as a process by which one sheet of atoms glides over another, the perfect arrangement being reformed each time the upper sheet moves on by one interatomic distance relative to the lower sheet. As the metal is hardened by progressive cold-work, the area over which each of these elementary slip movements can take place becomes reduced, since different portions of a crystal have crystalline orientations which do not correspond, like geological strata broken at intervals by faults. The greater elastic strain which can be borne by the distorted crystal is reasonably ascribed to this blocking of the run of each elementary process of slip. In any expression for the elastic limit, two quantities of the dimensions of length thus play an essential part. The first of these, denoted by s in this paper, is the unit of slip. It is the shortest distance between neighbouring atoms, and represents the amount by which one plane moves over another in the elementary slip process. Glide of one plane over another must take place by an integral number of unit jumps of s. The other constant, which may only be expected to have an average value, represents the length over which the slip can run before being interrupted by some discontinuity in the crystal structure of the cold-worked metal. This constant is denoted here by t. The formula for the strength which is deduced below expresses it in terms of the dimensionless variable t/s and the elastic constants.

3. The most thorough treatment of plastic flow in metals is that of G. I. Taylor (1). Taylor considers slip to take place by the movement of a line 'dislocation' across a slip plane. A dislocation is a place where there is an excess or defect of one atom in each row above the plane in which it occurs, as compared with those below this plane. If a

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dislocation starts at one side of a crystal, and runs across to the other, the final result will be a shear of the one part of the crystal by one interatomic distance relative to the other part. If a single dislocation occurs in an otherwise perfect crystal subject to no constraints, a vanishingly small force will cause it to run in one direction or the other. The presence of other dislocations or faults in the structure, however, sets up strains which act as barriers to the run of a dislocation, and an external stress is necessary to enable the dislocation to move through them, so that a definite elastic limit is attained.

4. The problem is attacked in the present paper from a rather different angle. In common with Taylor's treatment, it will be assumed that the run of a slip is confined to a length t, and takes place by a jump s. Instead, however, of considering the stress barrier through which the dislocations move, the problem is viewed in the following way. A polycrystalline mass is subjected to a general shear strain θ. It is possible for a given crystal element to slip by an amount s and for a distance t along a direction which for simplicity will be taken to be parallel to the shearing forces. When this slip takes place, the strain in the neighbourhood is altered. The criterion adopted is that slip will take place if it results in a local reduction of strain energy. Such a treatment takes no account of the forces which initiate a dislocation and cause it to run along the plane over the strain barriers which it has to pass; it is supposed that there is some triggering mechanism sufficient to bring the slip about, if its final result is a diminution of energy. A simple calculation shows that thermal fluctuations are far too small; some other cause of stress concentration must be the agent in enabling the slip to start.

It might appear at first sight that a slip would always relieve part of the strain and so reduce the energy, but this is not so owing to the fact that slip must take place by a whole unit s. The author gave a much simplified treatment of this problem in a note to *Nature* (2). If a small block of perfect crystal of dimensions t each way is gripped by its upper and lower surfaces and sheared to an extent θ, the difference of its energy before and after a slip s will be (unit dimensions perpendicular to the plane of the diagram being assumed)

$$\tfrac{1}{2}n\theta^2t^2 - \tfrac{1}{2}n\left(\theta - \frac{s}{t}\right)^2 t^2$$

or

$$\tfrac{1}{2}n(2s\theta t - s^2). \tag{1}$$

Therefore if this results in a reduction in energy, $\theta > s/2t$. If $\theta < s/2t$, the shear strain is greater after slip than before (see Fig. 1).

Even on this simplified basis, the elastic limit will be somewhat greater than $s/2t$, since the direction of slip in individual crystallites is not in general parallel to the average shear strain. In this first rough approximation, we may say

$$\theta \doteqdot s/t. \tag{2}$$

The smaller the crystal, the higher is its elastic limit.

Now this represents, in a very simplified way, the condition of a small mosaic element of perfect crystal embedded in the matrix of the surrounding polycrystalline metal. We may therefore expect the limiting strength of metals to be of the magnitude Gs/t, where G = elastic modulus for shear. It was shown in the *Nature* note that this expression gives values of the magnitude found experimentally for a number of metals, the values for t deduced by Wood from X-ray line broadening (3) being assumed.

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5. The above treatment clearly leads only to a very rough approximation. When the crystal element slips, it remains surrounded by a matrix of other crystals to which it must conform, and account must be taken of the misfits produced at either end of the range of slip, with their consequent strain energy.

The calculations by W. R. Dean and A. H. Wilson (4), and in the subsequent paper by W. R. Dean and E. Mann which appears in the *Proceedings* with the present paper, make it possible to allow for this energy and to modify the equation accordingly. Treating the problem as two-dimensional, we suppose a homogeneous material to be subjected to a uniform shear strain θ. It is then supposed to slip by a constant amount s along a line of length t between two points O_1, O_2, the line being parallel to the direction of shear. In order to avoid infinite values of the strain, the material is supposed to be removed inside circles of radius R surrounding the points O_1, O_2, R being small

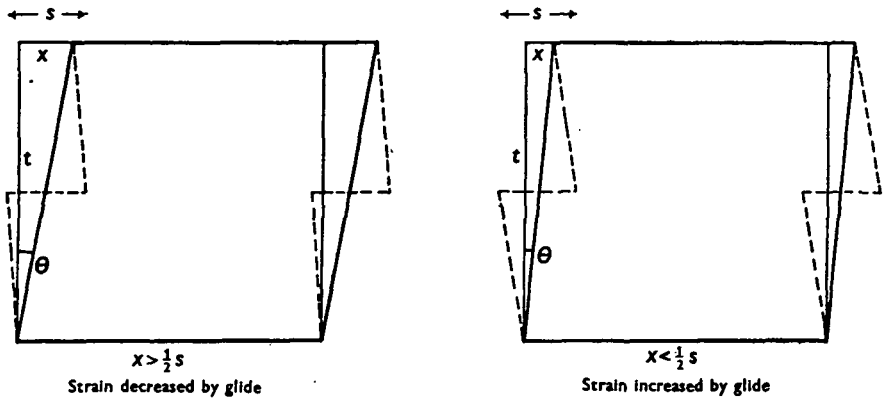


Fig. 1

compared with t. If we neglect energy inside these circles, the process of slip reduces the total energy by an amount

$$\frac{\mu(\lambda+\mu)}{\lambda+2\mu}s\left[\frac{\theta t}{2}-\frac{s}{\pi}\log\frac{t}{R}\right]. \tag{3}$$

The factor $\mu(\lambda+\mu)/(\lambda+2\mu)$ is equivalent to $\frac{1}{2}G/(1-\sigma)$, σ being Poisson's ratio and λ and μ the usual elastic constants.

The approximate criterion for slip to take place therefore becomes

$$\theta > \frac{2}{\pi}\frac{s}{t}\log\frac{t}{R}, \tag{4}$$

as compared with the criterion $\theta > s/2t$ obtained above.

6. To evaluate this expression, it is necessary to discuss what value should be assigned to R. The strain at a distance R from a dislocation of amount s is of the order of $s/\pi R$ (as calculated by Timpe; see (1), p. 376). As R approaches zero, the integrated energy has the form $\frac{1}{2}n\int 2\pi R(s/\pi R)^2\,dR$ and approaches infinity logarithmically. From a physical point of view, the energy cannot approach infinity, and as a reasonable assumption we may take it that the strain energy at any point cannot

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exceed that which corresponds to the metal being thrown into complete disorder, in other words the energy of latent heat. According to this assumption, R in the Dean and Mann formula may be taken to be the point where

$$\tfrac{1}{2}G\left(\frac{s}{\pi R}\right)^2 = L\rho, \tag{5}$$

where L is the latent heat per gram, and ρ the density. The additional energy inside the two circles of radius R is then reckoned to be the energy of latent heat, given by

$$2\pi R^2 L\rho. \tag{6}$$

The total energy dissipated by a slip s is thus

$$\frac{\mu(\lambda+\mu)}{\lambda+2\mu}s\left\{\tfrac{1}{2}\theta t - \frac{s}{\pi}\log\frac{t}{R}\right\} - 2\pi R^2 L\rho, \tag{7}$$

where R is defined by (5).

7. According to equation (5), $s/R = \pi\sqrt{(2L\rho/G)}$. In Table 1 the last column gives $L\rho/G$ for a number of common metals. It is approximately constant, with an average value of 0·036. This is another way of stating a relation pointed out long ago by Gulberg, that the latent heat per unit volume of a metal is approximately proportional to Young's modulus E. Since $G = \frac{1}{2}E/(1+\sigma)$ and σ does not differ greatly for these metals, it follows that $L\rho/G$ is also approximately constant.

Substituting $L\rho/G = 0{\cdot}036$ in equation (5), we get $R/s = 1{\cdot}16$. In other words, the strain energy due to a dislocation s is equal to the latent heat energy at a value of R of slightly more than one atomic diameter.

Since the energy ceases to be reckoned as strain energy and is reckoned as latent heat energy at a point where these are equal, the exact value chosen for R produces very little difference to the final result. To an amply sufficient approximation, we can put $R = s$ in equation (7) and obtain for the condition for slip

$$\frac{\mu(\lambda+\mu)}{\lambda+2\mu}s\left\{\tfrac{1}{2}\theta t - \frac{s}{\pi}\log\frac{t}{s}\right\} - 2\pi s^2 L\rho \geqslant 0 \tag{8}$$

or

$$\theta \geqslant \frac{s}{t}\left\{\frac{2}{\pi}\log\frac{t}{s} + 4\pi\frac{L\rho(\lambda+2\mu)}{\mu(\lambda+\mu)}\right\} \tag{9}$$

The second expression in the brackets is small compared with the first, so we only need its approximate value. Substituting

$$L\rho/\mu = L\rho/G = 0{\cdot}036, \quad (\lambda+2\mu)/(\lambda+\mu) = 2(1-\sigma) = 1{\cdot}26,$$

we get a value of 0·57. To a sufficient approximation, the criterion becomes

$$\theta \geqslant \frac{s}{t}\left(\frac{2}{\pi}\log\frac{t}{s} + 0{\cdot}6\right), \tag{10}$$

or, on expressing the logarithm in more convenient form,

$$\theta \geqslant \frac{s}{t}\left(1{\cdot}46\log_{10}\frac{t}{s} + 0{\cdot}6\right). \tag{11}$$

In order to give an indication of the magnitude of the quantities involved, estimates for t/s in cold-worked metals range from 1000 for copper to 10,000 for aluminium. The corresponding factors in the brackets are $\{3(1{\cdot}46) + 0{\cdot}6\}$ or 5·0, and $\{4(1{\cdot}46) + 0{\cdot}6\}$ or 6·4. To illustrate how little the factor depends on the exact value chosen for R, the quantity

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in brackets only varies by 10 % for the range $R/s = 0{\cdot}5$ to $R/s = 1{\cdot}5$ in the case of copper, with a minimum at $R/s = 1{\cdot}2$.

8. It is interesting to speculate on the significance of the relation

$$L\rho/G = 0{\cdot}036. \tag{12}$$

To state this in another way, we may calculate the average shear strain $\bar{\theta}$ in the molten state by the equation

$$\tfrac{1}{2}G\bar{\theta}^2 = L\rho, \tag{13}$$

which gives

$$\bar{\theta} = 0{\cdot}27.$$

This is a not unreasonable average from a physical point of view. It is equivalent to saying that the average state of strain is that due to a dislocation about one atomic distance away. This is only a very rough approximation because L is measured at the melting point and G at room temperature, all strain is assumed to be shear strain, and G is assumed to hold for such large shears. It does, however, indicate a similarity of the molten state in these typical metals, in that the state of confusion into which they are thrown by melting is geometrically much the same for them all.

9. To calculate the limiting shear strain and hence the ultimate shear strength of the metals, estimates of the value of t/s of the cold-worked metals are necessary. When metals are cold-worked they yield X-ray powder photographs with diffuse lines. Line-broadening may be due either to small particle size or to unhomogeneity of strain in the material. A number of investigators have attacked this problem, amongst them Lipson and Stokes (5), and Brindley and Ridley (6), by measuring the relative widths of X-ray lines of different indices, and have concluded that the major cause of line broadening is variation of strain. The following general argument leads to the same conclusion. If an X-ray reflexion at an angle θ is measured, and the line has a breadth $2\Delta\theta$, it can be shown that

$$\Delta\theta_{(\text{strain})} = \frac{da}{a}\tan\theta, \tag{14}$$

if the broadening is entirely due to strain, and

$$\Delta\theta_{(\text{size})} = \frac{\lambda}{2t\cos\theta}, \tag{15}$$

if the broadening is due to particle size t. Picture now a mass of crystallites of size t in a metal which has been strained beyond the yield point. The crystallites will range from those which have just yielded by a slip causing a change of strain of at least s/t, to those which are just about to yield because they are strained to the limit. The mean strain $\bar{\theta}$ will range from at least $\bar{\theta} + \frac{1}{2}s/t$ to $\bar{\theta} - \frac{1}{2}s/t$. This should be so independently of any theory of the slip process; it is a result of the general feature that strain in the neighbourhood of a slip plane jumps by an amount of magnitude s/t every time slip takes place. Hence the average value of da/a above should be of magnitude s/t. Thus

$$\frac{\Delta\theta_{(\text{strain})}}{\Delta\theta_{(\text{size})}} = \frac{s}{t}\tan\theta\,\frac{2t\cos\theta}{\lambda} = \frac{2s\sin\theta}{\lambda}, \tag{16}$$

and as line-broadening is generally measured at $\theta \doteqdot 90°$

$$\frac{\Delta\theta_{(\text{strain})}}{\Delta\theta_{(\text{size})}} = \frac{2s}{\lambda}. \tag{17}$$

Now $2s$ must always be greater than λ if X-ray reflexion is to take place so that line-broadening due to strain swamps that due to size. As a corollary, if an estimate of t

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has been made by ascribing line-broadening to the particle size of a cold-worked metal, we must increase it in the ratio $2s/\lambda$ if it ought to be ascribed to strain.

Wood (3) gave values of particle size for some cold-worked metals on the assumption that line-broadening was entirely due to particle size, while making it clear that strain was also a possible contributing cause. By means of relation (17) we can translate Wood's figures into estimates of t, assuming the line width to be due to a range of strains. The corresponding values of the ultimate shear strength S calculated by formula (11) are given in the last two columns:

	t (Wood)	t/s (strain)	$G \times 10^{-11}$ c.g.s units	S (dynes/cm.^{2})	S (tons/cm.^{2})
Cu	$0{\cdot}7 \times 10^{-5}$	900	4·55	$2{\cdot}55 \times 10^9$	16·5
Fe	$3{\cdot}2 \times 10^{-5}$	4,200	7·20	$1{\cdot}03 \times 10^9$	6·7
Ag	$0{\cdot}8 \times 10^{-5}$	1,040	2·86	$1{\cdot}41 \times 10^9$	9·1
Ni	$1{\cdot}2 \times 10^{-5}$	1,500	7·70	$2{\cdot}66 \times 10^9$	17·3
Al	$1{\cdot}0 \times 10^{-4}$	13,000	2·28	$1{\cdot}18 \times 10^8$	0·76

An estimate for da/a in cold-rolled copper is given by Dehlinger and Kochendörfer (7). It appears to rise rapidly to a value of 0·2 % as rolling proceeds. On the above interpretation, this corresponds to a value of t/s of 500, the corresponding strength being 23 tons/sq.in. The experimental value for copper, according to G. I. Taylor, is 14·1 tons/sq.in. Much more work must be done in order to obtain satisfactory estimates of t, and the above figures can only be regarded as showing that the formula gives estimates of the strength of the right magnitude.

Table 1

	L (calories)	ρ	$L\rho$ (cal.cm.$^{-3}$)	$G \times 10^{-11}$ (c.g.s.)	σ	$L\rho/G$ (c.g.s. units)
Aluminium	93	2·70	251	2·67	0·34	0·034
Copper	42	8·94	376	4·55	0·35	0·035
Gold	16	19·32	309	2·76	0·42	0·047
Iron	49–65	7·87	386–510	7·82	0·28	0·021–0·027
Lead	5·5	11·34	62·4	0·76	0·45	0·034
Nickel	73	8·9	650	7·16	0·31	0·038
Palladium	36	12·0	432	5·11	0·39	0·035
Platinum	27·2	21·45	584	6·10	0·39	0·040
Silver	24	10·5	252	2·88	0·38	0·038

REFERENCES

(1) TAYLOR, G. I. *Proc. Roy. Soc.* A, 145 (1934), 362.
(2) BRAGG, W. L. *Nature, London*, 149 (9 May, 1942), 511.
(3) WOOD, W. A. *Proc. Roy. Soc.* A, 172 (1939), 231.
(4) DEAN, W. R. and WILSON, A. H. *Proc. Cambridge Phil. Soc.* 43 (1947), 205.
(5) LIPSON, H. and STOKES, A. R. *Nature, London*, 152 (1943), 20.
(6) BRINDLEY, G. W. and RIDLEY, P. *Proc. Phys. Soc.* 51 (1939), 432.
(7) DEHLINGER, U. and KOCHENDÖRFER, A. *Z. Kristallogr.* 101 (1939), 134.

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