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Circle maps and C^*-algebras

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Abstract. We consider a construction of C^*-algebras from continuous piecewise monotone maps on the circle which generalizes the crossed product construction for homeomorphisms and more generally the construction of Renault, Deaconu and Anantharaman-Delaroche for local homeomorphisms. Assuming that the map is surjective and not locally injective we give necessary and sufficient conditions for the simplicity of the C^*-algebra and show that it is then a Kirchberg algebra. We provide tools for the calculation of the K-theory groups and turn them into an algorithmic method for Markov maps.

1. *Introduction*

There are by now a wealth of ways to associate a C^*-algebra to specific classes of dynamical systems, both reversible and irreversible. There are several different approaches to the construction, but the most successful is arguably the one which uses a groupoid as an intermediate step. Via the groupoid the study of how the C^*-algebra depends on the dynamical system and which features it captures can be broken into smaller steps, relating the dynamical system to the groupoid and the groupoid to the C^*-algebra. Since the C^*-algebra arises as the convolution algebra of the groupoid there are by now a large reservoir of results which can be exploited for this purpose. As concrete examples of very successful studies which have followed this general recipe we mention only the work of Putnam and Spielberg [**PS**] and the work of Deaconu and Shultz [**DS**].

Another important aspect of the groupoid approach is the interpretation which the construction is given in the non-commutative geometry of Connes, where the algebra is seen as a substitute for the poorly behaved quotient of the unit space by the equivalence relation coming from the action of the groupoid. In this picture the classical crossed product algebra coming from the action of a group by homeomorphisms is the non-commutative space representing the space of orbits under the action, and the groupoid which serves as the stepping stone from the dynamical system to the C^*-algebra is the transformation groupoid.

As a purely algebraic object the transformation groupoid of a homeomorphism can easily be described such that the invertibility of the dynamics is insignificant. It is

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therefore tempting and natural to try to base a generalization of the crossed product by a homeomorphism on the transformation groupoid. The problem is to equip this groupoid with a sufficiently nice topology which will allow the construction of the convolution algebra. It was shown by Renault [**Re**], Deaconu [**De**] and Anantharaman-Delaroche [**An**], in increasing generality, that for a local homeomorphism there is a canonical topology on the transformation groupoid which turns it into an étale locally compact Hausdorff groupoid, which is the ideal setting for the formation of the convolution algebra. In [**Th2**] the second author introduced a way to describe the transformation groupoid of a homeomorphism, and more generally a local homeomorphism, in such a way that it leads to an étale groupoid and hence a C^*-algebra for certain dynamical systems that are not local homeomorphisms. In [**Th2, Th3**] this generalization of the crossed product for homeomorphisms was used and investigated for holomorphic maps of Riemann surfaces. Such maps are open, but fail to be injective in any open set containing a critical point. The purpose of the present paper is to show how the method from [**Th2**] can be applied and what can be said about the resulting algebras, in a case where the maps are neither locally injective nor open. The class of maps we consider is the class of continuous piecewise monotone maps of the circle, and there are at least two natural ways in which the viewpoint from [**Th2**] can be applied. We make a thorough study of one of them.

Specifically, from a continuous piecewise monotone map $\phi : \mathbb{T} \to \mathbb{T}$ we construct an étale second countable locally compact Hausdorff groupoid Γ_ϕ^+ which is the usual transformation groupoid when ϕ is an orientation-preserving homeomorphism and is equal to the groupoid of Renault [**Re**], Deaconu [**De**] and Anantharaman-Delaroche [**An**], when the map is an orientation-preserving local homeomorphism. The C^*-algebra we study is then the reduced C^*-algebra $C_r^*(\Gamma_\phi^+)$ of the groupoid Γ_ϕ^+, cf. [**Re**]. Alternatively, it can be considered as a special case of the construction of étale groupoids and C^*-algebras from pseudogroups introduced by Arzumanian and Renault [**AR**].

We describe now our results. We assume that ϕ is surjective and not locally injective. Without surjectivity the necessary and sufficient condition for simplicity of $C_r^*(\Gamma_\phi^+)$ will be more messy than the one we obtain for surjective maps, and the lack of generality seems insignificant at this point. Assuming that ϕ is not locally injective means that we exclude the surjective local homeomorphisms. This is done with good conscience because $C_r^*(\Gamma_\phi^+)$ is equal to the usual C^*-algebra of a local homeomorphism when ϕ is a local homeomorphism of positive degree, and equal to that of ϕ^2 when the degree is negative. In addition, the simple C^*-algebras that arise from the transformation groupoid of a surjective local homeomorphism of the circle are known, cf. [**AT**], and there is therefore nothing lost by ignoring them here. We then establish the following facts.

(1) When ϕ is transitive, $C_r^*(\Gamma_\phi^+)$ is purely infinite (i.e. every non-zero hereditary C^*-subalgebra contains an infinite projection). (See Proposition 4.5.)

(2) The reduced C^*-algebra $C_r^*(\Gamma_\phi^+)$ is simple if and only if ϕ is exact (or, equivalently, totally transitive) and has no non-critical fixed point x such that $\phi^{-1}(x)\backslash\{x\}$ only contains critical points. (See Theorem 5.21.)

(3) The reduced C^*-algebra $C_r^*(\Gamma_\phi^+)$ is unital, separable, nuclear and satisfies the universal coefficient theorem (UCT) of Rosenberg and Schochet. (See Corollary 6.7.)

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It follows from (1)–(3) that when ϕ is exact without an exceptional fixed point the C^*-algebra $C_r^*(\Gamma_\phi^+)$ is classified by its K-theory, thanks to the Kirchberg–Phillips classification result [**Ph**]. To determine the algebra it suffices therefore to calculate its K-theory; a task which is far from trivial. We obtain here a six-terms exact sequence which can be applied for the purpose (Theorem 6.5), and we turn it into an effective algorithm for the calculation when ϕ is Markov in the sense that it takes critical points to critical points (§7).

As a couple of key references for our work we mention the work of Shultz [**S**] and the work of Katsura [**Ka**]. The work of Shultz is used to show that a continuous piecewise monotone and transitive map on the circle is conjugate to a piecewise linear map with slopes that are constant in absolute value; an important step towards the proof of the pure infiniteness of $C_r^*(\Gamma_\phi^+)$. His work is also used to show that total transitivity and exactness are equivalent in our setting; a fact which is important on the way to establish nuclearity and the UCT. For this purpose we use also the work of Katsura in much the same way it was used in [**Th3**], namely to show that $C_r^*(\Gamma_\phi^+)$ can be realized as a Cuntz–Pimsner algebra when it is simple. This is an important point. While the groupoid picture is the best approach for the study of many of the connections between properties of $C_r^*(\Gamma_\phi^+)$ and the dynamical properties of ϕ, it is the realization of the algebra as a Cuntz–Pimsner algebra which provides the decisive tools for the K-theory calculations.

The construction of Γ_ϕ^+ depends on the choice of a pseudo-group on $\mathbb{T}$; see §2. This freedom is present already for homeomorphisms and local homeomorphisms, and this is why the C^*-algebra $C_r^*(\Gamma_\phi^+)$ only generalizes the crossed product arising from a homeomorphism when it is orientation-preserving. While the choice of pseudo-group seems canonical for homeomorphisms and local homeomorphisms, this is much less so in our case where at least two choices seem equally natural; either one can work with the pseudo-group of all locally defined local homeomorphisms on $\mathbb{T}$, or one can restrict to those that preserve the orientation of the circle. In the construction of Γ_ϕ^+ we choose the latter and we postpone the study of the algebra which results when the largest pseudo-group is chosen.

2. *Transformation groupoid C^*-algebras*

Let G be an étale second countable locally compact Hausdorff groupoid with unit space $G^{(0)}$. Let $r : G \to G^{(0)}$ and $s : G \to G^{(0)}$ be the range and source maps, respectively. For $x \in G^{(0)}$ put $G^x = r^{-1}(x)$, $G_x = s^{-1}(x)$ and $\mathrm{Is}_x = s^{-1}(x) \cap r^{-1}(x)$. Note that Is_x is a group, the *isotropy group* at x. The space $C_c(G)$ of continuous compactly supported functions is a $*$-algebra when the product is defined by

$$(f_1 f_2)(g) = \sum_{h \in G^{r(g)}} f_1(h) f_2(h^{-1}g)$$

and the involution by $f^*(g) = \overline{f(g^{-1})}$. Let $x \in G^{(0)}$. There is a representation π_x of $C_c(G)$ on the Hilbert space $l^2(G_x)$ of square-summable functions on G_x given by

$$\pi_x(f)\psi(g) = \sum_{h \in G^{r(g)}} f(h)\psi(h^{-1}g). \tag{2.1}$$

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The *reduced groupoid C*-algebra* $C_r^*(G)$ (see [**Re**]) is the completion of $C_c(G)$ with respect to the norm

$$\|f\| = \sup_{x \in G^{(0)}} \|\pi_x(f)\|.$$

In [**Th2**] the second named author introduced an amended transformation groupoid for a self-map on a locally compact space which in certain cases allows us to topologize the transformation groupoid, or a groupoid closely related to the transformation groupoid, in such a way that the result is a well-behaved étale groupoid. It is this construction we shall consider in this paper for piecewise monotone maps of the circle. We begin therefore by reviewing the construction.

Let X be a locally compact Hausdorff space and $\psi : X \to X$ a map. Let $\mathcal{P}$ be a pseudo-group on X. More specifically, $\mathcal{P}$ is a collection of homeomorphisms $\eta : U \to V$ between open subsets of X such that:

(i) for every open subset U of X the identity map $\mathrm{id} : U \to U$ is in $\mathcal{P}$;
(ii) when $\eta : U \to V$ is in $\mathcal{P}$ then so is $\eta^{-1} : V \to U$; and
(iii) when $\eta : U \to V$ and $\eta_1 : U_1 \to V_1$ are elements in $\mathcal{P}$ then so is $\eta_1 \circ \eta : U \cap \eta^{-1}(V \cap U_1) \to \eta_1(V \cap U_1)$.

For each $k \in \mathbb{Z}$ we denote by $\mathcal{T}_k(\psi)$ the elements $\eta : U \to V$ of $\mathcal{P}$ with the property that there are natural numbers n, m such that $k = n - m$ and

$$\psi^n(z) = \psi^m(\eta(z)) \quad \text{for all } z \in U. \tag{2.2}$$

The elements of $\mathcal{T} = \bigcup_{k \in \mathbb{Z}} \mathcal{T}_k(\psi)$ will be called *local transfers* for ψ. We denote by $[\eta]_x$ the germ at a point $x \in X$ of an element $\eta \in \mathcal{T}_k(\psi)$. Set

$$\mathcal{G}_\psi = \{(x, k, \eta, y) \in X \times \mathbb{Z} \times \mathcal{P} \times X : \eta \in \mathcal{T}_k(\psi), \eta(x) = y\}.$$

We define an equivalence relation $\sim$ in $\mathcal{G}_\psi$ such that $(x, k, \eta, y) \sim (x', k', \eta', y')$ when:

(i) $x = x',\ y = y',\ k = k'$; and
(ii) $[\eta]_x = [\eta']_x$.

Let $[x, k, \eta, y]$ denote the equivalence class represented by $(x, k, \eta, y) \in \mathcal{G}_\psi$. The quotient space $G_\psi(\mathcal{P}) = \mathcal{G}_\psi/\!\sim$ is a groupoid such that two elements $[x, k, \eta, y]$ and $[x', k', \eta', y']$ are composable when $y = x'$ and their product is

$$[x, k, \eta, y][y, k', \eta', y'] = [x, k + k', \eta' \circ \eta, y'].$$

The inversion in $G_\psi(\mathcal{P})$ is defined such that $[x, k, \eta, y]^{-1} = [y, -k, \eta^{-1}, x]$. The unit space of G_ψ can be identified with X via the map $x \mapsto [x, 0, \mathrm{id}, x]$, where id is the identity map on X. When $\eta \in \mathcal{T}_k(\psi)$ we set

$$U(\eta) = \{[z, k, \eta, \eta(z)] : z \in U\}, \tag{2.3}$$

where U is the domain of η. It is straightforward to verify that by varying k, η and U the sets (2.3) constitute a base for a topology on $G_\psi(\mathcal{P})$. In general this topology is not Hausdorff and to amend this we now make the following additional assumption.

Assumption 2.1. When $x \in X$ and $\eta(x) = \xi(x)$ for some $\eta, \xi \in \mathcal{T}_k(\psi)$, then the implication

$$x \text{ is not isolated in } \{y \in X : \eta(y) = \xi(y)\} \Rightarrow [\eta]_x = [\xi]_x$$

holds.

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Then $G_\psi(\mathcal{P})$ is Hausdorff: let $[x, k, \eta, y]$ and $[x', k', \eta', y']$ be different elements of $G_\psi(\mathcal{P})$. There are then open neighborhoods W of x and W' of x' such that $U(\eta|_W) = \{[z, k, \eta, \eta(z)] : z \in W\}$ and $U'(\eta'|_{W'}) = \{[z, k', \eta', \eta'(z)] : z \in W'\}$ are disjoint. This is trivial when $(x, k, y) \neq (x', k', y')$ while it is a straightforward consequence of Assumption 2.1 when $(x, k, y) = (x', k', y')$.

Since the range and source maps are homeomorphisms from $U(\eta)$ onto U and $\eta(U)$, respectively, it follows that $G_\psi(\mathcal{P})$ is a locally compact Hausdorff space because X is. It is also straightforward to show that the groupoid operations are continuous so that we can conclude the following.

THEOREM 2.2. *Let Assumption 2.1 be satisfied. Then $G_\psi(\mathcal{P})$ is an étale locally compact Hausdorff groupoid.*

Compared with the notation used in [**Th2**], here we have emphasized the pseudo-group since there is more than one natural choice when ψ is a piecewise monotone map on the circle.

3. *The oriented transformation groupoid of a piecewise monotone map on the circle*

Let $\mathbb{T}$ be the unit circle in the complex plane. We consider $\mathbb{T}$ as an oriented space with the canonical counter-clockwise orientation. Consider a continuous map $\phi : \mathbb{T} \to \mathbb{T}$. There is then a unique continuous map $f : [0, 1] \to \mathbb{R}$ such that $f(0) \in [0, 1[$ and

$$\phi(e^{2\pi i t}) = e^{2\pi i f(t)}$$

for all $t \in [0, 1]$. We will refer to f as *the lift of* ϕ. Note that $f(1) - f(0)$ is an integer, *the degree of* ϕ. We say that ϕ is *piecewise monotone* when there are points $0 = c_0 < c_1 < \cdots < c_N = 1$ such that f is either strictly increasing or strictly decreasing on the intervals $]c_{i-1}, c_i[$, $i = 1, 2, \ldots, N$. When $\phi : \mathbb{T} \to \mathbb{T}$ is piecewise monotone and $t \in \mathbb{T}$ we define *the ϕ-valency* $\mathrm{val}(\phi, t)$ of t to be the element of

$$\mathcal{V} = \{(+, +), (-, -), (+, -), (-, +)\}$$

such that: $\mathrm{val}(\phi, t) = (+, +)$ when ϕ is strictly increasing in all sufficiently small neighborhoods of t; $\mathrm{val}(\phi, t) = (-, -)$ when ϕ is strictly decreasing in all sufficiently small open neighborhoods of t; $\mathrm{val}(\phi, t) = (+, -)$ when ϕ is strictly increasing in all sufficiently small intervals to the left of t and strictly decreasing in all sufficiently small intervals to the right of t; and, finally, $\mathrm{val}(\phi, t) = (-, +)$ when ϕ is strictly decreasing in all sufficiently small intervals to the left of t and strictly increasing in all sufficiently small intervals to the right of t.

The set $\mathcal{V}$ is a monoid with the composition table shown in Table 1.

This monoid structure will be important here because of the following observation.

LEMMA 3.1. *Let $\phi, \varphi : \mathbb{T} \to \mathbb{T}$ be piecewise monotone, and let $x \in \mathbb{T}$. Then*

$$\mathrm{val}(\phi \circ \varphi, x) = \mathrm{val}(\phi, \varphi(x)) \bullet \mathrm{val}(\varphi, x).$$

Let $\mathcal{P}^+$ be the pseudo-group of all locally defined homeomorphisms of $\mathbb{T}$ that are orientation-preserving. The elements of $\mathcal{P}^+$ consist of open subsets U, V in $\mathbb{T}$ and a homeomorphism $\eta : U \to V$ such that $s, t \in U, s < t \Rightarrow \eta(s) < \eta(t)$.

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TABLE 1. The composition table for $\bullet$.

$x \bullet y$	$y = (+, +)$	$y = (+, -)$	$y = (-, +)$	$y = (-, -)$
$x = (+, +)$	$(+, +)$	$(+, -)$	$(-, +)$	$(-, -)$
$x = (+, -)$	$(+, -)$	$(+, -)$	$(+, -)$	$(+, -)$
$x = (-, +)$	$(-, +)$	$(-, +)$	$(-, +)$	$(-, +)$
$x = (-, -)$	$(-, -)$	$(-, +)$	$(+, -)$	$(+, +)$

LEMMA 3.2. *Let $\phi, \varphi : \mathbb{T} \to \mathbb{T}$ be continuous and piecewise monotone maps and $x, y \in \mathbb{T}$ points such that $\phi(x) = \varphi(y)$. It follows that there is an $\eta \in \mathcal{P}^+$ such that:*
(a) $\eta(x) = y$; *and*
(b) $\phi(t) = \varphi(\eta(t))$ *for all t in a neighborhood of x;*
if and only if $\mathrm{val}(\phi, x) = \mathrm{val}(\varphi, y)$.

When this is the case, the germ $[\eta]_x$ of η at x is unique.

Proof. The proof is straightforward. □

When $x, y \in \mathbb{T}$ and $k \in \mathbb{Z}$ we write $x \overset{k}{\sim} y$ when $k = n - m$ for some $n, m \in \mathbb{N}$ such that $\phi^n(x) = \phi^m(y)$ and $\mathrm{val}(\phi^n, x) = \mathrm{val}(\phi^m, y)$. Set

$$\Gamma_\phi^+ = \{(x, k, y) \in \mathbb{T} \times \mathbb{Z} \times \mathbb{T} : x \overset{k}{\sim} y\}.$$

Then Γ_ϕ^+ is a groupoid where the composable pairs are

$$\Gamma_\phi^{+(2)} = \{((x, k, y), (x', k', y')) \in \Gamma_\phi^{+2} : y = x'\}$$

and the product is

$$(x, k, y)(y, k', y') = (x, k + k', y').$$

The inversion is given by $(x, k, y)^{-1} = (y, -k, x)$. This groupoid is identical with the groupoid denoted by $G_\phi(\mathcal{P}^+)$ in §2. To see this we denote, as in [**Th2**], by $\mathcal{T}_k(\phi)$ the set of elements $\eta \in \mathcal{P}^+$ with the property that for some $n, m \in \mathbb{N}$ such that $n - m = k$, the equality

$$\phi^n(z) = \phi^m(\eta(z))$$

holds for all z in the domain of η. It follows then from the last statement in Lemma 3.2 that the implication $\eta, \eta' \in \mathcal{T}_k(\phi),\ \eta(x) = \eta'(x) \Rightarrow [\eta]_x = [\eta']_x$ holds. We deduce therefore that the map

$$G_\phi(\mathcal{P}^+) \ni [x, k, \eta, y] \mapsto (x, k, y) \in \Gamma_\phi^+$$

is a bijection. It follows from Theorem 2.2 that Γ_ϕ^+ is an étale locally compact Hausdorff groupoid in the topology for which a base is given by sets of the form

$$\Omega(\eta, U) = \{(z, k, \eta(z)) : z \in U\}, \tag{3.1}$$

where $\eta \in \mathcal{T}_k(\phi)$ and U is an open subset of η's domain. However, there is an alternative description of this topology which we now present. Among others it has the virtue that it is obviously second countable.

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When $k \in \mathbb{Z}$, $n \in \mathbb{N}$ and $n + k \geq 1$, $n \geq 1$, set

$$\Gamma_\phi^+(k, n) = \{(x, l, y) \in \Gamma_\phi^+ : l = k,\ \phi^{k+n}(x) = \phi^n(y),\ \mathrm{val}(\phi^{k+n}, x) = \mathrm{val}(\phi^n, y)\}$$

and

$$\Gamma_\phi(k, n) = \{(x, l, y) \in \mathbb{T} \times \mathbb{Z} \times \mathbb{T} : l = k,\ \phi^{k+n}(x) = \phi^n(y)\}.$$

Note that $\Gamma_\phi(k, n)$ is closed in $\mathbb{T} \times \mathbb{Z} \times \mathbb{T}$.

LEMMA 3.3. *The set $\Gamma_\phi^+(k, n)$ is the intersection of a closed and an open subset of $\mathbb{T} \times \mathbb{Z} \times \mathbb{T}$.*

Proof. Write $\mathbb{T}$ as the union of non-degenerate closed intervals $\mathbb{T} = \bigcup_i I_i^+ \cup \bigcup_i I_i^-$ such that ϕ^{n+k} is increasing on each I_i^+ for all i and decreasing on I_i^- for all i, and such that none of the intervals overlap in more than one point. Similarly, write $\mathbb{T}$ as the union of non-degenerate closed intervals $\mathbb{T} = \bigcup_j J_j^+ \cup \bigcup_j J_j^-$ such that ϕ^n is increasing on each J_j^+ for all j and decreasing on J_j^- for all j, and such that none of the intervals overlap in more than one point. Then

$$\Gamma_\phi^+(k, n) = (A \cap \Gamma_\phi(k, n)) \backslash B,$$

where

$$A = \bigcup_{i,j} (I_i^+ \times \{k\} \times J_j^+) \cup \bigcup_{i,j} (I_i^- \times \{k\} \times J_j^-)$$

and B is the finite set consisting of elements $(x, k, y) \in \Gamma_\phi(n, k)$ such that $\mathrm{val}(\phi^{n+k}, x) \in \{(+, -), (-, +)\}$ or $\mathrm{val}(\phi^n, y) \in \{(+, -), (-, +)\}$ while $\mathrm{val}(\phi^{n+k}, x) \neq \mathrm{val}(\phi^n, y)$. □

It follows from Lemma 3.3 that $\Gamma_\phi^+(k, n)$ is a locally compact Hausdorff space in the relative topology inherited from $\mathbb{T} \times \mathbb{Z} \times \mathbb{T}$.

LEMMA 3.4. *A subset W of $\Gamma_\phi^+(k, n)$ is open in the relative topology inherited from $\mathbb{T} \times \mathbb{Z} \times \mathbb{T}$ if and only for all $(x, k, y) \in W$ there is an element $\eta \in \mathcal{T}_k(\phi)$ and an open subset U of the domain of η such that $(x, k, y) \in \Omega(\eta, U) \subseteq W$.*

Proof. Assume first that $W \subseteq \Gamma_\phi^+(k, n)$ is open in the relative topology inherited from $\mathbb{T} \times \mathbb{Z} \times \mathbb{T}$ and consider a point $(x, k, y) \in W$. It follows from Lemma 3.2 that there is an element $\eta \in \mathcal{T}_k(\phi)$ such that $\eta(x) = y$ and $\phi^{k+n}(z) = \phi^n(\eta(z))$ for all z in a neighborhood U of x. By continuity of η we can shrink U to arrange that $(x, k, y) \in \Omega(\eta, U) \subseteq W$. This establishes one implication. To prove the other, let $\eta \in \mathcal{T}_k(\phi)$ and let U be an open subset of the domain of η. We must show that $\Omega(\eta, U) \cap \Gamma_\phi^+(k, n)$ is open in the relative topology of $\Gamma_\phi^+(k, n)$ inherited from $\mathbb{T} \times \mathbb{Z} \times \mathbb{T}$. Let $(x, k, y) \in \Omega(\eta, U) \cap \Gamma_\phi^+(k, n)$. It follows from Lemma 3.2 that $\phi^{n+k}(z) = \phi^n(\eta(z))$ for all z sufficiently close to x. If (z, z') is sufficiently close to (x, y) in $\mathbb{T} \times \mathbb{T}$ and $(z, k, z') \in \Gamma_\phi^+(k, n)$, the conditions $\mathrm{val}(\phi^{n+k}, z) = \mathrm{val}(\phi^n, z')$ and $\phi^{n+k}(z) = \phi^n(z')$ imply that $z \leq x \Leftrightarrow z' \leq y$. Combined with the fact that $\phi^{n+k}(z) = \phi^n(\eta(z)) = \phi^n(z')$ we conclude from this that $z' = \eta(z)$ when (z, z') is sufficiently close to (x, y) and $(z, k, z') \in \Gamma_\phi^+(k, n)$. That is, there is an open neighborhood V of (x, k, y) in $\mathbb{T} \times \mathbb{Z} \times \mathbb{T}$ such that $V \cap \Gamma_\phi^+(k, n) \subseteq \Omega(\eta, U) \cap \Gamma_\phi^+(k, n)$. □

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In combination with Lemma 3.2 it follows from Lemma 3.4 that $\Gamma_\phi^+(k, n)$ is an open subset of $\Gamma_\phi^+(k, n+1)$. Therefore,

$$\Gamma_\phi^+(k) = \bigcup_{n \geq -k+1} \Gamma_\phi^+(k, n)$$

is locally compact and Hausdorff in the inductive limit topology, and the disjoint union

$$\Gamma_\phi^+ = \bigsqcup_{k \in \mathbb{Z}} \Gamma_\phi^+(k)$$

is a locally compact Hausdorff in the topology where each $\Gamma_\phi^+(k)$ is closed and open and has the topology just defined. By Lemma 3.4 this topology is identical with the one we obtain from Theorem 2.2. Note that the topology is second countable since the topology of $\mathbb{T} \times \mathbb{Z} \times \mathbb{T}$ is. This proves the following lemma.

LEMMA 3.5. *The groupoid Γ_ϕ^+ is a second countable locally compact Hausdorff étale groupoid.*

We assume now and throughout the paper that $\phi : \mathbb{T} \to \mathbb{T}$ is continuous and piecewise monotone. The objective is to investigate the structure of $C_r^*(\Gamma_\phi^+)$; in particular, when it is simple.

Let us first consider the case where ϕ is a local homeomorphism so that the groupoid Γ_ϕ of Renault [**Re**], Deaconu [**De**] and Anantharaman-Delaroche [**An**] is defined.

LEMMA 3.6. *Assume that ϕ is a local homeomorphism. If the degree of ϕ is positive, there is an isomorphism $\Gamma_\phi^+ \simeq \Gamma_\phi$ of topological groupoids. If the degree of ϕ is negative there is an isomorphism $\Gamma_\phi^+ \simeq \Gamma_{\phi^2}$ of topological groupoids.*

Proof. This follows straightforwardly from the observation that $\mathrm{val}(\phi, x) = (+, +)$ for all $x \in \mathbb{T}$ when the degree is positive and $\mathrm{val}(\phi, x) = (-, -)$ for all $x \in \mathbb{T}$ when the degree is negative. □

It follows from Lemma 3.6 that for a local homeomorphism ϕ we have the equality $C_r^*(\Gamma_\phi^+) = C_r^*(\Gamma_\phi)$ when the degree of ϕ is positive, and $C_r^*(\Gamma_\phi^+) = C_r^*(\Gamma_{\phi^2})$ when it is negative. The simple C^*-algebras of the form $C_r^*(\Gamma_\psi)$ for a surjective, local homeomorphism ψ of the circle were all described in [**AT**] and we will therefore here assume that ϕ is not locally injective. Thus, for the remaining part of the paper ϕ will be continuous, piecewise monotone, surjective and not locally injective.

4. *Transitivity implies piecewise linearity and pure infiniteness*

Let $s > 0$. A continuous function $g : [0, 1] \to \mathbb{R}$ is *uniformly piecewise linear with slope s* when there are points $0 = a_0 < a_1 < a_2 < \cdots < a_N = 1$ such that g is linear with slope $\pm s$ on each interval $[a_{i-1}, a_i]$, $i = 1, 2, \ldots, N$. We say that ϕ is uniformly piecewise linear with slope s when its lift $f : [0, 1] \to \mathbb{R}$ is.

Recall that a continuous map $h : X \to X$ on a compact metric space X is *transitive* when for each pair U, V of non-empty open sets in X there is an $n \in \mathbb{N}$ such that $h^n(U) \cap V \neq \emptyset$. When h is surjective, as in the case we consider, transitivity is equivalent to the existence of a point with dense forward orbit.

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THEOREM 4.1. *Assume that ϕ is transitive. It follows that there is an orientation-preserving homeomorphism $h : \mathbb{T} \to \mathbb{T}$ such that $h \circ \phi \circ h^{-1}$ is uniformly piecewise linear with slope $s > 1$.*

Proof. We will show how the theorem follows from the work of Shultz [**S**] on discontinuous piecewise monotone maps of the interval.

After conjugation by a rotation of the circle we can assume that $\phi(1) \neq 1$ and $1 \notin \phi(\mathcal{C}_1)$. (Indeed, since ϕ is piecewise monotone and transitive there are λ in $\mathbb{T}$ arbitrary close to 1 such that $\phi(\lambda) \neq \lambda$. Choose one of them such that $\lambda \notin \phi(\mathcal{C}_1)$. Then $\phi_1(t) = \lambda^{-1}\phi(\lambda t)$ is conjugate to ϕ, does not fix 1 and all its critical values are different from 1.) Let $\mu : \mathbb{T} \to [0, 1[$ be the inverse map of $[0, 1[\ni t \mapsto e^{2\pi i t}$. Then

$$\tau(t) = \mu \circ \phi(e^{2\pi i t})$$

is piecewise monotone in the sense of Shultz [**S**]. Since ϕ is surjective and $1 \notin \phi(\mathcal{C}_1 \cup \{1\})$, it follows that τ is discontinuous at a point in $]0, 1[$ and $\tau([0, 1]) = [0, 1[$. We claim that τ is transitive in the sense of [**S**, Definition 2.6]; that is, we claim that for every open non-empty subset $U \subseteq [0, 1]$ there is an $n \in \mathbb{N}$ such that

$$\bigcup_{i=0}^{n} \widehat{\tau}^k(U) = [0, 1]. \tag{4.1}$$

Here $\widehat{\tau}$ is the possibly multivalued map on $[0, 1]$ which associates to each $x \in [0, 1]$ the left- and right-hand limits of τ at x. By construction this union is either $\{\tau(x)\}$ or $\{1, 0\}$. In the latter case $0 = \tau(x)$. It follows therefore that $\widehat{\tau}(A)\backslash\{1\} = \tau(A)$ for every subset $A \subseteq [0, 1]$. Thus,

$$\widehat{\tau}^k(U) \supseteq \tau^k(U)$$

for all k. The strong transitivity of ϕ implies that $\bigcup_{i=0}^{n-1} \tau^k(U) = [0, 1[$ for some $n \in \mathbb{N}$. As observed above τ is discontinuous at a point in $]0, 1[$. It follows therefore that $1 \in \widehat{\tau}([0, 1[)$ and hence that (4.1) holds since

$$\bigcup_{i=0}^{n} \widehat{\tau}^i(U) \supseteq \widehat{\tau}\left(\bigcup_{i=0}^{n-1} \widehat{\tau}^i(U)\right) \supseteq \widehat{\tau}\left(\bigcup_{i=0}^{n-1} \tau^i(U)\right) = \widehat{\tau}([0, 1[) = [0, 1].$$

It follows now from [**S**, Propositions 4.3 and 3.6] that there is a homeomorphism $h : [0, 1] \to [0, 1]$ such that $f = h \circ \tau \circ h^{-1}$ is uniformly piecewise linear. From [**S**, proof of Proposition 3.6] we see that h is increasing. Since ϕ is not locally injective there are non-empty open intervals $I, I' \subseteq \mathbb{T}\backslash\{1\}$ such that $I \cap I' = \emptyset$ and $\phi(I) = \phi(I')$. Then $J = \mu(I)$ and $J' = \mu(I')$ are non-empty open intervals in $[0, 1[$ such that $J \cap J' = \emptyset$ and $\tau(J) = \tau(J')$, i.e. τ is not essentially injective in the sense of [**S**, Definition 4.1]. Hence, the slope s of the linear pieces of f is >1 by [**S**, Proposition 4.3].

Since $h(0) = 0$, $h(1) = 1$ and $\tau(0) = \tau(1)$ we find that $f(0) = f(1)$ and we can therefore define $\varphi : \mathbb{T} \to \mathbb{T}$ such that $\varphi(e^{2\pi i t}) = e^{2\pi i f(t)}$, $t \in [0, 1]$. Then $\varphi = g \circ \phi \circ g^{-1}$, where $g = \mu^{-1} \circ h \circ \mu$. Then $g(1) = 1 = \lim_{\lambda \to 1} g(\lambda)$. Hence, g is continuous and an orientation-preserving homeomorphism on $\mathbb{T}$. It follows that φ is a continuous map on $\mathbb{T}$ and conjugate to ϕ. By construction φ is uniformly piecewise linear with slope $s > 1$. □

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We say that a p-periodic point $x \in \mathbb{T}$ is *repelling* when there is an open interval I in $\mathbb{T}$ and an $r > 1$ such that $x \in I$ and $|\phi^p(y) - x| \geq r|y - x|$ for all $y \in I$.

LEMMA 4.2. *Assume that ϕ is transitive and uniformly piecewise linear with slope $s > 1$. Then the periodic points of ϕ are dense in $\mathbb{T}$ and they are all repelling.*

Proof. Since ϕ is transitive there is a point in $\mathbb{T}$ with dense forward orbit, cf. [**W**, Theorem 5.9]. It follows therefore from [**AK**, Corollary 2] that ϕ has periodic points, and then by [**CM**, Corollary 3.4] that the periodic points are dense. For each $n \in \mathbb{N}$ the map ϕ^n is uniformly piecewise linear with slope $s^n > 1$. Therefore, all periodic points of ϕ are repelling. □

COROLLARY 4.3. *Assume that ϕ is transitive. It follows that Γ_ϕ^+ is locally contractive in the sense of* [**An**].

Proof. By Theorem 4.1 we may assume that ϕ is piecewise linear with slope $s > 1$. Let U be an open non-empty subset of $\mathbb{T}$. By Lemma 4.2 there is in U a point which is periodic and repelling. Since there are only finitely many critical points there are also only finitely many periodic orbits which contain a critical point. Hence, U contains a periodic point x, say of period n, which is repelling and whose orbit does not contain a critical point. Then $\mathrm{val}(\phi^{2n}, x) = (+, +)$ and there is therefore an open neighborhood $W \subseteq U$ of x and a $\kappa > 1$ such that $\mathrm{val}(\phi^{2n}, y) = (+, +)$ and $|\phi^{2n}(y) - x| \geq \kappa|y - x|$ for all $y \in W$. The proof is then completed exactly as [**Th4**, proof of Proposition 4.1]. □

LEMMA 4.4. *Assume that ϕ is transitive. It follows that Γ_ϕ^+ is essentially free in the sense of* [**An**]*, i.e. the points in $\mathbb{T}$ with trivial isotropy group in Γ_ϕ^+ are dense in $\mathbb{T}$.*

Proof. A point in $\mathbb{T}$ has non-trivial isotropy group only when it is pre-periodic. It suffices therefore to show that the set of pre-periodic points has empty interior in $\mathbb{T}$; a fact which follows easily from the assumed transitivity of ϕ. □

PROPOSITION 4.5. *Assume that ϕ is transitive. It follows that $C_r^*(\Gamma_\phi^+)$ is purely infinite in the sense that every non-zero hereditary C*-subalgebra contains an infinite projection.*

Proof. This follows from Lemma 4.4 and Corollary 4.3, thanks to [**An**, Proposition 2.4]. □

There is one more fact about piecewise monotone circle maps which can be deduced from the work of Shultz [**S**], and which we shall use below. Recall that a continuous map $h : X \to X$ on a compact Hausdorff space X is *totally transitive* when h^n is transitive for all $n \in \mathbb{N}$, and *exact* when for all open non-empty subsets $U \subseteq X$ there is an $N \in \mathbb{N}$ such that $h^N(U) = X$.

LEMMA 4.6. *The map ϕ is exact if and only if ϕ is totally transitive.*

Proof. It is obvious that exactness implies total transitivity. To prove the converse, return to the notation introduced in the proof of Theorem 4.1 and assume that ϕ is totally transitive. As observed in that proof, τ is then transitive and not essentially injective in the sense of [**S**]. It follows therefore from [**S**, Corollary 4.7] that there is an $N \in \mathbb{N}$ and closed sets K_i, $i = 1, 2, \ldots, N$, with mutually disjoint non-empty interiors $\mathrm{Int}\, K_i$ in $[0, 1]$ such

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that τ^N maps Int K_i onto Int K_i and is exact on K_i for each i. The image of Int K_i in $\mathbb{T}$ is open and invariant under ϕ^N and must, therefore, be all of $\mathbb{T}$ since ϕ is totally transitive. This implies that $N = 1$, which means that τ is exact. It follows that ϕ is exact as well. □

5. *Simplicity*

For $x \in \mathbb{T}$, let $\mathrm{RO}^+(x)$ be the Γ_ϕ^+-orbit of x, i.e.

$$\mathrm{RO}^+(x) = \{y \in \mathbb{T} : \phi^n(x) = \phi^m(y), \mathrm{val}(\phi^n, x) = \mathrm{val}(\phi^m, y) \text{ for some } n, m \in \mathbb{N}\}.$$

A subset $A \subseteq \mathbb{T}$ will be called *restricted orbit invariant* or RO^+*-invariant* when $x \in A \Rightarrow \mathrm{RO}^+(x) \subseteq A$. Let $Y \subseteq \mathbb{T}$ be a closed RO^+-invariant subset. Then the reduction

$$\Gamma_\phi^+|_Y = \{(x, k, y) \in \Gamma_\phi^+ : x, y \in Y\}$$

is a closed subgroupoid of Γ_ϕ^+ and an étale groupoid in the topology inherited from Γ_ϕ^+. The same is true for the reduction

$$\Gamma_\phi^+|_{\mathbb{T}\setminus Y} = \{(x, k, y) \in \Gamma_\phi^+ : x, y \in \mathbb{T}\setminus Y\}.$$

The restriction map $C_c(\Gamma_\phi^+) \to C_c(\Gamma_\phi^+|_Y)$ extends to a $*$-homomorphism $\pi_Y : C_r^*(\Gamma_\phi^+) \to C_r^*(\Gamma_\phi^+|_Y)$ and the inclusion $C_c(\Gamma_\phi^+|_{\mathbb{T}\setminus Y}) \subseteq C_c(\Gamma_\phi^+)$ extends to an embedding $C_r^*(\Gamma_\phi^+|_{\mathbb{T}\setminus Y}) \subseteq C_r^*(\Gamma_\phi^+)$ which realizes $C_r^*(\Gamma_\phi^+|_{\mathbb{T}\setminus Y})$ as an ideal in $C_r^*(\Gamma_\phi^+)$. It is straightforward to adapt [**Th3**, proof of Lemma 3.2] to obtain the following.

LEMMA 5.1. *Let Y be a closed* RO^+*-invariant subset of* $\mathbb{T}$. *It follows that*

$$0 \longrightarrow C_r^*(\Gamma_\phi^+|_{\mathbb{T}\setminus Y}) \longrightarrow C_r^*(\Gamma_\phi^+) \xrightarrow{\pi_Y} C_r^*(\Gamma_\phi^+|_Y) \longrightarrow 0$$

is exact.

In particular, $C_r^*(\Gamma_\phi^+)$ is not simple when there are non-trivial closed RO^+-invariant subsets of $\mathbb{T}$. We aim now to show that this is the only obstruction.

For the statement of the next lemma recall that the *full orbit* of a point $x \in \mathbb{T}$ is the set $\{y \in \mathbb{T} : \phi^n(y) = \phi^m(x) \text{ for some } n, m \in \mathbb{N}\}$. For each $j \in \mathbb{N}$ we let $\mathcal{C}_j$ denote the critical points of ϕ^j, i.e.

$$\mathcal{C}_j = \{t \in \mathbb{T} : \mathrm{val}(\phi^j, t) \in \{(+, -), (-, +)\}\}.$$

The elements of $\bigcup_{n=0}^\infty \phi^{-n}(\mathcal{C}_1)$ are then the *pre-critical* points.

LEMMA 5.2. *Assume that there is a point $x \in \mathbb{T}$ whose full orbit is dense in $\mathbb{T}$. It follows that there is a point in $\mathbb{T}$ which is neither pre-periodic nor pre-critical.*

Proof. Let Per_n be the set of points in $\mathbb{T}$ of minimal period n. Assume for a contradiction that

$$\mathbb{T} = \bigcup_{n,k\in\mathbb{N}} \phi^{-k}(\mathrm{Per}_n \cup \mathcal{C}_1).$$

By the Baire category theorem this implies that there are $k, n \in \mathbb{N}$ such that $\phi^{-k}(\mathrm{Per}_n \cup \mathcal{C}_1)$ contains a non-degenerate interval. Since ϕ^k is piecewise monotone and $\mathcal{C}_1$ finite this

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implies that Per_n contains a non-degenerate interval. Then Per_n also contains two non-empty open intervals I_+, I_- such that $\bigcup_{i=0}^{n} \phi^i(I_+)$ and $\bigcup_{i=0}^{n} \phi^i(I_-)$ are disjoint. It follows that

$$\left(\bigcup_{j=0}^{\infty} \phi^j(I_+)\right) \cap \left(\bigcup_{j=0}^{\infty} \phi^j(I_-)\right) = \emptyset. \tag{5.1}$$

By assumption there is a point x with dense full orbit. Since both I_+ and I_- contain an element from this orbit it follows that there is a $k \in \mathbb{N}$ such that

$$\phi^k(x) \in \left(\bigcup_{j=0}^{\infty} \phi^j(I_+)\right) \cap \left(\bigcup_{j=0}^{\infty} \phi^j(I_-)\right).$$

This contradicts (5.1). □

LEMMA 5.3. *The C*-algebra $C_r^*(\Gamma_\phi^+)$ is simple if and only of $\mathrm{RO}^+(x)$ is dense in $\mathbb{T}$ for all $x \in \mathbb{T}$.*

Proof. The simplicity of $C_r^*(\Gamma_\phi^+)$ implies that $\mathrm{RO}^+(x)$ is dense for all x by Lemma 5.1. For the converse assume that $\mathrm{RO}^+(x)$ is dense for all x. By [**Th1**, Corollary 2.18] the simplicity of $C_r^*(\Gamma_\phi^+)$ will follow if we can show that not all points of $\mathbb{T}$ have non-trivial isotropy in Γ_ϕ^+. Since a point with non-trivial isotropy is pre-periodic it suffices to show that not all points of $\mathbb{T}$ are pre-periodic under ϕ. This follows from Lemma 5.2. □

Lemma 5.3 can also be obtained from results in [**AR**] rather than [**Th1**].

LEMMA 5.4. *Assume that $C_r^*(\Gamma_\phi^+)$ is simple. Then ϕ is transitive.*

Proof. Let $E \subseteq \mathbb{T}$ be closed, with non-empty interior and ϕ-invariant in the sense that $\phi(E) \subseteq E$. By [**W**, Theorem 5.9] it suffices to show that $E = \mathbb{T}$. For each $n, m \geq 1$ set

$$U_{n,m} = \{x \in \mathbb{T} : \phi^n(x) = \phi^m(y), \mathrm{val}(\phi^n, x) = \mathrm{val}(\phi^m, y) \text{ for some } y \in \mathrm{Int}\, E\},$$

where $\mathrm{Int}\, E$ is the interior of E. Note that $U_{n,m}$ is open and non-empty and that $\bigcup_{n,m} U_{n,m}$ is RO^+-invariant. It follows therefore from Lemma 5.3 that $\bigcup_{n,m} U_{n,m} = \mathbb{T}$. By compactness there is an $N \in \mathbb{N}$ such that $\mathbb{T} = \bigcup_{n,m=1}^{N} U_{n,m}$. Since $U_{n,m} \subseteq \phi^{-n}(E)$ we find then that

$$\mathbb{T} = \phi^N(\mathbb{T}) \subseteq \phi^N\left(\bigcup_{n=1}^{N} \phi^{-n}(E)\right) \subseteq E.$$ □

The converse of Lemma 5.4 is not true in general; transitivity of ϕ does not imply that $C_r^*(\Gamma_\phi^+)$ is simple. A necessary and sufficient condition for simplicity of $C_r^*(\Gamma_\phi^+)$ will be given in Theorem 5.21.

5.1. *Finite RO^+-orbits and quotients of $C_r^*(\Gamma_\phi^+)$.* The elements in $\phi(\mathcal{C}_1)$ are *the critical values* and the elements of $\bigcup_{n=1}^{\infty} \phi^n(\mathcal{C}_1)$ are the *post-critical points*. Note that a critical point is pre-critical, but not necessarily post-critical.

LEMMA 5.5. *Assume that ϕ is transitive. Let $A \subseteq \mathbb{T}$ be a non-empty RO^+-invariant subset which is not dense in $\mathbb{T}$. It follows that A is finite and consists of points that are post-critical and not pre-critical.*

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Proof. By assumption there is an open non-empty interval $J \subseteq \mathbb{T}$ such that

$$A \cap J = \emptyset. \tag{5.2}$$

By [**Y**, Corollary 4.2] ϕ is not only transitive, but also strongly transitive. There is therefore an $N \in \mathbb{N}$ such that

$$\bigcup_{i=0}^{N} \phi^i(J) = \mathbb{T}. \tag{5.3}$$

If $x \in A$ and $\mathrm{val}(\phi^j, x) \in \{(+,-),(-,+)\}$ for some $j \geq 1$ we can choose $y \in J$ such that $\phi^k(y) = x$ for some $k \in \{1, 2, \ldots, N\}$. It follows from Table 1 for $\bullet$ that $\mathrm{val}(\phi^{k+j}, y) = \mathrm{val}(\phi^j, x) \bullet \mathrm{val}(\phi^k, y) = \mathrm{val}(\phi^j, x)$. Hence, $y \in \mathrm{RO}^+(x) \subseteq A$, contradicting (5.2). It follows that $\mathrm{val}(\phi^j, x) \in \{(+,+),(-,-)\}$ for all $j \in \mathbb{N}$ when $x \in A$; i.e A consists of points that are not pre-critical.

Since ϕ is not locally injective there is a $z \in \mathbb{T}$ such that $\mathrm{val}(\phi, z) \in \{(+,-),(-,+)\}$. Choose $z_0 \in J$ and $k \in \{1, 2, \ldots, N\}$ such that $\phi^k(z_0) = z$ and note that $\mathrm{val}(\phi^{k+1}, z_0) \in \{(+,-),(-,+)\}$. There are therefore subintervals J_+, J_- of J such that $\mathrm{val}(\phi^{k+1}, y) = (+,+)$ when $y \in J_+$, $\mathrm{val}(\phi^{k+1}, y) = (-,-)$ when $y \in J_-$, and $\phi^{k+1}(J_+) = \phi^{k+1}(J_-) \stackrel{def}{=} I$. Since ϕ is strongly transitive there is a $K \in \mathbb{N}$ such that $\bigcup_{i=1}^{K} \phi^i(I) = \mathbb{T}$. Set $M_i = I \cap \mathcal{C}_i$. Let $a \in \mathbb{T}$ be a non-critical element, i.e. $\mathrm{val}(\phi, a) \in \{(+,+),(-,-)\}$. Assume that $a \notin \bigcup_{i=1}^{K} \phi^i(M_i)$. We claim that $\mathrm{RO}^+(a) \cap J \neq \emptyset$. To see this note that there is an $i \in \{1, 2, \ldots, K\}$ and a $y' \in I \backslash M_i$ such that $\phi^i(y') = a$. Then $\mathrm{val}(\phi^i, y') \in \{(+,+),(-,-)\}$ and there is also an element $y \in J_+ \cup J_-$ such that $\phi^{k+1}(y) = y'$ and $\mathrm{val}(\phi^{i+k+2}, y) = \mathrm{val}(\phi^{i+1}, y') \bullet \mathrm{val}(\phi^{k+1}, y) = \mathrm{val}(\phi, a)$. It follows that $y \in \mathrm{RO}^+(a) \cap J$, proving the claim.

The last two paragraphs show that $A \subseteq \bigcup_{i=1}^{K} \phi^i(M_i)$. This completes the proof because $\bigcup_{i=1}^{K} \phi^i(M_i)$ is finite and consists of post-critical points. □

Call a point $x \in \mathbb{T}$ *exposed* when $\mathrm{RO}^+(x)$ is finite. By Lemmas 5.3 and 5.5 it is the possible presence of exposed points which is the only obstruction for simplicity of $C_r^*(\Gamma_\phi^+)$.

COROLLARY 5.6. *Assume that ϕ is transitive. Then $C_r^*(\Gamma_\phi^+)$ is simple if and only if there are no exposed points.*

5.1.1. $|\deg\phi| \geq 2$.

LEMMA 5.7. *Assume that ϕ is transitive and that $|\deg\phi| \geq 2$. It follows that $\mathrm{RO}^+(x)$ is dense in $\mathbb{T}$ for all $x \in \mathbb{T}$.*

Proof. Let $n \in \mathbb{N}$. By looking at the graph of a lift $f : [0, 1] \to \mathbb{R}$ of ϕ^{2n} one sees that for any $x \in \mathbb{T}$, the set

$$A_n = \{y \in \mathbb{T} : \phi^{2n}(y) = x,\ \mathrm{val}(\phi^{2n}, y) = (+,+)\}$$

contains at least $\deg \phi^{2n}$ elements. Since $A_n \subseteq \mathrm{RO}^+(x)$ we conclude that $\mathrm{RO}^+(x)$ is infinite for all $x \in \mathbb{T}$. It follows then from Lemma 5.5 that $\mathrm{RO}^+(x)$ is dense for all x. □

PROPOSITION 5.8. *Assume that ϕ is transitive and that $|\deg\phi| \geq 2$. It follows that $C_r^*(\Gamma_\phi^+)$ is simple.*

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5.1.2. $|\deg\phi| = 1$. Before we specialize to the case where the degree is 1 or -1 we need a couple of more general facts.

LEMMA 5.9. *Assume that ϕ is transitive, but not totally transitive. It follows that there is a $p > 1$ and closed intervals I_i, $i = 0, 1, 2, \ldots, p-1$, such that:*

(1) $\phi(I_i) = I_{i+1}$ *(addition mod p);*
(2) $I_i \cap \operatorname{Int} I_j = \emptyset$, $i \neq j$;
(3) $\bigcup_{i=0}^{p-1} I_i = \mathbb{T}$;
(4) $\phi^p|_{I_i}$ *is totally transitive for each i.*

Proof. This is a special case of [**AdRR**, Corollary 2.7]. □

Note that the number p and the collection $\{I_0, I_1, \ldots, I_{p-1}\}$ of intervals in Lemma 5.9 are unique. We will refer to p as *the global period of ϕ*, and say that it is 1 when ϕ is totally transitive. In the following we denote the set of endpoints of the intervals I_i from Lemma 5.9 by $\mathcal{E}$.

LEMMA 5.10. *Assume that ϕ is transitive but not totally transitive. Then*

$$\phi^{-1}(\mathcal{E})\backslash\mathcal{C}_1 = \mathcal{E}. \tag{5.4}$$

Proof. Assume for a contradiction that $e \in \mathcal{E}$, but $\phi(e) \notin \mathcal{E}$. There are then intervals I_i, $I_{i'}$, I_j as in Lemma 5.9 such that $i \neq i'$, $e \in I_i \cap I_{i'}$ and $\phi(e) \in \operatorname{Int} I_j$. By continuity of ϕ and condition (1) from Lemma 5.9 it follows that $I_{i+1} \cap \operatorname{Int} I_j \neq \emptyset$ and $I_{i'+1} \cap \operatorname{Int} I_j \neq \emptyset$. Since $i + 1 \neq i' + 1$ this violates condition (2). Thus,

$$\phi(\mathcal{E}) \subseteq \mathcal{E}. \tag{5.5}$$

If $e \in \mathcal{E}$ is a critical point the images $I_{i+1} = \phi(I_i)$ and $I_{i'+1} = \phi(I_{i'})$ of the two intervals I_i, $I_{i'}$ containing e will both have non-trivial intersection with the same interval I_j containing $\phi(e)$; contradicting (2) again. Hence,

$$\mathcal{E} \cap \mathcal{C}_1 = \emptyset. \tag{5.6}$$

Consider then an element $x \in \phi^{-1}(\mathcal{E})$ and assume that $x \notin \mathcal{C}_1$. Let I_i and $I_{i'}$ be the two intervals among the intervals from Lemma 5.9 which contain $\phi(x)$. If $x \notin \mathcal{E}$ there is a third interval I_j which contains x in its interior. Since x is not critical it follows that $\phi(I_j) = I_{j+1}$ has non-trivial intersection with both Int I_i and Int $I_{i'}$, contradicting (2) once more. Hence,

$$\phi^{-1}(\mathcal{E})\backslash\mathcal{C}_1 \subseteq \mathcal{E}. \tag{5.7}$$

This completes the proof since (5.4) is equivalent to (5.5)–(5.7). □

LEMMA 5.11. *Assume that ϕ is transitive but not totally transitive. When* $\deg\phi = 1$ *the set $\mathcal{E}$ is a p-periodic orbit where p is the global period of ϕ, and* $\operatorname{val}(\phi, x) = (+, +)$ *for all $x \in \mathcal{E}$. When* $\deg\phi = -1$ *the global period of ϕ is 2 and $\mathcal{E}$ consists of two distinct fixed points of valency $(-, -)$.*

Proof. Let I_i, $i = 0, 1, \ldots, p-1$, be the intervals from Lemma 5.9, and let e_i^-, be the left endpoint of I_i, defined using the orientation of $\mathbb{T}$. When $\deg\phi = 1$ we see by looking at the

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graph of a lift of ϕ that $\mathrm{val}(\phi, e_i^-) = (+, +)$ and $\phi(e_i^-) = e_{i+1}^-$ (addition mod p). It follows that $\mathcal{E} = \mathrm{RO}^+(e_0^-)$, and that this is also the (forward) orbit of e_0^-. When $\deg\phi = -1$ observe first ϕ has a fixed point x. This fixed point lies in one of the intervals I_i. Since x also lies in I_{i+1} and I_{i+2} it follows that two of the intervals I_i, I_{i+1} and I_{i+2} must be the same, i.e. $p = 2$. By looking at the graph of a lift of ϕ we see that $\mathcal{E}$ consists of two fixed points of valency $(-, -)$. □

LEMMA 5.12. *If* $\deg\phi = 1$ *there is for all* $x \in \mathbb{T}$ *an element* $y \in \phi^{-1}(x)$ *such that* $\mathrm{val}(\phi, y) = (+, +)$. *If* $\deg\phi = -1$ *there is for all* $x \in \mathbb{T}$ *an element* $y \in \phi^{-1}(x)$ *such that* $\mathrm{val}(\phi, y) = (-, -)$.

Proof. Look at the graph of a lift of ϕ. □

LEMMA 5.13. *Assume that* $\deg\phi \in \{1, -1\}$. *Then* $\mathrm{RO}^+(x)$ *is infinite for all* $x \in \mathbb{T}$ *that are not periodic under* ϕ.

Proof. Let $x \in \mathbb{T}$. It follows from Lemma 5.12 that there are sequences $\{n_i\}$ in $\mathbb{N}$ and $\{x_i\}$ in $\mathbb{T}$ such that $\phi^{n_1}(x_1) = x$, $\phi^{n_i}(x_i) = x_{i-1}$, $i \geq 2$, and $\mathrm{val}(\phi^{n_i}, x_i) = (+, +)$ for all i. Then $x_i \in \mathrm{RO}^+(x)$ for all i. The set $\{x_i : i \in \mathbb{N}\}$ is infinite when x is not periodic. □

LEMMA 5.14. *Assume that* $\deg\phi \in \{-1, 1\}$ *and* ϕ *is transitive but not totally transitive. Then* $\mathcal{E}$ *is the set of exposed points for* ϕ.

Proof. Let $e \in \mathcal{E}$ and $y \in \mathrm{RO}^+(e)$. There are natural numbers $i, j \in \mathbb{N}$ such that $\phi^i(e) = \phi^j(y)$ and $\mathrm{val}(\phi^i, e) = \mathrm{val}(\phi^j, y)$. It follows from (5.4) that $\phi^j(y) = \phi^i(x) \in \mathcal{E}$ and that $\mathrm{val}(\phi^i, e) \in (\pm, \pm)$ since $e \in \mathcal{E}$. This implies first that $\mathrm{val}(\phi, \phi^k(y)) \in (\pm, \pm)$ for all $k \leq j - 1$ and then that $\phi^{j-1}(y) \in \phi^{-1}(\mathcal{E})\backslash\mathcal{C}_1 = \mathcal{E}$. But then $\phi^{j-2}(y) \in \phi^{-1}(\mathcal{E})\backslash\mathcal{C}_1 = \mathcal{E}$ and so on. After j steps we conclude that $y \in \mathcal{E}$. This shows that $\mathcal{E}$ is RO^+-invariant.

It remains to show that $\mathcal{E}$ contains all exposed points. Assume therefore that y_0 is an exposed point. It follows from Lemma 5.13 that all exposed points are periodic. Since they are also post-critical by Lemma 5.5 and there are only finitely many critical points it follows that there are only finitely many exposed points. Let $m \in \mathbb{N}$ be an even number divisible by the global period p and by all the periods of exposed points. Then $\phi^m(y_0) = y_0$. Furthermore, if $z \in \phi^{-m}(y_0)$ and $\mathrm{val}(\phi^m, z) = (+, +)$ we see that $z \in \mathrm{RO}^+(y_0)$ and, hence, z is exposed. By definition of m this implies that $\phi^m(z) = z$, i.e. $z = y_0$. To see that there can not be any $z \in \phi^{-m}(y_0)$ with $\mathrm{val}(\phi^m, z) = (-, -)$ observe by looking at the graph of the lift of ϕ^m, that since $\deg\phi^m = 1$ the existence of such a z would imply the existence of a $z' \in \phi^{-m}(y_0)\backslash\{y_0\}$ with $\mathrm{val}(\phi^m, z') = (+, +)$ which is impossible as we have just seen. Now assume for a contradiction that $y_0 \notin \mathcal{E}$. Then y_0 lies in the interior of one of the intervals from Lemma 5.9, say I_i. We can then write I_i as the union $I_i = J_1 \cup J_2$ of two closed non-degenerate intervals such that $J_1 \cap J_2 = \{y_0\}$. As we have just seen an element of $I_i \cap (\phi^{-m}(y_0)\backslash\{y_0\})$ must be critical for ϕ^m and it follows therefore that $\phi^m(J_1) = J_1$. This contradicts the total transitivity of $\phi^p|_{I_i}$. □

LEMMA 5.15. *Assume that* ϕ *is totally transitive and that* $\deg\phi \in \{-1, 1\}$. *It follows that there is at most a single exposed point, and it must be a fixed point* e *such that* $\phi^{-1}(e)\backslash\mathcal{C}_1 = \{e\}$.

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Proof. Let m be the same number as in the proof of Lemma 5.14. In that proof it was shown that

$$\phi^{-m}(y)\backslash\{y\} \subseteq \mathcal{C}_m \tag{5.8}$$

for every exposed point y. It follows that if there are two exposed points, say e_1 and e_2, we could write $\mathbb{T} = J_1 \cup J_2$, where J_1 and J_2 are non-degenerate closed intervals such that $J_1 \cap J_2 = \{e_1, e_2\}$ and $\phi^{2m}(J_i) = J_i$, $i = 1, 2$. This contradicts the assumed total transitivity of ϕ. Therefore, there is at most a single exposed point e, and it is fixed by ϕ^m. When $\deg \phi = 1$ it follows from Lemma 5.12 that there is an element $z \in \phi^{-1}(e)$ such that $\mathrm{val}(\phi, z) = (+, +)$. Then z is exposed (since $z \in \mathrm{RO}^+(e)$) and the uniqueness of e implies that $z = e$, proving that e is a fixed point for ϕ.

To reach the same conclusion when $\deg \phi = -1$ it suffices to consider the case where $\mathrm{val}(\phi, e) = (-, -)$. By Lemma 5.12 there are elements $z_1, z \in \mathbb{T}$ such that $\phi(z_1) = z$, $\phi(z) = e$ and $\mathrm{val}(\phi, z_1) = \mathrm{val}(\phi, z) = (-, -)$. Then $z_1 \in \mathrm{RO}^+(e)$ and, hence, $z_1 = e$ because e is the only exposed point. It follows that $z = \phi(e)$, i.e. $\phi^2(e) = e$. Note that $\mathrm{val}(\phi, \phi(e)) = (-, -)$. We claim that

$$\phi^{-1}(\{e, \phi(e)\})\backslash\mathcal{C}_1 = \{e, \phi(e)\}. \tag{5.9}$$

To show this let $x \in \phi^{-1}(e)\backslash\mathcal{C}_1$. If $\mathrm{val}(\phi, x) = (+, +)$ we find that $x = e$ since e is the only exposed point. If $\mathrm{val}(\phi, x) = (-, -)$ an application of Lemma 5.12 shows that $x = \phi(e)$. Consider then an element $y \in \phi^{-1}(\phi(e))\backslash\mathcal{C}_1$. If $\mathrm{val}(\phi, y) = (-, -)$ it follows that $y \in \mathrm{RO}^+(e)$ and, hence, $y = e$ by uniqueness of e. If instead $\mathrm{val}(\phi, y) = (+, +)$ an application of Lemma 5.12 shows that $y = \phi(e)$. Having established (5.9) note that it implies that $\phi(e)$ is exposed, whence equal to e.

To show that $\phi^{-1}(e)\backslash\mathcal{C}_1 = \{e\}$ we may assume that $\deg \phi = 1$, since the other case follows from (5.9). Furthermore, it suffices to show that $\phi^{-1}(e)\backslash\mathcal{C}_1 \subseteq \{e\}$ since exposed points are not critical by Lemma 5.5. Consider therefore an element $x \in \phi^{-1}(e)\backslash\mathcal{C}_1$. If $\mathrm{val}(\phi, x) = (+, +)$ it follows that $x \in \mathrm{RO}^+(e)$ and, hence, x is exposed. Since e is the only exposed point this shows that $x = e$. Assume then that $\mathrm{val}(\phi, x) = (-, -)$. If $x \neq e$ a look at the graph for a lift of ϕ shows that there is then also a point $y \in \phi^{-1}(e)\backslash\{e\}$ with $\mathrm{val}(\phi, y) = (+, +)$ which we have just seen is not possible. Hence, $x = e$. □

To formulate the next proposition we call a point $e \in \mathbb{T}$ an *exceptional fixed point* when $\phi^{-1}(e)\backslash\mathcal{C}_1 = \{e\}$.

PROPOSITION 5.16. *Assume that ϕ is transitive and that* $\deg \phi \in \{-1, 1\}$. *Then $C_r^*(\Gamma_\phi^+)$ is simple unless either:*

(1) *ϕ is not totally transitive; or*

(2) *ϕ is totally transitive and there is an exceptional fixed point.*

In case (2) *there is an extension*

$$0 \longrightarrow B \longrightarrow C_r^*(\Gamma_\phi^+) \longrightarrow C(\mathbb{T}) \longrightarrow 0\,, \tag{5.10}$$

where B is simple and purely infinite. When ϕ is not totally transitive and $\deg \phi = 1$ *there is an extension*

$$0 \longrightarrow B \longrightarrow C_r^*(\Gamma_\phi^+) \longrightarrow C(\mathbb{T}) \otimes M_p(\mathbb{C}) \longrightarrow 0\,, \tag{5.11}$$

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where p is the global period of ϕ and B is simple and purely infinite. When ϕ is not totally transitive and $\deg \phi = -1$ *there is an extension*

$$0 \longrightarrow B \longrightarrow C_r^*(\Gamma_\phi^+) \longrightarrow C(\mathbb{T}) \oplus C(\mathbb{T}) \longrightarrow 0\,, \tag{5.12}$$

where B is simple and purely infinite.

Proof. Assume that none of the two cases (1) or (2) occur. It follows from Lemma 5.15 that there are no exposed points, and from Corollary 5.6 that $C_r^*(\Gamma_\phi^+)$ is simple.

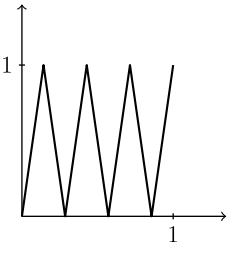
In case (2) it follows from Lemma 5.15 that there is exactly one exposed point, e, which is a fixed point. From Lemma 5.1 we then get the extension

$$0 \longrightarrow B \longrightarrow C_r^*(\Gamma_\phi^+) \longrightarrow C_r^*(\Gamma_\phi^+|_{\{e\}}) \longrightarrow 0\,, \tag{5.13}$$

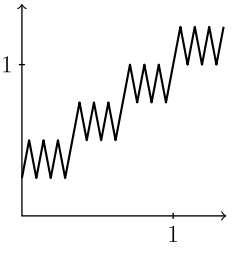
where $B = C_r^*(\Gamma_\phi^+|_{\mathbb{T}\setminus\{e\}})$. It is easy to see, cf. [**Th3**, proof of Lemma 4.11], that $C_r^*(\Gamma_\phi^+|_{\{e\}}) \simeq C(\mathbb{T})$. Furthermore, B is purely infinite because B is an ideal in $C_r^*(\Gamma_\phi^+)$ which is purely infinite by Proposition 4.5. To conclude that B is simple we argue as in [**Th3**, proof of Proposition 4.10]: the elements of $\mathbb{T}\setminus\{e\}$ with non-trivial isotropy in $C_r^*(\Gamma_\phi^+|_{\mathbb{T}\setminus\{e\}})$ are pre-periodic. It follows from Theorem 4.1 that the pre-periodic points are countable, whence $\mathbb{T}\setminus\{e\}$ must contain a point with trivial isotropy. By [**Th1**, Corollary 2.18] it suffices therefore to show that $\mathbb{T}\setminus\{e\}$ does not contain any non-trivial (relatively) closed RO^+-invariant subsets. Let therefore L be such a set. Then $L \cup \{e\}$ is closed and RO^+-invariant in $\mathbb{T}$ and, hence, either equal to $\mathbb{T}$ or contained in $\{e\}$ by Lemmas 5.5 and 5.15. It follows that $L = \emptyset$ or $L = \mathbb{T}\setminus\{e\}$. This completes the proof in case (2).

In case (1) we argue as above, except that we use Lemma 5.14 to replace Lemmas 5.15 and 5.11 to determine $C_r^*(\Gamma_\phi^+|_{\mathcal{E}})$. □

To show by example that all of the cases mentioned in Proposition 5.16 can occur consider the following graph.

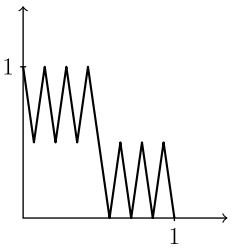


The graph describes the lift of an exact, and hence totally transitive map ϕ of the circle of degree one with an exceptional fixed point. The corresponding C^*-algebra $C_r^*(\Gamma_\phi^+)$ is an extension as in (5.10). To show that also the extensions (5.11) occur consider the following graph.



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This is the graph of the lift of a transitive, but not totally transitive circle map ϕ of degree one for which $C_r^*(\Gamma_\phi^+)$ is an extension as in (5.11) (with $p = 4$). In the same way the following graph describes a transitive circle map of degree -1 which is not totally transitive and for which $C_r^*(\Gamma_\phi^+)$ is an extension as in (5.12).



5.1.3. $\deg \phi = 0$. A point $z \in \mathbb{T}$ will be called *an exceptional critical value* when $\phi^{-1}(z) \subseteq \mathcal{C}_1$.

LEMMA 5.17. *Assume that* $\deg \phi = 0$ *and that* ϕ *is surjective. There is at most one exceptional critical value, and for all other elements* $x \in \mathbb{T}$ *there are points* $y_\pm \in \phi^{-1}(x)$ *such that* $\mathrm{val}(\phi, y_\pm) = (\pm, \pm)$.

Proof. Look at the graph of a lift of ϕ. □

LEMMA 5.18. *Assume that* ϕ *is transitive and that* $\deg \phi = 0$. *If* $y \in \mathbb{T}$ *is an exposed point there is an exceptional critical value* $e \in \mathbb{T}$ *such that* $\phi^2(e) = \phi(e) \neq e$, $\mathrm{RO}^+(y) = \{e, \phi(e)\}$ *and* $\phi^{-1}(\phi(e))\backslash\mathcal{C}_1 = \{e, \phi(e)\}$.

Proof. The main part of the proof will be to show that there is an exceptional critical value e such that one of the following holds:

(i) $\phi^2(e) = \phi(e) \neq e$, $\mathrm{val}(\phi, e) = (-, -)$ and $\mathrm{RO}^+(y) = \{\phi(e)\}$;
(ii) $\phi^2(e) = \phi(e) \neq e$, $\mathrm{val}(\phi, \phi(e)) = (+, +)$ and $\mathrm{RO}^+(y) = \{e\}$;
(iii) $\phi^2(e) = \phi(e) \neq e$, $\mathrm{RO}^+(y) = \{e, \phi(e)\}$.

Assume first that $\mathrm{RO}^+(y)$ does not contain an exceptional critical value. Let $z \in \mathrm{RO}^+(y)$. By using Lemma 5.17 we can construct y_k, $k = 0, 1, 2, 3, \ldots$, such that $y_0 = z$, $\phi(y_k) = y_{k-1}$ and $\mathrm{val}(\phi, y_k) = (+, +)$, $k \geq 1$. Then $y_k \in \mathrm{RO}^+(y)$ for all k so there are $k \neq k'$ such that $y_k = y_{k'}$. It follows that z is periodic and that $\mathrm{val}(\phi, u) = (+, +)$ for all u in the orbit $\mathrm{Orb}(z)$ of z. Hence, $\mathrm{Orb}(z) \subseteq \mathrm{RO}^+(y)$. Since this conclusion holds for all $z \in \mathrm{RO}^+(y)$ and since the forward orbits of elements from $\mathrm{RO}^+(y)$ must intersect we conclude that $\mathrm{RO}^+(y) = \mathrm{Orb}(y)$ and $\mathrm{val}(\phi, \phi^k(y)) = (+, +)$ for all $k \in \mathbb{N}$. Let $z \in \mathrm{RO}^+(y)$. Using Lemma 5.17 again we find $u_1, v_1 \in \phi^{-1}(z)$ such that $\mathrm{val}(\phi, u_1) = (+, +)$ and $\mathrm{val}(\phi, v_1) = (-, -)$. Then $u_1 \in \mathrm{RO}^+(y)$ and u_1 is therefore an element of the orbit of y. Since $v_1 \neq u_1$ (or since $\mathrm{val}(\phi, v_1) = (-, -)$), it follows that v_1 is not in the orbit of y. If v_1 is not an exceptional critical value we can find $v_2 \in \phi^{-1}(v_1)$ such that $\mathrm{val}(\phi, v_2) = (-, -)$. It follows that $v_2 \in \mathrm{RO}^+(y)$ and v_2 must therefore be an element of $\mathrm{Orb}(y)$. This contradicts that v_1 is not, and we conclude that v_1 must be an exceptional critical value e, which by Lemma 5.17 is unique. This shows that $z = \phi(e)$ and we conclude therefore that case (i) occurs.

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We consider then the case where $\mathrm{RO}^+(y)$ contains an exceptional critical value e. By looking at the graph of a lift of ϕ we see that a non-critical exceptional critical value e cannot be fixed since the degree is zero. Thus, $\phi(e) \neq e$ since exposed points are not critical. To see that $\phi(e)$ is a fixed point assume that it is not. Consider first the case where $\phi(e)$ is periodic, say of period $p > 1$. Since $\phi(e) \neq e$ it follows from Lemma 5.17 that there is a point $b_1 \in \phi^{-1}(\phi(e))$ such that $\mathrm{val}(\phi, b_1) \neq \mathrm{val}(\phi, e)$. Then $b_1 \notin \{e, \phi(e)\}$ and we use Lemma 5.17 again to find $b_2 \in \phi^{-1}(b_1)$ such that $\mathrm{val}(\phi, b_2) \neq \mathrm{val}(\phi, \phi(e))$. It follows that $b_2 \notin \{e, \phi(e), b_1\}$. By requiring in each step that $\mathrm{val}(\phi, b_i) \neq \mathrm{val}(\phi, \phi(e))$ we obtain through repeated application of Lemma 5.17 elements b_i, $i = 1, 2, \ldots, p+1$, such that $\phi(b_{k+1}) = b_k$ and $b_{k+1} \notin \{e, \phi(e), b_1, b_2, \ldots, b_k\}$ for all $k = 1, 2, \ldots, p$. Then, for $j > p+1$ we require in each step instead that $\mathrm{val}(\phi^j, b_j) = \mathrm{val}(\phi, e)$. It is then still automatic that $b_{k+1} \notin \{e, b_1, b_2, \ldots, b_k\}$ for all k, while the fact that $b_j \neq \phi(e)$ follows for $j \geq p+1$ because j is larger than the period of $\phi(e)$. Since $b_j \in \mathrm{RO}^+(e) = \mathrm{RO}^+(y)$ when $j > p+1$, we have contradicted the assumed finiteness of $\mathrm{RO}^+(y)$. To get the same contradiction when $\phi(e)$ is not assumed to be periodic we proceed in the same way, except that the steps between b_1 and b_{p+1} can be bypassed. In any case we conclude that $\phi^2(e) = \phi(e)$. We next argue, in a similar way, that $\phi^{-1}(\phi(e))\backslash\mathcal{C}_1 \subseteq \{e, \phi(e)\}$. Indeed, if $b_1 \in \phi^{-1}(\phi(e))\backslash(\{e, \phi(e)\} \cup \mathcal{C}_1)$ we use Lemma 5.17 to get a sequence b_i such that $\phi(b_{i+1}) = b_i$, $i \geq 1$, and $\mathrm{val}(\phi, b_i) = (-, -)$, $i \geq 2$. Then $i \neq i' \Rightarrow b_i \neq b_{i'}$, and $b_i \in \mathrm{RO}^+(e)$ for infinitely many i; again contradicting the infiniteness of $\mathrm{RO}^+(e)$. Since e is not pre-critical by Lemma 5.5 we have shown that $\phi^{-1}(\phi(e))\backslash\mathcal{C}_1 = \{e, \phi(e)\}$. If $\mathrm{val}(\phi, e) = (-, -)$ and $\mathrm{val}(\phi, \phi(e)) = (+, +)$ we find now easily that $\mathrm{RO}^+(y) = \mathrm{RO}^+(e) = \{e\}$, which is case (ii), and in all other cases that $\mathrm{RO}^+(y) = \mathrm{RO}^+(e) = \{e, \phi(e)\}$, which is case (iii).

Finally we argue that the cases (i) and (ii) are impossible. Indeed, in both cases we must have that $\mathrm{val}(\phi, e) = (-, -)$ and $\mathrm{val}(\phi, \phi(e)) = (+, +)$ since otherwise $e \in \mathrm{RO}^+(\phi(e))$. But then the two closed intervals J_1 and J_2 defined such that $J_1 \cap J_2 = \{e, \phi(e)\}$ and $J_1 \cup J_2 = \mathbb{T}$ are both ϕ-invariant, which contradicts the transitivity of ϕ. It follows that only case (iii) can occur. □

LEMMA 5.19. *Assume that ϕ is transitive and* $\deg \phi = 0$. *Then there are exposed points if and only if ϕ is not totally transitive.*

Proof. If ϕ is not totally transitive there are exposed points by (the proof of) Lemma 5.14. Conversely, if there are exposed points it follows from Lemma 5.18 that there is an exceptional critical value e such that $e \neq \phi(e) = \phi^2(e)$ and $\{e, \phi(e)\}$ is the set of exposed points. Furthermore, $\phi^{-1}(\phi(e))\backslash\mathcal{C}_1 = \{e, \phi(e)\}$. The points e and $\phi(e)$ define closed intervals J_1 and J_2 such that $\mathbb{T} = J_1 \cup J_2$, $J_1 \cap J_2 = \{e, \phi(e)\}$ and $\phi(J_i) = J_i$, $i = 1, 2$, or $\phi(J_1) = J_2$ and $\phi(J_2) = J_1$. The first case is ruled out by transitivity, and the second implies that ϕ is not totally transitive. □

PROPOSITION 5.20. *Assume that ϕ is transitive and that* $\deg \phi = 0$. *Then $C_r^*(\Gamma_\phi)$ is simple if and only if ϕ is totally transitive. When ϕ is not totally transitive there is an extension*

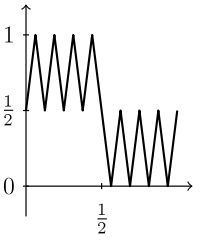
$$0 \longrightarrow B \longrightarrow C_r^*(\Gamma_\phi^+) \longrightarrow C(\mathbb{T}) \otimes M_2(\mathbb{C}) \longrightarrow 0\,, \tag{5.14}$$

where B is simple and purely infinite.

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Proof. With Lemmas 5.19 and 5.18 at hand, all of the necessary arguments can be found in the proof of Proposition 5.16. □

The following graph describes a transitive circle map of degree zero which is not totally transitive and for which $C_r^*(\Gamma_\phi^+)$ is an extension as in (5.14).



5.1.4. *Simplicity.* We can now finally give a necessary and sufficient condition for simplicity of $C_r^*(\Gamma_\phi^+)$.

THEOREM 5.21. *The following conditions are equivalent:*
(i) $C_r^*(\Gamma_\phi^+)$ *is simple;*
(ii) ϕ *is totally transitive and there is no exceptional fixed point;*
(iii) ϕ *is exact and there is no exceptional fixed point.*

Proof. (i) ⇒ (ii): Transitivity of ϕ follows from Lemma 5.4. And then ϕ must be totally transitive since otherwise the set $\mathcal{E}$ considered in Lemma 5.10 will be non-empty, finite and RO^+-invariant, as shown in the first paragraph of the proof of Lemma 5.14, and this contradicts simplicity by Lemma 5.3. The absence of an exceptional fixed point follows also from Lemma 5.3 since an exceptional fixed point is its own RO^+-orbit.

The implication (ii) ⇒ (i) follows from Propositions 5.20, 5.16 and 5.8, and the implication (iii) ⇒ (ii) is trivial. The implication (ii) ⇒ (iii) follows from Lemma 4.6. □

6. *Nuclearity, UCT and a six-terms exact sequence*

6.1. *The Cuntz–Pimsner picture of* $C_r^*(\Gamma_\phi^+)$. To simplify notation, set

$$R_\phi^+ = \Gamma_\phi^+(0).$$

Then $C_r^*(R_\phi^+)$ is the fixed point algebra of *the gauge action* $\beta = \beta^c$ on $C_r^*(\Gamma_\phi^+)$ arising from the homomorphism $c : \Gamma_\phi^+ \to \mathbb{Z}$ defined such that $c(x, k, y) = k$ (cf. [**Re**]). For $n \in \mathbb{N}$, set

$$R_\phi^+(n) = \Gamma_\phi^+(0, n)$$

which is an open sub-groupoid of R_ϕ^+. Then $R_\phi^+ = \bigcup_n R_\phi^+(n)$ and

$$C_r^*(R_\phi^+) = \overline{\bigcup_n C_r^*(R_\phi^+(n))}. \tag{6.1}$$

LEMMA 6.1. *Assume that* $C_r^*(\Gamma_\phi^+)$ *is simple. Let* $x \in \mathbb{T}\backslash\mathcal{C}_1$. *It follows that there are elements* $z, z' \in \mathbb{T}\backslash\mathcal{C}_1$ *such that* $(x, 1, z)$, $(z', 1, x) \in \Gamma_\phi^+$.

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Proof. If $\phi^k(x) \in \mathcal{C}_1$ for some $k \geq 1$, set $z = \phi(x)$ and let z' be any element of $\phi^{-1}(x)$. Then $(x, 1, z)$, $(z', 1, x) \in \Gamma_\phi^+$. Assume therefore now that $\phi^n(x) \notin \mathcal{C}_1$ for all $n \geq 1$. Since ϕ is not locally injective there are open non-empty intervals $I_\pm$ and I such that $\mathrm{val}(\phi, z) = (+, +)$ for all $z \in I_+$, $\mathrm{val}(\phi, z) = (-, -)$ for all $z \in I_-$ and $\phi(I_-) = \phi(I_+) = I$. It follows from Theorem 5.21 that ϕ is exact and there is therefore an $N \in \mathbb{N}$ such that $\phi^{N-1}(I) = \mathbb{T}$.

We consider first the case where x is pre-periodic to a finite orbit $\mathcal{O}$ of period p. Let $d \in \{-1, 1\}$. If there is an $M \in \mathbb{N}$ such that $\phi^{-j}(\mathcal{O}) \subseteq \mathcal{C}_j \cup \bigcup_{k=0}^{j-1} \phi^{-k}(\mathcal{O})$ for all $j > M$, it follows that $\mathrm{RO}^+(x) \subseteq \bigcup_{j=0}^{M} \phi^{-j}(\mathcal{O})$ which is a finite set. This is impossible since $C_r^*(\Gamma_\phi^+)$ is simple, cf. Corollary 5.6. Since $\mathcal{O}$ is finite it follows that there is a $z \in \mathcal{O}$ and an $n_0 \in \mathbb{N}$ such that $\phi^{-j}(z) \backslash (\mathcal{C}_j \cup \bigcup_{k=0}^{j-1} \phi^{-k}(\mathcal{O})) \neq \emptyset$ for all $j \geq n_0$. Since $z \in \mathcal{O}$ there are $k, l \in \mathbb{N}$ such that $\phi^l(z) = \phi^{kp+d}(x)$. Note that

$$j \neq j' \Rightarrow \left(\phi^{-j}(z) \Big\backslash \left(\mathcal{C}_j \cup \bigcup_{k=0}^{j-1} \phi^{-k}(\mathcal{O})\right)\right) \cap \left(\phi^{-j'}(z) \Big\backslash \left(\mathcal{C}_{j'} \cup \bigcup_{k=0}^{j'-1} \phi^{-k}(\mathcal{O})\right)\right) = \emptyset.$$

Since $\phi^N(\mathcal{C}_N)$ is a finite set there is therefore a $k' \geq k$ such that $k'p - N - l \geq n_0$ and

$$\phi^{-k'p+N+l}(z) \backslash (\mathcal{C}_{k'p-N-l} \cup \phi^N(\mathcal{C}_N)) \neq \emptyset.$$

Choose an element $a \in \phi^{-k'p+N+l}(z) \backslash (\mathcal{C}_{k'p-N-l} \cup \phi^N(\mathcal{C}_N))$. Then $a = \phi^N(y_\pm)$, where $y_\pm \in I_+ \cup I_-$ and $\mathrm{val}(\phi^N, y_+) = (+, +)$, $\mathrm{val}(\phi^N, y_-) = (-, -)$. It follows that $\phi^{k'p}(y_\pm) = \phi^{kp+d}(x) = \phi^{k'p+d}(x)$ and that either $(x, d, y_+) \in \Gamma_\phi^+$ or $(x, d, y_-) \in \Gamma_\phi^+$. When $d = 1$ this gives us z such that $(x, 1, z) \in \Gamma_\phi^+$ and when $d = -1$ it gives us z' such that $(x, -1, z') \in \Gamma_\phi^+$.

Consider then the case where x is not pre-periodic. If there are infinitely many $n \in \mathbb{N}$, $n > N$, such that the only elements of $\phi^{-N}(\phi^n(x)) \backslash \{\phi^{n-N}(x)\}$ are critical points for ϕ^N, it follows that $\phi^n(x) \in \phi^N(\mathcal{C}_N)$ for infinitely many n. Since $\phi^N(\mathcal{C}_N)$ is a finite set this contradicts that x is not pre-periodic. It follows that when x is not pre-periodic there is an $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$ there is an element $y_n \in \phi^{-N}(\phi^n(x))$ which is not in the forward orbit $\{\phi^j(x) : j \in \mathbb{N}\}$ of x and also not critical for ϕ^N. Note that

$$j \neq j' \Rightarrow \left(\bigcup_{k \in \mathbb{N}} \phi^{-kN}(y_j)\right) \cap \left(\bigcup_{k \in \mathbb{N}} \phi^{-kN}(y_{j'})\right) = \emptyset$$

when $j, j' \geq n_0$. Since $\phi^N(\mathcal{C}_N)$ is a finite set there is therefore an $m > 2$ such that $mN + d > n_0$ and

$$\left(\bigcup_{k \in \mathbb{N}} \phi^{-kN}(y_{mN+d})\right) \cap \phi^N(\mathcal{C}_N) = \emptyset.$$

Let $a \in \phi^{-(m-2)N}(y_{mN+d})$. Choose $y_\pm \in I_\pm$ such that $\phi^N(y_\pm) = a$. Then $\phi^{mN}(y_\pm) = \phi^{mN+d}(x)$ and for one of the elements v in $\{y_+, y_-\}$ it holds that $\mathrm{val}(\phi^{mN}, v) = \mathrm{val}(\phi^{mN+d}, x)$. Then $(x, 1, v) \in \Gamma_\phi^+$ when $d = 1$ and $(x, -1, v) \in \Gamma_\phi^+$ when $d = -1$. □

LEMMA 6.2. *Assume that $C_r^*(\Gamma_\phi^+)$ is simple. Let $x \in \mathcal{C}_1$ and let U be an open non-empty subset of $\mathbb{T}$. There are elements $\mu_1, \mu_2, \ldots, \mu_N \in \Gamma_\phi^+(1)$ such that $\gamma = \mu_1\mu_2\mu_3 \cdots \mu_N$ is defined, $s(\gamma) = x$ and $r(\gamma) \in U \backslash \mathcal{C}_1$.*

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Proof. Since ϕ is exact the backward orbit $\bigcup_{j=1}^{\infty}\phi^{-j}(x)$ is dense in $\mathbb{N}$. So there is an $N\in\mathbb{N}$ and a $z\in\phi^{-N}(x)\cap U\backslash\mathcal{C}_1$. Set $\mu_i=(\phi^{i-1}(z),1,\phi^i(z))$. □

LEMMA 6.3. *Assume that $C_r^*(\Gamma_\phi^+)$ is simple. Then the first spectral subspace for the gauge action, i.e. the set*

$$E_1=\{a\in C_r^*(\Gamma_\phi^+):\beta_\lambda(a)=\lambda a\ \forall\lambda\in\mathbb{T}\},$$

generates $C_r^*(\Gamma_\phi^+)$ as a C-algebra.*

Proof. The gauge action β on $C_r^*(\Gamma_\phi^+)$ restricts to an action on $C_r^*(\Gamma_\phi^+|_{\mathbb{T}\backslash\mathcal{C}_1})$. Let $V_1=E_1\cap C_r^*(\Gamma_\phi^+|_{\mathbb{T}\backslash\mathcal{C}_1})$ be the first spectral subspace of the restricted action. We claim that $V_1V_1^*$ spans a dense subspace in the fixed point algebra $C_r^*(\Gamma_\phi^+|_{\mathbb{T}\backslash\mathcal{C}_1})^\beta$. To show this observe that since the closed span of $V_1V_1^*$ is an ideal in $C_r^*(\Gamma_\phi^+|_{\mathbb{T}\backslash\mathcal{C}_1})^\beta$ it suffices to show that the span of $V_1V_1^*$ contains an approximate unit for $C_r^*(\Gamma_\phi^+|_{\mathbb{T}\backslash\mathcal{C}_1})$. Hence, it suffices to show that $C_c(\mathbb{T}\backslash\mathcal{C}_1)\subseteq\operatorname{Span}V_1V_1^*$. Let $f\in C_c(\mathbb{T}\backslash\mathcal{C}_1)$. Let $x\in\operatorname{supp}f$. It follows from Lemma 6.1 that there is a bisection $U\subseteq\Gamma_\phi^+(1)\cap\Gamma_\phi^+|_{\mathbb{T}\backslash\mathcal{C}_1}$ such that $x\in r(U)$. There is therefore a function $g\in C_c(U)$ such that $gg^*\in C_c(\mathbb{T}\backslash\mathcal{C}_1)$ and $gg^*(x)=1$. In this way we get a finite collection g_i, $i=1,2,\dots,N$, in $C_c(\Gamma_\phi^+(1)\cap\Gamma_\phi^+|_{\mathbb{T}\backslash\mathcal{C}_1})$ such that $g_ig_i^*\in C_c(\mathbb{T}\backslash\mathcal{C}_1)$ for all i and $\sum_{i=1}^N g_ig_i^*(y)>0$ for all $y\in\operatorname{supp}f$. There is then a function $h\in C_c(\mathbb{T}\backslash\mathcal{C}_1)$ such that $fh\sum_{i=1}^N g_ig_i^*=f$. Since fhg_i and g_i are elements of V_1 for all i this shows that $f\in\operatorname{Span}V_1V_1^*$.

A similar argument shows that also $V_1^*V_1$ spans a dense subspace of $C_r^*(\Gamma_\phi^+|_{\mathbb{T}\backslash\mathcal{C}_1})^\beta$. Thus, the restriction of the gauge action is full on $C_r^*(\Gamma_\phi^+|_{\mathbb{T}\backslash\mathcal{C}_1})$ and it follows that $C_r^*(\Gamma_\phi^+|_{\mathbb{T}\backslash\mathcal{C}_1})$ is generated by V_1, whence $C_r^*(\Gamma_\phi^+|_{\mathbb{T}\backslash\mathcal{C}_1})$ is contained in the C^*-subalgebra of $C_r^*(\Gamma_\phi^+)$ generated by E_1.

Note that it follows from Lemma 6.2 that there are elements g_x, $x\in\mathcal{C}_1$, in the $*$-algebra generated by $C_c(\Gamma_\phi^+(1))$ such that $g_{x'}^*g_x=0$ when $x\neq x'$, $g_x^*g_x\in C(\mathbb{T})$, $g_x^*g_x(x)=1$ and g_x is supported in $r^{-1}(\mathbb{T}\backslash\mathcal{C}_1)$ for all x. Consider then an element $f\in C_c(\Gamma_\phi^+)$. Write

$$f=f-\sum_{x\in\mathcal{C}_1}g_x^*g_xf+\sum_{x\in\mathcal{C}_1}g_x^*g_xf=f-\sum_{x\in\mathcal{C}_1}g_x^*g_xf+\sum_{x\in\mathcal{C}_1}g_x^*h,$$

where $h=\sum_{x\in\mathcal{C}_1}g_xf$. Note that $f-\sum_{x\in\mathcal{C}_1}g_x^*g_xf$ and h are both supported in $r^{-1}(\mathbb{T}\backslash\mathcal{C}_1)$. To conclude that f is contained in the C^*-algebra generated by E_1 we may therefore assume that f is supported in $r^{-1}(\mathbb{T}\backslash\mathcal{C}_1)$. Under this assumption we write

$$f=f-\sum_{x\in\mathcal{C}_1}fg_x^*g_x+\sum_{x\in\mathcal{C}_1}fg_x^*g_x,\tag{6.2}$$

and note that $f-\sum_{x\in\mathcal{C}_1}fg_x^*g_x$ and $\sum_{x\in\mathcal{C}_1}fg_x^*$ are elements of $C_c(\Gamma_\phi^+|_{\mathbb{T}\backslash\mathcal{C}_1})$. It follows therefore from the first part of the proof that both are elements of the C^*-algebra generated by E_1. Since

$$\sum_{x\in\mathcal{C}_1}fg_x^*g_x=\sum_{x'\in\mathcal{C}_1}\Bigl(\sum_{x\in\mathcal{C}_1}fg_x^*\Bigr)g_{x'}$$

it follows from (6.2) that f is in the C^*-algebra generated by E_1. □

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LEMMA 6.4. *Assume that $C_r^*(\Gamma_\phi^+)$ is simple. It follows that $E_1^*E_1$ has dense span in $C_r^*(R_\phi^+)$.*

Proof. Since the closed span of $E_1^*E_1$ is an ideal in $C_r^*(R_\phi^+)$ it suffices to show that $E_1^*E_1$ contains $1 \in C(\mathbb{T})$. Let $f \in C(\mathbb{T})$. Let $x \in \mathbb{T}$. Provided that $x \notin \mathcal{C}_1$ it follows from Lemma 6.1 that there is an element $\gamma \in \Gamma_\phi^+(1)$ such that $s(\gamma) = x$. When $x \in \mathcal{C}_1$ set $\gamma = (z, 1, x)$, where $z \in \phi^{-1}(x)$. Then $s(\gamma) = x$. Hence, regardless of which x we consider there is a bisection $U \subseteq \Gamma_\phi^+(1)$ such that $x \in s(U)$. It follows that there is a function $g \in C_c(U)$ such that $g^*g \in C(\mathbb{T})$ and $g^*g(x) = 1$. In this way we get a finite collection g_i, $i = 1, 2, \ldots, N$, in $C_c(\Gamma_\phi^+(1))$ such that $g_i^*g_i \in C(\mathbb{T})$ for all i and $\sum_{i=1}^N g_i^*g_i > 0$. There is then a function $h \in C(\mathbb{T})$ such that $h\sum_{i=1}^N g_i^*g_i = 1$. Since g_ih^* and g_i are elements of E_1 for all i, this completes the proof. □

It should be observed that the gauge action is generally not full. Even when $C_r^*(\Gamma_\phi^+)$ is simple it can easily happen that the ideal in $C_r^*(R_\phi^+)$ generated by $E_1E_1^*$ is a proper ideal. Assume, for example, that ϕ has two critical points whose forward orbits do not intersect, and that one of them, say c, has infinite forward orbit. Then $r^{-1}(c) \cap \Gamma_\phi^+(1) = \emptyset$. Let ω be the state on $C_r^*(R_\phi^+)$ such that $\omega(f) = f(c, 0, c)$ when $f \in C_c(R_\phi^+)$. Then $\omega(fg^*) = 0$ for all $f, g \in C_c(\Gamma_\phi^+(1))$ and it follows that ω annihilates the ideal in $C_r^*(R_\phi^+)$ generated by $E_1E_1^*$. This asymmetry in the gauge action is responsible for some intriguing features of the KMS states.

Let $E_{-1} = E_1^*$ be the first negative spectral subspace. Then E_{-1} is a $C_r^*(R_\phi^+)$-correspondence and we can consider the associated Cuntz–Pimsner C^*-algebra $\mathcal{O}_{E_{-1}}$ (cf. [**Ka**]). We can now show that this is another version of $C_r^*(\Gamma_\phi^+)$ when the latter algebra is simple.

THEOREM 6.5. *Assume that $C_r^*(\Gamma_\phi^+)$ is simple. It follows that $C_r^*(\Gamma_\phi^+) \simeq \mathcal{O}_{E_{-1}}$ and there is an exact sequence,*

$$\begin{array}{ccccc} K_0(C_r^*(R_\phi^+)) & \xrightarrow{\mathrm{id}-[E_{-1}]_0} & K_0(C_r^*(R_\phi^+)) & \xrightarrow{\iota_0} & K_0(C_r^*(\Gamma_\phi^+)) \\ \uparrow & & & & \downarrow \\ K_1(C_r^*(\Gamma_\phi^+)) & \xleftarrow[\iota_1]{} & K_1(C_r^*(R_\phi^+)) & \xleftarrow[\mathrm{id}-[E_{-1}]_1]{} & K_1(C_r^*(R_\phi^+)) \end{array} \tag{6.3}$$

where $\iota : C_r^*(R_\phi^+) \to C_r^*(\Gamma_\phi^+)$ is the inclusion map and

$$[E_{-1}] \in KK(C_r^*(R_\phi^+), C_r^*(R_\phi^+))$$

is the KK-theory element represented by E_{-1}.

Proof. The inclusions $C_r^*(R_\phi^+) \subseteq C_r^*(\Gamma_\phi^+)$ and $E_{-1} \subseteq C_r^*(\Gamma_\phi^+)$ define a representation of the C^*-correspondence E_{-1} in $C_r^*(\Gamma_\phi^+)$ as defined by Katsura in [**Ka**, Definition 2.1]. Observe that it follows from Lemma 6.4 that the representation is injective in the sense of Katsura, and that

$$\psi_t(C_r^*(R_\phi^+)) = \mathcal{K}(E_{-1}), \tag{6.4}$$

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in the notation from [**Ka**]. It follows now from [**Ka**, Proposition 3.3] that our representation of E_{-1} is covariant in the sense of [**Ka**, Definition 3.4]. Then the isomorphism $\mathcal{O}_{E_{-1}} \simeq C_r^*(\Gamma_\phi^+)$ follows from Lemma 6.3 above and [**Ka**, Theorem 6.4]. Thanks to (6.4) we get now the stated six-terms exact sequence from [**Ka**, Theorem 8.6]. □

6.2. *The structure of $C_r^*(R_\phi^+(k))$ and some consequences.* Fix a natural number k and let $\mathcal{D}$ be a finite subset of $\mathbb{T}$ such that $\mathcal{C}_k \subseteq \phi^{-k}(\mathcal{D})$. Let $c_1 < c_2 < \cdots < c_N$ be a numbering of the elements in $\mathcal{D}$ and set $c_0 = c_N$. Let $I_i =]c_{i-1}, c_i[$, $i = 1, 2, \ldots, N$. For each i we fix a homeomorphism $\psi_{I_i} :]0, 1[\to]c_{i-1}, c_i[$ such that $\lim_{t\to 0} \psi_{I_i}(t) = c_{i-1}$ and $\lim_{t\to 1} \psi_{I_i}(t) = c_i$. Let $\mathcal{I}_k$ be the set of connected components of $\mathbb{T}\backslash\phi^{-k}(\mathcal{D})$. Since $\mathcal{C}_k \subseteq \phi^{-k}(\mathcal{D})$, the map $x \mapsto \mathrm{val}(\phi^k, x)$ is constant on each $I \in \mathcal{I}_k$ and we set $\mathrm{val}(\phi^k, I) = \mathrm{val}(\phi^k, x)$, $x \in I$. Set

$$\mathcal{I}_k^{(2)} = \{(I, J) \in \mathcal{I}_k \times \mathcal{I}_k : \phi^k(I) = \phi^k(J), \mathrm{val}(\phi^k, I) = \mathrm{val}(\phi^k, J)\}.$$

Let $\mathbb{B}_k$ denote the finite-dimensional C^*-algebra generated by the matrix units $e_{I,J}$, where $(I, J) \in \mathcal{I}_k^{(2)}$. Similarly, we let $\mathbb{A}_k$ be the finite-dimensional C^*-algebra generated by the matrix units $e_{x,y}$, where $x, y \in \phi^{-k}(\mathcal{D})$, $\phi^k(x) = \phi^k(y)$ and $\mathrm{val}(\phi^k, x) = \mathrm{val}(\phi^k, y)$.

When $x \in \phi^{-k}(\mathcal{D})$ and $I \in \mathcal{I}_k$, write $I > x$ when $x \in \overline{I}$ and $y > x$ for all $y \in I$, and $I < x$ when $x \in \overline{I}$ and $y < x$ for all $y \in I$. Define a $*$-homomorphism $I_k : \mathbb{A}_k \to \mathbb{B}_k$ such that

$$I_k(e_{x,y}) = \sum_{I,J} e_{I,J},$$

where we sum over the set of pairs $(I, J) \in \mathcal{I}_k^{(2)}$ with the properties that $x < I$, $y < J$ and $\mathrm{val}(\phi^k, I) = (+, +)$, or $I < x$, $J < y$ and $\mathrm{val}(\phi^k, I) = (-, -)$.

Similarly, we define a $*$-homomorphism $U_k : \mathbb{A}_k \to \mathbb{B}_k$ such that

$$U_k(e_{x,y}) = \sum_{I,J} e_{I,J},$$

where we sum over the set of pairs $(I, J) \in \mathcal{I}_k^{(2)}$ with the properties that $I < x$, $J < y$ and $\mathrm{val}(\phi^k, I) = (+, +)$, or $x < I$, $y < J$ and $\mathrm{val}(\phi^k, I) = (-, -)$. Let $(I, J) \in \mathcal{I}_k^{(2)}$ such that $\phi^k(I) = \phi^k(J) = I_i$. Let $\lambda_I : I_i \to I$ be the inverse of $\phi^k : I \to I_i$, and similarly $\lambda_J : I_i \to J$ the inverse of $\phi^k : J \to I_i$. Then

$$(\lambda_I \circ \psi_{I_i}(t), 0, \lambda_J \circ \psi_{I_i}(t)) \in R_\phi^+(k)$$

for all $t \in]0, 1[$. Note that the limits

$$\overline{\lambda_I}(c_i) = \lim_{x\to c_i} \lambda_I(x) \quad \text{and} \quad \overline{\lambda_I}(c_{i-1}) = \lim_{x\to c_{i-1}} \lambda_I(x)$$

both exist. Let $f \in C_c(R_\phi^+(k))$. Then the function

$$]0, 1[\ni t \mapsto f(\lambda_I \circ \psi_{I_i}(t), 0, \lambda_J \circ \psi_{I_i}(t))$$

has a unique continuous extension $f_{I,J} : [0, 1] \to \mathbb{C}$. This is because

$$\lim_{t\to 1} f(\lambda_I \circ \psi_{I_i}(t), 0, \lambda_J \circ \psi_{I_i}(t)) = 0$$

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when $(\overline{\lambda_I}(c_i), 0, \overline{\lambda_J}(c_i)) \notin R_\phi^+(k)$ and

$$\lim_{t\to 1} f(\lambda_I \circ \psi_{I_i}(t), 0, \lambda_J \circ \psi_{I_i}(t)) = f(\overline{\lambda_I}(c_i), 0, \overline{\lambda_J}(c_i))$$

when $(\overline{\lambda_I}(c_i), 0, \overline{\lambda_J}(c_i)) \in R_\phi^+(k)$. Similarly,

$$\lim_{t\to 0} f(\lambda_I \circ \psi_{I_i}(t), 0, \lambda_J \circ \psi_{I_i}(t)) = 0$$

when $(\overline{\lambda_I}(c_{i-1}), 0, \overline{\lambda_J}(c_{i-1})) \notin R_\phi^+(k)$, and

$$\lim_{t\to 0} f(\lambda_I \circ \psi_{I_i}(t), 0, \lambda_J \circ \psi_{I_i}(t)) = f(\overline{\lambda_I}(c_{i-1}), 0, \overline{\lambda_J}(c_{i-1}))$$

when $(\overline{\lambda_I}(c_{i-1}), 0, \overline{\lambda_J}(c_{i-1})) \in R_\phi^+(k)$.

We can then define a $*$-homomorphism $b : C_c(R_\phi^+(k)) \to C([0, 1], \mathbb{B}_k)$ such that

$$b(f) = \sum_{I,J} f_{I,J} e_{I,J}.$$

We can also define a $*$-homomorphism $a : C_c(R_\phi^+(k)) \to \mathbb{A}_k$ such that

$$a(f) = \sum_{(x,y)\in\mathcal{A}_k} f(x, 0, y) e_{x,y},$$

where $\mathcal{A}_k = \{(x, y) \in \mathbb{T}^2 : (x, 0, y) \in R_\phi^+(k)\}$. By construction $I_k(a(f)) = b(f)(0)$ and $U_k(a(f)) = b(f)(1)$.

For $x \in \mathbb{T}$, let π_x be the $*$-representation used to define the norm on $C_r^*(R_\phi^+(k))$, cf. (2.1). When $x \notin \phi^{-k}(\mathcal{D})$ there is an i and a t such that $\phi^k(x) = \psi_{I_i}(t)$. Then

$$\|\pi_x(f)\| = \left\| \sum_{(I,J)\in B} f_{I,J}(t) e_{I,J} \right\|, \tag{6.5}$$

where $B = \{(I, J) \in \mathcal{I}_k^{(2)} : \phi^k(I) = \phi^k(J) = I_i\}$. When $x \in \phi^{-k}(\mathcal{D})$, we find that

$$\|\pi_x(f)\| = \left\| \sum_{(z,y)\in A} f(z, 0, y) e_{z,y} \right\|, \tag{6.6}$$

where $A = \{(z, y) \in \mathbb{T}^2 : (z, 0, y) \in R_\phi^+(k), \phi^k(z) = \phi^k(x), \mathrm{val}(\phi^k, z) = \mathrm{val}(\phi^k, x)\}$. By combining (6.5) and (6.6) we find that $f \to (a(f), b(f))$ is isometric and extends to an injective $*$-homomorphism

$$\mu_k : C_r^*(R_\phi^+(k)) \to \{(a, b) \in \mathbb{A}_k \oplus C([0, 1], \mathbb{B}_k) : I_k(a) = b(0), U_k(a) = b(1)\}.$$

LEMMA 6.6. *We have that μ_k is an isomorphism.*

Proof. It remains to show that μ_k is surjective. Let $(a, b) \in \mathbb{A}_k \oplus C([0, 1], \mathbb{B}_k)$ have the properties that $I_k(a) = b(0)$, $U_k(a) = b(1)$. Then $a = \sum_{(x,y)\in A} \lambda_{x,y} e_{x,y}$, where

$$A = \{(x, y) \in \phi^{-k}(\mathcal{D}) \times \phi^{-k}(\mathcal{D}) : (x, 0, y) \in R_\phi^+(k)\}$$

and $\lambda_{x,y} \in \mathbb{C}$. Since A is a finite set there is a function $g \in C_c(R_\phi^+(k))$ such that $g(x, 0, y) = \lambda_{x,y}$ for all $(x, y) \in A$. Then $(a, b) = \mu_k(g) + (0, b')$, where $b' \in C_0(]0, 1[) \otimes \mathbb{B}_k$. Write $b' = \sum_{(I,J)\in\mathcal{I}_k^2} b'_{I,J} e_{I,J}$, where $b'_{I,J} \in C_0(]0, 1[)$. For each (I, J)

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in the last sum there is an interval I_i such that $\phi^k(I) = \phi^k(J) = I_i$. Then

$$\mathcal{U} = \{(\lambda_I \circ \psi_{I_i}(t),\, 0,\, \lambda_J \circ \psi_{I_i}(t)) : t \in]0,\, 1[\}$$

is an open subset of $R_\phi^+(k)$ and we can define a function $h_{I,J}$ on $\mathcal{U}$ such that

$$h_{I,J}(\lambda_I \circ \psi_{I_i}(t),\, 0,\, \lambda_J \circ \psi_{I_i}(t)) = b'_{I,J}(t).$$

Then $h_{I,J} \in C_0(\mathcal{U})$ and we can choose a sequence $\{h_n\} \subseteq C_c(\mathcal{U})$ such that $\lim_{n\to\infty} h_n = h_{I,J}$, uniformly on $R_\phi^+(k)$. Since $\mathcal{U}$ is a bisection in R_ϕ^+ it follows that $\lim_{n\to\infty} \mu_k(h_n) = (0,\, b'_{I,J} e_{I,J})$ (cf. [**Th1**, Lemma 2.4]), proving that $b'_{I,J} e_{I,J}$ is in the range of μ_k for all $(I,\, J)$. It follows that $(a,\, b)$ is in the range of μ_k. □

It follows from Lemma 6.6 that $C_r^*(R_\phi^+(k))$ is a recursive sub-homogeneous C^*-algebra in the sense of N.C. Phillips. In combination with (6.1) it follows that $C_r^*(R_\phi^+)$ is an approximately sub-homogeneous (ASH) algebra with no dimension growth, cf. [**T**].

COROLLARY 6.7. *The C^*-algebra $C_r^*(\Gamma_\phi^+)$ is nuclear and satisfies the UCT of Rosenberg and Schochet* [**RS**].

Proof. Since $C_r^*(R_\phi^+)$ is the inductive limit of a sequence of sub-homogeneous C^*-algebras, it is nuclear and satisfies the UCT. The corollary follows therefore from Theorem 6.5 and [**Ka**] when $C_r^*(\Gamma_\phi^+)$ is simple. To obtain the same conclusion in general we must use a different and even more indirect path. First observe that the nuclearity of $C_r^*(R_\phi^+)$ implies that R_ϕ^+ is (topologically) amenable by [**A-DR**, Corollary 6.2.14 (ii) and Theorem 3.3.7], and then from [**A-DR**, Proposition 6.1.8] that $C_r^*(R_\phi^+)$ equals the full groupoid C^*-algebra of R_ϕ^+. Since $C_r^*(R_\phi^+)$ is the fixed-point algebra of the gauge action it follows then from a recent result of Spielberg [**Sp**, Proposition 9.3] that Γ_ϕ^+ is also (topologically) amenable. Then [**A-DR**, Corollary 6.2.14 (i)] shows that $C_r^*(\Gamma_\phi^+)$ is nuclear, and the work of Tu [**Tu**] (more precisely, [**Tu**, Lemme 3.5 and Proposition 10.7]) implies that it satisfies the UCT. □

We do not consider the full groupoid C^*-algebra here [**Re**], but observe in passing that the previous proof established the topological amenability of Γ_ϕ^+; a fact which implies that the full and reduced C^*-algebras of Γ_ϕ^+ are canonically isomorphic by [**A-DR**, Proposition 6.1.8].

7. *Markov maps*

We say that ϕ is *Markov* when $\phi(\mathcal{C}_1) \subseteq \mathcal{C}_1$.

LEMMA 7.1. *Assume that ϕ is Markov. Then $C_r^*(\Gamma_\phi^+)$ is simple if and only if ϕ is transitive.*

Proof. The Markov condition implies that all post-critical points are critical and it follows therefore from Lemma 5.5 that a transitive Markov map has no exposed points and therefore also that $C_r^*(\Gamma_\phi^+)$ is simple when ϕ is transitive and Markov. □

LEMMA 7.2. *Assume that ϕ is Markov and transitive. There is a natural number $k \in \mathbb{N}$ with the property that when $j \geq k$ and $x \in \phi(\mathbb{T}\backslash\mathcal{C}_1)$, there are elements $y_\pm \in \mathbb{T}$ such that $\phi^j(y_\pm) = x$, $\mathrm{val}(\phi^j,\, y_+) = (+,\, +)$ and $\mathrm{val}(\phi^j,\, y_-) = (-,\, -)$.*

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Proof. Since ϕ is not locally injective there are non-empty open intervals $I_\pm$ such that $\mathrm{val}(\phi, x) = (+, +)$ for all $x \in I_+$, $\mathrm{val}(\phi, y) = (-, -)$ for all $y \in I_-$ and $I = \phi(I_+) = \phi(I_-)$ is an open non-empty interval. Since ϕ is exact (by Lemma 7.1 and Theorem 5.21) there is an $N \in \mathbb{N}$ such that $\phi^N(I) = \mathbb{T}$. Set $k = N + 2$ and let $j \geq k$. Consider an element $x \in \phi(\mathbb{T}\backslash\mathcal{C}_1)$. Then $x = \phi(u)$ for some $u \in \mathbb{T}\backslash\mathcal{C}_1$. There is an element $z \in I$ such that $\phi^{j-2}(z) = u$ and elements $z_\pm \in I_\pm$ such that $z_\pm \in I_\pm$ and $\phi(z_\pm) = z$. Then $\phi^j(z_\pm) = x$ and the Markov condition implies that $z_\pm$ are not critical for ϕ^{j-1} since $u \notin \mathcal{C}_1$. Note that $\mathrm{val}(\phi^j, z_\pm) = \mathrm{val}(\phi^{j-1}, z) \bullet \mathrm{val}(\phi, z_\pm)$. If $\mathrm{val}(\phi^{j-1}, z) = (+, +)$, set $y_+ = z_+$ and $y_- = z_-$, and if $\mathrm{val}(\phi^{j-1}, z) = (-, -)$, set $y_+ = z_-$ and $y_- = z_+$. □

In the following we say that a Markov map ϕ is of *order* k when the conclusion of Lemma 7.2 holds, i.e. when:

(a) for all $j \geq k$ and all $x \in \phi(\mathbb{T}\backslash\mathcal{C}_1)$ there are elements $y_\pm \in \mathbb{T}$ such that $\phi^j(y_\pm) = x$, $\mathrm{val}(\phi^j, y_+) = (+, +)$ and $\mathrm{val}(\phi^j, y_-) = (-, -)$.

In addition to our standing assumptions (that $\phi : \mathbb{T} \to \mathbb{T}$ is continuous, piecewise monotone and not locally injective), we now also assume that ϕ is a Markov map of order k.

LEMMA 7.3. *Assume that* $(x, l, y) \in \Gamma_\phi^+(l, n)$, *where* $l \in \mathbb{Z}$, $n \in \mathbb{N}$ *and* $l + n \geq k + 1$. *It follows that there is an element* $z \in \mathbb{T}$ *such that* $(x, 1, z) \in \Gamma_\phi^+(1, k)$ *and* $(z, l - 1, y) \in \Gamma_\phi^+(l - 1, n)$.

Proof. If $\phi^k(x) \in \mathcal{C}_1$, set $z = \phi(x)$. It follows from Lemma 3.1 and Table 1 for $\bullet$ that

$$\begin{aligned}\mathrm{val}(\phi^{k+1}, x) &= \mathrm{val}(\phi, \phi^k(x)) \bullet \mathrm{val}(\phi^k, x)\\ &= \mathrm{val}(\phi, \phi^k(x)) \bullet \mathrm{val}(\phi^{k-1}, \phi(x)) = \mathrm{val}(\phi^k, \phi(x)),\end{aligned}$$

showing that $(x, 1, z) \in \Gamma_\phi^+(1, k)$. Now note that it follows from Lemma 3.1 and Table 1 for $\bullet$ first that $\phi^k(x)$ is critical for ϕ^{n+l-k} and then, as above, that

$$\begin{aligned}\mathrm{val}(\phi^n, y) &= \mathrm{val}(\phi^{n+l}, x) = \mathrm{val}(\phi^{n+l-k}, \phi^k(x)) \bullet \mathrm{val}(\phi^k, x)\\ &= \mathrm{val}(\phi^{n+l-k}, \phi^k(x)) \bullet \mathrm{val}(\phi^{k-1}, z) = \mathrm{val}(\phi^{n+l-1}, z).\end{aligned}$$

This shows that $(z, l - 1, y) \in \Gamma_\phi^+(l - 1, n)$.

If instead $\phi^k(x) \notin \mathcal{C}_1$ it follows from condition (a) that there is a $z \in \mathbb{T}$ such that $\phi^{k+1}(x) = \phi^k(z)$ and $\mathrm{val}(\phi^k, z) = \mathrm{val}(\phi^{k+1}, x)$. Then $(x, 1, z) \in \Gamma_\phi^+(1, k)$ and $(z, l - 1, y) \in \Gamma_\phi^+(l - 1, n)$ since

$$\begin{aligned}\mathrm{val}(\phi^{l-1+n}, z) &= \mathrm{val}(\phi^{l+n-k-1}, \phi^k(z)) \bullet \mathrm{val}(\phi^k, z)\\ &= \mathrm{val}(\phi^{l+n-k-1}, \phi^{k+1}(x)) \bullet \mathrm{val}(\phi^{k+1}, x) = \mathrm{val}(\phi^{l+n}, x) = \mathrm{val}(\phi^n, y)\end{aligned}$$

when $l + n \geq k + 2$, while

$$\mathrm{val}(\phi^{l-1+n}, z) = \mathrm{val}(\phi^k, z) = \mathrm{val}(\phi^{k+1}, x) = \mathrm{val}(\phi^{l+n}, x) = \mathrm{val}(\phi^n, y)$$

when $l + n = k + 1$. □

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LEMMA 7.4. *Assume that* $n \geq k+1$. *Let* $(x, 0, y) \in \Gamma_\phi^+(0, n)$. *There are elements* $z_1, z_2 \in \mathbb{T}$ *such that* $(z_1, 0, z_2) \in \Gamma_\phi^+(0, n-1)$, $(x, 1, z_1)$, $(y, 1, z_2) \in \Gamma_\phi^+(1, k)$ *and*

$$(x, 0, y) = (x, 1, z_1)(z_1, 0, z_2)(z_2, -1, y)$$

in Γ_ϕ^+.

Proof. We obtain from Lemma 7.3 an element $z_1 \in \mathbb{T}$ such that $(x, 1, z_1) \in \Gamma_\phi^+(1, k)$, $(z_1, -1, y) \in \Gamma_\phi^+(-1, n)$ and $(x, 0, y) = (x, 1, z_1)(z_1, -1, y)$. Then $(y, 1, z_1) \in \Gamma_\phi^+(1, n-1)$ and a second application of Lemma 7.3 gives a $z_2 \in \mathbb{T}$ such that $(y, 1, z_2) \in \Gamma_\phi^+(1, k)$, $(z_2, 0, z_1) \in \Gamma_\phi^+(0, n-1)$ and $(y, 1, z_1) = (y, 1, z_2)(z_2, 0, z_1)$. □

Let E be the closure of $C_c(\Gamma_\phi^+(1, k))$ in $C_r^*(\Gamma_\phi^+)$. Then

$$E^*E \subseteq C_r^*(R_\phi^+(k)), \quad C_r^*(R_\phi^+(k+1))E \subseteq E, \quad EC_r^*(R_\phi^+(k)) \subseteq E.$$

We can therefore, in a natural way, consider E as a C^*-correspondence on $C_r^*(R_\phi^+(k))$ in the sense of Katsura [**Ka**]. Let $\mathcal{O}_E$ denote the corresponding C^*-algebra (cf. [**Ka**, Definition 3.5]). We aim now to show that $\mathcal{O}_E$ is isomorphic to $C_r^*(\Gamma_\phi^+)$.

LEMMA 7.5. *We have the following:*

(1) $\overline{\operatorname{Span} EE^*} = C_r^*(R_\phi^+(k+1))$;

(2) $C_r^*(R_\phi^+(n)) \subseteq EC_r^*(R_\phi^+(n-1))E^*$ *when* $n \geq k+1$;

(3) $C_r^*(\Gamma_\phi^+)$ *is generated by* E.

Proof. (1) It is clear that $EE^* \subseteq C_r^*(R_\phi^+(k+1))$. Let $h \in C_c(R_\phi^+(k+1))$. To show that h is contained in $\overline{\operatorname{Span} EE^*}$ we may assume that h is supported in a bisection U. Set $K = r(\operatorname{supp} h) \subseteq \mathbb{T}$. Let $x \in K$. There is then a unique $y \in \mathbb{T}$ such that $(x, 0, y) \in U$. It follows from Lemma 7.4 that there is a $z \in \mathbb{T}$ such that $(x, 1, z)$, $(y, 1, z) \in \Gamma_\phi^+(1, k)$. Choose open bisections $W_1 \subseteq \Gamma_\phi^+(1, k)$, $W_2 \subseteq \Gamma_\phi^+(-1, k+1)$ containing $(x, 1, z)$ and $(z, -1, y)$, respectively, such that $W_1W_2 \subseteq U$. We can then define functions $f \in C_c(W_1)$, $g \in C_c(W_2)$ such that $f(\gamma) = h(\gamma')$ in a neighborhood of $(x, 1, z)$, where $\gamma' \in U$ is determined by the condition that $r(\gamma) = r(\gamma')$, and $g(\gamma) = 1$ in a neighborhood of $(z, -1, y)$. There is then an open neighborhood V_x of x such that $fg(\gamma) = h(\gamma)$ for all $\gamma \in r^{-1}(V_x)$. By compactness of K we get in this way a finite collection of functions $\psi_i \in C(\mathbb{T})$, $f_i, g_i \in C_c(\Gamma_\phi^+(1, k))$ such that $h = \sum_i \psi_i f_i g_i$. Since $\psi_i f_i, g_i^* \in E$ this shows that h is in the span of EE^*.

(2) This follows from Lemma 7.4 in a similar way.

(3) By arguments similar to those used for (1) it follows from Lemma 7.3 that $C_c(\Gamma_\phi^+(l)) \subseteq \operatorname{Span} EC_c(\Gamma_\phi^+(l-1))$ for all $l \in \mathbb{Z}$. Since $C_c(\Gamma_\phi^+(l))^* = C_c(\Gamma_\phi^+(-l))$ and since $\bigcup_{l\in\mathbb{Z}} C_c(\Gamma_\phi^+(l))$ is dense in $C_r^*(\Gamma_\phi^+)$ it suffices then to show that $C_r^*(R_\phi^+)$ is contained in the C^*-algebra generated by E. This follows from (1), (2) and (6.1). □

Let $\mathbb{L}(E)$ denote the C^*-algebra of adjointable operators on the Hilbert $C_r^*(R_\phi^+(k))$-module E and $\mathbb{K}(E)$ the ideal in $\mathbb{L}(E)$ consisting of the 'compact' operators.

LEMMA 7.6. *Define* $\pi : C_r^*(R_\phi^+(k+1)) \to \mathbb{L}(E)$ *such that* $\pi(a)b = ab$. *Then* π *is injective and* $\pi(C_r^*(R_\phi^+(k+1))) = \mathbb{K}(E)$.

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Proof. This follows from (1) in Lemma 7.5. □

THEOREM 7.7. *Assume that ϕ is Markov of order k. Then*

$$C_r^*(\Gamma_\phi^+) \simeq \mathcal{O}_E, \tag{7.1}$$

and there is a six-terms exact sequence

$$\begin{array}{ccccc} K_0(C_r^*(R_\phi^+(k))) & \xrightarrow{\mathrm{id}-[E]_0} & K_0(C_r^*(R_\phi^+(k))) & \xrightarrow{\iota_0} & K_0(C_r^*(\Gamma_\phi^+)) \\ \uparrow & & & & \downarrow \\ K_1(C_r^*(\Gamma_\phi^+)) & \xleftarrow{\iota_1} & K_1(C_r^*(R_\phi^+(k))) & \xleftarrow{\mathrm{id}-[E]_1} & K_1(C_r^*(R_\phi^+(k))) \end{array}, \tag{7.2}$$

where $\iota : C_r^*(R_\phi^+(k)) \to C_r^*(\Gamma_\phi^+)$ is the inclusion map and $[E]$ is the KK-theory element defined from $\pi : C_r^*(R_\phi^+(k)) \to \mathbb{K}(E)$.

Proof. Let $t : E \to C_r^*(\Gamma_\phi^+)$ and $\iota : C_r^*(R_\phi^+(k)) \to C_r^*(\Gamma_\phi^+)$ be the embeddings. It follows then from Lemma 7.6 above and [**Ka**, Proposition 3.3] that (ι, t) is covariant in the sense of [**Ka**, Definition 3.4]. The isomorphism (7.1) follows then from (3) in Lemma 7.5 and [**Ka**, Theorem 6.4]. Then the six-terms exact sequence (7.2) follows from Lemma 7.6 above and [**Ka**, Theorem 8.6]. □

7.1. *The linking groupoid.*

LEMMA 7.8. *Let $(x, 0, y) \in R_\phi^+(k)$. There is an element $z \in \mathbb{T}$ such that*

$$(z, 1, x), (z, 1, y) \in \Gamma_\phi^+(1, k)$$

and $(x, 0, y) = (x, -1, z)(z, 1, y)$ in Γ_ϕ^+.

Proof. If $\phi^{k-1}(x) \in \mathcal{C}_1$, choose any $z \in \phi^{-k}(\phi^{k-1}(x))$ and note that $\mathrm{val}(\phi^{k+1}, z) = \mathrm{val}(\phi, \phi^{k-1}(x)) = \mathrm{val}(\phi^k, x)$. It follows that $(z, 1, x), (z, 1, y) \in \Gamma_\phi^+(1, k)$.

If $\phi^{k-1}(x) \notin \mathcal{C}_1$ it follows from the Markov condition and the surjectivity of ϕ that $\phi^{k-1}(x) \in \phi(\mathbb{T}\backslash\mathcal{C}_1)$. Furthermore, the Markov condition also ensures that $\mathrm{val}(\phi^{k-1}, x) \in (\pm, \pm)$. Since ϕ is Markov of order k there is therefore an element $z \in \mathbb{T}$ such that $\mathrm{val}(\phi^k, z) = \mathrm{val}(\phi^{k-1}, x)$ and $\phi^k(z) = \phi^{k-1}(x)$. It follows that $(z, 1, x), (z, 1, y) \in \Gamma_\phi^+(1, k)$. □

It follows from Lemmas 7.3 and 7.8 that $\Gamma_\phi^+(1, k)$ is a $R_\phi^+(k+1) - R_\phi^+(k)$-equivalence in the sense of [**SW**] and, hence, from [**SW**, Theorem 13] that $C_r^*(R_\phi^+(k))$ is Morita equivalent to $C_r^*(R_\phi^+(k+1))$. We need a detailed description of the corresponding linking algebra. To this end we introduce the linking groupoid $\mathcal{L}$ as follows, cf. [**SW**, Lemma 3]. As a topological space $\mathcal{L}$ is the disjoint union

$$\mathcal{L} = R_\phi^+(k+1) \sqcup R_\phi^+(k) \sqcup \Gamma_\phi^+(1, k) \sqcup \Gamma_\phi^+(-1, k+1).$$

To give $\mathcal{L}$ a groupoid structure, define $r, s : \mathcal{L} \to \mathbb{T} \sqcup \mathbb{T} \subseteq R_\phi^+(k+1) \sqcup R_\phi^+(k) \subseteq \mathcal{L}$ to be the maps coming from the range and source maps on Γ_ϕ^+, but such that r

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takes $R_\phi^+(k+1) \sqcup \Gamma_\phi^+(1,k)$ to the first copy of $\mathbb{T}$, and $R_\phi^+(k) \sqcup \Gamma_\phi^+(-1,k+1)$ to the second while s takes $R_\phi^+(k+1) \sqcup \Gamma_\phi^+(-1,k+1)$ to the first copy of $\mathbb{T}$, and $R_\phi^+(k) \sqcup \Gamma_\phi^+(1,k)$ to the second. We then set $\mathcal{L}^{(2)} = \{(\gamma,\gamma') \in \mathcal{L} \times \mathcal{L} : \ s(\gamma) = r(\gamma')\}$ and define the product $\gamma\gamma'$ for $(\gamma,\gamma') \in \mathcal{L}^{(2)}$ to be the same as the product in Γ_ϕ^+. It is then straightforward to verify that $\mathcal{L}$ is a second countable étale locally compact Hausdorff groupoid.

The C^*-algebra of the reduction of $\mathcal{L}$ to the first copy of $\mathbb{T}$ is in a natural way identified with $C_r^*(R_\phi^+(k+1))$ and in the same way the C^*-algebra of the reduction of $\mathcal{L}$ to the second copy of $\mathbb{T}$ is identified with $C_r^*(R_\phi^+(k))$. These identifications give us embeddings $a : C_r^*(R_\phi^+(k+1)) \to C_r^*(\mathcal{L})$ and $b : C_r^*(R_\phi^+(k)) \to C_r^*(\mathcal{L})$ onto full corners of $C_r^*(\mathcal{L})$. To see how a and b are related to $[E]$, let $f \in C_c(\mathcal{L})$ and write $f = f_{11} + f_{12} + f_{21} + f_{22}$, where $f_{11} \in C_c(R_\phi^+(k+1))$, $f_{12} \in C_c(\Gamma_\phi^+(1,k))$, $f_{21} \in C_c(\Gamma_\phi^+(-1,k+1))$ and $f_{22} \in C_c(R_\phi^+(k))$. We can then define $\Psi(f) \in \mathbb{L}(E \oplus R_\phi^+(k))$ such that

$$\Psi(f)(e,g) = (f_{11}e + f_{12}g,\ f_{21}e + f_{22}g).$$

It follows from [**SW**, Theorem 13] and [**RW**, Corollary 3.21] that Ψ extends to a $*$-isomorphism

$$C_r^*(\mathcal{L}) \to \mathbb{K}(E \oplus C_r^*(R_\phi^+(k))).$$

Let $\mathbb{K}(E) \to \mathbb{K}(E \oplus C_r^*(R_\phi^+(k)))$ and $C_r^*(R_\phi^+(k)) \to \mathbb{K}(E \oplus C_r^*(R_\phi^+(k)))$ be the canonical embeddings. Then

$$\begin{array}{ccccc} C_r^*(R_\phi^+(k+1)) & \xrightarrow{\ a\ } & C_r^*(\mathcal{L}) & \xleftarrow{\ b\ } & C_r^*(R_\phi^+(k)) \\ \downarrow{\scriptstyle \pi} & & \downarrow{\scriptstyle \Psi} & & \| \\ \mathbb{K}(E) & \longrightarrow & \mathbb{K}(E \oplus C_r^*(R_\phi^+(k))) & \longleftarrow & C_r^*(R_\phi^+(k)) \end{array}$$

commutes, when π is the isomorphism from Lemma 7.6. By relating this diagram to the description of $[E]_*$ given by [**Ka**, Definition 8.3 and Appendix A] we find that

$$[E]_* = b_*^{-1} \circ a_* \circ \rho_*. \tag{7.3}$$

To study $[E]_*$ further we need information on the structure of $C_r^*(\mathcal{L})$ and, hence, the following observation.

LEMMA 7.9. *We have $\phi^j(\mathcal{C}_j) = \phi^{j+1}(\mathcal{C}_{j+1})$ for all $j \geq 1$.*

Proof. The inclusion $\phi^j(\mathcal{C}_j) \subseteq \phi^{j+1}(\mathcal{C}_{j+1})$ is a general fact and follows from the observation that $\phi^{-1}(\mathcal{C}_j) \subseteq \mathcal{C}_{j+1}$. For the converse, note that $\mathcal{C}_{j+1} = \mathcal{C}_1 \cup \phi^{-1}(\mathcal{C}_1) \cup \cdots \cup \phi^{-j}(\mathcal{C}_1)$ and by the Markov property $\phi(\mathcal{C}_{j+1}) \subseteq \phi(\mathcal{C}_1) \cup \phi(\phi^{-1}(\mathcal{C}_1)) \cup \cdots \cup \phi(\phi^{-j}(\mathcal{C}_1)) \subseteq \mathcal{C}_1 \cup \mathcal{C}_1 \cup \phi^{-1}(\mathcal{C}_1) \cup \cdots \cup \phi^{-j+1}(\mathcal{C}_1) = \mathcal{C}_j$, which implies the desired inclusion. □

COROLLARY 7.10. *We have $\phi^j(\mathcal{C}_j) = \phi(\mathcal{C}_1)$ when $j \geq 1$.*

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Set $\mathcal{D} = \phi(\mathcal{C}_1)$. Then $\mathcal{C}_j \subseteq \phi^{-j}(\mathcal{D})$ by Corollary 7.10. Note that $\phi^{-k-1}(\mathcal{D}) \sqcup \phi^{-k}(\mathcal{D})$ is an $\mathcal{L}$-invariant subset of $\mathbb{T} \sqcup \mathbb{T}$. We therefore obtain an extension

$$0 \longrightarrow C_r^*(\mathcal{L}|_{(\mathbb{T}\sqcup\mathbb{T})\setminus(\phi^{-k-1}(\mathcal{D})\sqcup\phi^{-k}(\mathcal{D}))}) \longrightarrow C_r^*(\mathcal{L}) \longrightarrow C_r^*(\mathcal{L}|_{\phi^{-k-1}(\mathcal{D})\sqcup\phi^{-k}(\mathcal{D})}) \longrightarrow 0\,. \tag{7.4}$$

By using the same procedure as in §6.2 we can give the following alternative description of this extension. As in §6.2 let $\mathcal{I}_{k+1}$ be the set of intervals of connected components in $\mathbb{T}\setminus\phi^{-k-1}(\mathcal{D})$, considered as subsets of the first copy of $\mathbb{T}$ in $\mathbb{T} \sqcup \mathbb{T}$, and $\mathcal{I}_k$ the set of intervals of connected components of $\mathbb{T}\setminus\phi^{-k}(\mathcal{D})$, considered as subsets of the second copy of $\mathbb{T}$ in $\mathbb{T} \sqcup \mathbb{T}$. Let $\mathbb{B}_{\mathcal{L}}$ be the finite-dimensional C^*-algebra generated by the matrix units $e_{I,J}$, $I, J \in \mathcal{I}_{k+1} \cup \mathcal{I}_k$, subject to the conditions that:

(a) $\phi^{k+1}(I) = \phi^{k+1}(J)$ and $\mathrm{val}(\phi^{k+1}, I) = \mathrm{val}(\phi^{k+1}, J)$ when $I, J \in \mathcal{I}_{k+1}$;
(b) $\phi^{k}(I) = \phi^{k}(J)$ and $\mathrm{val}(\phi^{k}, I) = \mathrm{val}(\phi^{k}, J)$ when $I, J \in \mathcal{I}_{k}$;
(c) $\phi^{k+1}(I) = \phi^{k}(J)$ and $\mathrm{val}(\phi^{k+1}, I) = \mathrm{val}(\phi^{k}, J)$ when $I \in \mathcal{I}_{k+1}$ and $J \in \mathcal{I}_k$;
(d) $\phi^{k}(I) = \phi^{k+1}(J)$ and $\mathrm{val}(\phi^{k}, I) = \mathrm{val}(\phi^{k+1}, J)$ when $I \in \mathcal{I}_{k}$ and $J \in \mathcal{I}_{k+1}$.

Then

$$C_r^*(\mathcal{L}|_{(\mathbb{T}\sqcup\mathbb{T})\setminus(\phi^{-k-1}(\mathcal{D})\sqcup\phi^{-k}(\mathcal{D}))}) \simeq S\mathbb{B}_{\mathcal{L}},$$

where we have used the notation SD for the suspension of a C^*-algebra D, i.e. $SD = C_0(\mathbb{R}) \otimes D \simeq C_0(]0, 1[) \otimes D$.

Similarly, we can describe the C^*-algebra $C_r^*(\mathcal{L}|_{\phi^{-k-1}(\mathcal{D})\sqcup\phi^{-k}(\mathcal{D})})$ as the finite-dimensional C^*-algebra $\mathbb{A}_{\mathcal{L}}$ generated by the matrix units $e_{x,y}$, $x, y \in \phi^{-k-1}(\mathcal{D}) \sqcup \phi^{-k}(\mathcal{D}) \subseteq \mathbb{T} \sqcup \mathbb{T}$, subject to the conditions that:

(e) $\phi^{k+1}(x) = \phi^{k+1}(y)$ and $\mathrm{val}(\phi^{k+1}, x) = \mathrm{val}(\phi^{k+1}, y)$ when $x, y \in \phi^{-k-1}(\mathcal{D})$;
(f) $\phi^{k}(x) = \phi^{k}(y)$ and $\mathrm{val}(\phi^{k}, x) = \mathrm{val}(\phi^{k}, y)$ when $x, y \in \phi^{-k}(\mathcal{D})$;
(g) $\phi^{k+1}(x) = \phi^{k}(y)$ and $\mathrm{val}(\phi^{k+1}, x) = \mathrm{val}(\phi^{k}, y)$ when $x \in \phi^{-k-1}(\mathcal{D})$ and $y \in \phi^{-k}(\mathcal{D})$;
(h) $\phi^{k}(x) = \phi^{k+1}(y)$ and $\mathrm{val}(\phi^{k}, x) = \mathrm{val}(\phi^{k+1}, y)$ when $x \in \phi^{-k}(\mathcal{D})$ and $y \in \phi^{-k-1}(\mathcal{D})$.

Then

$$C_r^*(\mathcal{L}|_{\phi^{-k-1}(\mathcal{D})\sqcup\phi^{-k}(\mathcal{D})}) \simeq \mathbb{A}_{\mathcal{L}},$$

and the extension (7.4) takes the form

$$0 \longrightarrow S\mathbb{B}_{\mathcal{L}} \longrightarrow C_r^*(\mathcal{L}) \longrightarrow \mathbb{A}_{\mathcal{L}} \longrightarrow 0\,. \tag{7.5}$$

This extension is compatible with the description of $C_r^*(R_\phi^+(k))$ and $C_r^*(R_\phi^+(k+1))$ coming from §6.2 in the sense that we get a commutative diagram of $*$-homomorphisms:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & S\mathbb{B}_{k+1} & \longrightarrow & C_r^*(R_\phi^+(k+1)) & \longrightarrow & \mathbb{A}_{k+1} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow a & & \downarrow & & \\
0 & \longrightarrow & S\mathbb{B}_{\mathcal{L}} & \longrightarrow & C_r^*(\mathcal{L}) & \longrightarrow & \mathbb{A}_{\mathcal{L}} & \longrightarrow & 0 \\
 & & \uparrow & & \uparrow b & & \uparrow & & \\
0 & \longrightarrow & S\mathbb{B}_{k} & \longrightarrow & C_r^*(R_\phi^+(k)) & \longrightarrow & \mathbb{A}_{k} & \longrightarrow & 0
\end{array} \tag{7.6}$$

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From the six-terms exact sequences of the upper and lower extensions in (7.6) we get for $j = k$ and $j = k + 1$ the identifications

$$K_0(C_r^*(R_\phi^+(j))) = \ker((I_j)_0 - (U_j)_0) \tag{7.7}$$

and

$$K_1(C_r^*(R_\phi^+(j))) = \operatorname{coker}((I_j)_0 - (U_j)_0), \tag{7.8}$$

where $I_j, U_j : \mathbb{A}_j \to \mathbb{B}_j$ are the $*$-homomorphisms introduced in §6.2. Then $b_0^{-1} \circ a_0$ is realized as a homomorphism

$$b_0^{-1} \circ a_0 : \ker((I_{k+1})_0 - (U_{k+1})_0) \to \ker((I_k)_0 - (U_k)_0)$$

and $b_1^{-1} \circ a_1$ as a homomorphism

$$b_1^{-1} \circ a_1 : \operatorname{coker}((I_{k+1})_0 - (U_{k+1})_0) \to \operatorname{coker}((I_k)_0 - (U_k)_0).$$

Note that the full matrix summands in each of the C^*-algebras $\mathbb{B}_k$, $\mathbb{B}_{\mathcal{L}}$ and $\mathbb{B}_{k+1}$ are in one-to-one correspondence with $\mathcal{I}(\pm) = \mathcal{I} \times \{(+, +), (-, -)\}$, where $\mathcal{I}$ are the intervals of connected components in $\mathbb{T}\backslash\mathcal{D}$. In $\mathbb{B}_k$, for example, the element $(I, (+, +)) \in \mathcal{I}(\pm)$ labels the C^*-subalgebra of $\mathbb{B}_k$ generated by the matrix units $e_{I',J'}$ where I', J' are intervals in $\mathcal{I}_k$ such that $\phi^k(I') = \phi^k(J') = I$ and $\operatorname{val}(\phi^k, I') = \operatorname{val}(\phi^k, J') = (+, +)$, while the element $(I, (-, -)) \in \mathcal{I}(\pm)$ labels the C^*-subalgebra of $\mathbb{B}_k$ generated by the matrix units $e_{I',J'}$, where I', J' are intervals in $\mathcal{I}_k$ such that $\phi^k(I') = \phi^k(J') = I$ and $\operatorname{val}(\phi^k, I') = \operatorname{val}(\phi^k, J') = (-, -)$. Similarly, the full matrix summands in $\mathbb{A}_k$, $\mathbb{A}_{\mathcal{L}}$ and $\mathbb{A}_{k+1}$ are in one-to-one correspondence with the subset $\mathcal{D}(\pm)$ of $\mathcal{D} \times \mathcal{V}$ consisting of the pairs $(d, v) \in \mathcal{D} \times \mathcal{V}$ for which there exists an element $x \in \phi^{-k}(d)$ such that $\operatorname{val}(\phi^k, x) = v$. By using these labels we get isomorphisms

$$K_0(\mathbb{A}_k) \simeq K_0(\mathbb{A}_{k+1}) \simeq K_0(\mathbb{A}_{\mathcal{L}}) \simeq \mathbb{Z}^{\mathcal{D}(\pm)}$$

and

$$K_0(\mathbb{B}_k) \simeq K_0(\mathbb{B}_{k+1}) \simeq K_0(\mathbb{B}_{\mathcal{L}}) \simeq \mathbb{Z}^{\mathcal{I}(\pm)} \simeq \mathbb{Z}^{\mathcal{I}} \oplus \mathbb{Z}^{\mathcal{I}}$$

with the property that by making them identifications the maps ${b_0}^{-1} \circ a_0$ and ${b_1}^{-1} \circ a_1$ become identities. Therefore, in order to obtain the desired description of $[E]_*$ we must determine what the map $\rho_* : K_*(C_r^*(R_\phi^+(k))) \to K_*(C_r^*(R_\phi^+(k+1)))$ becomes under these identifications.

7.2. *A description of* $\rho : C_r^*(R_\phi^+(k)) \to C_r^*(R_\phi^+(k+1))$. Using the notation from §6.2, set

$$\mathbb{D}_j = \{(a, f) \in \mathbb{A}_j \oplus C([0, 1], \mathbb{B}_j) : f(0) = I_j(a),\ f(1) = U_j(a)\}$$

when $j \geq 1$. It follows from Lemma 6.6 that there are isomorphisms $\mu_k : C_r^*(R_\phi^+(k)) \to \mathbb{D}_k$ and $\mu_{k+1} : C_r^*(R_\phi^+(k+1)) \to \mathbb{D}_{k+1}$. There is therefore a unique $*$-homomorphism $\Phi : \mathbb{D}_k \to \mathbb{D}_{k+1}$ such that

$$\begin{array}{ccc} C_r^*(R_\phi^+(k)) & \xrightarrow{\mu_k} & \mathbb{D}_k \\ \rho \downarrow & & \downarrow \Phi \\ C_r^*(R_\phi^+(k+1)) & \xrightarrow{\mu_{k+1}} & \mathbb{D}_{k+1} \end{array} \tag{7.9}$$

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commutes. The $*$-homomorphism Φ is given by the formula

$$\Phi(a,\, f) = (\chi(a) + \mu(f),\, \varphi(f)), \tag{7.10}$$

where

$$\chi : \mathbb{A}_k \to \mathbb{A}_{k+1}, \quad \mu : C([0,\,1],\, \mathbb{B}_k) \to \mathbb{A}_{k+1}, \quad \varphi : C([0,\,1],\, \mathbb{B}_k) \to C([0,\,1],\, \mathbb{B}_{k+1})$$

are $*$-homomorphisms such that χ and μ have orthogonal ranges. We describe them individually. The easiest to define is χ; it is simply given by the formula

$$\chi(e_{x,y}) = e_{x,y}$$

when $\phi^k(x) = \phi^k(y) \in \mathcal{D}$ and $\mathrm{val}(\phi^k,\, x) = \mathrm{val}(\phi^k,\, y)$.

To define μ and φ we use the homeomorphisms $\psi_J :]0,\,1[\to J \in \mathcal{I}$ that were introduced in §6.2. Consider $f \in C[0,\,1]$ and a pair $(I,\, J) \in \mathcal{I}_k^{(2)}$. Then

$$\mu(f \otimes e_{I,J}) = \sum_{(x,y)\in N_{I,J}} f({\psi_{\phi^k(I)}}^{-1}(\phi^k(x)))e_{x,y},$$

where

$$N_{I,J} = \{(x,\, y) \in \phi^{-k-1}(\mathcal{D})^2 : x \in I,\, y \in J,\, \phi^k(x) = \phi^k(y) \notin \mathcal{D}\}.$$

Similarly, φ is given by

$$\varphi(f \otimes e_{I,J}) = \sum_{(I_1,J_1)\in M_{I,J}} f_{I_1} \otimes e_{I_1,J_1}, \tag{7.11}$$

where

$$M_{I,J} = \{(I_1,\, J_1) \in \mathcal{I}_{k+1}^{(2)} : I_1 \subseteq I,\ J_1 \subseteq J\}$$

and $f_{I_1} \in C[0,\,1]$ is the continuous extension of the function

$$]0,\,1[\ni t \mapsto f \circ \psi^{-1}_{\phi^k(I)} \circ (\phi|_{\phi^k(I)\cap\phi^{-1}(\phi^{k+1}(I_1))})^{-1} \circ \psi_{\phi^{k+1}(I_1)}(t).$$

(We remark that this description of ρ is almost identical to that given in [**Th5**, Lemma 3.5] in a different setting.)

Define $p_j : \mathbb{D}_j \to \mathbb{A}_j$ such that $p_j(a,\, f) = a$, and note that $p_{k+1} \circ \Phi$ is homotopic to $(\chi + \mu \circ I_k) \circ p_k$ so that

$$\begin{array}{ccc} \mathbb{D}_k & \xrightarrow{p_k} & \mathbb{A}_k \\ \downarrow{\scriptstyle \Phi} & & \downarrow{\scriptstyle \chi+\mu\circ I_k} \\ \mathbb{D}_{k+1} & \xrightarrow{p_{k+1}} & \mathbb{A}_{k+1} \end{array} \tag{7.12}$$

commutes up to homotopy. It follows that

$$\rho_0 = \chi_0 + \mu_0 \circ (I_k)_0$$

on $K_0(\mathbb{D}_k) = \ker((I_k)_0 - (U_k)_0)$.

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7.3. *An algorithm for the calculation of* $K_*(C_r^*(\Gamma_\phi^+))$. For each $(d, v) \in \mathcal{D}(\pm)$ let $[d, v]$ be the corresponding element in the standard basis for $\mathbb{Z}^{\mathcal{D}(\pm)}$. For each $d \in \mathcal{D}$ we let I_d^+ be the interval $I_d^+ \in \mathcal{I}$ such that $d < I_d^+$. We can then define a homomorphism $A : \mathbb{Z}^{\mathcal{D}(\pm)} \to \mathbb{Z}^{\mathcal{D}(\pm)}$ such that

$$A[d, v] = [\phi(d), \mathrm{val}(\phi, d)]$$

when $v = (+, -)$,

$$A[d, v] = [\phi(d), \mathrm{val}(\phi, d)] + \sum_{z \in I_d^+ \cap \phi^{-1}(\mathcal{D})} ([\phi(z), \mathrm{val}(\phi, z) \bullet (+, +)] + [\phi(z), \mathrm{val}(\phi, z) \bullet (-, -)])$$

when $v = (-, +)$,

$$A[d, v] = [\phi(d), \mathrm{val}(\phi, d)] + \sum_{z \in I_d^+ \cap \phi^{-1}(\mathcal{D})} [\phi(z), \mathrm{val}(\phi, z) \bullet (+, +)]$$

when $v = (+, +)$ and, finally,

$$A[d, v] = [\phi(d), \mathrm{val}(\phi, d)] + \sum_{z \in I_d^+ \cap \phi^{-1}(\mathcal{D})} [\phi(z), \mathrm{val}(\phi, z) \bullet (-, -)]$$

when $v = (-, -)$.

Under our identification of $K_0(\mathbb{A}_k)$ and $K_0(\mathbb{A}_{k+1})$ with $\mathbb{Z}^{\mathcal{D}(\pm)}$, the homomorphism $\rho_0 = \chi_0 + \mu_0 \circ (I_k)_0$ is given by A. In the following we let $\tilde{A}$ denote the restriction of A to $\ker((I_k)_0 - (U_k)_0) \subseteq \mathbb{Z}^{\mathcal{D}(\pm)}$.

Since $(i_j)_1 : K_1(S\mathbb{B}_j) \to K_1(\mathbb{D}_j)$ is surjective and $K_1(S\mathbb{B}_k)$ is free there is automatically a homomorphism $B : K_1(S\mathbb{B}_k) \to K_1(S\mathbb{B}_{k+1})$ such that the diagram

$$\begin{array}{ccc} K_1(S\mathbb{B}_k) & \xrightarrow{(i_k)_1} & K_1(\mathbb{D}_k) \\ {\scriptstyle B}\downarrow & & \downarrow{\scriptstyle \Phi} \\ K_1(S\mathbb{B}_{k+1}) & \xrightarrow{(i_{k+1})_1} & K_1(\mathbb{D}_{k+1}) \end{array} \tag{7.13}$$

commutes. In fact, there are generally many choices since $(i_{k+1})_1$ has a kernel. We describe now a way to choose B such that it is easy to determine from ϕ. To this end we need some notation. For each $I \in \mathcal{I}$, $v \in \{(+, +), (-, -)\}$, let $[I, v]$ be the corresponding standard basis element in $\mathbb{Z}^{\mathcal{I}(\pm)}$. For $v \in \{(+, +), (-, -)\}$, set

$$(-1)^v = \begin{cases} 1 & \text{when } v = (+, +), \\ -1 & \text{when } v = (-, -). \end{cases}$$

Let $J \in \mathcal{I}$ and choose $J_1 \in \mathcal{I}$ such that $J_1 \subseteq \phi(J)$. We choose J_1 such that $J_1 = \phi(J_X)$, where $J_X \subseteq J$ is an open subinterval and $J_X \cap \mathcal{C}_1 = \emptyset$. Let $I \in \mathcal{I}_k$ be such that $\phi^k(I) = J$ and $\mathrm{val}(\phi^k, I) = v \in (\pm, \pm)$. Since ϕ^k is injective on I there is a unique open subinterval $I_1 \subseteq I$ such that $\phi^k(I_1) = J_X$. Note that ϕ^{k+1} is injective on I_1 and $\phi^{k+1}(I_1) = J_1$. In particular, $I_1 \in \mathcal{I}_{k+1}$. Let $u \in C_c(I_1) \subseteq C_c(R_\phi^+(k))$ be a positively oriented path in $\mathbb{T} - 1$ of degree one. Then

$$(u \circ \lambda_I \circ \psi_J) \otimes e_{I,I}$$

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represents $(-1)^v[J, v] \in \mathbb{Z}^{\mathcal{I}(\pm)} \simeq K_1(S\mathbb{B}_k)$. By using (7.10) and (7.11) we find that $\mu_{k+1} \circ \rho(u) \in S\mathbb{B}_{k+1} \subseteq \mathbb{D}_{k+1}$, and in $S\mathbb{B}_{k+1}$ is the element

$$(u \circ \lambda_I \circ (\phi|_{\phi^k(I)\cap\phi^{-1}(\phi^{k+1}(I_1))})^{-1} \circ \psi_{J_1}) \otimes e_{I_1,I_1} = (u \circ \lambda_{I_1} \circ \psi_{J_1}) \otimes e_{I_1,I_1}$$

which corresponds to $(-1)^v(-1)^w[J_1, w \bullet v]$ in $\mathbb{Z}^{\mathcal{I}(\pm)}$, where $w = \mathrm{val}(\phi, J_X)$. Therefore, a recipe for a construction of B reads as follows: for each $J \in \mathcal{I}$ choose an open interval $J' \subseteq J$ such that $J' \cap \mathcal{C}_1 = \emptyset$ and $\phi(J') \in \mathcal{I}$. Then

$$B[J, v] = (-1)^{\mathrm{val}(\phi, J')}[\phi(J'), \mathrm{val}(\phi, J') \bullet v].$$

The construction of B generally depends on many choices which lead to different homomorphisms. But the commutativity of (7.13) guarantees that they all leave $\mathrm{im}((I_k)_0 - (U_k)_0) \subseteq \mathbb{Z}^{\mathcal{I}(\pm)}$ globally invariant and induce the same endomorphism $\tilde{B}$ of $\mathrm{coker}((I_k)_0 - (U_k)_0)$.

In view of the six-terms exact sequence (7.2) it now follows that the endomorphisms

$$\tilde{A} : \ker((I_k)_0 - (U_k)_0) \to \ker((I_k)_0 - (U_k)_0),$$
$$\tilde{B} : \mathrm{coker}((I_k)_0 - (U_k)_0) \to \mathrm{coker}((I_k)_0 - (U_k)_0)$$

determine $K_0(C_r^*(\Gamma_\phi^+))$ and $K_1(C_r^*(\Gamma_\phi^+))$ in the sense that there are extensions

$$0 \longrightarrow \mathrm{coker}(1 - \tilde{A}) \longrightarrow K_0(C_r^*(\Gamma_\phi^+)) \longrightarrow \ker(1 - \tilde{B}) \longrightarrow 0 \tag{7.14}$$

and

$$0 \longrightarrow \mathrm{coker}(1 - \tilde{B}) \longrightarrow K_1(C_r^*(\Gamma_\phi^+)) \longrightarrow \ker(1 - \tilde{A}) \longrightarrow 0. \tag{7.15}$$

Note that the last extension is always split and, hence,

$$K_1(C_r^*(\Gamma_\phi^+)) \simeq \mathrm{coker}(1 - \tilde{B}) \oplus \ker(1 - \tilde{A}).$$

To identify the C^*-algebra from its K-theory groups it is important to know which element of $K_0(C_r^*(\Gamma_\phi^+))$ represents the unit 1 of $C_r^*(\Gamma_\phi^+)$. Note, therefore, that $[1] \in K_0(C_r^*(\Gamma_\phi^+))$ is the image of $[1] \in K_0(C_r^*(R_\phi^+(k)))$ under the map ι_0 in (7.2). Under the identification $K_0(\mathbb{A}_k) = \mathbb{Z}^{\mathcal{D}(\pm)}$ we have that

$$[1] = \sum_{(d,v)\in\mathcal{D}(\pm)} m(d, v)[d, v]$$

in $K_0(\mathbb{A}_k)$, where $m(d, v) = \#\{x \in \phi^{-k}(d) : \mathrm{val}(\phi^k, x) = v\}$. Since I_k and U_k are unital $*$-homomorphisms this element is always in $\ker((I_k)_0 - (U_k)_0)$ and gives therefore rise to an element of $\mathrm{coker}(1 - \tilde{A})$ which under the embedding $\mathrm{coker}(1 - \tilde{A}) \subseteq K_0(C_r^*(\Gamma_\phi^+))$ from (7.14) gives us the element representing $[1] \in K_0(C_r^*(\Gamma_\phi^+))$.

Example 7.11. In this example we show with a fair amount of detail how to complete the K-theory calculations for a family of Markov maps by using the recipe described above. Let $m, k \in \mathbb{N}$ and let $g_{m,k} : [0, 1] \to \mathbb{R}$ be the continuous piecewise linear map with the properties that $g_{m,k}(0) = 0$, $g_{m,k}$ has slope $2m$ on $[0, \frac{1}{2}]$ and slope $-2k$ on $[\frac{1}{2}, 1]$.

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Then $g_{m,k}$ is the lift of a piecewise monotone map $\phi_{m,k}$ on the circle which is exact and not locally injective. To simplify the calculations, we exclude the cases $m = 1$, $k = 2$ and $m = 2$, $k = 1$. Then $\phi_{m,k}$ is Markov of order one and $C_r^*(\Gamma^+_{\phi_{m,k}})$ is simple, unital, separable, purely infinite and satisfies the UCT. By the Kirchberg–Phillips classification result the algebra is therefore determined by its K-theory groups and the position of the unit in the K_0-group.

Assume first that both m and k are at least two. Note that 1 is the only critical value and that

$$\mathcal{D}(\pm) = 1 \times \mathcal{V} = \{(1, (-, +)), (1, (+, -)), (1, (+, +)), (1, (-, -))\}.$$

There is only one interval in $\mathcal{I}$, namely $I = \mathbb{T}\backslash\{1\}$, and $\mathcal{I}(\pm) = \{(I, (+, +)), (I, (-, -)\}$. When we take the elements of $\mathcal{D}(\pm)$ and $\mathcal{I}(\pm)$ in that order we find that

$$(I_1)_0 = \begin{pmatrix} 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{pmatrix}, \quad (U_1)_0 = \begin{pmatrix} 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{pmatrix},$$

It follows that

$$\ker((I_1)_0 - (U_1)_0) = \{(z, z, u, v) \in \mathbb{Z}^4 : z, u, v \in \mathbb{Z}\} \simeq \mathbb{Z}^3,$$

and

$$\operatorname{coker}((I_1)_0 - (U_1)_0) \simeq \mathbb{Z}.$$

The matrix A is

$$A = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 2 & 0 & 1 & 1 \\ m+k-2 & 0 & m-1 & k-1 \\ m+k-2 & 0 & k-1 & m-1 \end{pmatrix}$$

and its restriction $\tilde{A}$ to $\ker((I_1)_0 - (U_1)_0) = \mathbb{Z}^3$ is

$$\tilde{A} = \begin{pmatrix} 2 & 1 & 1 \\ m+k-2 & m-1 & k-1 \\ m+k-2 & k-1 & m-1 \end{pmatrix}.$$

Hence,

$$\tilde{A} - 1 = \begin{pmatrix} 1 & 1 & 1 \\ m+k-2 & m-2 & k-1 \\ m+k-2 & k-1 & m-2 \end{pmatrix}.$$

After a couple of row-operations we get from $\tilde{A} - 1$ the matrix

$$\begin{pmatrix} 1 & 1 & 1 \\ 0 & k & m-1 \\ 0 & m-1 & k \end{pmatrix}.$$

Hence, $\ker(\tilde{A} - 1) = 0$ when $k \neq m - 1$ while $\ker(\tilde{A} - 1) \simeq \mathbb{Z}$ when $k = m - 1$. To determine $\operatorname{coker}(\tilde{A} - 1)$ note that this cokernel is the same as the cokernel of

$$\begin{pmatrix} k & m-1 \\ m-1 & k \end{pmatrix}.$$

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Let g be the greatest common divisor of k and $m-1$, and choose $x, y \in \mathbb{Z}$ such that $x(k/g) + y((m-1)/g) = 1$. Then

$$\begin{pmatrix} x & y \\ -\dfrac{m-1}{g} & \dfrac{k}{g} \end{pmatrix} \in GL_2(\mathbb{Z})$$

and

$$\begin{pmatrix} x & y \\ -\dfrac{m-1}{g} & \dfrac{k}{g} \end{pmatrix} \begin{pmatrix} k & m-1 \\ m-1 & k \end{pmatrix} = \begin{pmatrix} g & 0 \\ 0 & g \end{pmatrix} \begin{pmatrix} 1 & x\dfrac{m-1}{g} + y\dfrac{k}{g} \\ 0 & \dfrac{k^2-(m-1)^2}{g^2} \end{pmatrix}.$$

After a final column operation this shows that the Smith normal form of $\left(\begin{smallmatrix} k & m-1 \\ m-1 & k \end{smallmatrix}\right)$ is

$$\begin{pmatrix} g & 0 \\ 0 & \dfrac{k^2-(m-1)^2}{g} \end{pmatrix}.$$

Hence,

$$\operatorname{coker}(\tilde{A} - 1) \simeq \mathbb{Z}_g \oplus \mathbb{Z}_{(k^2-(m-1)^2)/g}.$$

From the recipe for the matrix B it is easily seen that we can choose it to be the identity matrix in this case. It follows that the endomorphism $\tilde{B}$ of $\operatorname{coker}((I_1)_0 - (U_1)_0)$ is the identity and, hence, $\ker(\tilde{B} - 1) \simeq \operatorname{coker}(\tilde{B} - 1) \simeq \operatorname{coker}((I_1)_0 - (U_1)_0) \simeq \mathbb{Z}$. All in all we get the conclusion that

$$K_0(C_r^*(\Gamma^+_{\phi_{m,k}})) \simeq \mathbb{Z}^2 \oplus \mathbb{Z}_{m-1}, \quad K_1(C_r^*(\Gamma^+_{\phi_{m,k}})) \simeq \mathbb{Z}^2$$

when $k = m - 1 = g$ and

$$K_0(C_r^*(\Gamma^+_{\phi_{m,k}})) \simeq \mathbb{Z} \oplus \mathbb{Z}_g \oplus \mathbb{Z}_{(k^2-(m-1)^2)/g}, \quad K_1(C_r^*(\Gamma^+_{\phi_{m,k}})) \simeq \mathbb{Z}$$

when $k \neq m - 1$. To find the position of the unit in the K_0-group, we note first that

$$[1] = \sum_{(d,v)\in\mathcal{D}(\pm)} m(d, v)[d, v] = (1, 1, m-1, k-1)$$

in $K_0(\mathbb{A}_k)$, corresponding to the element $(1, m-1, k-1)$ in $\ker((I_1)_0 - (U_1)_0)$. To calculate the image of the unit in the K_0-group, we have to perform the same row operations on $(1, m-1, k-1)$ as on the matrix of $\tilde{A} - 1$. The first operations, after throwing away the first coordinate, yields the vector $(k-1, m-1)$ and, hence, the final result becomes

$$\begin{pmatrix} x & y \\ -\dfrac{m-1}{g} & \dfrac{k}{g} \end{pmatrix} \begin{pmatrix} k-1 \\ m-1 \end{pmatrix} = \begin{pmatrix} -x+g \\ -\dfrac{m-1}{g} \end{pmatrix}.$$

Calculating in $\mathbb{Z}_g \oplus \mathbb{Z}_{(k^2-(m-1)^2)/g}$, we get that the unit in $K_0(C_r^*(\Gamma^+_{\phi_{m,k}}))$ is represented by $(-x, (m-1)/g) \in \mathbb{Z}_g \oplus \mathbb{Z}_{(k^2-(m-1)^2)/g}$. Note that $g = m - 1$ when $k = m - 1$ and that we can then take $x = 0$. It follows that the unit is represented by $1 \in \mathbb{Z} \subseteq \mathbb{Z}^2 \oplus \mathbb{Z}_{m-1}$ in this case.

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Finally, when $m = k = 1$, we mimic our previous calculations and get that the matrix of $\tilde{A} - 1$ is row equivalent to the identity matrix, hence both ker $(\tilde{A} - 1)$ and coker $(\tilde{A} - 1) = 0$ vanishes. Since $\ker(\tilde{B} - 1) \simeq \operatorname{coker}(\tilde{B} - 1) \simeq \mathbb{Z}$, we get

$$K_0(C_r^*(\Gamma^+_{\phi_{m,k}})) \simeq K_1(C_r^*(\Gamma^+_{\phi_{m,k}})) \simeq \mathbb{Z}.$$

Let $\phi_{-m,-k}(z) = \overline{\phi_{m,k}(z)}$. Then $\phi_{-m,-k}$ is also exact and Markov, and the calculation of $K_*(C_r^*(\Gamma^+_{\phi_{-m,-k}}))$ can proceed exactly as above. In comparison the roles of k and m are interchanged and we find that

$$K_0(C_r^*(\Gamma^+_{\phi_{-m,-k}})) \simeq \mathbb{Z}^2 \oplus \mathbb{Z}_{k-1}, \quad K_1(C_r^*(\Gamma^+_{\phi_{-m,-k}})) \simeq \mathbb{Z}^2$$

when $m = k - 1$ and

$$K_0(C_r^*(\Gamma^+_{\phi_{-m,-k}})) \simeq \mathbb{Z} \oplus \mathbb{Z}_{g'} \oplus \mathbb{Z}_{(m^2-(k-1)^2)/g'}, \quad K_1(C_r^*(\Gamma^+_{\phi_{-m,-k}})) \simeq \mathbb{Z}$$

when $m \neq k - 1$, where g' is now the greatest common divisor of m and $k - 1$, or $g' = m$ when $k = 1$.

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