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Two limit theorems for ergodically regenerated stochastic processes

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Abstract. There is a subtle difference between the recurrent event of Feller and the regenerative phenomenon of Kingman: The former regenerates an entire ambient process, whereas the latter regenerates only itself. This paper generalizes Feller's definition to a discrete regenerative phenomenon E in association with an arbitrary discrete-time stochastic process X. Two limit theorems of a general character are proved for the process X when it is regenerated by an ergodic phenomenon E. The first implies that X becomes strictly stationary at a uniform rate which is determined solely by the asymptotic behaviour of E. The second is essentially a mixing convergence theorem which implies the asymptotic independence of regenerating and regenerated events. Applications include (1) the outright independence of the regenerating events and the regenerated events which occur 'at the end of time', and (2) the identification of the transient features of X with the null sets of the stationary limiting probability. Numerous open questions are posed.

1. *Introduction.* Feller's original definition of a recurrent event ((8), definition 1, p. 308) was stated in terms of a given (discrete, discrete-time) stochastic process. This ambient process was abandoned in Kingman's generalization of that definition (introduced in ((14), p. 268), developed in (15), and later christened a (discrete) *regenerative phenomenon* (see ((16), p. 15))). Kingman's generalization necessarily entails a subtle shift of emphasis: The occurrence of the nth event in a regenerative phenomenon causes the *phenomenon* to regenerate, whereas the occurrence of a recurrent event at time n causes the *process* to regenerate. In this paper we shall (within the context of a general probability measure space) shift the emphasis back to the regenerated process, thereby creating a distinction between those events which *regenerate*, and those which are *regenerated*.

Accordingly, in Section 2, we discuss what it means for a regenerative phenomenon E to *regenerate* an arbitrary discrete-time process X, and we establish a few basic facts which will be used in the subsequent analysis. The remainder of the paper is devoted to a study of the asymptotic behaviour of X when it is regenerated *ergodically* (i.e. at frequent, irregular intervals – see Definition 2·11). The main theorem of Section 3, our first limit theorem, implies that X becomes stationary at a uniform rate, and that this rate is determined solely by the rate at which the (ergodic) phenomenon E stabilizes itself. This theorem can be regarded as a generalization of ((8), theorem 2, p. 319), the one result in Feller's theory of recurrent events where he endeavours to exploit the full power of his definition! Section 3 concludes with a few applications of the main theorem, including (1) the independence of the regenerating and the 'ultimate'

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regenerated events (Theorem 3.9), and (2) a characterization (in Theorem 3·13) of the transient features of X. In Section 4 it is seen that the regenerating events are asymptotically independent of 'penultimate' regenerated events. This fact emerges (in Remark 4·2) as a special case of our second limit theorem, which proposes various general conditions under which the conditional expectation of a regenerating function f with respect to a σ-algebra of sufficiently 'penultimate' regenerated events is arbitrarily closely approximated by just Ef. (However, since these conditions are not definitive, the second limit theorem has been divided into three separate partial results.) Section 5 rounds out the discussion with a number of open problems.

For ready reference we present now a list of the symbols which will recur ergodically in the following pages.

$\mathbb{R}$ = the real line.

$\mathscr{B}_{\mathbb{R}}$ = the Borel subsets of $\mathbb{R}$.

$\mathbb{R}^{\mathbb{N}}$ = the cartesian product of countably many copies of $\mathbb{R}$. A typical element of $\mathbb{R}^{\mathbb{N}}$ will be written as $x = (x_1, x_2, \ldots)$.

$\mathscr{B}$ = the Borel subsets of $\mathbb{R}^{\mathbb{N}}$, also identified as the product σ-algebra of $\mathscr{B}_{\mathbb{R}}$ on each copy of $\mathbb{R}$.

π_n = the projection onto the nth factor of $\mathbb{R}^{\mathbb{N}}$, for $n = 1, 2, \ldots$. Thus, $\pi_n(x) = x_n$ for all $x \in \mathbb{R}^{\mathbb{N}}$.

τ_n = the translation by n units in $\mathbb{R}^{\mathbb{N}}$, for $n = 0, 1, 2, \ldots$. Thus,

$$\tau_n(x) = (x_{n+1}, x_{n+2}, \ldots)$$

for all $x \in \mathbb{R}^{\mathbb{N}}$.

$(\Omega, \mathscr{F}, P)$ = an arbitrary countably additive probability measure space.

E = the expectation and conditional expectation operator: $Ef = \int f dP$.

X_n = a real-valued, $\mathscr{F}$-measurable random variable, for $n = 1, 2, \ldots$.

$X = \{X_n\}_{n=1}^{\infty}$, regarded as a discrete-time stochastic process with state space $\mathbb{R}$. We shall also regard X as a (measurable) transformation from Ω to $\mathbb{R}^{\mathbb{N}}$ defined by $X(\omega) = (X_1(\omega), X_2(\omega), \ldots)$ for all $\omega \in \Omega$.

P_X = the distribution measure of X, defined by $P_X(S) = P(X \in S)$ for all $S \in \mathscr{B}$.

$\mathscr{A}_n$ = the σ-algebra generated by $X_{n+1}, X_{n+2}, \ldots$, for $n = 0, 1, 2, \ldots$.

$\mathscr{A} = \mathscr{A}_0$.

$\mathscr{U} = \cap_{n=0}^{\infty} \mathscr{A}_n$, the tail σ-algebra.

$E = \{E_n\}_{n=1}^{\infty}$, a sequence of sets in $\mathscr{F}$ which constitutes a *regenerative phenomenon* in the sense of the following definition:

(1·1) $\quad P(E_{n_1} E_{n_2} \ldots E_{n_k}) = P(E_{n_1}) P(E_{n_2 - n_1} \ldots E_{n_k - n_1})$ whenever

$$1 \leqslant n_1 < n_2 < \ldots < n_k.$$

$I_E = \{I_{E_n}\}_{n=1}^{\infty}$, where I_{E_n} is the indicator function of E_n. We regard this sequence as a distinguished special case of X.

$\mathscr{R}$ = the σ-algebra generated by $E_1, E_2, \ldots$.

$\mathscr{R}_n$ = the (finite) σ-algebra generated by $E_1, \ldots, E_{n-1}$, for $n = 1, 2, \ldots$

$(\mathscr{R}_1 = \{\varnothing, \Omega\})$.

W = 'the time at which E first appears'. Thus, W is defined by the identities $(W = 1) = E_1$, and $(W = n) = E_1^c E_2^c \ldots E_{n-1}^c E_n$ for $n = 2, 3, \ldots$.

$\mu = EW$.

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For X, and similarly for other measurable transformations, we shall use the notations $X^{-1}(S)$ and the more standard $(X \in S)$ interchangeably, in accordance with the dictates of clarity and economy. In particular, the X^{-1} notation will be employed when we wish to emphasize the fact that X^{-1}, regarded as a set mapping of $\mathscr{B}$ *onto* $\mathscr{A}$, constitutes a σ-algebra homomorphism (and in fact preserves all Boolean operations).

It will be important to view the phenomenon E and the process X as inducing an incompletely defined notion of 'time' into the sample space Ω. (Thus, we think of X_n as the 'state of the system at time n', of $\mathscr{R}_n$ as the σ-algebra of *regenerating* events which occur before time n, of $\mathscr{U}$ as the σ-algebra of *ultimate* events, and so on.) Since we are aiming in particular to achieve a precise interpretation of Feller's intuitive notion of 'starting over from scratch' (the notion which underlies the whole idea of regeneration), we should like to have, given an event $A \in \mathscr{A}$, a well-defined concept of the 'same' or the 'corresponding' event n time units *later*. For certain events, such as $A = (X_1 \in H)$, where $H \in \mathscr{B}_{\mathbb{R}}$, it seems obvious that the corresponding event should be $(X_{n+1} \in H)$. But even in this case, if it happened coincidentally to be true that $A = \varnothing$, or that $A = (X_5 \in K)$, we might wish the corresponding event to be $\varnothing$, or to be $(X_{n+5} \in K)$, depending upon the context of the discussion. So it is clear that, in itself, the event A does not uniquely determine a 'corresponding event n time units later'. (But see Theorem 2·3.) Notice, however, that there is always at least one set $S \in \mathscr{B}$ such that $A = (X \in S)$. We shall think of S as a kind of index set which serves to *fix A in time*. The corresponding event n time units later can now unambiguously be defined to be the event $(\tau_n X \in S)$. (For example, when $S = (\pi_1 \in H)$, then we have $(X \in S) = (X_1 \in H)$ (since $X_1 = \pi_1 X$), while $(\tau_n X \in S) = (X_{n+1} \in H)$ (since $X_{n+1} = \pi_{n+1} X = \pi_1 \tau_n X$).) This definition reveals our motive for making use of the 'alternate sample space' $\mathbb{R}^{\mathbb{N}}$ with its well-defined notion of translation, and it also inspires the following terminology.

1·2. *Definition.* Let $A \in \mathscr{A}$. Then any set $S \in \mathscr{B}$ such that $A = (X \in S)$ will be called a *time frame* for A. The sequence $X_S = \{A_n\}_{n=0}^{\infty}$ defined by $A_n = (\tau_n X \in S)$ for $n = 0, 1, 2, \ldots$ will be termed the *feature* of the process X which is determined by the time frame S. ∎

In general there will be many choices of a time frame S for an event $A \in \mathscr{A}$. Is it possible to make a natural, or at least a consistent choice of S? The following result in this direction is an immediate consequence of a classical theorem of von Neumann and Stone ((19), theorem 17, p. 369).

1·3. THEOREM. *Assume the continuum hypothesis. Then there exists a Boolean algebra homomorphism $\rho: \mathscr{A} \to \mathscr{B}$ which is a right inverse for the σ-algebra homomorphism $X^{-1}: \mathscr{B} \to \mathscr{A}$. Thus, for all $A \in \mathscr{A}$, we have*

$$A = X^{-1}(\rho(A)) = (X \in \rho(A)). \tag{1·3·1}$$

Moreover, any homomorphism ρ_0 which is defined on a subalgebra $\mathscr{A}_0$ of $\mathscr{A}$, and which satisfies (1·3·1) for all $A \in \mathscr{A}_0$, may be extended to a right inverse homomorphism ρ for X^{-1} which is defined on all of $\mathscr{A}$. ∎

In plain language, the mapping ρ *selects* a time frame $S = \rho(A)$ for each event $A \in \mathscr{A}$, and the selection is moreover consistent in the sense that finite Boolean operations are preserved. But we are not yet prepared to assert that any of these selections is 'natural,' and so our results will be addressed to arbitrary features of X

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(in the sense of Definition 1·2). Various choices of time frames will then yield information about the asymptotic behaviour of X, and about a few other matters besides.

The theory of Dynkin classes will be frequently invoked in what follows. A good reference for this is ((6), § 1·6, pp. 44–46).

2. *Basic definitions and results.* In this section we evolve a notion of the regeneration of a process X by a phenomenon E. The key ingredient of this notion will be a concept of the regeneration of X at a specific time n by a specific event F.

2·1. *Definition.* Let $F \in \mathscr{F}$, and let a positive integer n be fixed. We shall say that the event F *regenerates the process* X *at time* n if, for all choices of indices

$$n < n_1 < n_2 < \ldots < n_k,$$

and of Borel sets $H_1, H_2, \ldots, H_k \subseteq \mathbb{R}$, we have

$$P(F, X_{n_1} \in H_1, X_{n_2} \in H_2, \ldots, X_{n_k} \in H_k) = P(F)\, P(X_{n_1-n} \in H_1, X_{n_2-n} \in H_2, \ldots, X_{n_k-n} \in H_k). \quad |$$

A standard Dynkin class argument shows that it is sufficient to restrict the sets H_i to any generating set for $\mathscr{B}_{\mathbb{R}}$ which is closed under finite intersections. The following result further extends the key identity.

2·2. PROPOSITION. *Assume that the event* F *regenerates the process* X *at time* n. *Then, for every set* $S \in \mathscr{B}$, *we have*

$$P(F, \tau_n X \in S) = P(F)\, P(X \in S).$$

Proof. This, too, follows from a standard Dynkin class argument once it is verified for an arbitrary generating set of the form $S = (X_{n_1} \in H_1, \ldots, X_{n_k} \in H_k)$, where $1 \leqslant n_1 < \ldots < n_k$, and where $H_1, \ldots, H_k \in \mathscr{B}_{\mathbb{R}}$. Since $X_{n_i+n} = \pi_{n_i+n} X = \pi_{n_i} \tau_n X$, it follows that $(\tau_n X \in S) = (X_{n_1+n} \in H_1, \ldots, X_{n_k+n} \in H_k)$, and from this the desired identity is an immediate consequence of Definition 2.1. |

If Definition 2·1 is 'correct', it should follow that, conditional upon F, the process X will 'start over from scratch' at time n. An explicit mathematical interpretation of this statement (that is to say, a mathematical description of the effect of a regeneration on the process) seems not yet to have been noted, and so we offer one in the next theorem. For this theorem we introduce the following notation: Given a σ-algebra $\mathscr{G} \subseteq \mathscr{F}$ and an event $G \in \mathscr{G}$, we let $[G]_P$ denote the equivalence class of G modulo the P-null sets, and we let $[\mathscr{G}]_P$ denote the (abstract) σ-algebra consisting of the P-equivalence classes of events $G \in \mathscr{G}$. We make $[\mathscr{G}]_P$ into a measure algebra by transferring P to $[\mathscr{G}]_P$ in the obvious way (see, for example, ((20), p. 54) for more details).

2·3. THEOREM. *Assume that the event* $F \in \mathscr{F}$ *regenerates the process* X *at time* n. *Assume also that* $P(F) > 0$, *and define*

$$P_F(A) = P(A \mid F)$$

for all $A \in \mathscr{A}_n$. *Then, for* $S \in \mathscr{B}$, *the identity*

$$\Phi[(X \in S)]_P = [(\tau_n X \in S)]_{P_F}$$

defines a measure-preserving, measure algebra isomorphism $\Phi: [\mathscr{A}]_P \to [\mathscr{A}_n]_{P_F}$.

Remark. Thus, so far as the probability P_F is concerned, there is essentially only one event 'n time units later' corresponding to a given event (or equivalence class) from $\mathscr{A}$.

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Moreover, the P_F-probability of the corresponding even coincides with the P-probability of the original event, and the Boolean structure of the totality of corresponding events coincides with that of the totality of original events.

Proof. This result is a routine consequence of Proposition 2·2, but is perhaps most clearly seen if the X^{-1} notation is utilized. Let us verify, for illustration, that the map Φ is well defined. To this end, we must specify two time frames $S, S' \in \mathscr{B}$ such that the corresponding events $A = (X \in S) = X^{-1}(S)$ and $A' = (X \in S')$ are P-equivalent, and we must verify that the events $A_n = (\tau_n X \in S) = X^{-1}(\tau_n^{-1}(S))$ and $A_n' = (\tau_n X \in S')$ are P_F-equivalent. But now $A \triangle A' = X^{-1}(S) \triangle X^{-1}(S') = X^{-1}(S \triangle S') = (X \in S \triangle S')$, and likewise $A_n \triangle A_n' = (\tau_n X \in S \triangle S')$. Since $P(A \triangle A') = 0$, the desired identity

$$P_F(A_n \triangle A_n') = 0$$

follows as an immediate consequence of Proposition 2·2 (applied to $S \triangle S'$). ∎

We are now prepared for the principal definition of this paper.

2·4. *Definition.* The regenerative phenomenon E will be said to *regenerate* the process X if, for every choice of indices $1 \leqslant n_1 < n_2 < \ldots < n_k$, the event $F = E_{n_1} E_{n_2} \ldots E_{n_k}$ regenerates X at time n_k. ∎

When this definition is satisfied, we shall, in a loose sense, think of $\mathscr{R}$ as the σ-algebra of *regenerating* events and of $\mathscr{A}$ as the σ-algebra of *regenerated* events. It is easy to see that if Feller's '$P\{E_{j_1}, \ldots, E_{j_n}\}$' probabilities are interpreted to be our probabilities of the form $P(X_1 = j_1, \ldots, X_n = j_n)$, then a recurrent event ((8), definition 1, p. 308) will regenerate the ambient (discrete) process in the sense of Definition 2·4. (See also ((5), § 8, pp. 39–43).) Two noteworthy examples of Definition 2·4 are the following: (1) Let X be a Markov chain (with stationary transition probabilities) which starts from a certain state i. Then the phenomenon E defined by $E_n = (X_n = i)$ for $n = 1, 2, \ldots$ will regenerate X by virtue of satisfying Feller's definition of a recurrent event. (2) The phenomenon E should certainly be self-regenerative, i.e. it should regenerate the process I_E. This follows as a special case of Theorem 2·7 below, but is essentially well known.

Definition 2·4 lies midway in strength between two equally attractive alternative conditions, the stronger of which is also satisfied by Feller's recurrent events.

(2·5) For every integer $n = 1, 2, \ldots$, the event E_n regenerates the process X at time n.

(2.6) For every choice of indices $1 \leqslant n < n_1 < \ldots < n_k$, we have the identity

$$P(E_n F_{n_1} \ldots F_{n_k}) = P(E_n)\, P(F_{n_1 - n} \ldots F_{n_k - n}),$$

where, for each $i = 1, 2, \ldots, k$, it is *either* the case that $F_{n_i} = E_{n_i}$ (in which case $F_{n_i - n} = E_{n_i - n}$), *or* that $F_{n_i} = (X_{n_i} \in H_i)$ for some set $H_i \in \mathscr{B}_{\mathbb{R}}$ (in which case

$$F_{n_i - n} = (X_{n_i - n} \in H_i)).$$

Its admirable simplicity and generality notwithstanding, condition (2·5) does not seem to imply anything special about the asymptotic behaviour of X, whereas condition (2·6) asserts more than is required for the theorems of this paper. We shall discuss the sort of result which can be obtained from condition (2·6) at the end of Section 4.

The following theorem, which is a consequence of Definition 2·4 as it stands, goes

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a bit further than we shall require towards the sort of 'mixed' regeneration asserted by condition (2·6). It is based upon the special case of (2·6) in which every regenerating event is required to *precede* every regenerated event (that is to say, whenever $F_{n_i} = E_{n_i}$ and $F_{n_j} = (X_{n_j} \in H_j)$, then necessarily $n_i < n_j$), and it extends this special case in much the same way that Proposition 2·2 extends Definition 2·1.

2·7. THEOREM. *Assume that E regenerates X, and let the following items be given:*
$R \in \mathscr{R}_n$, for fixed $n = 1, 2, \ldots$;
$G \in \mathscr{R}_{m+1}$, for fixed $m = 0, 1, 2, \ldots$;
$\mathscr{B}_m$ = the σ-subalgebra of $\mathscr{B}$ generated by $\pi_1, \pi_2, \ldots, \pi_m$ $(\mathscr{B}_0 = \{\varnothing, \mathbb{R}^{\mathbb{N}}\})$;
$M \in \mathscr{B}_m$ a time frame for G with respect to the process I_E (so that $G = (I_E \in M)$); and
$S \in \mathscr{B}$.

For $j = 0, 1, 2, \ldots$, define

$$A_j = (\tau_j X \in S), \quad \text{and} \quad G_j = (\tau_j I_E \in M).$$

Then we have

$$P(RE_n G_n A_{n+m}) = P(RE_n)\, P(GA_m). \tag{2·7·1}$$

From this it follows, for every event $D \in \mathscr{A}_{m+n}$, that

$$P(RE_n G_n D)\, P(E_n) = P(RE_n)\, P(E_n G_n D). \tag{2·7·2}$$

Note. The proof below will overlook a few special cases (e.g. the case $m = 0$). These are handled in a similar fashion.

Proof. Take R to be of the form $R = E_{n_1} \ldots E_{n_k}$, where $1 \leqslant n_1 < \ldots < n_k < n$, and take M to be of the form $M = (\pi_1 \in H_1, \ldots, \pi_m \in H_m)$, where $H_i \in \mathscr{B}_{\mathbb{R}}$, and $0 \notin H_i$, for each $i = 1, 2, \ldots, m$. It follows that G is of the form $G = E_{m_1} \ldots E_{m_l}$, where

$$1 \leqslant m_1 < \ldots < m_l \leqslant m,$$

and that $G_n = E_{m_1+n} \ldots E_{m_l+n}$. Successive applications of Proposition 2·2 and condition (1·1) yield the identities

$$\begin{aligned} P(E_{n_1} \ldots E_{n_k} E_n E_{m_l+n} \ldots E_{m_1+n} A_{n+m}) &= P(E_{n_1} \ldots E_n \ldots E_{m_l+n})\, P(A_{m-m_l}) \\ &= P(E_{n_1} \ldots E_{n_k} E_n)\, P(E_{m_1} \ldots E_{m_l})\, P(A_{m-m_l}) = P(RE_n)\, P(GA_m). \end{aligned}$$

This establishes (2·7·1) for the specified choices of R and M. The general case is obtained from two Dynkin class arguments, the first applied to M for R fixed as above, and the second applied to R for fixed $M \in \mathscr{B}_m$.

Since every event $D \in \mathscr{A}_{m+n}$ is clearly of the form $D = A_{m+n} = (\tau_{n+m} X \in S)$ for some set $S \in \mathscr{B}$, identity (2·7·2) follows at once from two applications of (2·7·1) (the second with $R = \Omega$). ∎

We shall use precisely the last two of the following three special cases of Theorem 2·7. Each of these may be proved separately, and without I_E time framing. The first constitutes the natural analogue in this setting of ((15), proposition 2, p. 182); it implies that past regenerating events are independent of future regenerated events, conditional upon present regeneration (i.e. upon E_n, provided that $P(E_n) > 0$).

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2·8. COROLLARY. *Assume that E regenerates X, let n be a positive integer, let $R \in \mathscr{R}_n$, and let $D \in \mathscr{A}_n$. Then we have*

$$P(RE_nD)\,P(E_n) = P(RE_n)\,P(E_nD).$$

Proof. This follows from (2·7·2) with $m = 0$, and with $M = \mathbb{R}^{\mathbb{N}}$ (so that $G = G_n = \Omega$). ∎

2·9. COROLLARY. *Assume that E regenerates X, let n be a positive integer and let $R \in \mathscr{R}_n$. Then the event RE_n regenerates X at time n.* ∎

2·10. COROLLARY. *Assume that E regenerates X, and let $X_S = \{A_n\}_{n=0}^{\infty}$ be an arbitrary feature of X (see Definition 1·2). Then, for $m, n = 1, 2, \ldots$, we have*

$$P(E_nE_{n+1}^c \ldots E_{n+m}^c A_{n+m}) = P(E_n)\,P(E_1^c \ldots E_m^c A_m).$$

Proof. This follows from (2·7·1) with $R = \Omega$, and with $M = (\pi_1 = 0, \ldots, \pi_m = 0)$ (so that $G = E_1^c \ldots E_m^c$, and $G_n = E_{n+1}^c \ldots E_{n+m}^c$). The direct proof seems to require a somewhat tedious application of the generalized addition rule ((8), theorem, p. 99). ∎

We do not expect a transient regeneration to influence the long-term behaviour of X, and so we record the following definition (cf. ((8), Definition 3, p. 389)).

2·11. *Definition.* The regenerative phenomenon E is *ergodic* if it is persistent (i.e. W is proper), positive-recurrent (i.e. $\mu = EW < \infty$), and aperiodic (i.e. $\lambda = 1$, where λ, the *period* of E, is the greatest common divisor of the set $\{n: P(E_n) > 0\}$). The process X will be termed *ergodically regenerated* if there exists an ergodic regenerative phenomenon which regenerates X. ∎

Thus, with probability one, an ergodically regenerated process will 'start over from scratch' infinitely often, and, moreover, the times between successive regenerations will be independent and identically distributed ((11), p. 22). One is clearly tempted to ask how close such a process might come to being stationary, and this theme will be pursued in the next section. (See also Problem 5·8.)

3. *The first limit theorem: asymptotic stationarity.* The principal result of this section is a generalization of ((8), theorem 2, p. 319) which asserts that the features (Definition 1·2) of an ergodically regenerated process become stable in probability at a uniform rate. Moreover, this rate is governed solely by the 'stabilization' rate of the regenerating phenomenon. Therefore it is, at least in principle, explicitly calculable. We shall clarify this point before proceeding on to the theorem itself. The following proposition is an elaboration of ((8), theorem 1, p. 318), and its elementary proof is omitted.

3·1. PROPOSITION. *Let $\{u_n\}_{n=0}^{\infty}$, $\{b_n\}_{n=0}^{\infty}$, and $\{x_n\}_{n=0}^{\infty}$ be sequences of nonnegative real numbers such that*

$$u_n \leqslant 1 \quad \textit{for all } n, \quad \textit{and} \quad u_n \to u;$$

$$b_n \leqslant x_n \quad \textit{for all } n;$$

and

$$b = \sum_{n=0}^{\infty} b_n \leqslant x = \sum_{n=0}^{\infty} x_n < \infty.$$

For $n = 0, 1, 2, \ldots$, define

$$r_n = x_{n+1} + x_{n+2} + \ldots,$$

and

$$v_n = b_0u_n + b_1u_{n-1} + \ldots + b_nu_0.$$

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Then $v_n \to bu$, and moreover, the rate of this convergence is determined by that of the u_n to u, and of the r_n to 0. Specifically, given any number $\epsilon > 0$ (and eliminating the trivial case $x = 0$), let k be fixed so that $r_k < \min(\epsilon/2, \epsilon/2x)$, and then let n_0 be chosen so that, for all indices $n \geqslant n_0 - k$, we have $|u_n - u| < \epsilon/2x$. Then we obtain $|v_n - bu| < \epsilon$ for all $n \geqslant n_0$.

In the following example we elucidate an empirical method for obtaining the index n_0 as above for an ergodic regenerative phenomenon E.

3·2. *Example.* Assume that E is ergodic. Define $u_0 = x_0 = 1$, define $f_0 = P(W = 0) = 0$, and, for $n = 1, 2, \ldots$, define

$$u_n = P(E_n),$$

$$f_n = P(W = n),$$

and

$$x_n = P(W > n) = f_{n+1} + f_{n+2} + \ldots,$$

so that $x = \Sigma_{n=0}^{\infty} x_n = EW = \mu < \infty$. And by the Erdös–Feller–Pollard Theorem ((8), theorem 3, p. 313) (see also ((11), § 1·1·2·2, pp. 21–27)), we have $u_n \to u = \mu^{-1}$. Much study has been devoted to the form and asymptotic behaviour of the sequence $\{u_n\}_{n=0}^{\infty}$, the classical result about the rate of convergence being that of Karlin ((13), p. 238). However, there seem to be no results which are addressed specifically to the conditions of Proposition 3·1.

Let us assume that the value of μ is explicitly known, and that the u_n and the f_n may be calculated recursively. (The feasibility of these requirements in specific cases is illustrated by the elementary examples in chapters XIII and XVI of (8).) We now indicate how to choose the index n_0 of Proposition 3·1 corresponding to a given number $0 < \epsilon < 1$. Starting with $r_0 = \mu - 1$, the r_n of Proposition 3·1 are easily obtained from the recurrence relation

$$r_{n+1} = r_n - x_{n+1} = r_n - \left(1 - \sum_{i=1}^{n+1} f_i\right).$$

Accordingly, the first step will be:

(3·2·1) Proceed with the recursion until the smallest index k is reached such that

$$r_{k-1} \leqslant \epsilon/2(2\mu + 2 + \epsilon).$$

This step guarantees that $r_k \leqslant r_{k-1} < \epsilon/2\mu \leqslant \epsilon/2$. The second step is:

(3·2·2) Continue with the recursion until the first k *successive* indices n are obtained such that $|u_n - \mu^{-1}| \leqslant \epsilon/4\mu$. Define n_0 to be one greater than the largest of these indices.

It remains to be verified that $|u_n - \mu^{-1}| < \epsilon/2\mu$ for all $n \geqslant n_0 - k$. This is certainly true when $n < n_0$. When $n = n_0$, we define

$$\phi = \sum_{i=1}^{k} f_i = 1 - x_k > 0,$$

and

$$w = \phi^{-1} \sum_{i=1}^{k} f_i u_{n-i},$$

so that

$$\begin{aligned} u_n &= (f_1 u_{n-1} + f_2 u_{n-2} + \ldots + f_k u_{n-k}) + (f_{k+1} u_{n-k-1} + \ldots + f_n u_0) \\ &= \phi w + f_{k+1} u_{n-k-1} + \ldots. \end{aligned}$$

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The key observation is that w is a *convex* combination of $u_{n-1}, \ldots, u_{n-k}$. Therefore $|w-\mu^{-1}| \leqslant \epsilon/4\mu$. It follows that

$$\begin{aligned} |u_n - w| &\leqslant |\phi w - w| + f_{k+1} + f_{k+2} + \ldots \\ &= x_k(w+1) \\ &< x_k\left(\frac{1}{\mu} + \frac{\epsilon}{2\mu} + 1\right) = x_k\left(\frac{2\mu+2+\epsilon}{2\mu}\right), \end{aligned}$$

so that

$$|u_n - \mu^{-1}| < \frac{\epsilon}{4\mu} + x_k\left(\frac{2\mu+2+\epsilon}{2\mu}\right) = \alpha.$$

The proof continues by induction. At the next stage we observe that the numbers $u_n, u_{n-1}, \ldots, u_{n-k}$ all belong to the open interval about μ^{-1} with radius α, and we use the fact that any convex combination of these numbers must belong to the same interval. Reasoning successively as above permits the conclusion that, for $i = 0, 1, 2, \ldots$ (and $n = n_0$), we have

$$\begin{aligned} |u_{n+i} - \mu^{-1}| &< \frac{\epsilon}{4\mu} + \left(\frac{2\mu+2+\epsilon}{2\mu}\right)(x_k + x_{k+1} + \ldots + x_{k+i}) \\ &\leqslant \frac{\epsilon}{4\mu} + \left(\frac{2\mu+2+\epsilon}{2\mu}\right) r_{k-1} \\ &\leqslant \frac{\epsilon}{2\mu}, \end{aligned}$$

as desired.

The table below contains some numerical illustration of the method for the phenomena determined by 'random Morse code messages' in the tossing of a fair coin ((8), example (*e*), p. 305; § 7, pp. 322–326; example (*c*), p. 328). The symbol H stands for 'heads', and T stands for 'tails'.

phenomenon	μ	ϵ	k	n_0
H	2	0.01	13	13
		0.001	16	16
HTH	10	0.01	83	94
HHH	14	0.01	136	146
HHHTTTHHH	526	0.1	8397	8412
		0.05	8756	8771
		0.01	9589	9606

The f_n may be calculated recursively either from the (rational) generating function $F(s) = \Sigma_{n=0}^{\infty} f_n s^n$, or from the renewal equation $f_n = u_n - f_1 u_{n-1} - \ldots - f_{n-1} u_1$. In the case of the 'S.O.S.' message HHHTTTHHH, the n_0 value of 9606 is achieved, via the first approach, after approximately 100 000 calculations, whereas the second approach requires some 46 000 000 calculations!

The motive behind all of this fuss is revealed in the proof of our first limit theorem, which follows immediately.

3·3. Theorem. *Assume that the process X is ergodically regenerated. Then, for every set $S \in \mathscr{B}$, the limit*

$$Q(S) = \lim_{n\to\infty} P(\tau_n X \in S)$$

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exists uniformly in S and hence determines a countably additive probability measure Q on $\mathscr{B}$.

Proof. Only a standard recurrence argument remains. Let E regenerate X ergodically, and let the notation of Example 3·2 be adopted. Define $b_0 = v_0 = P(X \in S)$, and, for $n = 1, 2, \ldots$, define $v_n = P(\tau_n X \in S)$ and $b_n = P(E_1^c \ldots E_n^c, \tau_n X \in S)$. Note that $b_0 \leqslant 1 = x_0$, and that $b_n \leqslant P(E_1^c \ldots E_n^c) = x_n$. If the event $(\tau_n X \in S)$ is partitioned (in the usual fashion) by the sets $E_n, E_{n-1}E_n^c, \ldots, E_1 E_2^c \ldots E_n^c, E_1^c \ldots E_n^c$, then Proposition 2·2 and Corollary 2·10 yield the recurrence relation

$$v_n = b_0 u_n + b_1 u_{n-1} + \ldots + b_n u_0,$$

which is valid for all integers $n \geqslant 0$. The desired result now follows from Proposition 3·1, which also gives the following formula for Q:

$$Q(S) = \frac{1}{\mu}\left[P(X \in S) + \sum_{n=1}^{\infty} P(E_1^c \ldots E_n^c, \tau_n X \in S)\right] \tag{3·3·1}$$

for all $S \in \mathscr{B}$. The countable additivity of Q follows either from the uniformity of the convergence in the statement of the theorem, or from an interchange of summation in formula (3·3·1). ∎

3·4. *Remark.* The conclusion of Theorem 3·3 can be restated as follows: For $n = 0, 1, 2, \ldots$, define a probability P_n on $\mathscr{B}$ by

$$P_n(S) = P(\tau_n X \in S) = P_X(\tau_n^{-1}(S))$$

for all $S \in \mathscr{B}$ (so that $P_0 = P_X$). Then we have

$$\|P_n - Q\| \to 0,$$

where $\|\cdot\|$ denotes the total variation norm ((7), definition 4, p. 97; example 16, p. 240). ∎

3·5. *Remark.* When the period, λ, of E is > 1, similar reasoning yields the existence, for $j = 0, 1, \ldots, \lambda - 1$, of a probability measure Q_j on $\mathscr{B}$ such that

$$P(\tau_{n\lambda+j} X \in S) \to Q_j(S)$$

uniformly in $S \in \mathscr{B}$, or, equivalently, such that

$$\|P_{n\lambda+j} - Q_j\| \to 0.$$

The sequence $\{P_n\}_{n=0}^{\infty}$ itself converges in Cesaro mean to the probability $Q = \lambda^{-1}\Sigma_{j=0}^{\lambda-1} Q_j$, i.e.

$$\left\| n^{-1} \sum_{i=0}^{n-1} P_i - Q \right\| \to 0.$$

(The above definitions of Q and the Q_j do not extend beyond this remark.) ∎

Let us now clarify what Theorem 3·3 is saying about the asymptotic behaviour of X. Let Q_1 be the image measure of Q under the map π_1, i.e.

$$Q_1(H) = Q(\pi_1 \in H) \tag{3·6}$$

for all $H \in \mathscr{B}_{\mathbf{R}}$. Then, for $n = 0, 1, 2, \ldots$, we have

$$P(X_{n+1} \in H) = P(X^{-1}(\tau_n^{-1}(\pi_1^{-1}(H)))) = P(\tau_n X \in (\pi_1 \in H)) \to Q_1(H)$$

uniformly in $H \in \mathscr{B}_{\mathbf{R}}$, i.e. the X_n converge uniformly in distribution to Q_1. (Therefore, in particular, we have $Eg(X_n) \to \int g\, dQ_1$ for every *bounded* Borel measurable function

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g on $\mathbb{R}$, and the convergence is uniform over the family of those functions g which are bounded by a fixed constant.) A similar remark applies to the joint distributions of any subfamily, finite or infinite, of the X_n taken together with the time-translates of that subfamily. We summarize this in the following corollary.

3·7. COROLLARY. *An ergodically regenerated process X is uniformly asymptotically strictly stationary.*

The following example illustrates the failure of Theorem 3·3 when the phenomenon E is not ergodic.

3·8. *Example.* For $n = 1, 2, \ldots$, let X_n be the 'S_{2n}' random variable of the classical symmetric random walk ((8), example (a), p. 304; example (b), p. 313), and let $E_n = (X_n = 0)$. Then the sequence E, thus defined, constitutes a persistent, aperiodic, null-recurrent regenerative phenomenon which regenerates the process X. Clearly we have $P(X_n \in H) \to 0$ for every bounded (Borel) subset H of $\mathbb{R}$. It is therefore an immediate consequence of ((7), theorem 5, p. 308) that there exists at least one set $H \in \mathscr{B}_{\mathbb{R}}$ for which the limit $Q(\pi_1 \in H) = \lim_{n\to\infty} P(X_n \in H)$ does not exist.

The following theorem forms the basis of the results in Section 4. It asserts that the regenerating events are independent of the ultimate regenerated events. It thus provides a logical extension to Corollary 2·8, although it does not seem to follow directly from that result. Instead we shall exploit the uniform convergence of Theorem 3·3 in a curious way.

3·9. THEOREM. *Let X be regenerated by the ergodic phenomenon E. Then the σ-algebras $\mathscr{R}$ and $\mathscr{U}$ are independent.*

Proof. Let $U \in \mathscr{U}$, and, for each $n = 0, 1, 2, \ldots$, let any set $S_n \in \mathscr{B}$ be selected such that $U = (\tau_n X \in S_n)$. Let $R \in \mathscr{R}$ be of the form $R = E_{n_1} \ldots E_{n_k}$, where

$$1 \leqslant n_1 < n_2 < \ldots < n_k,$$

and define $n = n_k$. Since E is persistent, the event $E_{n+1} \cup E_{n+2} \cup \ldots$ has probability one, and so

$$P(RU) = P(RE_{n+1}U) + \sum_{j=2}^{\infty} P(RE_{n+1}^c \ldots E_{n+j-1}^c E_{n+j} U).$$

By Corollary 2·9 and Proposition 2·2, we have, for $m = 0, 1, 2, \ldots$, that

$$\begin{aligned} P(RE_{n+1}^c \ldots E_{n+j-1}^c E_{n+j} U) &= P(RE_{n+1}^c \ldots E_{n+j-1}^c E_{n+j})\, P(\tau_m X \in S_{m+n+j}) \\ &= P(R)\, P(E_1^c \ldots E_{j-1}^c E_j)\, P(\tau_m X \in S_{m+n+j}). \end{aligned}$$

Therefore, when all terms are nonzero, the probabilities $P(\tau_m X \in S_{m+n+j})$ are constant, independent of m. We recall from Theorem 3·3 that the convergence of $P(\tau_m X \in S)$ to $Q(S)$ is uniform in S, so that, for suitable choices of m, the approximations

$$P(\tau_m X \in S_{m+n+j}) \doteq Q(S_{m+n+j}) \doteq P(\tau_{m+n+j} X \in S_{m+n+j}) = P(U)$$

become arbitrarily close. It therefore must be the case that $P(\tau_m X \in S_{m+n+j}) = P(U)$! With a straightforward treatment of some remaining details, we conclude that

$$\begin{aligned} P(RU) &= P(R)\, P(E_1)\, P(U) + \sum_{j=2}^{\infty} P(R)\, P(E_1^c \ldots E_{j-1}^c E_j)\, P(U) \\ &= P(R)\, P(U). \end{aligned}$$

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The same conclusion for an arbitrary event $R \in \mathscr{R}$ now follows from the well-known Dynkin class argument. |

We finish off this section with two noteworthy properties of the limit measure in Theorem 3·3.

3·10. THEOREM. *The probability measure Q of Theorem 3·3 is translation-invariant, i.e. we have $Q(\tau_n^{-1}(S)) = Q(S)$ for all $S \in \mathscr{B}$, and for all $n = 0, 1, 2, \ldots$.*

Proof. From the definitions, we have

$$Q(\tau_n^{-1}(S)) = \lim_{k\to\infty} P(\tau_k X \in \tau_n^{-1}(S)) = \lim_{k\to\infty} P(\tau_{n+k} X \in S) = Q(S). \quad |$$

3·11. COROLLARY. *The coordinate representation process $\pi = \{\pi_n\}_{n=1}^{\infty}$ is strictly stationary with respect to Q.* |

And of course π is uniformly asymptotically strictly stationary with respect to the distribution measure P_X.

3·12. *Definition.* The feature $X_S = \{A_n\}_{n=0}^{\infty}$ of the process X is said to be *persistent* if the A_n occur infinitely often with probability one, and it is said to be *transient* if the A_n occur finitely often with probability one. |

The following theorem identifies the transient features of X with the Q-null subsets of $\mathbb{R}^{\mathbb{N}}$, and it also establishes that the sets of Q-probability one determine persistent features of X. The fact that a set $S \in \mathscr{B}$ with $Q(S) < 1$ may also determine a persistent feature is illustrated by the sequence $E = \{E_{n+1}\}_{n=0}^{\infty}$, which is really just a distinguished feature of the process I_E.

3·13. THEOREM. *Under the assumptions of Theorem 3·3, let $S \in \mathscr{B}$, let $X_S = \{A_n\}_{n=0}^{\infty}$ be the feature of X which is determined by S, and let $U \in \mathscr{U}$ denote the event that infinitely many of the A_n occur. Then X_S is transient if and only if $Q(S) = 0$, whereas if $Q(S) = 1$, or if $Q(S) > 0$ and it happens that $U \in \mathscr{R}$, then X_S is persistent.*

Proof. Write $U = \cap_{n=1}^{\infty} \cup_{k=n}^{\infty} A_k = (X \in S^*)$, where $S^* = \cap_{n=1}^{\infty} \cup_{k=n}^{\infty} \tau_k^{-1}(S)$. Observe that, for $m = 0, 1, 2, \ldots$, we have

$$\tau_m^{-1}(S^*) = \bigcap_{n=1}^{\infty} \bigcup_{k=n}^{\infty} \tau_{m+k}^{-1}(S) = S^*.$$

It is now clear from Theorem 3·10 that $P(U) = Q(S^*) \geqslant Q(S)$. In view also of Theorem 3·9, the stated assertions follow easily. |

3·14. *Remark.* It is worthwhile to point out what these results are saying in the special case that $\mathscr{U} \subseteq \mathscr{R}$. (This obtains, for example, when $X = I_E$.) First, Theorem 3·9 asserts that the tail σ-algebra $\mathscr{U}$ is trivial. The usual felicitous consequences for X flow from this (for example, the a.s. convergence of $E(f|\mathscr{A}_n)$ to Ef for every P-integrable function f: $\Omega \to \mathbb{R}$). Second, Theorem 3·13 asserts that every feature of the process X is either transient or persistent. Moreover, Q itself is trivial on the invariant σ-subalgebra of $\mathscr{B}$ (since, in the notation of the proof of Theorem 3·13, the condition $\tau_n^{-1}(S) = S$ for all n implies that $Q(S) = Q(S^*) = P(X \in S) = P(U) \in \{0, 1\}$). It follows that the Q-measure-preserving transformation $T = \tau_1$ is *ergodic* in the sense of ((2), definition 6·14, p. 109). |

4. *The second limit theorem: asymptotic independence.* Theorem 3·9 asserts the independence of the regenerating events and the regenerated events which occur 'at

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the end of time'. This section is devoted to the question of asymptotic independence. Specifically, we shall investigate the asymptotic behaviour of regenerating (i.e. $\mathscr{R}$-measurable) functions when they are conditioned by σ-algebras of regenerated events which occur progressively later in time. Since we do not have a complete description of this behaviour, we shall apportion our second limit theorem into three separate results: Theorem 4·1, Theorem 4·6, and Proposition 4·8. These are to be considered as special cases of the yet-to-be-proved definitive theorem.

The first result collects together the information which can be easily obtained from Theorem 3·9, and it provides a convenient springboard for further investigation.

4·1. THEOREM. *Assume that the process X is regenerated by the ergodic phenomenon E. For $n = 0, 1, 2, \ldots$, let $\mathscr{G}_n$ be a σ-subalgebra of $\mathscr{A}_n$. Let $r \geqslant 1$, and let $f: \Omega \to \mathbb{R}$ be an $\mathscr{R}$-measurable function such that $E|f|^r < \infty$. Then we have*

$$E(f|\mathscr{G}_n) \to Ef \quad \text{in } L_r, \tag{4·1·1}$$

and, if the G_n are decreasing, we also have

$$E(f|\mathscr{G}_n) \to Ef \quad \text{a.s.} \tag{4·1·2}$$

Note. It is in fact sufficient for f to be measurable with respect to any σ-algebra $\mathscr{V}$ which is independent of $\mathscr{U}$.

Proof. Both the L_r convergence (convergence in the rth mean) and the a.s. convergence of the sequence $E(f|\mathscr{A}_n)\}_{n=0}^{\infty}$ to the limit $E(f|\mathscr{U})$ follow from the submartingales closure theorem ((17), theorem B, p. 394). By Theorem 3·9 and ((17), § 25·2·2, p. 349), we have $E(f|\mathscr{U}) = Ef$ a.s. Because the limit is thus trivial, condition (4·1·1) follows from the smoothing ((17), § 25·2·4, p. 350) and the contraction ((17), § 25·1·2, p. 348) properties of conditional expectation.

When the $\mathscr{G}_n$ are decreasing, the sequence $\{E(f|\mathscr{G}_n)\}_{n=0}^{\infty}$ is itself a reversed martingale, and so condition (4·1·2) follows. ∎

In particular, we have $E(f|X_n) \to Ef$ in L_r, but it is not yet clear in this case when almost sure convergence obtains.

4·2. *Remark.* The literature contains various notions of *mixing* convergence (for example, see ((1), pp. 328–329) and ((20), definition 1·5, p. 40)). The proof of Theorem 4·1 reveals, in a very simple way, the following general fact about mixing convergence: If the process X is quite arbitrary, if $\mathscr{V}$ is any σ-subalgebra of $\mathscr{F}$ which is independent of the tail σ-algebra $\mathscr{U}$, if $f: \Omega \to \mathbb{R}$ is any $\mathscr{V}$-measurable, P-integrable function, and if it happens, for a particular set $S \in \mathscr{B}$, that the limit $Q(S) = \lim_{n\to\infty} P(\tau_n X \in S)$ exists, then we have

$$\int_{(\tau_n X \in S)} f\,dP \to Q(S) \int f\,dP. \tag{4·2·1}$$

(Simply take $\mathscr{G}_n$ to be the σ-algebra generated by the single set $(\tau_n X \in S)$.) When X is ergodically regenerated, and when $\mathscr{V} = \mathscr{R}$, statement (4·2·1) may be established (without martingale theory) by a straightforward Dynkin class argument. Two noteworthy special cases of (4·2·1) in the present setting are the following:

(4·2·2). For every event $R \in \mathscr{R}$, and for every set $H \in \mathscr{B}_{\mathbf{R}}$, we have

$$P(R, X_n \in H) \to P(R)\, Q_1(H).$$

(See (3·6).)

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(4·2·3) For every event $F \in \mathscr{F}$, we have

$$P(F|E_n) \to P(F).$$

Statement (4·2·3) follows because the tail σ-algebra of the process I_E is trivial (see Remark 3·14), and of course it remains valid with an arbitrary feature of I_E in place of the sequence E. ∎

It remains to ascertain precisely when X is asymptotically independent of $\mathscr{R}$ in the sense of a.s. convergence (that is, in the sense of condition (4·1·2)). Our next theorem (Theorem 4·6) presents a characterization of almost sure asymptotic independence in the presence of a slight restriction upon the ergodic phenomenon E.

4·3. *Definition.* The regenerative phenomenon E is said to be *strongly ergodic* if the random variable W has finite variance. ∎

The proof of Theorem 4·6 hinges upon an upgrading of the convergence in Theorem 3·3 which is occasioned by strong ergodicity. The following lemmas are addressed to this upgrading.

4·4. LEMMA. *Assume, in Proposition* 3·1, *that the series* $\Sigma|u_n - u|$ *and* Σr_n *are both convergent. For* $n = 0, 1, 2, \ldots$, *define* $c_n = \Sigma_{k=0}^{n} x_k |u_{n-k} - u| + u r_n$. *Then the series* Σc_n *is convergent, and we have* $|v_n - bu| \leqslant c_n$ *for all* n.

Proof. The sequence $\{c_n\}_{n=0}^{\infty}$ is a combination, by convolution and addition, of three sequences for which the corresponding series are convergent. Hence Σc_n is convergent. The inequality $|v_n - bu| \leqslant c_n$ is clear from the definitions. ∎

4·5. LEMMA. *Let* X *be regenerated by the strongly ergodic phenomenon* E. *Then, in the notation of Theorem* 3·3 *and Remark* 3·4, *we have* $\Sigma_{n=1}^{\infty} \|P_n - Q\| < \infty$.

Proof. When $\operatorname{Var} W < \infty$, it follows from ((13), theorem 4, pp. 237–238) that the series $\Sigma|u_n - \mu^{-1}|$ is convergent, and it follows from ((8), problem 19, p. 340) that the series Σr_n is convergent. (Recall from Example 3·2 that $u_n = P(E_n)$, that $x_n = P(W > n)$, and that $r_n = x_{n+1} + x_{n+2} + \ldots$.) Thus the assumptions of Lemma 4·4 are satisfied (where $u = \mu^{-1}$). It is clear from the proof of Theorem 3·3 that, for every $n = 0, 1, 2, \ldots$, and for every set $S \in \mathscr{B}$, the number $|P_n(S) - Q(S)|$ is of the form $|v_n - bu|$. Therefore Lemma 4·4 yields the inequality $\|P_n - Q\| \leqslant 2c_n$, and the desired convergence follows. ∎

4·6. THEOREM. *Assume that the process* X *is regenerated by the strongly ergodic phenomenon* E. *For* $n = 0, 1, 2, \ldots$, *let* $\mathscr{G}_n$ *be a* σ*-subalgebra of* $\mathscr{A}_n$. *Let* $f: \Omega \to \mathbb{R}$ *be an* $\mathscr{R}$*-measurable function such that* $E|f| < \infty$, *and define* $\mathscr{R}_0 = \cup_{n=1}^{\infty} \mathscr{R}_n$. *Then we have*

$$E(f|\mathscr{G}_n) \to Ef \quad a.s.$$

if and only if:

(4·6·1). *For every number* $\epsilon > 0$, *there exists an integer* n_0 *and a simple* $\mathscr{R}_0$*-measurable function* g *such that* $|Ef - Eg| < \epsilon$, *and such that* $P(\sup_{n \geqslant n_0} |E(f - g|\mathscr{G}_n)| > \epsilon) < \epsilon$.

Moreover, if condition (4·6·1) *is satisfied for a particular simple* $\mathscr{R}_0$*-measurable function* g_0, *then it is satisfied for every simple* $\mathscr{R}_0$*-measurable function* g *such that* $|Ef - Eg| < \epsilon$.

Note. Since $\mathscr{R}$ is the σ-algebra generated by $\mathscr{R}_0$, there is always at least one simple $\mathscr{R}_0$-measurable function g such that $|Ef - Eg| < \epsilon$.

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Proof. Let $R = E_{n_1} \dots E_{n_k}$, where $1 \leqslant n_1 < n_2 < \dots < n_k$, and define $f_n = E(I_R|\mathscr{G}_n)$ for $n = 1, 2, \dots$. We shall establish, to begin with, that $\Sigma_{n=1}^{\infty} E|f_n - P(R)| < \infty$. To this end, let $n > n_k$ be arbitrary, define $B = (f_n > P(R)) \in \mathscr{G}_n$, and select any set $S \in \mathscr{B}$ such that $B = (\tau_n X \in S)$. From the definition of conditional expectation, and from Proposition 2·2, we obtain

$$\left|\int_B (f_n - P(R))\, dP\right| = \left|\int_B (I_R - P(R))\, dP\right| = |P(RB) - P(R)\,P(B)|$$
$$= P(R)\,|P_{n-n_k}(S) - P_n(S)|.$$

Likewise, we have

$$\left|\int_{B^c} (f_n - P(R))\, dP\right| = P(R)\,|P_{n-n_k}(S^c) - P_n(S^c)|,$$

so that

$$E|f_n - P(R)| \leqslant P(R)\,\|P_{n-n_k} - P_n\| \leqslant P(R)\,(\|P_{n-n_k} - Q\| + \|P_n - Q\|).$$

The desired convergence now follows from Lemma 4·5.

We shall establish next that $E(f|\mathscr{G}_n) \to Ef$ a.s. whenever f is a simple $\mathscr{R}_0$-measurable function. A trivial Dynkin class argument (applied to each $\mathscr{R}_n$ separately) shows that the convergence of the last paragraph holds for every event $R \in \mathscr{R}_0$. From this, and from the linearity of conditional expectation, we obtain

$$(4\cdot6\cdot2) \quad \sum_{n=1}^{\infty} E|E(f|\mathscr{G}_n) - Ef| < \infty \text{ for every simple } \mathscr{R}_0\text{-measurable function } f.$$

The fact that $E(f|\mathscr{G}_n) \to Ef$ a.s. now follows as usual from the Markov inequality and the Borel–Cantelli lemma.

Suppose next that the given function f satisfies (4·6·1). Let $\epsilon > 0$ be arbitrary, and let a simple $\mathscr{R}_0$-measurable function g be chosen to satisfy (4·6·1) with $\epsilon/3$ in place of ϵ. Since $E(g|\mathscr{G}_n) \to Eg$ almost uniformly, there is an index n_1 such that

$$|E(g|\mathscr{G}_n)(\omega) - Eg| < \epsilon/3$$

for all $n \geqslant n_1$, except for some points ω in a set of P-measure $< \epsilon/2$. In view of (4·6·1), we may exploit the triangle inequality to obtain

$$|E(f|\mathscr{G}_n)(\omega) - Ef| < \epsilon$$

for all $n \geqslant \max(n_0, n_1)$, except for ω in a set of P-measure $< \epsilon$. This is sufficient to guarantee that $E(f|\mathscr{G}_n) \to Ef$ almost uniformly, and hence almost surely.

To complete the proof of the theorem, it remains to assume that $E(f|\mathscr{G}_n) \to Ef$ a.s., to let g be any simple $\mathscr{R}_0$-measurable function such that $|Ef - Eg| < \epsilon$, and to obtain an integer n_0 such that $P(\sup_{n \geqslant n_0} |E(f-g|\mathscr{G}_n)| > \epsilon) < \epsilon$. To this end, select any number $\delta > 0$ such that $|Ef - Eg| < \epsilon - 2\delta$. Then choose n_0 so that, for all $n \geqslant n_0$, we have simultaneously $|E(f|\mathscr{G}_n)(\omega) - Ef| < \delta$ and $|E(g|\mathscr{G}_n)(\omega) - Eg| < \delta$, except for ω in a set of P-measure $< \epsilon$. That n_0 is as desired follows once again from the triangle inequality. ∎

4·7. Corollary. *Assume, in Theorem* 4·6, *that we have* $\sup_n |E(f|\mathscr{G}_n)| < \infty$ *a.s. for every choice of an* $\mathscr{R}$*-measurable function* f *such that* $E|f| < \infty$. *Then* $E(f|\mathscr{G}_n) \to Ef$ *a.s. for all such functions* f.

Proof. This follows immediately from Theorem 4·6 together with a classical theorem of Banach. The space $L_1(\Omega, \mathscr{R}, P|_{\mathscr{R}})$ will play the role of the space $\mathscr{X}$ in the exposition of ((7), theorem 2, p. 332), and the conditional expectation operator $E(\cdot|\mathscr{G}_n)$ (restricted to $\mathscr{X}$) will play the role of the linear map T_n. Theorem 4·6 ensures that the assumptions

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of Banach's theorem are satisfied, and the theorem then asserts the a.s. convergence, for all $f \in \mathscr{X}$, of the sequence $\{E(f|\mathscr{G}_n)\}_{n=1}^{\infty}$. That the limit reduces to the constant Ef is clear from Theorem 4·1. ∎

Owing to the technical nature of condition (4·6·1), it is important to clarify when it can be satisfied easily. It is obviously sufficient for f to be the (a.s.) *uniform* limit of a sequence of simple $\mathscr{R}_0$-measurable functions. We may think of such a function f as 'essentially occurring within a finite period of time'. More precisely, f is determined to within an arbitrary positive constant by the regenerating events which occur prior to a certain fixed time. The fixed time possibly depends upon the constant.

When the $\mathscr{G}_n$ are decreasing, condition (4·6·1) is readily deduced from the (elementary) maximal inequality for martingales ((17), theorem A(ii), p. 391). In the light of this, of condition (4·6·2), and of the inequality $|a|^r \leqslant |a|$ (for $|a| \leqslant 1$ and $r \geqslant 1$), it is not difficult to see that Theorem 4·6 actually implies Theorem 4·1, *provided that E is assumed to be strongly ergodic*. Thus, in this case, Theorem 4·1 is obtainable without resort to the heavier tools of martingale theory.

A substantial extension of the almost sure asymptotic independence beyond 'regenerating functions which essentially occur within a finite period of time' is problematical because the regenerative influence of such a function only takes effect *after* it has 'occurred', and because the almost sure convergence appears to arise precisely from this regenerative influence. At the same time it is already clear from Theorem 4·1, and from the little result to follow, that Theorem 4·6 does not disclose the final word about almost sure asymptotic independence.

4·8. PROPOSITION. *Assume that X is regenerated by the ergodic phenomenon E. Let* $f: \Omega \to \mathbb{R}$ *be an* $\mathscr{R}$*-measurable function such that* $E|f| < \infty$, *and let* $\mathscr{B}_0$ *be a* finite σ*-subalgebra of* $\mathscr{B}$. *For* $n = 0, 1, 2, \ldots$, *define* $f_n = E(f|X^{-1}(\tau_n^{-1}(\mathscr{B}_0)))$. *Then we have* $f_n \to Ef$ *a.s.*

Proof. Let $\{S_1, \ldots, S_m\}$ be the finite partition of $\mathbb{R}^{\mathbb{N}}$ which generates $\mathscr{B}_0$. Then

$$f_n = \sum_{i=1}^{m} \left[P(\tau_n X \in S_i)^{-1} \int_{(\tau_n X \in S_i)} f\, dP \right] I_{(\tau_n X \in S_i)}$$

(a.s.), so that, by Theorem 3·3 and Remark 4·2, we have $f_n \to Ef$ *uniformly* if $Q(S_i) > 0$ for all i. But by Theorem 3·13 the time frame $S = \cup \{S_i : Q(S_i) = 0\}$ determines a transient feature of X, and so, in any event, $f_n \to Ef$ a.s. ∎

4·9. COROLLARY. *Assume, in Proposition* 4·8, *that the state space of X is finite. Then we have* $E(f|X_n) \to Ef$ *a.s.* ∎

We do not know whether this corollary extends even to a discrete process X.

4·10. *Remark.* In view of ((3), theorem 2, p. 144; theorem 5, p. 147) (see also ((4), theorem 4, p. 28), it is easy to check that the results of this section remain valid when the $\mathscr{R}$-measurable function f assumes its values in an arbitrary Banach space. ∎

We finish off this section with a second look at condition (2·6) as a possible alternative to Definition 2·4. Since the phenomenon E regenerates itself as well as regenerating X, one might wish to regard the σ-algebra generated by $\mathscr{A}$ and $\mathscr{R}$, call it $\mathscr{G}$, as the 'σ-algebra of regenerated events'. The results of this paper can be applied to $\mathscr{G}$ in place of $\mathscr{A}$ by means of the following artificial device: Define a process $Y = \{Y_n\}_{n=1}^{\infty}$ by

$$Y_n = \exp(X_n) - \exp(I_{E_n})$$

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for all n. Then $\mathscr{G}$ constitutes the σ-algebra generated by Y, and condition (2·6) ensures that the phenomenon E will regenerate Y in the sense of Definition 2·4.

5. *Open problems.* There is a wide range of questions about regenerated processes which remain to be answered. In this section we collect together a dozen of the more obvious of these.

5·1. *Problem.* Develop a parallel theory for a continuous-time process $X = \{X(t)\}_{t \geqslant 0}$ which is regenerated by a continuous regenerative phenomenon $E = \{E(t)\}_{t \geqslant 0}$. ‖

5·2. *Question.* We have treated the X_n more as measurable transformations than as random variables. The major results of this paper all remain valid when the range space $(\mathbb{R}, \mathscr{B}_{\mathbb{R}})$ of X_n is replaced by an arbitrary measurable space $(\mathbb{M}, \mathscr{M})$, provided that $\mathscr{B}$ is defined to be the product σ-algebra of $\mathscr{M}$ on each copy of $\mathbb{M}$. Does Theorem 1·3 generalize to this setting? ‖

5·3. *Problem.* Refine and elaborate the method of Example 3·2. In particular, when $\sigma^2 = \operatorname{Var} W < \infty$, an application of the Markov inequality will yield a crude estimate, in terms of μ and σ^2, of the index k in step (3·2·1). Since it does not seem that the index n_0 in step (3·2·2) will greatly exceed k, we ask: Can n_0 be effectively estimated, without recursive calculations, from ϵ and from (some of) the moments of W? ‖

5·4. *Problem.* Describe the asymptotic behaviour of a process X which is regenerated by a persistent, aperiodic, null-recurrent phenomenon E. In particular, do we have $P(X_n \in H) \to 0$ for every bounded Borel subset H of $\mathbb{R}$? (An affirmative answer to this question is suggested in the last line of ((8), theorem 2, p. 319), but no proof is indicated. See also Example 3·8.) In view of ((18), lemma 2 and theorem 1, p. 572), we also ask: Do the ratios $P(\tau_{n+1} X \in S)/P(\tau_n X \in S)$ converge for very many or for all sets $S \in \mathscr{B}$? ‖

5·5. *Question.* Does Theorem 4·6 remain true when E is ergodic but not strongly ergodic? ‖

5·6. *Question.* For which infinite σ-algebras $\mathscr{B}_0 \subseteq \mathscr{B}$ does Proposition 4·8 remain valid? ‖

5·7. *Question.* In Theorem 3·3, the Dynkin class

$$\mathscr{D} = \{S \in \mathscr{B} \colon P_X(S) = Q(S)\}$$

contains the σ-algebra $\mathscr{I}_X = \{S \in \mathscr{B} \colon (\tau_n X \in S) = (X \in S) \text{ for all } n\}$ (see ((2), definition 6·26, p. 118)), and equals $\mathscr{B}$ precisely when X is strictly stationary to begin with. Does $\mathscr{D}$ provide any useful information about X? When is $\mathscr{D}$ a σ-algebra? When is $\mathscr{D} = \mathscr{I}_X$? ‖

5·8. *Problem.* Explore the possible relationships between strict stationarity and ergodic regeneration. Is there a natural analogue for ergodically regenerated processes of the fundamental theorem of strictly stationary processes ((2), Theorem 6.28, p. 118)? When is an asymptotically strictly stationary process ergodically regenerated? ‖

5·9. *Question.* When is the tail σ-algebra $\mathscr{U}$ of an ergodically regenerated process X trivial? (See Remark 3·14.) When is the invariant σ-algebra $X^{-1}(\mathscr{I}_X)$ trivial? (See Question 5·7.) In any event, is an arbitrary feature of X necessarily either persistent or transient? ‖

5·10. *Question.* If X is ergodically regenerated, does there exist a probability measure Q^* on $(\Omega, \mathscr{A})$ (or even on $(\Omega, \mathscr{F})$) with respect to which X is strictly stationary? (See Theorem 3·10 and Corollary 3·11.) ‖

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An affirmative answer to the next question would provide at least a finitely additive probability Q^* in answer to Question 5·10.

5·11. *Question.* Does the σ-algebra homomorphism X^{-1} possess a *translation-invariant* right inverse? (See Theorem 1·3.) We are referring here to an algebra homomorphism $\rho: \mathscr{A} \to \mathscr{B}$ with the following properties:

(5·11·1) $X^{-1}(\rho(A)) = A$ for all $A \in \mathscr{A}$;

(5·11·2) $\rho(X^{-1}(\tau_n^{-1}(S))) = \tau_n^{-1}(\rho(X^{-1}(S)))$ [i.e. $\rho(\tau_n X \in S) = (\tau_n \in \rho(X \in S))$] for all $S \in \mathscr{B}$, and for all $n = 1, 2, \ldots$.

(Of course it suffices to obtain (5·11·2) only for $n = 1$.)

5·12. *Question.* Does the measure Q of Theorem 3·3 admit a *translation-invariant lifting*? (See Theorem 3·10.) By this we mean a lifting ρ for Q such that

$$\rho(\tau_n^{-1}(S)) = \tau_n^{-1}(\rho(S))$$

for all $S \in \mathscr{B}$, and for all $n = 1, 2, \ldots$. (For a general background on liftings, see (10), and for the derivation of other types of translation-invariant liftings, see (9) and (12).)

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